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When to Attack an Oppressive Government?

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Abstract: Initiating a conflict is an investment in social, political or economic change. The decision to attack is sequential in time, irreversible and, more important, includes highly uncertain and erratic threats and opportunities yet completely disregarded in conflict theory. In this dynamic model of decision making we focus on the time dimension of an escalating conflict. In order to cover the effects of high uncertainties we extend methods in real option theory by introducing a discontinuous Ito-Lévy Jump Diffusion processes. We analytically derive a threshold that triggers the attack and determine the expected time of action. With this new discontinuous processs we are able to show that an increasing number and intensity of oppressive government actions may lead to an earlier outbreak of conflict. However, even if latent conflicts are not immediately solved policies can prolong the peace period to find a long term solution to the conflict.

JEL classifications: D74, D81, C61
Keywords: non-systematic risk of social conflicts,
uncertain investment in conflict, theory of conflict,
decision to attack

1 Introduction

Group discrimination, repressive government actions, and deteriorating living conditions cause frustration, feed latent conflicts and may eventually provoke an insurgence between discriminated groups and an oppressive government.[1] Launching a violent conflict becomes an instrument to improve conditions for the rebel or his social group. Since for a rebel attacking the oppressive government is a strategic action, the decision whether and when to attack is a major decision under high uncertainty[2]. How do more frequent or more severe major threats like waves of political persecution, or arbitrary detention and torture of group members affect the decision and timing to attack; and how does the uncertainty in the aftermath of an attack affect this decision?

Since Dahrendorf (1958) conflict theory has been regarded as an important part of social science. While there are a large number of sometimes conflicting hypotheses and empirical studies, consistent closed formal theories remain rather limited, on both the macro and the micro level.[3]

At the micro level, which is the relevant level of this paper, it is taken for granted that conflicts have economic roots. In game theory, conflicts are the result of strategic interactions between conflict partners. In other words, conflicts are the outcomes of games where two competing agents, usually a rebel and the government, are not able to negotiate. In this context there may be several reasons for the conflict. For instance, Skaperdas (1992) identifies a lack of property rights as the leading source, since no single agent can be prevented from coercing another agent. Furthermore, incomplete and insufficient information can lead to conflict. Following Fearon (1995) leaders of two competing groups are not able to bargain because of a lack of information on the other party's relative military power. In particular, countries tend to misrepresent private information in order to gain a better deal. Even if there is no deliberate misrepresentation, Powell (2002, 2006) concludes that conflicts are possibly

[1]For instance, Piazza (2011) examines that countries featuring minority group economic discrimination are more prone to suffer from social conflicts, especially domestic terrorism.

[2]Bloom (2009) show that uncertainty shocks, e.g. major events like the assissination of John F. Kennedy, the OPEC oil price shock and the 9/11 terrorist attacks have an impact on economic decisions in that they promote a "wait and see attitude". This, in fact, is the behaviour which can be captured by the real option approach described in this paper.

[3]See the survey by Blattman and Miguel (2010).

due to the agent's inability to estimate the opponent's ability to win, or due to commitment problems. The static game theory approach to explaining social conflicts is extended by Yared (2010) who suggests repeated games with asymmetric information and limited commitment. Although these scenarios seem suitable for explaining certain kinds of social conflict, they only shed light on explicit bilateral interaction with mutual strategic behavior of clearly defined and strictly controlled conflict parties, such as a well-organized, homogeneous group of rebels playing a game with the government.

However, many terrorist attacks are not the result of centralized decisions following a grand strategy, but are individual or small group reactions to exogenous developments with the hope of an effect. In addition, for a two-party game to happen there have to be factors that drive individuals into a conflict or a game. There must be an individual willingness to join an attacking group or an individual decision to attack.

Therefore, as far as individual decisions are driven by economic considerations, decisions about conflicts are based on the development of future costs and benefits. As an early contribution Morrow (1985) claimed that war decisions are based on the actors' utility of uncertain outcomes, and Collier and Höffler (1998) state that civil wars occur if the perceived benefits outweigh the costs of rebellion, namely the opportunity costs of labor and coordination costs. In the same context, Grossman (1991) considers insurrection and its deterrence or suppression as economic activities that compete with production for scarce resources. Hence, rulers and rebels are income maximizing agents that search for the best possible outcome, e.g. a conflict. He finds that the probability of a successful insurrection depends on the fraction of time that rebels devote to insurrection and soldiering. Using a dynamic model Tornell (1998) concludes that many economic reforms have taken place in the context of economic crisis and drastic political change. That is, agents have expectations about the economic situation after a turning point and decide whether to drive the economy into a crisis in order to make reforms possible. Blomberg et al. (2004), who combine the static model of Grossman (1991) and the dynamic model of Tornell (1998), suggest that terrorist activities or rebellion are rational if there is no other way to bring about drastic institutional changes. This decision is fundamentally driven by the state of a country's economy and the costs of conflict. That is, unsatisfying economic, social and political developments in a country trigger a conflict in order to change the status quo. All the mentioned approaches assume that the initiator of a conflict believes that the conflict is more beneficial than peace under present conditions. However, this belief involves extremely high non-marginal uncertainties. Many events connected to violent conflicts are no marginal phenomena. They are sudden significant incidents with a strong impact on the motivation of violence, or they are highly uncertain outcomes in the aftermath with substantial implications. For instance, rebels launching an attack hope to push for a major policy change and reform steps in favour of their group. Evidence that major uncertain events may have an effect on the decision to launch a conflict is discussed recently. For the special case of terrorism Berrebi and Ostwald (2011) show that conflicts may be a consequence of natural disasters like earthquakes and hurricanes. Such major events can increase vulnerability of a country which terrorist groups might exploit. In the aftermath of a conflict, however, rebels may become heroes, or they are just as likely to be killed or their families exposed to even more severe repression. However, taking risk is a major element of the decision and hence high uncertainty must be considered in a theory of launching a conflict. Morrow (1985) started to include uncertainty with a static cost-benefit view of social conflicts, and Reynal-Querol (2002) varies the model by Collier and Höffler (1998) by adding elements of uncertainty such that a group will rebel if the expected net utility of rebellion exceeds the utility of the status quo. However, the role of uncertainty in formal conflict theory is rather rudimentary, even if nothing is more likely to trigger a conflict than an unforeseeable major event, and nothing is more uncertain than the outcome of a conflict once it is triggered. Specifically, neither the dynamics of future uncertain processes nor large and non-systematic uncertainties generated by major fundamental events in future developments have been considered so far. In line with Blomberg et al. (2004) it is easy to think of illustrative examples. For instance, a social group feels discriminated by an oppressive government. A sudden wave of arrests of political, cultural or religious leaders of this group will affect the assessment of future prospects just as much as economically or culturally and discriminatory. By contrast, a new head of government, or government reforms, lifting oppressive laws, economic, political, cultural or religious liberalization, or even external pressure in the shape of international sanctions can have a major positive impact on the assessment of future developments. Before the outbreak of a conflict such events are indicators with substantial implications for the expectations of the group in question. Similarly, the aftermath of a conflict is also highly uncertain.

Reforms can be major or minor, political actions can be well executed or a complete failure. There can be a counter attack or a counterrevolution, a military coup, another rival group appearing, or even an attack from a foreign power.

Because uncertainties connected to violent conflicts are large and manifold, the purpose of this paper is to analyze how these large and fundamentally uncertain events affect the decision to launch the conflict. We focus on how large sudden uncertain shocks like massive threats or major opportunities before and after a conflict can affect the decision to attack. Furthermore, in a formal dynamic model of sequential decision making we can determine not only what drives the conflict, but also when we can expect the outbreak to be triggered. Before we can do that we fundamentally extend the real option approach, where the rebel is able to evaluate the attack run at different points in time and therefore to find the optimal timing structure of attack. Hence, the optimal time of attack after considering all conditions and effects of uncertainties like major uncertain events determining the decision can be found. Explicitly modeling the timing of the outbreak of conflict allows us to discuss important political implications. If there is a period during which a latent conflict matures and moves towards a triggering threshold, this period can be used for de-escalation or changing the underlying conditions. Even if the conflict can break out randomly at any moment, there is an expected time period that can be used for eliminating the sources of conflict.

In the model the conflict has two phases. First, we have a latent conflict with escalating tension, but the conflict is still non-violent. Later, the conflict may turn into violent actions and the attack marks the beginning of the second phase. This second phase describes the aftermath of the violent actions. Hence, in this model a violent attack is the result of a dynamic of deteriorating prospects for the oppressed group. Even if the escalation, driven by non-marginal shocks, appears to stay non-violent for a while, eventually a random event will trigger an outbreak. Hence, we determine for a latent but not yet violent conflict if and when a violent outbreak can be expected.

2 Model

2.1 Model Idea

Many attacks are individual violent actions by agents. In order to understand why violent situations emerge, we first have to examine what leads individuals to attack other persons or to be violent. Hence, irrespective of group dynamics or psychological, ethnic or sociological reasons we focus on the idea that starting a conflict can be regarded as an investment in a better future.[4] An attack is not something that unexpectedly enters the mind of a decision making individual. Rather, it is the result of a dynamic process in which the current path of development of the economic and social situation is evaluated and compared to the expected path of development after a potential attack, including conflict costs and all potential threats and opportunities. If current conditions are discriminating and dissatisfying, a latent conflict exists and an attack may be considered. One of the potential outcomes of such a situation is hence an attack.

However, even if an immediate attack may have some benefits, it is possible that a non-violent strategy comprising a potential later attack is the better option. Rebels act rationally; they decide whether to invest immediately and pay the costs by launching the conflict (attack), or to maintain the status quo, at least for a while. Since conditions may change even without an attack, sometimes simply waiting may be beneficial. For instance, a rebel who plans to assassinate an oppressive politician will only carry out the attack if he expects that his actions will have an overall positive effect. If the assassination is successful, political reforms could lead to more political participation and freedom but also to economic and welfare improvements. However, the hope for a non-violent change may postpone the attack. As time goes on, rebels repeatedly consider their living conditions, which in bad times become worse, and they repeatedly decide whether to arrest this process of deterioration by violent means. As each moment's conditions determine this decision, it is a sequential decision in time. The decision is also irreversible and once the attack is carried out, there is no return. All consequences have to be accepted, and the freedom and flexibility

[4] Besides the economic analysis of social conflicts there is an extensive discussion on psychological and sociological reasons for e.g. terrorist attacks. For instance, Victoroff (2005) summarizes relevant psychological theories of terrorism and conclude that previous approaches are very limited in explaining the causes. Since they do not apply scientific methodology to derive hypotheses, conclusions are not able to be tested.

to choose more moderate strategies to solve the conflict are no longer present.

With sequential decisions, high uncertainty and irreversibility as major components of the decision problem, real option theory is an appropriate methodology. More specifically, the decision to attack or remain peaceful is particularly difficult because both the current path of development of economic, social and political conditions and the expected path after an attack are highly uncertain. In order to capture large uncertainties like major threats (more repressive actions) and opportunities (like successful major reforms) we extend the standard model of marginal risk to include non-systematic risks. We describe the two risk components by discontinuous stochastic processes, namely Ito-Lévy Jump Diffusion processes.[5] Hence, this paper is the first to be able to formally model the effects of such major uncertain events on conflict decision. High uncertainty is also a strong factor when evaluating irreversibility and flexibility in the decision process. Not being tied to a violent strategy may be a major advantage if high uncertainty emerges and sudden major events may change current conditions significantly in some way. Hence, in this model the rebel maximizes his present discounted net value of the benefit of an attack (including the value of flexibility) by deriving a conflict threshold that determines the benefit level required to trigger the violent outbreak. Randomly reaching this threshold represents the straw that breaks the camel's back. The decision is determined by the sequential comparison of the net present value of the benefits of a potential attack with the value of postponing an attack and possibly attack later. Knowing the triggering threshold, we can also determine the 'first passage time', that is, the expected time of attack. However, even if our sequential process identifies an expected time of the outbreak of the conflict, the model is also able to suggest that a sudden random change in conditions may also lead to an unexpected attack at any moment.

[5] The importance of Jump Diffusions was first recognized in financial economics. For instance, Merton (1976) derived an option value of an European option similar to the Black Scholes formula. In the course of time some extensions of the Merton approach followed. Pure jump or Levy processes were analyzed by e.g. Geman (2002) and Elliot (2006). Options values for American options with more general Jump diffusions were derived by e.g. Pham (1997), Gukhal (2001), Mordecki (2007) and Bayraktar (2009). However, in line with this formal modelling we consider Jump diffusion processes in the context of real options.

2.2 Benefits of Conflict

As the attack is expected to change conditions, there are potentially two periods in conflict evaluation with two sets of conditions: first, the current period with a *set of current conditions* associated with a path of non-violent, but dissatisfying development; and second, a new *set of conditions* in the period *after the attack* that is expected to generate a better path of development. Both elements determine the evaluation of total benefits of the conflict and eventually, the decision to launch it.

Current Conditions and Path of Development: In this model current conditions lead to a time path that is not satisfying for a certain social group. A harsh set of conditions for this group provokes resistance and a start of a latent conflict between an oppressive government and rebels. Increasing repression, worsening economic restrictions or discrimination, growing inequality of chances and opportunities and an increasing threat of persecution may lead to greater frustration in the face of deteriorating opportunities, and will eventually increase his propensity to turn violent. Further, with each additional moment of waiting and not attacking, the worsening welfare of the rebels may generate an increasing current benefit of conflict. In other words, as the attack is a potential action, deteriorating living conditions increase the benefits of an attack. Expectation of a deteriorating current time path of welfare produces a sufficiently bleak outlook as to make an attack increasingly beneficial.

However, even if another period of not attacking can be expected to generate a marginal increase in the current benefit of attack by a rate of δ, in this model we would like to focus on uncertainty as an essential component of the decision. We distinguish between systematic and non-systematic risk. On the one hand there are marginal fluctuations, usually referred to as systematic risk, due to small variations in the economic and political situation. More importantly, however, economic disasters or sudden positive turns may lead to a dramatic and non-marginal change from one moment to the next. In particular, major disastrous events have negative effects on rebel's welfare and prospects, so that the current benefit of conflict may suddenly increase significantly. By contrast, positive political or economic turns may abruptly improve the situation leading to great opportunities for the rebel or his group such that an attack becomes less beneficial. In this case, the current benefit of conflict diminishes dramatically. Large upward or downward jumps in current benefits of an attack are regarded as a non-systematic risk because they represent both fundamental threats or great opportunities. In order to consider these major events we describe the development of current benefits of a potential attack during the period before the conflict is launched as an Ito-Lévy Jump Diffusion process.[6]

Definition 1 *Let U_1 be a Borel set whose closure does not contain 0. Further let W_1 be a standard Wiener process, N_1 a Poisson process with intensity λ_1 and constants $\delta, \sigma_1 \in \mathbb{R}_+$. Then the Ito-Lévy Jump Diffusion process $\tilde{Y}$, indicating the benefits of a potential attack during the period before the conflict, is defined by means of the following differential equation*

$$d\tilde{Y} = \delta\tilde{Y}dt + \sigma_1\tilde{Y}dW_1 + \tilde{Y}\int_{U_1} uN_1(t, du) \quad \text{for} \quad 0 < t < T. \tag{1}$$

Note that σ_1 denotes constant marginal volatility (systematic risk) and the constant $\delta > 0$ is the expected marginal and non-random differential in the net benefit level with respect to marginal waiting time and T denotes the start of conflict. More importantly, in this model we would like to focus on large uncertainties in the time path of development. Hence, the stochastic process includes a continuous and a discontinuous part through a combination of a geometric Brownian motion and a compound Poisson process. In addition to the normal systematic risk we need to describe exceptional stochastic events and conditions which affect the benefits of conflict. Hence, uncertain fundamental non-systematic events are modeled in the jump part (compound Poisson process) of the stochastic process through the integral $\int_{U_1} uN_1(t, dz)$. The integrand u denotes the step height of jumps which is uncertain but limited by U_1. This modeling enables an accumulation of non-marginal jumps which occur at random points in time.[7]

Conflict Benefits in the Aftermath: Since conflict is assumed to pay off somehow, an attack generates a new set of more positive conditions afterwards. The increasingly dissatisfying current situation of a rebel increases the benefits of conflict so that once the attack has been carried out, the rebel's living conditions are expected

[6] The Ito-Lévy Jump Diffusion process is a special case of geometric Lévy processes. For further information about Lévy processes see e.g. Oksendal and Sulem (2007) or Applebaum (2009).

[7] For a graphical illustration of Jump Diffusions see e.g. Cont an Tankov (2004) pp. 71.

to improve so that the resulting benefits of conflict can be realized in the aftermath. Turning back to our example, successfully assassinating an oppressive leader may lead to political and economic reforms that, in a next step, improve the welfare of the rebel. Hence, even if high uncertainty is involved, carrying out an attack is expected to lead to a satisfactory improvement in the rebel's social environment.

However, the path of future benefits of conflict is highly uncertain. A new process of uncertain developments starts and although living conditions may be expected to improve on average, unforeseen events for the better or worse may take place and must be considered when evaluating the benefits of an attack. While the abolition of repressive laws, effective economic and political reforms, or international investment booms may open up the expected major opportunities, the uncertain aftermath of a conflict can also be full of threatening incidents. The government may become even more oppressive, the rebel may be caught or even tortured, and reforms, counterattacks or military coups may fail. This are all examples for such major threats. As a result, again the future benefits not only involve random marginal changes due to usual variations in the economy; non-systematic large random jumps may also occur for the better or worse. Since developments of future benefits in the aftermath of an attack incorporate fundamental threats and opportunities, we model them as an Ito-Lévy Jump Diffusion process.

Definition 2 *Let U_2 be a Borel set whose closure does not contain 0. Further let W_2 be a standard Wiener process, N_2 a Poisson process with intensity λ_2 and constants $\alpha, \sigma_2 \in \mathbb{R}_+$. Then the Ito-Lévy Jump Diffusion process Y, indicating the future benefits in the aftermath of an attack, is defined by means of the following differential equation*

$$dY = \alpha Y dt + \sigma_2 Y dW_2 + Y \int_{U_2} z N_2(t, dz) \quad \text{for} \quad T < t. \tag{2}$$

Note that the constant $\alpha > 0$ represents an increasing drift and σ_2 denotes the constant volatility of the benefits. While the first part of the stochastic process is again a geometric Brownian motion, the second part allows for fundamental events in the aftermath of the conflict. As our focus is on these fundamental opportunities or threats we need to model them in detail. We describe major positive or negative events by an accumulation of jumps through the integral $\int_{U_2} z N_2(t, dz)$. Jumps with an uncertain step height z out of U_2 occur randomly and can be downward as well as upward. These parameters describing the path of non-systematic uncertain events allow to discuss the accumulated direction of events, their impact and the frequency.

2.3 Value of Conflict and the Option Value of Peacekeeping

***Net Present Value of Conflict*:** As conflicts may lead to an improvement in living conditions, an attack enables the realization of potential benefits of conflict. Once the attack is carried out, the dynamic development of benefits is given within the limits of a random process. The economic value of conflict consists solely of its future benefit stream. Each dynamic development of benefits generates its own value of conflict. For a risk neutral individual the *gross value of conflict* V^{gross} is determined by the expected present value of the benefit stream in the aftermath. As the outbreak of the conflict is a severe break, it enables a new and better path of development with major expected improvements for the rebel. The expected gross value of these benefits in the aftermath is given by this value.[8]

Lemma 3 *Let Y be an Ito-Lévy-Jump-Diffusion process, υ_2 a Lévy measure and r the risk-free interest rate. Further assume $r > \alpha + \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) + \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz)$ where $f(z) = ln(1+z)$. Then the gross value of benefits is*

$$V^{gross}(T) = \frac{Y(T)}{\left(r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha \right)}. \tag{3}$$

Proof. For a proof see Appendix 2.

Note that for simplicity the rebel has an infinite lifespan, which may be also motivated by idea that the rebel acts for his group as well as for future.

In order to determine the expected net value of conflict the expected gross value (3) has to be adjusted for the costs of attack I, which include all costs connected to the attack like cost for preparation, weapons etc.. Hence, the expected net value of

[8]A detailed solution to the SDE and the derivation of the expected value of the Ito-Lévy Jump Diffusion process, which is used for determining the gross value of conflict is presented in Appendix 1.

conflict is the gross stream V^{gross} minus investment costs:

$$V(T) = V^{gross}(T) - I.$$

Option Value of Peacekeeping: Even if an immediate attack may be profitable for the rebel, not attacking and waiting has its own value. As the path of development is uncertain, waiting may open up additional opportunities that could otherwise not have been foreseen and realized. On the one hand, this value of waiting may indicate that a later conflict will become profitable even if recent conditions indicate that an attack will not pay off. On the other hand, it may identify a better time of attack which leads to even greater benefits of conflict. Not attacking also protects rebels from the irreversible costs of attack. Having the freedom to choose between alternative policies has an extra value that is particularly obvious when talking about violent conflicts. With the violent attack the conflict is lifted to another level. It removes any opportunity to resolve problems with a large variety of peaceful measures. With a violent or even deadly attack, such as an assassination of a state representative, there is no turning back, like saying "sorry, we did not mean it". Once the attack is carried out, rebels cannot return; they are tied to the expected benefit track they have chosen. This logically corresponds to a firm's investment decision (Dixit (1989), and Dixit and Pindyck (1994)), where the option value of the freedom of choice (here, the option value of further using peaceful measures) is a measure of opportunities, that may open up in the future when an agent does not irreversibly embark on a particular benefit stream.

Accounting for the option value F for the Ito-Lévy Jump Diffusion Process (1), we apply dynamic programing to obtain the Hamilton-Jacobi-Bellman equation:[9]

$$rFdt = E(dF). \tag{4}$$

This equation indicates that for a time interval dt, the total expected return on the investment opportunity is equal to the expected rate of capital appreciation.

[9] For a detailed discussion of the option value see Appendix 2.

2.4 Decision to Attack

So far we have identified a latent conflict between rebels and an oppressive government, and have determined the expected net benefit of an attack V and the option value of a later attack F. The decision to attack straight away is a sequential decision where the rebel repeatedly considers his living conditions and evaluates if a conflict at this point in time is the best strategy. In order to solve the decision problem of launching the attack, the rebel compares the benefit of immediate conflict V with the option value of a later attack F. Therefore, the problem is solved by the solution to:

$$\max \left\{ V^{gross}(T) - I, F(T) \right\}. \tag{5}$$

At any time during the non-violent waiting period the rebel will compare the expected *net benefit of conflict* with the *option value* of an uncertain non-violent development with the freedom to attack later. If the net value of conflict is greater than the option value ($V^{gross}(T) - I \geq F(T)$), the rebel will carry out the attack. By contrast, if the option value of postponing the attack exists and is greater than the net benefit of attacking straight away he will not initiate the conflict and wait. Solving this continuous sequential decision problem (5) also allows us to determine the expected time of the attack.

3 Solving for the Expected Time of Conflict

To identify the conditions that eventually trigger the attack and also to determine the expected time of attack involves two steps.

First, for each non-violent period during which deteriorating living conditions generate increasing benefits of the attack, we need to determine the benefit value of conflict in the current period (Y^* *threshold*) that would trigger the outbreak. This threshold is the required current benefit level that would make the attack preferable. It marks a boundary at which conditions have become so bad that a conflict becomes unavoidable for the rebel. Reaching this threshold is the straw that breaks the camel's back. Then, the expected value of conflict exceeds the option value of peace and hence the attack becomes more profitable. Peaceful waiting even if uncertain positive events are still possible is no longer rational.

Second, as the threshold indicates the start of the conflict, rebels simultaneously observe the development of the current period's benefit $\tilde{Y}$. Under worsening conditions during the waiting period they compare the threshold $Y^*$with the corresponding current period's benefit level of conflict $\tilde{Y}$ and verify if the threshold has already been reached. Even if the hope for positive events that improve the living conditions will currently let the rebel to remain peaceful, the expected end of the peaceful period can be predicted. Hence, understanding this mechanism allows for an extension of peaceful episodes in latent conflicts and may help to generate more time to look for peaceful solutions.

3.1 Conflict Threshold

In order to determine the benefit value that triggers the conflict we need to consider the standard conditions of a stochastic dynamic programming problem. In addition to the *Hamilton-Jacobi-Bellman equation* for the option value F and applying Ito's lemma for jump diffusions to dF, we have to use the well known boundary conditions, namely (6), the *value matching* condition (7) and the *smooth pasting* condition (8)

$$F(0) = 0 \tag{6}$$

$$F(Y^*) = V^{gross}(Y^*) - I \qquad \textit{value matching condition,} \tag{7}$$

$$\frac{dF(Y^*)}{dY} = \frac{d(V^{gross}(Y^*) - I)}{dY} \qquad \textit{smooth pasting condition.} \tag{8}$$

to solve for the threshold benefit Y^*. The setting of the decision problem implies that the net benefits of conflict must be sufficiently large to launch the attack. In other words, the current benefit of conflict given by the Ito-Lévy Jump Diffusion process must be great enough. Reaching this threshold triggers a change in strategy from peace to conflict. Therefore, determining this threshold is the first part of a solution to the expected timing of attack.

Proposition 4 *Let I be constant costs of conflict,* (1) *a sequence of increasing current benefit levels while remaining peaceful, and* (2) *future benefit developments after the attack. Further let β be an implicit function resulting from the differential equation $rFdt = E(dF)$ with solution $F = B\tilde{Y}^{\beta}$. Then the threshold Y^* that would trigger a* *conflict is*

$$Y^* = \frac{\beta}{\beta - 1}(r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha)I. \tag{9}$$

Proof. For a proof see Appendix 3.

The threshold is the current benefit that an attack needs to generate as a minimum if all positive values of peaceful waiting are accounted for. It is the ultimate limit to what one can bear in terms of discrimination, oppression, or persecution. As long as this threshold is not reached, the latent conflict is not triggered and the rebel would somehow tolerate the conditions in the hope for improvements. The situation remains calm. However, it is a calm before the storm. Even if the rebel does not attack straight away, the expectations about the future suggest that there will be a point in time when the attack is beneficial enough to be launched. Once the threshold is reached, the benefit of an immediate attack pays off even if all positive values of peaceful waiting are accounted for, and the conflict breaks out. That is, knowing the value of the threshold and the random process of living conditions during the peaceful period the expected time of attack becomes predictable.

3.2 Expected Time of Conflict

In the next step the rebel determines the time at which he expects to reach the threshold (9) that indicates the beginning of the conflict. Once the rebel knows from the threshold at which current benefit level he should attack, the question is when he can expect to obtain this benefit for the first time.

As described above, the path of current benefits of conflict is a random process. For each additional moment of non-violence the rebel has expectations concerning the paths of development; a time path of conditions after the attack and the current time path of benefits without attacking. For the current non-violent period, the benefits of attack are expected to increase systematically by rate δ in (1) when t increases. As the path of current benefits is described by a random process and the threshold triggers the conflict, we are interested in the time in when the threshold is expected to be reached for the first time, which is referred to as the first passage time. However, since we consider Jump-diffusion processes there could be a overshooting which has to be taken into account. This means that the horizontal boundary marking the potential start of conflict does not have to be hit exactly, but can be overshot instead. The existence of an analytical solution for the first passage time can therefore only be ensured for a small number of jump size distributions. Accordingly, we utilize the double exponential distribution as a rare example for which an analytical solution exists. Similar to Kou and Wang (2003) the double exponential distribution is given by

$$h(z) = p\eta_1 e^{-\eta_1 z} 1_{\{z \geq 0\}} + q\eta_2 e^{\eta_2 z} 1_{\{z<0\}},$$

where p is the probability of a positive jump and q of a negative jump, respectively, with $p+q=1$. $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$ denote the means of the two exponential distributions. Each exponential distribution can be interpreted as a distribution of the waiting period until a positive or negative jump occurs. In other words, in this waiting period the occurrence of fundamental opportunities and threats affect the decision to attack.

By using the Girsanov theorem we can derive the probability density function of T^* [10] which is sometimes referred to as the *Inverse Gaussian Distribution.*[11]

Proposition 5 *Let $\tilde{Y}$ be an Ito-Lévy Jump Diffusion process in (1), Y^* a constant threshold in (9) and $h(z) = p\eta_1 e^{-\eta_1 z} 1_{\{z \geq 0\}} + q\eta_2 e^{\eta_2 z} 1_{\{z<0\}}$ be the density function of the double exponential distribution. Further, let p be the probability of a positive jump and q the probability of a negative jump respectively. Then the first hitting time of Y^* is*

$$E(T^*) = \frac{1}{\bar{u}} \left[Y^* + \frac{\mu_2^* - \eta_1}{\eta_1 \mu_2^*} (1 - e^{-Y^* \mu_2^*}) \right] \tag{10}$$

with $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$ being the means of the two exponential distributions. $\bar{u} = \delta + \lambda_1 \left(\frac{p}{\eta_1} - \frac{q}{\eta_2} \right)$ denotes the overall drift and μ_2^* is defined as the unique root of $G(\mu_2^*) = 0$ with $G(x) := x\delta + \frac{1}{2}x^2\sigma_1^2 + \lambda_1 \left(\frac{p\eta_1}{\eta_1 - x} + \frac{p\eta_2}{\eta_2 + x} - 1 \right)$ and $0 < \eta_1 < \mu_2^* < \infty$.

Proof. For a proof see Appendix 3.

[10] An extensive discussion is offered by Karatzas and Shreve (1991, p.196) and Karlin and Taylor (1975, p.363).

[11] The term "Inverse Gaussian Distribution" stems from the inverse relationship between the cumulant generating functions of these distributions and those of the Gaussian distributions. For a detailed discussion of the inverse Gaussian distribution see Johnson, Kotz, and Balakrishnan (1995) or Dixit (1993).

As suggested, in order to decide whether to attack the oppressive government the rebel sequentially considers the status quo, namely his present economic, social and political conditions, which are significantly determined by the government, and make expectations about how they will develop with and without a conflict. Based on these expectations he determines whether a conflict will pay off. After having determined the minimum level of current benefits making the attack profitable, the expected time of the attack $E(T^*)$ is the time when this threshold is expected to be reached for the first time. More precisely, $E(T^*)$ is the result of a sequential optimization and hence the solution to a timing problem.

However, this point in time is just an expectated value. In the course of time a major event can trigger an attack at any, even an unexpected, moment.

4 Determinants of the Expected Time of Attack

In the previous section we determined the expected timing of an attack. In particular, we were able to show that the dynamic structure of the problem with special regard to risk and irreversibility is an important ingredient of the decision problem. In this section we examine the effects of parameters describing non-systematic risk in the decision problem. During peace, where we can already identify a latent conflict, but when the rebels have not yet turned to violence, uncertainty describing parameters may expand the peaceful episode, and a latent conflict may yield more time to find peaceful solutions. In the aftermath of conflict and after a violent action, the situation of the rebel may evolve as expected, but, however, there could be uncertain events that notably affect the situation again for the better or the worse. In particular, we look at the frequency of threats and opportunities, and the effect of their magnitude given by the jump size. The effects of frequency and size of these massive uncertain events can be studied, both for changes in uncertainties during the time of latent conflict which is still non-violent, and changes in expectations about the aftermath of an outbreak.

In a first step, we look at large events during the non-violent period. What happens e.g. if the government intensifies the fight against the discriminated group and introduces massive oppressive actions like persecution of group members, waves of detention, or assassination of rebel leaders.

Proposition 6 *An increase in the frequency λ_1 of jumps during the latent conflict is generally ambiguous. However, an increase in λ_1 may lead to an earlier attack ET^* if $\frac{p}{\eta_1} - \frac{q}{\eta_2} > 0$, $\frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*} > 1$ and the sum of upward jumps (threats leading to higher benefits of insurgence) is sufficiently large to outweigh of the sum of downward jumps (opportunities leading to less benefits of an uprise) so that $\frac{\partial \beta}{\partial \lambda_1} < 0$.*

$$
\begin{aligned}
\frac{\partial E(T^*)}{\partial \lambda_1} \quad = \quad & -\frac{\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)}{\left[\delta + \lambda_1 \left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)\right]^2} \left[Y^* + \frac{\mu_2^* - \eta_1}{\eta_1 \mu_2^*} (1 - e^{-Y^* \mu_2^*}) \right] \\
& + \frac{1}{\delta + \lambda_1 \left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)} \left[\begin{array}{c} -\frac{\frac{\partial \beta}{\partial \lambda_1}}{(\beta - 1)^2} (r - \int\limits_{f^{-1}(U_2)} z \upsilon(dz) \\ - \int\limits_{U_2} \left[\ln(1+z) - z \right] \upsilon_2(dz) - \alpha) \end{array} \right] \\
& \cdot (1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*}) < 0.
\end{aligned}
$$

Proof. For a proof see Appendix 4.

The conditions $\frac{p}{\eta_1} - \frac{q}{\eta_2} > 0$ and $1 < \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*}$ are connected to the probabilities of threats or opportunities and the mean waiting times until a major event occurs. The decision whether to attack, if the frequency of threats or opportunities increases, strongly depends on the direction of those jumps. In general, an increase in λ_1 implies that more fundamental events are occurring that imply non-marginal changes in the expected path of benefits associated with the attack. An increasing number of upward jumps suggests that a sudden increase in benefits of the attack becomes more likely. Increasing sudden benefits of the attack can be due to sudden deteriorations in welfare conditions like unfavorable regime changes, increasing number of oppressive government actions, or even external disasters. In this case the rebel's situation worsens suddenly so that attacking becomes more favorable and an earlier attack is preferable. As we can see, large uncertainty and non-systematic risk affects the outbreak of conflict.

The political implications are quite simple. In a latent conflict, signals pointing to randomly deteriorating fundamental welfare conditions for rebels would escalate the situation. A policy to generate more significant opportunities for the group represented by the rebels may not terminate the conflict but postpone the attack; hence, an earlier attack would become less likely.

Proposition 7 *An increase in the size of sudden jumps, namely the extent of threats or opportunities (u) during the latent conflict is generally ambiguous. However, an* *increase in u leads to an earlier attack if $\frac{p}{\eta_1} - \frac{q}{\eta_2} > 0$, $\frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*} > 1$ and the sum of upward jumps (threats leading to higher benefits of insurgence) is sufficiently large to outweigh of the sum of downward jumps (opportunities leading to smaller benefits of an uprise) so that $\frac{\partial \beta}{\partial u} < 0$.*

$$\frac{\partial E(T^*)}{\partial u} = \frac{\left(1-\frac{\mu_2^*-\eta_1}{\eta_1}e^{-Y^*\mu_2^*}\right)}{\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}\left[\frac{-\frac{\partial\beta}{\partial u}}{(\beta-1)^2}\begin{pmatrix} r - \int\limits_{f^{-1}(U_2)} z\upsilon(dz) \\ -\int\limits_{U_2}\left[\ln(1+z)-z\right]\upsilon_2(dz)-\alpha \end{pmatrix} I\right] \cdot\left(1-\frac{\mu_2^*-\eta_1}{\eta_1}e^{-Y^*\mu_2^*}\right) < 0.$$

Proof. For a proof see Appendix 4.

An increase in u means that threats become larger and opportunities become smaller. Larger upward jumps suggest a greater benefit of the attack determined by deteriorating events that worsen the situation of the rebel. Hence, The rebel group faces more severe threats when following the current welfare path. For instance, torture instead of detention, or expropriation instead of taxation during the non-violent period will make the attack more profitable since the uprise can lead to a new regime change that is more beneficial for the rebel, and hence we can expect that an attack will be carried out sooner.

While the recent discussion looks at changes in non-systematik risk during the non-violent period of a conflict, sudden large events in the more fragile aftermath of an uprise will also affect the decision to attack. E.g. an insurrection was successful and dispossessed a dictator. However, a sudden appearance of a counter coup by military, or by fundamentalists may be such a large event which threatens the rebel group and dramatically changes the benefits of conflict. Therefore, we also need to account for such potential large events in the aftermath of a uprise in the decision to attack.

Proposition 8 *An increase in the frequency of jumps after the conflict is generally ambiguous. However, an increase in λ_2 may lead to an later attack ET^* if $\frac{p}{\eta_1} - \frac{q}{\eta_2} > 0$, $\frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*} > 1$ and the sum of downward jumps (opportunities leading to smaller benefits) is sufficiently small to outweigh of the sum of upward jumps (threats leading* *to higher benefits).*

$$\frac{\partial E(T^*)}{\partial \lambda_2} = \frac{1}{\delta + \lambda_1\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)}\left[\frac{\beta}{\beta-1}\begin{pmatrix} -\int\limits_{f^{-1}(U_2)} zh(dz) \\ -\int\limits_{U_2} [\ln(1+z) - z]\, h(dz) \end{pmatrix} I\right]$$
$$\cdot\left(1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^*\mu_2^*}\right) > 0.$$

Proof. For a proof see Appendix 4.

As discussed in Appendix 4 the sign of the derivative according to λ_2 is ambiguous. However, if we assume that threats are frequent and disastrous enough to outweigh the magnitude of positive events a later attack will be preferred. More negative jumps indicate that more threats and losses can be expected after the attack. If, for instance, a counter-revolution takes place in the aftermath of the attack the conditions for the rebel will worsen again. Although the attack is assumed to pay off somehow on average, uncertain events that worsen the welfare of the rebel are expected to occur anyway. Hence, the benefits of conflict are not expected to be satisfying. In this situation, waiting for the situation to improve and for further information about developments even without an attack is the better strategy.

Proposition 9 *An increase in the size of jumps after the attack is ambiguous. However, an increase in z leads to an earlier attack if $\frac{p}{\eta_1} - \frac{q}{\eta_2} > 0$, $\frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^*\mu_2^*} > 1$ and the sum of upward jumps (threats leading to higher benefits) is sufficiently large to outweigh the sum of downward jumps (opportunities leading to smaller benefits).*

$$\frac{\partial E(T^*)}{\partial z} = \frac{1}{\delta + \lambda\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)}\left[\begin{array}{c} -\underbrace{\frac{\beta}{\beta-1}}_{>0}\Big(\int\limits_{f^{-1}(U_2)} 1\upsilon_2(dz) \\ +\int\limits_{U_2}\left[\frac{1}{(1+z)} - 1\right]\upsilon_2(dz)\Big)\underbrace{I}_{>0} \end{array}\right]$$
$$\cdot\left(1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^*\mu_2^*}\right) < 0$$

Proof. For a proof see Appendix 4.

An increase in z indicates that the magnitude of opportunities becomes larger and the magnitude of threats becomes smaller. Larger upward jumps imply that more favorable events like political reforms and liberalizations will occur that improve the social, political or economic situation of the rebel. Hence, the benefits from opportunities provided by the attack increase. At the same time, smaller downward jumps reduce the loss in benefits generated by threats. In particular, the magnitude of threats is reduced. Hence, opportunities are more beneficial than threats so that an attack is expected to have a greater payoff. Waiting for a while becomes less attractive and an earlier attack is preferred.

5 Summary

In this model the decision to turn to violence is based on the idea that an attack is a kind of investment in a change of conditions. The major focus of this paper is on the impact of large (non-marginal) uncertainties like fundamental threats or grand opportunities on the decision to launch a conflict. In order to capture these major uncertainties, this paper for the first time extends standard concepts for measuring systematic risk to include a concept of the non-systematic risks associated with major positive or negative events. We describe the two risk components by discontinuous stochastic processes, namely Ito-Lévy Jump Diffusion processes.

Under dynamic conditions and especially when uncertainty is taken into account, an attack is the result of a process in which the current path of development in economic, social and political conditions is evaluated and compared to the expected conditions and path of development after a potential attack, including conflict costs. This decision is particularly difficult because both the current path of development and the expected path after an attack are highly uncertain. If current conditions are expected to develop sufficiently badly and hence an attack is increasingly beneficial, the attack can be considered and a latent conflict may be identified. The decision to attack is a sequential decision in the course of time that is taken under conditions of high uncertainty and subject to irreversibility. Hence, real option theory in terms of dynamic programming is an appropriate methodology. As a result there is a non-violent period with a latent conflict where rebels have expectations about future economic, political and social developments and benefits of the conflict, but have not attacked yet. In this period the benefits of the attack are still not sufficiently high. For this situation we are particularly interested in the effects of large uncertain events on the decision to attack. Therefore, we analytically derive the threshold that triggers the attack and determine the time this is expected to happen.

From comparative statics we can conclude that the behavior of a rebel is highly affected by these fundamental uncertainties. The direction (opportunities or threats), magnitude, and frequency of these uncertain fundamental events are important elements of the decision. Effects of upward or downward jumps during the non-violent period of a latent conflict are most important for the outbreak of the conflict since they indicate a sudden worsening or improvement in the rebel's welfare. More frequent and stronger downward jumps in welfare during the non-violent period shorten the expected duration of the peace period and make an early attack more likely. Improvements in conditions during the non-violent period will lead the rebels to postpone the attack. If welfare conditions become more threatening for the rebels, the benefits of an attack increase and it is expected to be carried out earlier.

By considering time and sequential decisions we can show, that even if latent conflicts are not immediately solved, certain policies are able to extend the peaceful period and provide more time to find a solution to the conflict.

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6 Appendix

Appendix 1: Solution and Expected Value of the SDE

Solution of the SDE for Y

In this section we determine the solution of the SDE for Y and derive the expected value. A more general formulation of this process can be found in Oksendal and Sulem (2007). They describe under which conditions a solution to these SDE exists and discuss some characteristics. For our purpose we assume that the existence conditions are fulfilled. A further discussion of Lévy processes and their characteristics can be found in e.g. Applebaum (2009) and in Cont and Tankov (2004). However, in order to obtain the respective results for $\tilde{Y}$ replace Y by $\tilde{Y}$.

Lemma 10 *The Ito-Lévy Jump Diffusion process described by the SDE*

$$dY = \alpha Y dt + \sigma_2 Y dW_2 + Y \int_{U_2} z N_2(t, dz) \quad \textit{for} \quad T < t,$$

has the solution

$$Y(t) = Y(0) \exp \left[\begin{array}{c} \left(\alpha - \frac{1}{2}\sigma_2^2\right) t + \sigma_2 W_2(t) + \int_0^t \int_{U_2} [\ln(1+z) - z]\, v_2(dz) ds \\ + \int_0^t \int_{U_2} \ln(1+z) N_2(dt, dz) \end{array} \right].$$

Proof. Similar to Oksendal and Sulem (2007) we define $X(t) = \ln Y(t)$ and use Ito's Lemma for jump processes to obtain the solution to the SDE. Note that according to $ln(1+z)$ the function $Y(t)$ is only defined for $z > -1$.

Lemma 11 *Let Y be an Ito-Lévy Jump Diffusion process. The expected value of Y is*

$$EY(t) = Y(0) \exp \left[t \left(\alpha + \int_{f^{-1}(U_2)} z v_2(dz) + \int_{U_2} [\ln(1+z) - z]\, v_2(dz) \right) \right].$$

Proof. Assume that the Wiener and the compound Poisson process, are independent. Then the expected value of Y can be decomposed into

$$\begin{aligned} EY(t) &= \underbrace{EY(0) \cdot E e^{\left(\alpha - \frac{1}{2}\sigma_2^2\right)t + \sigma_2 W_2(t)}}_{(1)} \cdot \underbrace{E\left[\exp\left(\int_0^t \int_{U_2} \ln(1+z) N_2(ds, dz)\right)\right]}_{(2)} \\ & \quad \underbrace{\cdot E\left[\exp\left(\int_0^t \int_{U_2} [\ln(1+z) - z]\, v_2(dz) ds\right)\right]}_{(3)}. \end{aligned}$$

In this case compute the respective values for all three components.

- The expectation value (1) is the same as for the geometric Brownian motion, which can be found in Dixit and Pindyck (1994)

$$EY(0)e^{\alpha t}.$$

- In order to compute the expected value (2) we use Theorem 2.3.7 (i) in Applebaum (2009). As $\int_0^t \int_{U_2} \ln(1+z) N_2(ds,dz)$ is compound Poisson distributed with the characteristic function

$$E\left[\exp\left(iu\int_0^t\int_{U_2}\ln(1+z)N_2(ds,dz)\right)\right]=\exp\left(t\int_{U_2}\left(e^{iuz}-1\right)\upsilon_{2f}(dz)\right)$$

it follows from $\upsilon_{2f}=\upsilon_2\circ f^{-1}$, $f=\ln(1+z)$ and $u=-i$

$$E\left[\exp\left(\int_0^t\int_{U_2}\ln(1+z)N_2(ds,dz)\right)\right]=\exp\left(t\int_{f^{-1}(U_2)}z\upsilon_2(dz)\right).$$

This result only holds for $\int_{U_2} e^{uz}\upsilon_2(dz)<\infty$. In contrast to Theorem 2.3.7 (i) in Applebaum (2009) where $u\in\mathbb{R}$ we assume u to be complex. This is possible since the according integral exists anyway.

- As the expected value (3) is given by

$$E\left[\exp\left(\int_0^t\int_{U_2}\left[\ln(1+z)-z\right]\upsilon_2(dz)ds\right)\right]=\exp\left(t\int_{U_2}\left[\ln(1+z)-z\right]\upsilon_2(dz)\right).$$

Accordingly, the resulting expected value for Y is

$$EY(t)=Y(0)\exp\left[t\left(\alpha+\int_{f^{-1}(U_2)}z\upsilon_2(dz)+\int_{U_2}\left[\ln(1+z)-z\right]\upsilon_2(dz)\right)\right].$$

$EY(t)$ is an increasing function for $\alpha+\int_{f^{-1}(U_2)}z\upsilon_2(dz)+\int_{U_2}\left[\ln(1+z)-z\right]\upsilon_2(dz)>0$, otherwise it decreases with t.

Appendix 2: Value of Conflict and Option Value of Peacekeeping

In order to determine the optimal investment in rebellion the value of conflict and the option value of waiting to attack are optimized each period.

Proof of Lemma 3. For the value of conflict all benefits per period are summarized through an integral. Similar to Dixit and Pindyck (1994) we obtain

$$V^{gross} = E\int_{T}^{\infty} Y e^{-r(t-T)} Y\, dt$$

Under the assumption $r > \alpha + \int_{f^{-1}(U_2)} z\upsilon_2(dz) + \int_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz)$ we obtain the result.

Corollary 12 *Let $\tilde{Y}$ be an Ito-Lévy Jump Diffusion process. Then the Hamilton-Jacobi-Bellman equation defined in Dixit and Pindyck (1994)*

$$rF = \frac{1}{dt}E(dF)$$

has the solution $F = B\tilde{Y}^{\beta}$ where β is defined implicitly.

Proof. From Ito's Lemma we know:

$$\begin{aligned} dF &= (\frac{\partial F}{\partial t} + \delta\tilde{Y}\frac{\partial F}{\partial\tilde{Y}} + \frac{1}{2}\sigma_1^2\tilde{Y}^2\frac{\partial^2 F}{\partial\tilde{Y}^2})dt + \sigma_1\tilde{Y}\frac{\partial F}{\partial\tilde{Y}}dW_1 \\ &+ \int_{U_1}\left[F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-)) - \frac{\partial F}{\partial\tilde{Y}}u\tilde{Y}(t^-)\right]\upsilon_1(du)dt \\ &+ \int_{U_1}\left[F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-))\right]N_1(dt, du). \end{aligned}$$

In order to determine $E(dF)$, we use Theorem 2.3.7 (ii) in Applebaum (2009). For the expectation value of

$$\int_{U_1}\left[F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-))\right]N_1(dt, du)$$

we obtain

$$\begin{aligned} & E\int_{U_1}\left[F(\tilde{Y}(t^-)+u\tilde{Y}(t^-))-F(\tilde{Y}(t^-))\right]N_1(dt,du) \\ = \; & t\int_{U_1}\left[F(\tilde{Y}(t^-)+u\tilde{Y}(t^-))-F(\tilde{Y}(t^-))\right]\upsilon_1(du) \end{aligned}$$

and with $E(dW_1)=0$ this leads us to

$$\begin{aligned} \Rightarrow \quad & E(dF)=(\frac{\partial F}{\partial t}+\delta\tilde{Y}\frac{\partial F}{\partial\tilde{Y}}+\frac{1}{2}\sigma_1^2\tilde{Y}^2\frac{\partial^2 F}{\partial\tilde{Y}^2})dt \\ & +2\int_{U_1}\left[F(\tilde{Y}(t^-)+u\tilde{Y}(t^-))-F(\tilde{Y}(t^-))-\frac{1}{2}\frac{\partial F}{\partial\tilde{Y}}u\tilde{Y}(t^-)\right]\upsilon_1(du)dt. \end{aligned}$$

From the Bellman and the last equation we obtain the following differential equation:

$$\Leftrightarrow \delta\tilde{Y}\frac{\partial F}{\partial\tilde{Y}}+\frac{1}{2}\sigma_1^2\tilde{Y}^2\frac{\partial F}{\partial\tilde{Y}^2}+2\int_{U_1}\left[\begin{array}{c} F(\tilde{Y}(t^-)+u\tilde{Y}(t^-))-F(\tilde{Y}(t^-)) \\ -\frac{1}{2}\frac{\partial F}{\partial\tilde{Y}}u\tilde{Y}(t^-) \end{array}\right]\upsilon_1(du)-rF=0.$$

This is a second-order homogenous ordinary differential equation with a free boundary. A general solution to this differential equation is of the form

$$F=B\tilde{Y}^{\beta}.$$

Hence,

$$\delta\beta+\frac{1}{2}\sigma_1^2\beta(\beta-1)+2\int_{U_1}\left[(1+u)^{\beta}-(1+\frac{1}{2}\beta u)\right]\upsilon_1(du)-r=0.$$

If we define

$$g(\beta):=\delta\beta+\frac{1}{2}\sigma_1^2\beta(\beta-1)+2\int_{U_1}\left[(1+u)^{\beta}-(1+\frac{1}{2}\beta u)\right]\upsilon_1(du)-r$$

with h being the distribution of the jump sizes, then it follows

$$\begin{aligned} g(1) &= \delta + \int_{U_1} u\upsilon(du) - r, \\ \lim_{\beta\to\infty} g(\beta) &= \infty. \end{aligned}$$

Accordingly, we can assume that $r > \delta + \int_{U_1} u\upsilon_1(du)$ leading to $g(1) < 0$. With the intermediate value theorem we find $\beta_1 \in (1, \frac{r}{\delta + \int_{U_1} u\upsilon_1(du)})$ such that $g(\beta_1) = 0$. It follows immediately that β_1 is a function of r, δ and $\int_{U_1} u\upsilon_1(du)$ determined as the implicit function of $g(\beta_1) = 0$, and $\beta_1 > 1$.

Corollary 13 *The derivatives of β_1 according to λ_1 and u are*

$$\frac{\partial\beta_1}{\partial\lambda_1} = \frac{2\int_{U_1}\left[-(1+u)^{\beta_1} + (1+\frac{1}{2}\beta_1 u)\right]h(du)}{\delta - \frac{1}{2}\sigma_1^2 + 2\lambda_1\int_{U_1}\left[\ln(1+u)(1+u)^{\beta_1} - \frac{1}{2}u\right]h(du)}$$

and

$$\frac{\partial\beta_1}{\partial u} = -\frac{2\beta\int_{U_1}\left[(1+u)^{\beta-1} - \frac{1}{2}\right]\upsilon_1(du)}{\delta - \frac{1}{2}\sigma_1^2 + 2\lambda_1\int_{U_1}\left[\ln(1+u)(1+u)^{\beta_1} - \frac{1}{2}u\right]\upsilon_1(du)}.$$

Proof. Apply the rules

$$\frac{\partial\beta_1}{\partial\lambda_1} = -\frac{\frac{\partial g}{\partial\lambda}}{\frac{\partial g}{\partial\beta_1}} \text{ and } \frac{\partial\beta_1}{\partial u} = -\frac{\frac{\partial g}{\partial u}}{\frac{\partial g}{\partial\beta_1}}$$

in order to obtain the derivatives. However, their sign is obtained by discussing whether positive jumps outweigh the negative. The numerator of $\frac{\partial\beta_1}{\partial\lambda_1}$ contains a measure integral $\int_U\left[-(1+u)^{\beta_1} + (1+\frac{1}{2}\beta_1 u)\right]h(du)$ with a measurable function f and a measure h. This integral is defined as

$$\int_{U_1} fh(du) = \int_{U_1} f^+h(du) - \int_{U_1} f^-h(du) \tag{11}$$

and is a measure integral where, depending on U_1, f^+ is the positive and f^- the negative part of f. The difference between the two integrals in (11) depends on the size of each integral leading to an ambiguous sign of (11). Hence, if the negative jumps outweigh the positive jumps the sign of the integral will be negative and otherwise positive. In the denominator we have

$$\delta - \frac{1}{2}\sigma_1^2 + 2\lambda_1 \int\limits_{U_1} \ln(1+u)(1+u)h(du) - \lambda_1 \int\limits_{U_1} uh(du)$$

consisting of

$$\delta - \frac{1}{2}\sigma_1^2 > 0$$

and a jump component

$$2\lambda_1 \int\limits_{U_1} \ln(1+u)(1+u)h(du) - \lambda_1 \int\limits_{U_1} uh(du)$$

where the logarithm is only defined for $u > -1$. Again, if the positive jumps outweigh the negative jumps the integral will be positive, otherwise it is negative. Assuming the denominator and nominator are positive, the derivative according to λ_1 becomes negative.

Similarly, the derivative according to z becomes positive with a positive integral in the nominator and if $\int\limits_{U_1} \left[\ln(1+u)(1+u) - \frac{1}{2}u\right] v_1(du) > 0$ and $\delta > \frac{1}{2}\sigma_1^2$.

Appendix 3: Investment Threshold and Expected Time of Conflict

Proof of Proposition 4. Apply the boundary conditions

$$\begin{aligned} F(0) &= 0 \\ F(Y^*) &= V^{gross}(Y^*) - I \qquad \textit{value matching condition,} \\ \frac{dF(Y^*)}{dY} &= \frac{d(V^{gross}(Y^*) - I)}{dY} \qquad \textit{smooth pasting condition.} \end{aligned}$$

and solve the equation system for Y^*.

Proof of Proposition 5. For the jump diffusion case the first passage problem can be solved analytically if we assume an explicit distribution of the jump sizes.

According to Kou and Wang (2003) the moment generating function for $\tilde{Y}(t)$ with $\theta \in (-\eta_2, \eta_1)$ is

$$\phi(\theta, t) := E(e^{\theta \tilde{Y}(t)}) = \exp(G(\theta)t)$$

where the function G is defined as

$$G(x) := x\delta + \frac{1}{2}x^2\sigma_1^2 + \lambda_1 \left(\frac{p\eta_1}{\eta_1 - x} + \frac{p\eta_2}{\eta_2 + x} - 1 \right).$$

For jump diffusion processes the study of first passage times has to consider the exact hit of a constant boundary as well as an overshoot. Accordingly, two cases have to be distinguished. The Laplace transformation of the first hitting time, when $\tilde{Y}(t)$ hits the boundary Y^* exactly[12] is :

$$E(e^{-\varepsilon \tilde{T}_i} 1_{\{\tilde{Y}(\tilde{T}_i)=Y^*\}}) = \frac{\eta_1 - \beta_{1,\varepsilon}}{\beta_{2,\varepsilon} - \beta_{1,\varepsilon}} e^{-Y^*\beta_{1,\varepsilon}} + \frac{\beta_{2,\varepsilon} - \eta_1}{\beta_{2,\varepsilon} - \beta_{1,\varepsilon}} e^{-Y^*\beta_{2,\varepsilon}}$$

with $\mu_{1,\varepsilon}$ and $\mu_{2,\varepsilon}$ being the only positive roots of $G(\beta) = \varepsilon$ and $0 < \beta_{1,\varepsilon} < \eta_1 < \beta_{2,\varepsilon} < \infty$. For every overshoot $\tilde{Y}(\tilde{T}) - Y^*$ the Laplace transformation is

$$E(e^{-\varepsilon \tilde{T}} 1_{\{\tilde{Y}(\tilde{T}_i)-Y^*>y\}}) = e^{-\eta_1 y} \frac{(\eta_1 - \mu_{1,\varepsilon})(\mu_{2,\varepsilon} - \eta_1)}{\eta_1 (\mu_{2,\varepsilon} - \mu_{1,\varepsilon})} \left(e^{-Y^*\mu_{1,\varepsilon}} - e^{-Y^*\mu_{2,\varepsilon}}\right) \quad \text{for all } y \geq 0.$$

The expectation of the first passage time is finite, i.e. $E(T^*) < \infty$, if and only if the overall drift of the jump diffusion process is positive. Hence,

$$E(T^*) < \infty \Leftrightarrow \bar{u} = \delta + \lambda_1 \left(\frac{p}{\eta_1} - \frac{q}{\eta_2} \right) > 0.$$

Now for $\bar{u} > 0$ we determine the first passage time as

$$E(T^*) = \frac{1}{\bar{u}} \left[Y^* + \frac{\mu_2^* - \eta_1}{\eta_1 \mu_2^*} (1 - e^{-Y^*\mu_2^*}) \right]$$

where μ_2^* is defined as the unique root of $G(\mu_2^*) = 0$ with $0 < \eta_1 < \mu_2^* < \infty$.

[12] See Kou and Wang (2003) Theorem 3.1.

Appendix 4: Determinants of the Expected Time of Conflict

Proof of Proposition 6.

$$\begin{aligned}\frac{\partial E(T^*)}{\partial \lambda_1} &= -\underbrace{\frac{\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}{\left[\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)\right]^2}}_{(1)}\underbrace{\left[Y^*+\frac{\mu_2^*-\eta_1}{\eta_1\mu_2^*}(1-e^{-Y^*\mu_2^*})\right]}_{(2)} \\ &\quad+\underbrace{\frac{1}{\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}}_{(3)}\underbrace{\left[\begin{array}{c}-\frac{\frac{\partial\beta}{\partial\lambda_1}}{(\beta-1)^2}(r-\int\limits_{f^{-1}(U_2)} zv(dz) \\ -\int\limits_{U_2}\left[\ln(1+z)-z\right]v_2(dz)-\alpha)I\end{array}\right]}_{(4)}\underbrace{(1-\frac{\mu_2^*-\eta_1}{\eta_1}e^{-Y^*\mu_2^*})}_{(5)}\end{aligned}$$

For the first term (1) we obtain

$$\frac{\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}{\left[\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)\right]^2} > 0 \Leftrightarrow \frac{p}{\eta_1}-\frac{q}{\eta_2} > 0 \quad \text{with } q = 1-p.$$

With the same condition, we obtain a positive sign also for term (3)

$$\frac{1}{\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)} > 0.$$

For the second term (2) it holds that

$$\underbrace{Y^*}_{>0}+\frac{\mu_2^*-\eta_1}{\underbrace{\eta_1\mu_2^*}_{>0}}\underbrace{(1-e^{-Y^*\mu_2^*})}_{\geq 0} > 0.$$

The sign of the fourth term (4) depends on whether $\frac{\partial\beta}{\partial\lambda_1}$ is positive or negative. Assuming $\frac{\partial\beta}{\partial\lambda_1} < 0$ then term (4) becomes

$$-\frac{\frac{\partial\beta}{\partial\lambda_1}}{(\beta-1)^2}\underbrace{(r-\int\limits_{f^{-1}(U_2)} zv(dz)-\int\limits_{U_2}\left[\ln(1+z)-z\right]v_2(dz)-\alpha)}_{>0}\underbrace{I}_{>0} > 0$$

The last term (5)

$$1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*}$$

is negative if

$$1 < \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*}.$$

Summarizing all conditions leads to $\frac{\partial E(T^*)}{\partial \lambda_1} < 0$.

Proof of Proposition 7.

$$\begin{aligned} \frac{\partial E(T^*)}{\partial u} &= \underbrace{\frac{1}{\delta + \lambda \left(\frac{p}{\eta_1} - \frac{q}{\eta_2} \right)}}_{(1)} \underbrace{\left[\begin{array}{c} -\frac{\frac{\partial \beta}{\partial u}}{(\beta-1)^2} (r - \int\limits_{f^{-1}(U_2)} z \upsilon(dz) \\ - \int\limits_{U_2} \left[\ln(1+z) - z \right] \upsilon_2(dz) - \alpha) I(T^*) \end{array} \right]}_{(2)} \\ & \quad \underbrace{\cdot (1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*})}_{(3)}. \end{aligned}$$

Similar to the last proof, the term (1) is positive for $p > \frac{\eta_1}{\eta_2 + \eta_1}$. Accordingly, the last component (3) is negative for $1 < \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*}$. The sign of (2) depends on whether $\frac{\partial \beta}{\partial u} \gtrless 0$. Assuming that $\frac{\partial \beta}{\partial u} < 0$ it follows that $\frac{\partial E(\tilde{T})}{\partial u} < 0$.

Proof of Proposition 8.

$$\begin{aligned} \frac{\partial E(T^*)}{\partial \lambda_2} &= \underbrace{\frac{1}{\delta + \lambda_1 \left(\frac{p}{\eta_1} - \frac{q}{\eta_2} \right)}}_{(1)} \underbrace{\left[\underbrace{\frac{\beta}{\beta - 1}}_{>0} \left(\begin{array}{c} - \int\limits_{f^{-1}(U_2)} z h(dz) \\ - \int\limits_{U_2} \left[\ln(1+z) - z \right] h(dz) \end{array} \right) \underbrace{I}_{>0} \right]}_{(2)} \\ & \quad \underbrace{\cdot \left(1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*} \right)}_{(3)}. \end{aligned}$$

From the conditions from above (1) is positive and (3) is negative. Hence, the sign of $\frac{\partial E(T^*)}{\partial \lambda_2}$ depends on the second term and especially on the sign of $- \int\limits_{f^{-1}(U_2)} z h(dz) - \int\limits_{U_2} \left[\ln(1+z) - z \right] h(dz)$. Assuming more negative than positive jumps lead to a positive sign.

Proof of Proposition 9.

$$\frac{\partial E(T^*)}{\partial z} = \underbrace{\frac{1}{\delta + \lambda\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right)}}_{(>0)} \underbrace{\left[\begin{array}{c} -\underbrace{\frac{\beta}{\beta-1}}_{>0}\Big(\int\limits_{f^{-1}(U_2)} 1\upsilon_2(dz) \\ +\int\limits_{U_2}\left[\frac{1}{(1+z)} - 1\right]\upsilon_2(dz)\Big)\underbrace{I}_{>0} \end{array}\right]}_{(2)}$$
$$\cdot\underbrace{\left(1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^*\mu_2^*}\right)}_{(<0)}$$

According to the above assumptions (1) is positive and (3) is negative. The second term again depends on the jump part. However, even if there are more negative than positive jumps the effect on the jump part is not so large as to make the sign negative. Therefore the sign of the derivative is negative.

Technical Note[13]

In the technical note we have fully written out all calculations in order to help the referee follow the calculations. The following paragraphs are an intuitive presentation of the proofs.

Technical Note 1: Solution and Expected Value of the SDE

Solution of the SDE

The benefits of an attack are modeled by a stochastic Ito-Lévy Jump Diffusion process according to a stochastic differential equation (SDE)

$$\frac{dY}{Y} = \alpha dt + \sigma_2 dW_2 + \int_{U_2} z N_2(t, dz)$$

where α, σ_2 are both constants $\in \mathbb{R}_+$. α denotes the drift and σ_2 is the volatility of the function Y. Furthermore, dW_2 represents the increment of the standard Wiener process. In addition to the geometric Brownian motion part we include a jump part $\int_{U_2} z N_2(t, dz)$ where $N_2(t, dz)$ denotes the Poisson process with intensity λ_2. Hence, non-marginal jumps which occur at random time with an uncertain step height out of U_2 are accumulated. U_2 itself is a Borel set whose closure does not contain 0.

A more general formulation of this process can be found in Oksendal and Sulem (2007). They describe under which conditions a solution to these SDE exists and discuss some characteristics. For our purpose we assume that the existence conditions are fulfilled. A further discussion of Lévy processes and their characteristics can be found in e.g. Applebaum (2009) and in Cont and Tankov (2004).

Similar to Oksendal and Sulem (2007) we define $X(t) = \ln Y(t)$ and use Ito's Lemma for jump processes to obtain the solution to the SDE:

[13] For comments and discussions concerning our methodology and solutions we are grateful to B. Schmalfuss from the Mathematics Department.

$$\begin{aligned}
dX(t) &= \frac{Y(t)}{Y(t)}\left[\alpha dt + \sigma_2 dW_2(t)\right] - \frac{1}{2}\sigma_2^2 \frac{Y(t)^2}{Y(t)^2} dt \\
&+ \int\limits_{U_2} \left[\ln(Y(t^-) + zY(t^-)) - \ln(Y(t^-)) - \frac{Y(t^-)z}{Y(t^-)}\right] \upsilon_2(dz)dt \\
&+ \int\limits_{U_2} \left[\ln(Y(t^-) + zY(t^-)) - \ln(Y(t^-))\right] N_2(dt, dz) \\
&= \left(\alpha - \frac{1}{2}\sigma_2^2\right) dt + \sigma dW_2(t) + \int\limits_{U_2} \left[\ln(1+z) - z\right] \upsilon_2(dz)dt \\
&+ \int\limits_{U_2} \ln(1+z) N_2(dt, dz)
\end{aligned}$$

which leads us to the solution of the above SDE for Y

$$Y(t) = Y(0)\exp\left[\begin{array}{c} \left(\alpha - \frac{1}{2}\sigma_2^2\right)t + \sigma_2 W_2(t) + \int\limits_0^t \int\limits_{U_2} \left[\ln(1+z) - z\right] \upsilon_2(dz)ds \\ + \int\limits_0^t \int\limits_{U_2} \ln(1+z) N_2(dt, dz) \end{array}\right].$$

Note that according to $ln(1+z)$ the function $Y(t)$ is only defined for $z > -1$.

However, in order to obtain these results for $\tilde{Y}$ replace Y by $\tilde{Y}$.

Expected value of $Y(t)$

Under the assumption that the Lévy processes included in the SDE, the Wiener and the compound Poisson process, are independent, the expected value of Y can be decomposed into

$$EY(t) = \underbrace{EY(0)\cdot Ee^{\left(\alpha-\frac{1}{2}\sigma_2^2\right)t+\sigma_2 W_2(t)}}_{(1)} \cdot \underbrace{E\left[\exp\left(\int_0^t\int_{U_2}\ln(1+z)N_2(ds,dz)\right)\right]}_{(2)}$$
$$\cdot \underbrace{E\left[\exp\left(\int_0^t\int_{U_2}\left[\ln(1+z)-z\right]\upsilon_2(dz)ds\right)\right]}_{(3)}.$$

so that we can compute the respective values for all three components.

- The expectation value (1) is the same as for the geometric Brownian motion, which can be found in Dixit and Pindyck (1994)

$$EY(0)e^{\alpha t}.$$

- In order to compute the expected value (2) we use Theorem 2.3.7 (i) in Applebaum (2009). As $\int_0^t\int_{U_2}\ln(1+z)N_2(ds,dz)$ is compound Poisson distributed with the characteristic function

$$E\left[\exp\left(iu\int_0^t\int_{U_2}\ln(1+z)N_2(ds,dz)\right)\right] = \exp\left(t\int_{U_2}\left(e^{iuz}-1\right)\upsilon_{2f}(dz)\right)$$

it follows from $\upsilon_{2f}=\upsilon_2\circ f^{-1}$, $f=\ln(1+z)$ and $u=-i$

$$\begin{aligned} E\left[\exp\left(\int_0^t\int_{U_2}\ln(1+z)N_2(ds,dz)\right)\right] &= \exp\left(t\int_{U_2}\left(e^{z}-1\right)\upsilon_{2f}(dz)\right) \\ &= \exp\left(t\int_{f^{-1}(U_2)} z\upsilon_2(dz)\right). \end{aligned}$$

This result only holds for $\int\limits_{U_2} e^{uz}\upsilon_2(dz) < \infty$. In contrast to Theorem 2.3.7 (i) where $u \in \mathbb{R}$ we assume u to be complex. This is possible since the according integral exists anyway.

- As the expected value (3) is given by

$$\begin{aligned} E\left[\exp\left(\int_0^t\int_{U_2} \left[\ln(1+z)-z\right]\upsilon_2(dz)ds\right)\right] &= E\left[\exp\left(t\int_{U_2} \left[\ln(1+z)-z\right]\upsilon_2(dz)\right)\right] \\ &= \exp\left(t\int_{U_2} \left[\ln(1+z)-z\right]\upsilon_2(dz)\right). \end{aligned}$$

Accordingly, the resulting expected value for Y is

$$\begin{aligned} EY(t) &= Y(0)e^{\alpha t}\cdot\exp\left(t\int_{f^{-1}(U)} z\upsilon_2(dz)\right)\cdot\exp\left(t\int_{U_2} \left[\ln(1+z)-z\right]\upsilon_2(dz)\right) \\ &= Y(0)\exp\left[t\left(\alpha+\int_{f^{-1}(U_2)} z\upsilon_2(dz)+\int_{U_2} \left[\ln(1+z)-z\right]\upsilon_2(dz)\right)\right]. \end{aligned}$$

$EY(t)$ is an increasing function for $\alpha + \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) + \int\limits_{U_2} \left[\ln(1+z)-z\right]\upsilon_2(dz) > 0$, otherwise it decreases with t.

However, in order to obtain these results for $\tilde{Y}$ replace Y by $\tilde{Y}$.

Technical Note 2: Value of Conflict and Option Value of Peacekeeping

In order to determine the optimal investment in rebellion the value of conflict and the option value of waiting to attack are optimized each period.

Present Value of Conflict

For the value of conflict all benefits per period are summarized through an integral. Similar to Dixit and Pindyck (1994) we obtain

$$\begin{aligned}
V^{gross} &= E\int_T^\infty Ye^{-r(t-T)}Y\,dt \\
&= \int_T^\infty e^{-r(t-T)}Y(t)\exp\left[(t-T)\left(\alpha+\int_{f^{-1}(U_2)} z\upsilon_2(dz)+\int_{U_2}[\ln(1+z)-z]\,\upsilon_2(dz)\right)\right]dt \\
&= \left[\begin{array}{c}\frac{1}{\alpha+\int_{f^{-1}(U_2)} z\upsilon_2(dz)+\int_{U_2}[\ln(1+z)-z]\upsilon_2(dz)-r} \\ \cdot e^{(\alpha+\int_{f^{-1}(U_2)} z\upsilon_2(dz)+\int_{U_2}[\ln(1+z)-z]\upsilon_2(dz)-r)(t-T)}\,Y\end{array}\right]_T^\infty .
\end{aligned}$$

Under the assumption $r>\alpha+\int_{f^{-1}(U_2)} z\upsilon_2(dz)+\int_{U_2}[\ln(1+z)-z]\,\upsilon_2(dz)$ we obtain

$$V^{gross}=\frac{Y(T)}{\left(r-\int_{f^{-1}(U_2)} z\upsilon_2(dz)-\int_{U_2}[\ln(1+z)-z]\,\upsilon_2(dz)-\alpha\right)}.$$

Option value of waiting and the derivatives of β

Option value: According to Dixit and Pindyck (1994) for the option value F the Hamilton-Jacobi-Bellman equation holds:

$$rF=\frac{1}{dt}E(dF).$$

From Ito's Lemma we know:

$$\begin{aligned}
dF &= \left(\frac{\partial F}{\partial t}+\delta\tilde{Y}\frac{\partial F}{\partial\tilde{Y}}+\frac{1}{2}\sigma_1^2\tilde{Y}^2\frac{\partial^2 F}{\partial\tilde{Y}^2}\right)dt+\sigma_1\tilde{Y}\frac{\partial F}{\partial\tilde{Y}}dW_1 \\
&\quad+\int_{U_1}\left[F(\tilde{Y}(t^-)+u\tilde{Y}(t^-))-F(\tilde{Y}(t^-))-\frac{\partial F}{\partial\tilde{Y}}u\tilde{Y}(t^-)\right]\upsilon_1(du)dt \\
&\quad+\int_{U_1}\left[F(\tilde{Y}(t^-)+u\tilde{Y}(t^-))-F(\tilde{Y}(t^-))\right]N_1(dt,du).
\end{aligned}$$

In order to determine $E(dF)$, we use Theorem 2.3.7 (ii) in Applebaum (2009). For the expectation value of

$$\int_{U_1} \left[F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-)) \right] N_1(dt, du)$$

we get

$$\begin{aligned} & E\int_{U_1} \left[F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-)) \right] N_1(dt, du) \\ = \; & t\int_{U_1} \left[F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-)) \right] \upsilon_1(du) \end{aligned}$$

and with $E(dW_1) = 0$ this leads us to

$$\begin{aligned} \Rightarrow \quad & E(dF) = (\frac{\partial F}{\partial t} + \delta\tilde{Y}\frac{\partial F}{\partial \tilde{Y}} + \frac{1}{2}\sigma_1^2\tilde{Y}^2\frac{\partial^2 F}{\partial \tilde{Y}^2})dt \\ & +\int_{U_1} \left[F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-)) - \frac{\partial F}{\partial \tilde{Y}}u\tilde{Y}(t^-) \right] \upsilon_1(du)dt \\ & +\int_{U_1} \left[F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-)) \right] \upsilon_1(du)dt \\ = \quad & (\frac{\partial F}{\partial t} + \delta\tilde{Y}\frac{\partial F}{\partial \tilde{Y}} + \frac{1}{2}\sigma_1^2\tilde{Y}^2\frac{\partial^2 F}{\partial \tilde{Y}^2})dt \\ & +2\int_{U_1} \left[F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-)) - \frac{1}{2}\frac{\partial F}{\partial \tilde{Y}}u\tilde{Y}(t^-) \right] \upsilon_1(du)dt. \end{aligned}$$

From the Bellman and the last equation we obtain the following differential equation:

$$\underbrace{\frac{\partial F}{\partial t}}_{=0} + \delta\tilde{Y}\frac{\partial F}{\partial \tilde{Y}} + \frac{1}{2}\sigma_1^2\tilde{Y}^2\frac{\partial F}{\partial \tilde{Y}^2} + 2\int_{U_1} \left[\begin{array}{c} F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-)) \\ -\frac{1}{2}\frac{\partial F}{\partial \tilde{Y}}u\tilde{Y}(t^-) \end{array} \right] \upsilon_1(du) - rF = 0$$

$$\Leftrightarrow \delta\tilde{Y}\frac{\partial F}{\partial \tilde{Y}} + \frac{1}{2}\sigma_1^2\tilde{Y}^2\frac{\partial F}{\partial \tilde{Y}^2} + 2\int_{U_1} \left[\begin{array}{c} F(\tilde{Y}(t^-) + u\tilde{Y}(t^-)) - F(\tilde{Y}(t^-)) \\ -\frac{1}{2}\frac{\partial F}{\partial \tilde{Y}}u\tilde{Y}(t^-) \end{array} \right] \upsilon_1(du) - rF = 0.$$

This is a second-order homogenous ordinary differential equation with a free boundary.

A general solution to this differential equation is of the form

$$F = B\tilde{Y}^{\beta}.$$

$B\tilde{Y}^{\beta}$ solves the homogenous differential equation.

$$\delta\tilde{Y}B\beta\tilde{Y}^{\beta-1}+\frac{1}{2}\sigma_1^2B\tilde{Y}^2\beta(\beta-1)\tilde{Y}^{\beta-2}+2\int\limits_{U_1}\left[\begin{array}{c} B\left(\tilde{Y}+u\tilde{Y}\right)^{\beta}-B\tilde{Y}^{\beta} \\ -\frac{1}{2}B\beta\tilde{Y}^{\beta-1}u\tilde{Y}(t^-)\end{array}\right]\upsilon_1(du)-rB\tilde{Y}^{\beta}=0$$

$$\Leftrightarrow \delta B\beta\tilde{Y}^{\beta}+\frac{1}{2}\sigma_1^2B\beta(\beta-1)\tilde{Y}^{\beta}+2B\int\limits_{U_1}\left[\begin{array}{c}\tilde{Y}^{\beta}(1+u)^{\beta}-\tilde{Y}^{\beta} \\ -\frac{1}{2}\beta\tilde{Y}^{\beta}u\end{array}\right]\upsilon_1(du)-rB\tilde{Y}^{\beta}=0$$

$$\Leftrightarrow \delta B\beta\tilde{Y}^{\beta}+\frac{1}{2}\sigma_1^2B\beta(\beta-1)\tilde{Y}^{\beta}+B\tilde{Y}^{\beta}2\int\limits_{U_1}\left[(1+u)^{\beta}-(1+\frac{1}{2}\beta u)\right]\upsilon_1(du)-rB\tilde{Y}^{\beta}=0$$

$$\Leftrightarrow \delta\beta+\frac{1}{2}\sigma_1^2\beta(\beta-1)+2\int\limits_{U_1}\left[(1+u)^{\beta}-(1+\frac{1}{2}\beta u)\right]\upsilon_1(du)-r=0.$$

If we define

$$\begin{aligned} g(\beta) \quad : &= \delta\beta+\frac{1}{2}\sigma_1^2\beta(\beta-1)+2\int\limits_{U_1}\left[(1+u)^{\beta}-(1+\frac{1}{2}\beta u)\right]\upsilon_1(du)-r \\ &= \delta\beta+\frac{1}{2}\sigma_1^2\beta(\beta-1)+2\lambda_1\int\limits_{U_1}\left[(1+u)^{\beta}-(1+\frac{1}{2}\beta u)\right]h(du)-r, \end{aligned}$$

with h being the distribution of the jump sizes, then it follows

$$g(1)=\delta+\int\limits_{U_1}u\upsilon(du)-r$$

and

$$\lim_{\beta\to\infty}g(\beta)=\infty.$$

Accordingly, we can assume that $r>\delta+\int\limits_{U_1}u\upsilon_1(du)$ leading to $g(1)<0$. With the intermediate value theorem we find $\beta_1 \in (1, \frac{r}{\delta + \int\limits_{U_1} uv_1(du)})$ such that $g(\beta_1) = 0$.

It follows immediately that β_1 is a function of r, δ and $\int\limits_{U_1} uv_1(du)$ determined as the implicit function of $g(\beta_1) = 0$, and $\beta_1 > 1$.

Derivatives of β: The derivatives of β will be used in order to determine the reactions of the expected time of conflict according to risk factors.

For the derivatives of β_1 we obtain

$$
\begin{aligned}
\frac{\partial \beta_1}{\partial \lambda_1} &= -\frac{\frac{\partial g}{\partial \lambda}}{\frac{\partial g}{\partial \beta_1}} \\
&= -\frac{2\int\limits_{U_1} \left[(1+u)^{\beta_1} - (1 + \frac{1}{2}\beta_1 u)\right] h(du)}{\delta + \frac{1}{2}\sigma_1^2(\beta_1 - 1) - \frac{1}{2}\sigma_1^2\beta_1 + 2\lambda_1 \int\limits_{U_1} \left[\ln(1+u)(1+u)^{\beta_1} - \frac{1}{2}u\right] h(du)} \\
&= -\frac{2\int\limits_{U_1} \left[(1+u)^{\beta_1} - (1 + \frac{1}{2}\beta_1 u)\right] h(du)}{\delta - \frac{1}{2}\sigma_1^2 + 2\lambda_1 \int\limits_{U_1} \left[\ln(1+u)(1+u)^{\beta_1} - \frac{1}{2}u\right] h(du)} \\
&= \frac{2\int\limits_{U_1} \left[-(1+u)^{\beta_1} + (1 + \frac{1}{2}\beta_1 u)\right] h(du)}{\delta - \frac{1}{2}\sigma_1^2 + 2\lambda_1 \int\limits_{U_1} \left[\ln(1+u)(1+u)^{\beta_1} - \frac{1}{2}u\right] h(du)}.
\end{aligned}
$$

Let us consider the numerator $2\int\limits_{U} \left[-(1+u)^{\beta_1} + (1 + \frac{1}{2}\beta_1 u)\right] h(du)$ first. As $\int\limits_{U} \left[-(1+u)^{\beta_1} + (1 + \frac{1}{2}\beta_1 u)\right] h(du)$ is a measure integral with a measurable function f and a measure h, the integral is defined as

$$
\int\limits_{U_1} f h(du) = \int\limits_{U_1} f^{+} h(du) - \int\limits_{U_1} f^{-} h(du) \tag{12}
$$

where, depending on U_1, f^+ is the positive and f^- the negative part of f. The difference between the two integrals in (12) depends on the size of each integral leading to an ambiguous sign of (12). Hence, if the negative jumps outweigh the positive jumps the sign of the integral will be negative and otherwise positive. In the denominator we have

$$\delta - \frac{1}{2}\sigma_1^2 + 2\lambda_1 \int\limits_{U_1} \ln(1+u)(1+u)h(du) - \lambda_1 \int\limits_{U_1} uh(du)$$

consisting of

$$\delta - \frac{1}{2}\sigma_1^2 > 0$$

and a jump component

$$2\lambda_1 \int\limits_{U_1} \ln(1+u)(1+u)h(du) - \lambda_1 \int\limits_{U_1} uh(du)$$

where in

$$\int\limits_{U_1} \ln(1+u)(1+u)h(du)$$

the logarithm is only defined for $u > -1$. Again, if the positive jumps outweigh the negative jumps the integral will be positive; otherwise it is negative. Assuming the denominator and nominator are positive, the derivative according to λ_1 becomes negative.

Similarly, the derivative according to z is

$$\begin{aligned}
\frac{\partial \beta_1}{\partial u} &= -\frac{\frac{\partial g}{\partial u}}{\frac{\partial g}{\partial \beta_1}} \\
&= -\frac{2\int\limits_{U_1} \left[\beta\left(1+u\right)^{\beta-1} - \frac{1}{2}\beta\right] \upsilon_1(du)}{\delta + \frac{1}{2}\sigma_1^2(\beta_1 - 1) - \frac{1}{2}\sigma_1^2\beta_1 + 2\int\limits_{U_1} \left[\ln(1+u)(1+u)^{\beta_1} - \frac{1}{2}u\right] \upsilon_1(du)} \\
&= -\frac{2\beta\int\limits_{U_1} \left[\left(1+u\right)^{\beta-1} - \frac{1}{2}\right] \upsilon_1(du)}{\delta - \frac{1}{2}\sigma_1^2 + 2\lambda_1\int\limits_{U_1} \left[\ln(1+u)(1+u)^{\beta_1} - \frac{1}{2}u\right] \upsilon_1(du)}.
\end{aligned}$$

With a positive integral in the nominator the derivative becomes positive if $\int\limits_{U_1} \left[\ln(1+u)(1+u) - \frac{1}{2}u\right] \upsilon_1(du) > 0$ and $\delta > \frac{1}{2}\sigma_1^2$. Assuming that the denominator is positive the derivative according to z becomes negative.

Technical Note 3: Expected Time of Conflict

Investment threshold Y^* At the investment trigger point Y^* the value of the option must equal the net value obtained by exercising the option (value of the active project minus sunk cost of the investment). Hence the following must hold:

$$\begin{aligned} F(Y^*) &= V^{gross}(Y^*) - I. \\ &= \frac{Y^*}{\left(r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha \right)} - I \end{aligned}$$

$$B(Y^*)^\beta = \frac{Y^*}{\left(r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha \right)} - I.$$

The smooth-pasting condition requires that the two value functions meet tangentially:

$$\begin{aligned} &(F(Y^*))' = (V^{gross}(Y^*))' \\ &\Leftrightarrow B\beta(Y^*)^{\beta-1} = \frac{1}{\left(r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha \right)}. \end{aligned}$$

This implies

$$B(Y^*)^\beta = \frac{Y^*}{(r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha)\beta}.$$

Now we compute the threshold Y^*:

$$\begin{aligned} &\frac{Y^*}{r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha} - I \\ = \; &\frac{Y^*}{(r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha)\beta} \end{aligned}$$

$$\Leftrightarrow \frac{Y^*\beta - Y^*}{(r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha)\beta} = I$$

$$\Leftrightarrow \quad Y^*(\beta - 1) = (r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha)\beta I$$

$$\Leftrightarrow \quad Y^* = \frac{\beta}{\beta - 1}(r - \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) - \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha)I$$

Note that Y^* is a positive function due to $r > \int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) + \int\limits_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) + \alpha$.

Expected Time of Conflict: We can determine the expected time $E(\tilde{T})$ needed to reach a certain benefit level Y^* for the first time, given the present value $\tilde{Y}(0)$.

For the jump diffusion case the first passage problem can be solved analytically if we assume an explicit distribution of the jump sizes. According to Kou and Wang (2003) we assume the double exponential distribution

$$h(z) = p\eta_1 e^{-\eta_1 z} 1_{\{z \geq 0\}} + q\eta_2 e^{\eta_2 z} 1_{\{z < 0\}}$$

for the jump sizes, where p is the probability of a positive jump and q for a negative jump respectively. $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$ are the means of the two exponential distributions. The moment generating function for $\tilde{Y}(t)$ with $\theta \in (-\eta_2, \eta_1)$ is

$$\phi(\theta, t) := E(e^{\theta \tilde{Y}(t)}) = \exp(G(\theta)t)$$

where the function G is defined as

$$G(x) := x\delta + \frac{1}{2}x^2\sigma_1^2 + \lambda_1 \left(\frac{p\eta_1}{\eta_1 - x} + \frac{p\eta_2}{\eta_2 + x} - 1 \right).$$

For jump diffusion processes the study of first passage times has to consider the exact hit of a constant boundary as well as an overshoot. Accordingly, two cases have to be distinguished. The Laplace transformation of the first hitting time, when $\tilde{Y}(t)$

hits the boundary Y^* exactly[14] is :

$$E(e^{-\varepsilon \tilde{T}_i} 1_{\{\tilde{Y}(\tilde{T}_i)=Y^*\}}) = \frac{\eta_1 - \beta_{1,\varepsilon}}{\beta_{2,\varepsilon} - \beta_{1,\varepsilon}} e^{-Y^* \beta_{1,\varepsilon}} + \frac{\beta_{2,\varepsilon} - \eta_1}{\beta_{2,\varepsilon} - \beta_{1,\varepsilon}} e^{-Y^* \beta_{2,\varepsilon}}$$

with $\mu_{1,\varepsilon}$ and $\mu_{2,\varepsilon}$ being the only positive roots of $G(\beta) = \varepsilon$ and $0 < \beta_{1,\varepsilon} < \eta_1 < \beta_{2,\varepsilon} < \infty$. For every overshoot $\tilde{Y}(\tilde{T}) - Y^*$ the Laplace transformation is

$$E(e^{-\varepsilon \tilde{T}} 1_{\{\tilde{Y}(\tilde{T}_i)-Y^*>y\}}) = e^{-\eta_1 y} \frac{\left(\eta_1 - \mu_{1,\varepsilon}\right)\left(\mu_{2,\varepsilon} - \eta_1\right)}{\eta_1 \left(\mu_{2,\varepsilon} - \mu_{1,\varepsilon}\right)} \left(e^{-Y^* \mu_{1,\varepsilon}} - e^{-Y^* \mu_{2,\varepsilon}}\right) \quad \text{for all } y \geq 0.$$

The expectation of the first passage time is finite, i.e. $E(T^*) < \infty$, if and only if the overall drift of the jump diffusion process is positive. Hence,

$$E(T^*) < \infty \Leftrightarrow \bar{u} = \delta + \lambda_1 \left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right) > 0.$$

Now for $\bar{u} > 0$ we determine the first passage time as

$$E(T^*) = \frac{1}{\bar{u}} \left[Y^* + \frac{\mu_2^* - \eta_1}{\eta_1 \mu_2^*} (1 - e^{-Y^* \mu_2^*}) \right] \tag{13}$$

where μ_2^* is defined as the unique root of $G(\mu_2^*) = 0$ with $0 < \eta_1 < \mu_2^* < \infty$.

[14] See Kou and Wang (2003) Theorem 3.1.

Technical Note 4: Determinants of the Expected Time of Conflict

The derivative of $E(\tilde{T})$ with respect to λ_1 is

$$\begin{aligned}
\frac{\partial E(T^*)}{\partial \lambda_1} &= -\frac{\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}{\left[\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)\right]^2}\left[Y^*+\frac{\mu_2^*-\eta_1}{\eta_1\mu_2^*}(1-e^{-Y^*\mu_2^*})\right] \\
&\quad +\frac{1}{\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}\left[\begin{array}{c} -\frac{\frac{\partial\beta}{\partial\lambda_1}}{(\beta-1)^2}(r-\int\limits_{f^{-1}(U_2)} z\upsilon_2(dz) \\ -\int\limits_{U_2}\left[\ln(1+z)-z\right]\upsilon_2(dz)-\alpha)I \\ -\frac{\partial Y^*}{\partial\lambda_1}\frac{\mu_2^*-\eta_1}{\eta_1}e^{-Y^*\mu_2^*} \end{array}\right] \\
&= -\underbrace{\frac{\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}{\left[\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)\right]^2}}_{(1)}\underbrace{\left[Y^*+\frac{\mu_2^*-\eta_1}{\eta_1\mu_2^*}(1-e^{-Y^*\mu_2^*})\right]}_{(2)} \\
&\quad +\frac{1}{\underbrace{\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}_{(3)}}\underbrace{\left[\begin{array}{c} -\frac{\frac{\partial\beta}{\partial\lambda_1}}{(\beta-1)^2}(r-\int\limits_{f^{-1}(U_2)} z\upsilon(dz) \\ -\int\limits_{U_2}\left[\ln(1+z)-z\right]\upsilon_2(dz)-\alpha)I \end{array}\right]}_{(4)} \\
&\quad \cdot\underbrace{(1-\frac{\mu_2^*-\eta_1}{\eta_1}e^{-Y^*\mu_2^*})}_{(5)}
\end{aligned}$$

For the first term (1) we obtain

$$\frac{\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}{\left[\delta+\lambda_1\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)\right]^2}>0$$

$$\Leftrightarrow \frac{p}{\eta_1}-\frac{q}{\eta_2}>0 \ \text{ with } q=1-p.$$

With the same condition, we obtain a positive sign also for term (3)

$$\frac{1}{\delta + \lambda_1 \left(\frac{p}{\eta_1} - \frac{q}{\eta_2} \right)} > 0.$$

For the second term (2) it holds that

$$\underbrace{Y^*}_{>0} + \underbrace{\frac{\mu_2^* - \eta_1}{\eta_1 \mu_2^*}}_{>0} \underbrace{(1 - e^{-Y^* \mu_2^*})}_{\geq 0} > 0.$$

The sign of the fourth term (4) depends on whether $\frac{\partial \beta}{\partial \lambda_1}$ is positive or negative. Assuming $\frac{\partial \beta}{\partial \lambda_1} < 0$ then term (4) becomes

$$-\frac{\frac{\partial \beta}{\partial \lambda_1}}{(\beta - 1)^2} \underbrace{(r - \int_{f^{-1}(U_2)} z \upsilon(dz) - \int_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha)}_{>0} \underbrace{I}_{>0} > 0$$

The last term (5)

$$1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*}$$

is negative if

$$1 < \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*}.$$

Summarizing all conditions leads to $\frac{\partial E(T^*)}{\partial \lambda_1} < 0$.

Now we consider the derivative of $E(T^*)$ according to u.

$$\frac{\partial E(T^*)}{\partial u} = \underbrace{\frac{1}{\delta + \lambda \left(\frac{p}{\eta_1} - \frac{q}{\eta_2} \right)}}_{(1)} \underbrace{\left[\begin{array}{c} -\frac{\frac{\partial \beta}{\partial u}}{(\beta-1)^2} (r - \int_{f^{-1}(U_2)} z \upsilon(dz) \\ -\int_{U_2} [\ln(1+z) - z]\, \upsilon_2(dz) - \alpha) I(T^*) \end{array} \right]}_{(2)}$$
$$\cdot \underbrace{(1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*})}_{(3)}.$$

As before, the term (1) is positive for $p > \frac{\eta_1}{\eta_2 + \eta_1}$. Accordingly, the last component

(3) is negative for $1 < \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*}$. The sign of (2) depends on whether $\frac{\partial \beta}{\partial u} \gtrless 0$. Assuming that $\frac{\partial \beta}{\partial u} < 0$ it follows that $\frac{\partial E(\tilde{T})}{\partial u} < 0$.

The derivative of $E(T^*)$ according to λ_2

$$
\begin{aligned}
\frac{\partial E(T^*)}{\partial \lambda_2} &= \frac{1}{\delta + \lambda_1 \left(\frac{p}{\eta_1} - \frac{q}{\eta_2} \right)} \left[\underbrace{\frac{\beta}{\beta - 1}}_{>0} \left(\begin{array}{c} - \int\limits_{f^{-1}(U_2)} z h(dz) \\ - \int\limits_{U_2} \left[\ln(1+z) - z \right] h(dz) \end{array} \right) \underbrace{I}_{>0} \right. \\
&\qquad \left. - \frac{\partial Y^*}{\partial \lambda_2} \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*} \right] \\
&= \underbrace{\frac{1}{\delta + \lambda_1 \left(\frac{p}{\eta_1} - \frac{q}{\eta_2} \right)}}_{(1)} \underbrace{\left[\underbrace{\frac{\beta}{\beta - 1}}_{>0} \left(\begin{array}{c} - \int\limits_{f^{-1}(U_2)} z h(dz) \\ - \int\limits_{U_2} \left[\ln(1+z) - z \right] h(dz) \end{array} \right) \underbrace{I}_{>0} \right]}_{(2)} \\
&\qquad \cdot \underbrace{\left(1 - \frac{\mu_2^* - \eta_1}{\eta_1} e^{-Y^* \mu_2^*} \right)}_{(3)}.
\end{aligned}
$$

From the conditions from above (1) is positive and (3) is negative. Hence, the sign of $\frac{\partial E(T^*)}{\partial \lambda_2}$ depends on the second term and especially on the sign of $- \int\limits_{f^{-1}(U_2)} z h(dz) - \int\limits_{U_2} \left[\ln(1+z) - z \right] h(dz)$. Assuming more negative than positive jumps lead to a positive sign.

For the derivative of $E(T^*)$ according to z we obtain

$$\frac{\partial E(T^*)}{\partial z} = \underbrace{\frac{1}{\delta+\lambda\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}}_{(1)} \underbrace{\left[\begin{array}{c} -\frac{\beta}{\beta-1}(\int\limits_{f^{-1}(U_2)} 1\upsilon_2(dz) \\ +\int\limits_{U_2}\left[\frac{1}{(1+z)}-1\right]\upsilon_2(dz))I \end{array}\right]}_{(2)} \underbrace{(1-\frac{\mu_2^*-\eta_1}{\eta_1}e^{-Y^*\mu_2^*})}_{(3)}$$

$$= \underbrace{\frac{1}{\delta+\lambda\left(\frac{p}{\eta_1}-\frac{q}{\eta_2}\right)}}_{(>0)} \underbrace{\left[\begin{array}{c} -\underbrace{\frac{\beta}{\beta-1}}_{>0}(\int\limits_{f^{-1}(U_2)} 1\upsilon_2(dz) \\ +\int\limits_{U_2}\left[\frac{1}{(1+z)}-1\right]\upsilon_2(dz))\underbrace{I}_{>0} \end{array}\right]}_{(2)}$$

$$\cdot\underbrace{(1-\frac{\mu_2^*-\eta_1}{\eta_1}e^{-Y^*\mu_2^*})}_{(<0)}$$

According to the above assumptions (1) is positive and (3) is negative. The second term again depends on the jump part. However, even if there are more negative than positive jumps the effect on the jump part is not so large as to make the sign negative. Therefore the sign of the derivative is negative.