

Hall, Peter; Härdle, Wolfgang; Kleinow, Torsten; Schmidt, Peter

Working Paper

Semiparametric bootstrap approach to hypothesis tests and confidence intervals for the hurst coefficient

SFB 373 Discussion Paper, No. 1999,62

Provided in Cooperation with:

Collaborative Research Center 373: Quantification and Simulation of Economic Processes, Humboldt University Berlin

Suggested Citation: Hall, Peter; Härdle, Wolfgang; Kleinow, Torsten; Schmidt, Peter (1999) : Semiparametric bootstrap approach to hypothesis tests and confidence intervals for the hurst coefficient, SFB 373 Discussion Paper, No. 1999,62, Humboldt University of Berlin, Interdisciplinary Research Project 373: Quantification and Simulation of Economic Processes, Berlin,
<http://nbn-resolving.de/urn:nbn:de:kobv:11-10046527>

This Version is available at:

<http://hdl.handle.net/10419/61786>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

**SEMIPARAMETRIC BOOTSTRAP APPROACH
TO HYPOTHESIS TESTS AND CONFIDENCE INTERVALS
FOR THE HURST COEFFICIENT**

Peter Hall¹ Wolfgang Härdle² Torsten Kleinow² Peter Schmidt³

ABSTRACT. A major application of rescaled adjusted range analysis (R–S analysis) is to the study of price fluctuations in financial markets. There, the value of the Hurst constant, H , in a time series may be interpreted as an indicator of the irregularity of the price of a commodity, currency or similar quantity. Interval estimation and hypothesis testing for H are central to comparative quantitative analysis. In this paper we propose a new bootstrap, or Monte Carlo, approach to such problems. Traditional bootstrap methods in this context are based on fitting a process chosen from a wide but relatively conventional range of discrete time series models, including autoregressions, moving averages, autoregressive moving averages and many more. By way of contrast we suggest simulation using a single type of continuous-time process, with its fractal dimension. We provide theoretical justification for this method, and explore its numerical properties and statistical performance by application to real data on commodity prices and exchange rates.

KEYWORDS. Box-counting method, commodity price, financial market, fractal dimension, fractional Brownian motion, Gaussian process, long-range dependence, Monte Carlo, R–S analysis, self affinity, self similarity.

SHORT TITLE. Bootstrap for Hurst constant.

¹Centre for Mathematics and its Applications, Australian National University, Canberra, ACT 0200, Australia

²Institut für Statistik und Ökonometrie, Humboldt-Universität zu Berlin, Spandauer Str. 1, D–10178 Berlin, Germany

³Quantitative Research, Bankgesellschaft Berlin AG, Germany

The research of this paper was carried out within the Sonderforschungsbereich 373 at Humboldt University Berlin and was printed using funds made available by the Deutsche Forschungsgemeinschaft

1. INTRODUCTION

R–S analysis has its roots in early work of the British hydrologist H.E. Hurst, who investigated dependence properties of phenomena such as levels of the River Nile. The Hurst constant H , as the index of dependence is often called, always lies between 0 and 1, and equals $\frac{1}{2}$ for processes that have independent increments. Particular interest focuses on the hypothesis that $H > \frac{1}{2}$, indicating relatively long-range dependence. For example, Hurst observed that $H = 0.91$ in the case of Nile data, indicating a strength of dependence that was well beyond what could be adequately explained assuming independent increments.

Today, a principal application of R–S analysis is to the study of fluctuations in financial markets, where the value of H is variously interpreted as an indicator of range of dependence, of irregularity and of nervousness. (Adler, 1981, coined the word ‘erraticism’ to denote a quantitative measure of ‘nervousness’.) To elucidate this point we note that the fractal dimension D of sample paths of a locally self-similar or self-affine random process increases monotonically with the irregularity of those paths; and that $D = 2 - H$ (see e.g. Berry and Hannay, 1978; Sayles and Thomas, 1978; Adler, 1981, Chapter 8; Mandelbrot, Passoja and Paullay, 1984; Hall, Matthews and Platen, 1996). Therefore, a process with higher Hurst constant is more regular, or less erratic, or less ‘nervous’ than one with a lower value. For example, a time series of commodity prices that is characterised by a larger Hurst constant enjoys greater stability, over at least short periods of time; and trade in that commodity might be said to be subject to less nervousness. See for example Peters (1994).

Thus, point and interval estimation of the Hurst constant can be basic to quantitative descriptions of market fluctuations. And testing for significant differences between two Hurst constants, or between one constant and the value

$\frac{1}{2}$, is fundamental to comparative quantitative analysis of market ‘nervousness’. In this paper we suggest bootstrap, or Monte Carlo, methods for constructing confidence intervals and hypothesis tests for Hurst indices.

Our methods are based on the estimator \hat{H} of H derived from R–S analysis, and involve simulating the sampled process using a time-adjusted version of fractional Brownian motion. We argue that, since the ‘S’ part of R–S analysis corrects for inhomogeneities in the data, it is unnecessary to reproduce them in the bootstrap algorithm.

This approach differs fundamentally from more traditional methods currently used for simulation, where the model is taken to be a relatively conventional discrete time series such as an autoregression, or moving average, or autoregressive moving average. See for example Peters (1994, Chapter 5). Instead, we suggest simulating a single type of continuous stochastic process, where the degree of irregularity is determined empirically through an estimator of H . We justify this approach through theoretical analysis, and assess its numerical and statistical properties using applications to real data on stock prices.

The idea of basing the bootstrap method on a continuous rather than a discrete stochastic process has been suggested before, but in the very different context of bootstrap methods for spatial samples of data on surface roughness (Davies and Hall, 1998). There, the ‘S’ part of R–S analysis is usually omitted, since the observed process is generally scale-homogeneous. Such bootstrap methods are nonstandard, not least because they conform to neither the parametric nor nonparametric bootstrap approaches. They fall midway between the two, and might fairly be said to be semiparametric bootstrap methods.

2. METHODOLOGY AND THEORY

2.1. R–S Analysis. We observe a stochastic process X_t at time points $t \in \mathcal{I} = \{0, \dots, N\}$. Let n be an integer that is small relative to N (asymptotically, as $n/N \rightarrow \infty$), and let A denote the integer part of N/n . Divide the ‘interval’ \mathcal{I} into A consecutive ‘subintervals’, each of length n and with overlapping endpoints. In every subinterval correct the original datum X_t for location, using the mean slope of the process in the subinterval, obtaining $X_t - (t/n)(X_{an} - X_{(a-1)n})$ for all t with $(a-1)n \leq t \leq an$ and for all $a = 1, \dots, A$. Over the a ’th subinterval $\mathcal{I}_a = \{(a-1)n, (a-1)n+1, \dots, an\}$, for $1 \leq a \leq A$, construct the smallest box (with sides parallel to the coordinate axes) such that the box contains all the fluctuations of $X_t - (t/n)(X_{an} - X_{(a-1)n})$ that occur within \mathcal{I}_a . Then, the height of the box equals

$$R_a = \max_{(a-1)n \leq t \leq an} \left\{ X_t - \frac{t}{n}(X_{an} - X_{(a-1)n}) \right\} - \min_{(a-1)n \leq t \leq an} \left\{ X_t - \frac{t}{n}(X_{an} - X_{(a-1)n}) \right\}$$

Let S_a denote the empirical standard error of the n variables $X_t - X_{t-1}$, for $(a-1)n+1 \leq t \leq an$. If the process X is stationary then S_a varies little with a ; in other cases, dividing R_a by S_a corrects for the main effects of scale inhomogeneity in both spatial and temporal domains.

The total area of the boxes, corrected for scale, is proportional in n to

$$\left(\frac{R}{S}\right)_n := A^{-1} \sum_{a=1}^A \frac{R_a}{S_a}. \quad (2.1)$$

The slope \hat{H} of the regression of $\log(R/S)_n$ on $\log n$, for k values of n , may be taken as an estimator of the Hurst constant H describing long-range dependence of the process X . See for example Beran (1994, Chapter 1) and Peters (1994, Chapters 4–6).

This R–S analysis, or ‘rescaled adjusted range’ analysis, dates from Hurst

(1951). If the process X is stationary then correction for scale is not strictly necessary, and we may take each S_a to be the constant 1. In that case the R–S statistic \hat{H} is a version of the box-counting estimator that is widely used in physical science applications; see for example Carter, Cawley and Mauldin (1988), Sullivan and Hunt (1988) and Hunt (1990). The box-counting estimator is related to the capacity definition of fractal dimension (Barnsley, 1988, p. 172ff), and the R–S estimator may be interpreted in the same way. Statistical properties of the box-counting estimator have been discussed by Hall and Wood (1993).

A more detailed analysis, exploiting dependence among the errors in the regression of $\log(R/S)_n$ on $\log n$, may be undertaken in place of R–S analysis. See Kent and Wood (1997) for a version of this approach in the case where scale correction is unnecessary. However, as Kent and Wood show, the advantages of the approach tend to be asymptotic in character, and sample sizes may need to be extremely large before real improvements are obtained.

2.2. Approximating the distribution of \hat{H} . Depending on the value of H , and on the nature of the stochastic process X , the asymptotic distribution of \hat{H} (as $N \rightarrow \infty$, for fixed k) can be Normal or Rosenblatt; the latter was introduced by Taqqu (1975), following work of Rosenblatt (1961). (More concisely, in the Rosenblatt case the asymptotic distribution of \hat{H} is that of a finite linear form in correlated Rosenblatt-distributed random variables, but for simplicity we shall refer to this as a Rosenblatt distribution.) Indeed, the asymptotic distribution of \hat{H} can be Rosenblatt for $3/4 < H < 1$ and Normal for $0 < H \leq 3/4$; see section 2.4. The Rosenblatt distribution that is relevant here is particularly complex, and its shape depends intimately on the unknown value of H . The distribution has not been tabulated.

If the value of k is large, i.e. the number of values of n for the linear regression is large then the Rosenblatt approximation becomes, by virtue of the central limit theorem, similar to the Normal approximation. However, the asymptotic variance is difficult to calculate. Moreover, it is known from work of Hall and Wood (1993) and Constantine and Hall (1994) that, due to long-range dependence, statistical performance of the estimator \hat{H} generally deteriorates for large k , and in fact optimal mean squared error properties are often achieved by keeping k fixed as N increases.

These considerations motivate Monte Carlo analysis, rather than more conventional asymptotic methods, in the range $3/4 < H < 1$. Even when H lies outside this interval there is much to be said for taking a Monte Carlo approach, however. Monte Carlo simulation can be expected to capture many of the penultimate, second-order effects that describe departure of the distribution of \hat{H} from its asymptotic limit, so that even if the limiting distribution were known, the Monte Carlo approach would be expected to provide somewhat greater accuracy than the conventional asymptotic approximation. The second-order effects arise from finiteness of N , and from the fact that stochastic fluctuations of the scale correction in R-S analysis influence the true distribution of \hat{H} even though they do not affect the limit distribution.

A more familiar example of the same phenomenon is use of Student's t distribution to approximate the distribution of a Studentised ratio, even when the sampled distribution is not exactly Normally distributed. The Student's t approximation represents a 'penultimate' form of the Normal 'ultimate' limiting distribution. Even for data from a skew distribution the Student's t approach generally captures finite-sample properties better than the Normal approximation, despite the fact that it does not capture all second-order departures from

Normality.

We shall show in section 2.4 that in many cases the limiting distribution of \widehat{H} depends only on H and a temporal scale factor. The spatial scale of the process X , and the process's potential heteroscedasticity and non-Gaussianity, do not feature in first-order asymptotic results. In large part this is a result of the 'S' component of R-S analysis. Therefore, the limiting distribution of \widehat{H} is the same as it would be if X_t were ζ_t , where ζ is an elementary self-similar Gaussian process.

The Gaussian process that we have in mind is fractional Brownian motion, defined by $P(\zeta_0 = 0) = 1$, $E(\zeta_t) = 0$ and $E(\zeta_{s+t} - \zeta_s)^2 = |t|^\alpha$ for all s and t , where $\alpha = 2H \in (0, 2)$. Equivalently, ζ_t is defined to be that Gaussian process with zero mean and covariance

$$\gamma(s, t) \equiv \text{cov}(\zeta_s, \zeta_t) = \frac{1}{2} (|s|^\alpha + |t|^\alpha - |s - t|^\alpha).$$

See for example Beran (1994, p. 51ff) and Peters (1994, p. 183ff).

We may simulate from a discrete approximation to ζ_t , say on the points $t_j = j/\nu$ for a large integer ν , by forming the $(2p\nu + 1) \times (2p\nu + 1)$ covariance matrix, M , of which the (i, j) 'th element is $\gamma(t_i, t_j)$ for $-p\nu \leq i, j \leq p\nu$ (p an integer); and then using the spectral decomposition of M to generate Gaussian random $(2p\nu + 1)$ -vectors with this covariance. Alternatively, methods of Davies and Harte (1987), or those of Wood and Chan (1994) or of the many authors whose work is surveyed by Wood and Chan, may be employed.

Denote the original data set $\{X_1, \dots, X_N\}$ by \mathcal{X} . Our bootstrap algorithm is as follows. Compute the estimator \widehat{H} , and in the steps below, take $\alpha = 2\widehat{H}$ when constructing the fractional Brownian motion ζ , conditional on \mathcal{X} . Let X_t^* , for $0 \leq t \leq N$, denote a realisation of the process ζ . Compute the corresponding

value \widehat{H}^* of \widehat{H} . Take the conditional distribution of \widehat{H}^* , given the data \mathcal{X} , to be a Monte Carlo approximation to the unconditional distribution of \widehat{H} ; or alternatively, take the conditional distribution of $\widehat{H}^* - \widehat{H}$ to approximate the unconditional distribution of $\widehat{H} - H$. These approaches give rise respectively to the two percentile methods discussed in section 2.3.

Some of the second-order properties that this approach does not capture may be addressed by fitting a smooth estimate of scale to the process ζ . For example, we might model the variance function $\sigma(t)^2 = \text{var}(X_t)$, and thereby compute an estimator $\hat{\sigma}(\cdot)$ of $\sigma(\cdot)$; and simulate from the process $\hat{\sigma}(t) |t|^{-\alpha/2} \zeta_t$ rather than from ζ_t . In this case we should translate the time interval so as to avoid the origin.

2.3. Confidence regions and hypothesis testing. Confidence intervals and hypothesis tests for H may be constructed using either of the two standard bootstrap percentile methods. For example, a nominal 95% confidence interval for H is given by $(\widehat{H}^{(1)}, \widehat{H}^{(2)})$, where $\widehat{H}^{(1)}$ and $\widehat{H}^{(2)}$ are defined by either $P(\widehat{H}^* \leq \widehat{H}^{(1)} | \mathcal{X}) = P(\widehat{H}^* \geq \widehat{H}^{(2)} | \mathcal{X}) = 0.025$ or $P(\widehat{H}^* - \widehat{H} \leq \widehat{H} - \widehat{H}^{(2)} | \mathcal{X}) = P(\widehat{H}^* - \widehat{H} \geq \widehat{H} - \widehat{H}^{(1)} | \mathcal{X}) = 0.025$. A test at the 5% level of the null hypothesis that $H = \frac{1}{2}$, corresponding to X being a random walk, is to reject the null if $(\widehat{H}^{(1)}, \widehat{H}^{(2)})$ does not contain the point $\frac{1}{2}$.

Given two independent samples from long-range dependent processes, leading to respective estimators \widehat{H}_1 and \widehat{H}_2 of Hurst constants, we may generate independent realisations from respective stochastic processes $\zeta^{(1)}$ and $\zeta^{(2)}$, and thereby compute a bootstrap approximation to the distribution of $\widehat{H}_1 - \widehat{H}_2$ or of $\widehat{H}_1 - \widehat{H}_2 - (H_1 - H_2)$. As before, this may be used as the basis of percentile-bootstrap confidence intervals and hypothesis tests for $H_1 - H_2$.

These techniques, being based on the percentile bootstrap, lack the piv-

otalness that bootstrap methods for confidence procedures should ideally enjoy. However, they have asymptotically correct levels, as N increases. Moreover, even when the statistic \hat{H} admits a Normal asymptotic distribution we lack a simple, computable variance estimator with which to correct for scale. And when the limiting distribution is Rosenblatt, rather than Normal, scale corrections are not sufficient to produce pivotalness, since the shape of the Rosenblatt distribution depends on the unknown Hurst constant through more than simply scale. For these reasons we argue that the percentile- t bootstrap, often suggested in simpler problems as a pivotal method for constructing confidence intervals and hypothesis tests with relatively accurate levels (see for example Hall, 1992, p. 14f; Efron and Tibshirani, 1993, p. 158f; Shao and Tu, 1995, p. 94f; Davison and Hinkley, 1997, p. 29f), is not appropriate in the present setting.

Instead, level accuracy may be enhanced by using the double bootstrap (see for example Hall, 1992, p. 20ff; Efron and Tibshirani, 1993, p. 263ff; Shao and Tu, 1995, p. 155ff; Davison and Hinkley, 1997, p. 103ff). However, the relatively high orders of accuracy associated with double-bootstrap confidence procedures in simpler problems cannot be expected to be generally available here, since our Gaussian model based on fractional Brownian motion does not necessarily reflect all second-order features of the distribution of the sampled stochastic process X . It seems difficult to improve on this situation without introducing relatively complex high-order models for X .

2.4. Theoretical properties. Suppose the data X_t , $t \in \mathcal{I}$, are generated as $X_t = g(Y_{et}, t)$, where

- (a) g is a smooth bivariate function,
- (b) Y is a Gaussian process whose sample paths have fractal dimension $D =$

$2 - H$, and

(c) ϵ denotes a small positive constant.

The function g represents a possibly nonlinear transformation of Y , implying in particular that the observed process X is not necessarily Gaussian. Importantly, it allows a wide range of different types of inhomogeneity. By taking ϵ small we ensure that even if t_1 is moderately distant from t_2 , X_{t_1} can be strongly correlated with X_{t_2} . This confers long-range dependence on the observed process. There is no difficulty in extending our results to the case where X is a function of a vector of Gaussian processes, say $X_t = g(Y_{\epsilon t}^{(1)}, \dots, Y_{\epsilon t}^{(k)}, t)$. Here the Hurst index that prevails equals 2 minus the fractal dimension of sample paths of the process $Y^{(j)}$ that has the roughest sample paths. It is also possible to incorporate a smooth, monotone, nonlinear transformation of the time variable t . However, the simpler setting prescribed by condition (a) conveys the important characteristics of these more complex models.

We claim that, under models of the type characterised by (a)–(c), \widehat{H} is consistent for H and has an asymptotic distribution that is either Normal or of the type introduced by Rosenblatt (1961). To formulate this assertion as a mathematical theorem we first elaborate on (a)–(c), as follows. Assume that

(A) the derivatives

$$g_{j_1 j_2}(y, t) = (\partial/\partial y)^{j_1} (\partial/\partial t)^{j_2} g(y, t)$$

are bounded for each $j_1, j_2 \geq 0$, and g_{10} does not vanish;

(B) the Gaussian process Y satisfies $E(Y_t) \equiv 0$, and for constants $c > 0$, $\alpha = 2H \in (\frac{1}{2}, 2)$ and $\beta > \min(\frac{1}{2}, 2 - \alpha)$, $E(Y_{s+t} - Y_s)^2 = c|t|^\alpha + O(|t|^{\alpha+\beta})$, uniformly in $s \in \mathcal{J} = [0, 1]$, as $t \rightarrow 0$; and

(C) $\epsilon = 1/N \rightarrow 0$,

and we define \widehat{H} by regression of $\log(R/S)_n$ on $\log n$ for a fixed number, k , of values $\ell_1 m, \dots, \ell_k m$ of n , where ℓ_1, \dots, ℓ_k are fixed and $m = m(\epsilon) \rightarrow \infty$ as $\epsilon \rightarrow 0$, in such a manner that $m^{-1} + m\epsilon = O(\epsilon^a)$ for some $a > 0$. Define $\xi = m\epsilon$ and

$$t_\xi = \begin{cases} \xi^{2(1-H)} & \text{if } 3/4 < H < 1 \\ (\xi \log \xi^{-1})^{1/2} & \text{if } H = 3/4 \\ \xi^{1/2} & \text{if } 0 < H < 3/4, \end{cases}$$

which converges to 0 as $\epsilon \rightarrow 0$. Then, we claim that $\widehat{H} - H$ may be expressed as $t_\xi Z_\xi$, where Z_ξ has a proper limiting distribution as $\epsilon \rightarrow 0$. (The regularity conditions may be relaxed in many circumstances. For example, the restriction in (B) that $\alpha > \frac{1}{2}$ may be dropped if $g(y, t) \equiv y$, and also in some other cases. The boundedness condition on derivatives of g may also be relaxed.)

Crucially, the limiting distribution of \widehat{H} depends only on H and ℓ_1, \dots, ℓ_k ; it does not depend on g or on the scale constant, c , appearing in the first-order approximation to covariance. The main effects of scale and heteroscedasticity, entering through g and c , have cancelled due to rescaling by the terms S_a in (2.1). The limiting distribution is Normal when $0 < H \leq 3/4$, and a finite linear combination of correlated Rosenblatt distributions when $3/4 < H < 1$. Outline proofs of all these assertions are given in the appendix.

The results are foreshadowed by those of Hall and Wood (1993) for box-counting estimators, of which \widehat{H} may be regarded as a scale-corrected version. We do not give the form of the limits, since it is complex (particularly in the Rosenblatt case), but it is of the type discussed by Hall and Wood (1993, p. 252). The relationships between statistical properties of a Gaussian process (e.g. Y), and of a smooth function of that process (e.g. X), have been addressed by Hall and Roy (1994).

The fact that the limiting distribution depends only on H and ℓ_1, \dots, ℓ_k justifies the bootstrap methods suggested in section 2.2. Specifically, since the bootstrap algorithm preserves the way in which H and ℓ_1, \dots, ℓ_k contribute to the limiting distribution, and since $\hat{H} \rightarrow H$ at a rate that is polynomial in ξ (indeed, at rate t_ξ), then the bootstrap produces confidence intervals and hypothesis tests that have asymptotically correct coverage. The fractional Brownian motion ζ , used as the basis for our simulations, is just one of many that could have been employed, satisfying condition (B) above.

Note particularly that we keep k fixed as ϵ decreases. If our regularity conditions were to allow $k = k(\epsilon)$ to diverge then the Rosenblatt limit would change to Normal, but as discussed by Constantine and Wood (1994), this would generally be at the expense of increased mean squared error of \hat{H} .

3. APPLICATION TO DATA

The aim of this section is to obtain an estimator \hat{H} of the Hurst coefficient H and to construct hypothesis tests and confidence intervals for H for the logarithm of the price process of certain German stocks.

Denote the logarithm of the price process of a stock (or index) by $\{X_t : 0 \leq t \leq T\}$. To estimate the Hurst coefficient H we applied R-S analysis, as described in section 2.1, to N discrete observations $\{X_n : n = 1, \dots, N\}$ of $\{X_t\}$ at times $t_1 \leq t_2 \leq \dots \leq t_N$,

For the empirical study we used 6900 observations ($N = 6900$) of 24 German blue chip stocks obtained from the Datastream/Primark's database from 8th of January 1973 to the 18th of June 1999. The blue chips are included in the DAX, an index comprising 30 German blue chip stocks. We analysed Datastream performance indices instead of prices in order to avoid jumps in the

respective time series due to dividend payments or rights issues. The obtained Hurst coefficients are shown in Table 1. Figure 1 shows the R–S plot for the price process of the stock of Volkswagen. The R–S plot also includes a line with slope 0.5, which correspond to Brownian motion. As one can see, the R–S line

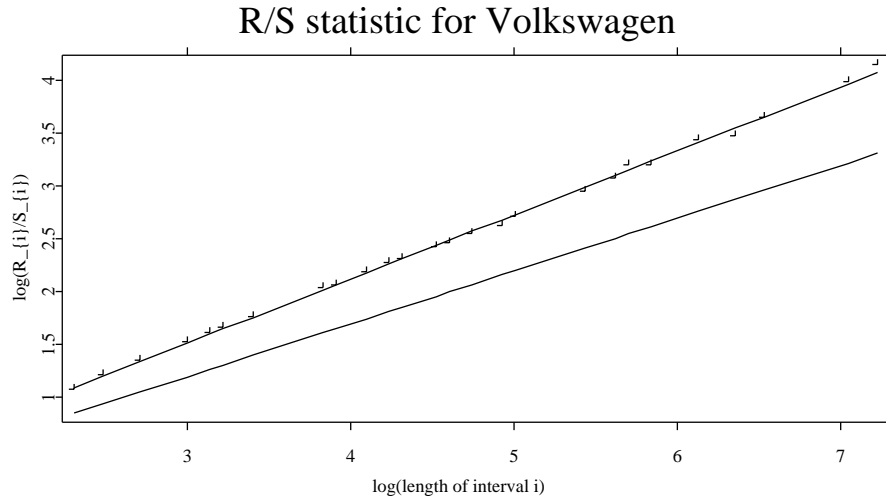


Figure 1: R–S plot for VW, $\hat{H} = 0.606$

has a different slope then it would have if the underlying process corresponded to a Brownian Motion.

In the first step of our empirical analysis we tested whether the Hurst coefficient of an asset was significantly different from 0.5 or not. A significant difference from 0.5 would indicate that X_t did not follow a Brownian Motion. In order to test the null hypothesis that $H = 0.5$, against the alternative $H \neq 0.5$, i.e.

$$h_0 : H = 0.5 \quad h_1 : H \neq 0.5,$$

we approximated the distribution of $\hat{H} - H$ conditional on the null hypothesis, and calculated the p-values, $P\{|\hat{H} - E\hat{H}| > |H_{\text{observed}} - E\hat{H}| \mid h_0\}$, of the

estimated \hat{H} . For this approximation the bootstrap algorithm described in Section 2 was used. For $H = 0.5$ the fractional Brownian Motion coincided with usual Brownian Motion, which we simulated as a random walk. An estimate of the conditional density of $\hat{H}^* - \hat{H}$, computed from 400 simulated random walks of length 6900, is shown in figure 2. Table 1 shows the p-values for the

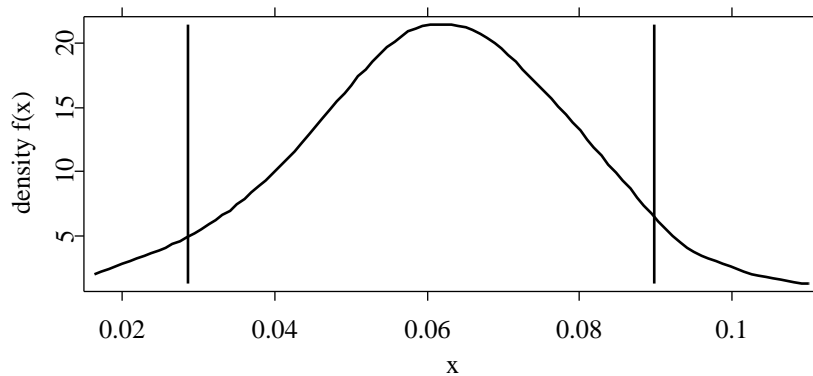


Figure 2: Estimated density of $\hat{H} - H$ for 400 simulated Brownian Motions with length 6900. The vertical lines determine the 0.05, 0.95 quantiles.

estimated Hurst coefficient of the stocks.

Our analysis suggests that the difference between the estimated Hurst index of the prices of BMW, Daimler, Mannesmann, Preussag, Siemens and Volkswagen, and the value the Hurst index would take if the stochastic process describing prices were Brownian motion, is so great that it cannot be adequately explained by stochastic fluctuations.

We studied the assets for which the estimated Hurst coefficient H was significantly different from 0.5. For our further analysis we assumed that the

asset	\hat{H}	p-value
Allianz	0.5642	0.6
BASF	0.5390	0.24
Bayer	0.5288	0.073
BMW	0.5851	0.05
Commerzbank	0.5536	0.88
Dt. Bank	0.5743	0.22
Daimler	0.5859	0.05
Degussa Hüls	0.5629	0.68
Dresdner Bank	0.5625	0.7
Hoechst	0.5420	0.37
HypoVereinsbank	0.5533	0.86
Karstadt	0.5552	0.95
Lufthansa	0.5584	0.89
Linde	0.5583	0.90
MAN	0.5605	0.79
Mannesmann	0.5856	0.05
Münchner Rück NA	0.5589	0.88
Preussag	0.5884	0.035
RWE	0.5398	0.29
Schering	0.5772	0.17
Siemens	0.6007	0
ThyssenKrupp	0.5794	0.13
Veba	0.5426	0.38
Volkswagen	0.6049	0

Table 1: Estimated Hurst coefficient of German stocks

logarithm of the price processes are self similar with stationary increments, i.e.

$$c^{-H}(X_{ct})_{t \in \mathbb{R}} =_d (X_t)_{t \in \mathbb{R}} \quad \text{for all } c > 0 \quad (1)$$

and for any $k \geq 1$ and any time points t_1, \dots, t_k ,

$$(X(t_1), \dots, X(t_k)) =_d (X(t_1 + c), \dots, X(t_k + c)) \quad \text{for all } c \in \mathbb{R} \quad (2)$$

Here, $Y =_d Z$ means that Y and Z have the same distribution. These assumptions are often made in literature on financial market analysis. A well known model is the Multifractal Model of Asset Returns (MMAR) introduced by Calvet, Fisher and Mandelbrot (1997). In this model the logarithms of prices are assumed to follow a fractional Brownian Motion, i.e.

$$X(t) - X(0) = B_H(\theta(t)),$$

where $\theta(t)$ is a multifractal process with continuous, non-decreasing paths and stationary increments.

Under assumptions 1 and 2 the autocorrelation function $\rho(k) = E[\{X(t) - EX(t)\} \{X(t+k) - EX(t+k)\}]$ of $X(t)$ is approximately of the form ck^{2H-2} . More precisely, the following holds (see Beran, 1994):

$$\frac{\rho(k)}{H(2H-1)k^{2H-2}} \longrightarrow 1 \quad 0 < H < 1, H \neq \frac{1}{2}, \quad k \longrightarrow \infty.$$

This means that for $\hat{H} > 0.5$, X_t has long memory. Stocks where long memory was detected are displayed in bold face in table 1.

The second step of our analysis was construction of confidence intervals. For this purpose we approximated the distribution of $\hat{H} - H$ by that of $\hat{H}^* - \hat{H}$, where \hat{H}^* denotes the estimated value of the Hurst coefficient of simulated fractional Brownian Motions with coefficient $\hat{\alpha} = 2\hat{H}$. That is, we computed the conditional (on $X(t)$) distribution of the bootstrap form of $\hat{H}^* - \hat{H}$, as

asset	0.9 confidence region	0.95 confidence region
BMW	[0.475, 0.579]	[0.466, 0.594]
Daimler	[0.476, 0.581]	[0.467, 0.596]
Mannesmann	[0.476, 0.580]	[0.467, 0.596]
Preussag	[0.481, 0.585]	[0.472, 0.601]
Siemens	[0.506, 0.610]	[0.497, 0.626]
Volkswagen	[0.514, 0.619]	[0.505, 0.634]

Table 2: Confidence regions for Hurst coefficients

an approximation to the unconditional distribution of $\hat{H} - H$. We applied the bootstrap method described in section 2. To simulate fractional Brownian motion we used methods from section 2.2 with $p = 1$ as well as the algorithm described in Beran (1994, p. 216). The latter is based on the finite Fourier transform of the autocovariance function of fractional Gaussian noise. Both methods lead to similar results.

The bootstrap densities for the different Hurst values of the assets which have significantly larger Hurst coefficient than a Brownian Motion were approximately the same except for the mean value. For this reason we calculated only the density of $\hat{H}^* - \hat{H}$ of the Volkswagen stock. It is shown in figure 3. The confidence intervals for the other assets were obtained by correcting this density for the different estimated Hurst coefficient. Table 2 shows the resulting confidence regions.

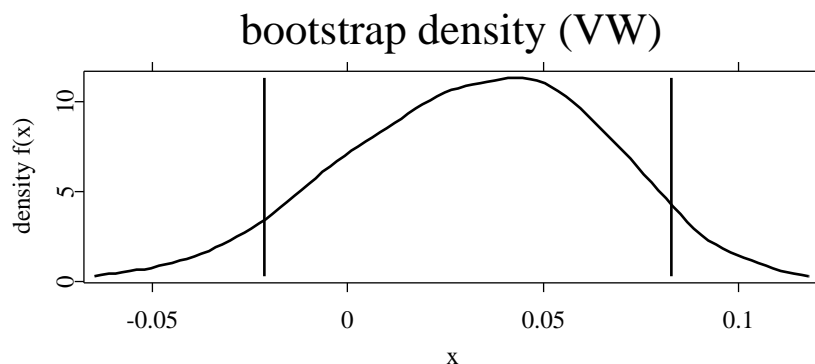


Figure 3: Bootstrap density of $\hat{H} - H$ for the Volkswagen stock. The vertical lines determine the 0.05, 0.95 quantiles.

REFERENCES

- ADLER, R.J. (1981). *The Geometry of Random Fields*. Wiley, New York.
- BARNESLEY, M. (1988). *Fractals Everywhere*. Academic Press, New York.
- BERAN, J. (1994). *Statistics for Long-Memory Processes*. Chapman and Hall, London.
- BERRY, M.V. AND HANNAY, J.H. (1978). Topography of random surfaces. *Nature* **273**, 573.
- CALVET, L., FISHER, A. AND MANDELBROT, B.B. (1997), A Multifractal Model of Asset Returns, Cowles Foundation Discussion Paper #1164
- CARTER, P.H., CAWLEY, R. AND MAULDIN, R.D. (1988). Mathematics of dimension measurements of graphs of functions. In: *Proc. Symp. Fractal Aspects of Materials, Disordered Systems*, Eds. D.A. Weitz, L.M. Sander and B.B. Mandelbrot, pp. 183–186. *Materials Research Society*, Pittsburgh, PA.
- CONSTANTINE, A.G. AND HALL, P. (1994). Characterizing surface smoothness via estimation of effective fractal dimension. *J. Roy. Statist. Soc. Ser. B* **56**, 97–113.
- DAVIES, S. AND HALL, P. (1998). Fractal analysis of surface roughness using spatial data. *J. Roy. Statist. Soc., Ser. B*, to appear.
- DAVIES, R.B. AND HARTE, D.S. (1987). Tests for Hurst effect. *Biometrika* **74**, 95–101.

- DAVISON, A.C. AND HINKLEY, D.V. (1997). *Bootstrap Methods and their Application*. Cambridge: Cambridge University Press.
- EFRON, B. AND TIBSHIRANI, R. (1993). *An Introduction to the Bootstrap*. Chapman and Hall, London.
- HALL, P. (1992). *The Bootstrap and Edgeworth Expansion*. Springer, New York.
- HALL, P., MATTHEWS, D. AND PLATEN, E. (1996). Algorithms for analyzing nonstationary time series with fractal noise. *J. Computat. Graph. Statist.* **5**, 351–364.
- HALL, P. and ROY, R. (1994). On the relationship between fractal dimension and fractal index for stationary stochastic processes. *Ann. Appl. Probab.* **4**, 241–253.
- HALL, P. and WOOD, A.T.A. (1993). On the performance of box-counting estimators of fractal dimension. *Biometrika* **80**, 246–252.
- HUNT, F. (1990). Error analysis and convergence of capacity dimension algorithms. *SIAM J. Appl. Math.* **50**, 307–321.
- HURST, H.E. (1951). Long-term storage capacity of reservoirs. *Trans. Amer. Soc. Civil Engineers* **116**, 770–799.
- KENT, J.T. AND WOOD, A.T.A. (1993). Estimating the fractal dimension of a locally self-similar Gaussian process by using increments. *J. Roy. Statist. Soc. Ser. B* **59**, 679–699.
- MANDELBROT, B.B., PASSOJA, D.E. AND PAULLAY, A.J. (1984). Fractal character of surfaces of metals. *Nature* **308**, 721–722.
- PETERS, E.E. (1994). *Fractal Market Analysis: Applying Chaos Theory to Investment and Economics*. Wiley, New York.
- ROSENBLATT, M. (1961). Independence and dependence. In: *Proc. 4th Berkeley Symp. Math. Statist. Probab.*, Ed. J. Neyman, pp. 411–433. University of California Press, Berkeley.
- SAYLES, R.S. AND THOMAS, T.R. (1978). Surface topography as a nonstationary random process. *Nature* **271**, 431–434.
- SHAO, J. AND TU, D. (1995). *The Jackknife and Bootstrap*. Springer, New York.
- SULLIVAN, F. AND HUNT, F. (1988). How to estimate capacity dimension. *Nuclear Phys. B. (Proc. Suppl.)* **5A**, 125–128.
- TAQQU, M.S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. Verw. Gebiete* **31**, 287–302.

WOOD, A.T.A. AND CHAN, G. (1994). Simulation of stationary Gaussian processes in $[0, 1]^d$. *J. Comput. Graph. Statist.* **3**, 409–432.

APPENDIX: Technical arguments

Put $Z_t = g(Y_t, t)$ and let $\mathcal{J} = [0, 1]$. Given $B > 0$, choose an integer B' so large that $B'\alpha > 2B$. Then, uniformly in $t_1, t_2 \in \mathcal{J}$,

$$Z_{t_1} - Z_{t_2} = \sum_{j_1=1}^{B'} \sum_{j_2=1}^{B'} \frac{1}{j_1!j_2!} (Y_{t_1} - Y_{t_2})^{j_1} (t_1 - t_2)^{j_2} g_{j_1 j_2}(Y_{t_2}, t_2) + O_p(|t_1 - t_2|^B). \quad (\text{A.1})$$

This formula provides the opportunity to develop Taylor expansions of quantities such as R_a/S_a . It turns out that only the first term in such expansions contributes to asymptotic results. Nevertheless, higher-order Taylor-expansion terms should be included since, prior to correction for their means and analysis of their size, they are potential first-order contributors to limit theory for $(R/S)_n$. In our work the contributions of these high-order terms will be denoted by Q_1, Q_2, \dots . For the sake of simplicity we ignore the mean correction in the definition of S_a .

Let $\mathcal{T} \subseteq \mathcal{J}$ denote a set of $n + 1$ equally-spaced points $t_0 < \dots < t_n$ within an interval of width $\delta = n\epsilon$, and write $S_{\mathcal{T}}$ and $U_{\mathcal{T}}$ for the empirical standard errors of the ‘samples’ $\{Z_{t_i} - Z_{t_{i-1}}, 1 \leq i \leq n\}$ and $\{Y_{t_i} - Y_{t_{i-1}}, 1 \leq i \leq n\}$, respectively. Then by (A.1), for all $\eta > 0$,

$$S_{\mathcal{T}}^2 = g_{10}(Y_{t_2}, t_2)^2 U_{\mathcal{T}}^2 + Q_1 + O_p(\epsilon^{(\alpha/2)+B-\eta}), \quad (\text{A.2})$$

$$\begin{aligned} R_{\mathcal{T}} &\equiv \max_{t \in \mathcal{T}} Z_t - \min_{t \in \mathcal{T}} Z_t \\ &= s |g_{10}(Y_{t_2}, t_2)| (Y_{T_{\mathcal{T}}} - Y_{T'_{\mathcal{T}}}) + Q_2 + O_p(\delta^B), \end{aligned} \quad (\text{A.3})$$

where $T_{\mathcal{T}} = \operatorname{argmax}_{t \in \mathcal{T}} Z_t$, $T'_{\mathcal{T}} = \operatorname{argmin}_{t \in \mathcal{T}} Z_t$, and s denotes the sign of g_{10} . Hence, for all $\eta > 0$,

$$\frac{R_{\mathcal{T}}}{S_{\mathcal{T}}} = \frac{s}{U_{\mathcal{T}}} (Y_{T_{\mathcal{T}}} - Y_{T'_{\mathcal{T}}}) + Q_3 + O_p(\delta^{\alpha/2} \epsilon^{B-(\alpha/2)-\eta} + \delta^B \epsilon^{-(\alpha/2)-\eta}), \quad (\text{A.4})$$

where Q_3 represents a series of ratios of terms, of the form $V/U_{\mathcal{T}}$, in Taylor expansions (in this sense, each summand is like the first term on the right-hand side of (A.4)), and the $O_p(\cdot)$ remainder is of the stated order uniformly in \mathcal{T} . Note particularly that in forming the leading ratio in (A.4) the contribution $g_{10}(Y_{t_2}, t_2)$ has cancelled from the leading terms in (A.2) and (A.3), and likewise the effect of the constant c (see condition (B) in section 2.4) may be seen to cancel. This results from the scaling aspect of R-S analysis, and explains why the process ζ from which we simulate when applying the bootstrap does not need to reflect either the properties of g or the value of c .

We deal with each ratio, V/W where $W = U_{\mathcal{T}}$, by expressing it as $w^{-1}(v + \Delta_V)(1 + \frac{1}{2}\Delta_W + \frac{3}{8}\Delta_W^2 + \dots)$, where $\Delta_V = V - v$, $\Delta_W = -(W^2 - w^2)/w^2$, $v = E(V)$ and $w^2 = E(W^2)$. For purposes of exposition we shall confine attention to the three main terms in such an expansion, i.e. to $(v/w) + (\Delta_V/w) + \frac{1}{2}v(\Delta_W/w)$, in the case $V = Y_{T_{\mathcal{T}}} - Y_{T'_{\mathcal{T}}}$ and $W = U_{\mathcal{T}}$. (Without loss of generality, $s = 1$.) Other terms may be treated similarly, although the argument is lengthy.

Let Δ_{V_a} , Δ_{W_a} , v_a and w_a denote versions of Δ_V , Δ_W , v and w when $\mathcal{T} = \mathcal{I}_a$, the latter defined in section 2.1. Note that, by condition (B), $w_a = w^0 \{1 + O(\xi^\beta)\}$ uniformly in a , where w^0 does not depend on a or n . Since $\beta > \min(\frac{1}{2}, 2-\alpha)$ (see condition (B)) then $\xi^\beta = o(t_\xi)$. Arguing thus it may be proved that A^{-1} times the sum over $1 \leq a \leq A$ of v_a/w_a equals $C \delta^{\alpha/2} (w^0)^{-1} \{1 + o(t_\xi)\}$, where $C > 0$ is a constant not depending on n .

Put $u = A^{-1} \delta^{-\alpha/2} w^0$, and let $S_\xi(n)$ equal u times the sum over $1 \leq$

$a \leq A$ of the term Δ_{V_a}/w_a . Methods of Hall and Wood (1993) may be used to show that $S_\xi(n)$ has variance asymptotic to a constant multiple of t_ξ^2 , and that for the k values of n being considered, the variables $S_\xi(n)/t_\xi$ have a joint asymptotic distribution which is k -variate Normal when $0 < H \leq 3/4$, and k -variate Rosenblatt (Rosenblatt, 1961; Taqqu, 1975) when $3/4 < H < 1$.

By considering properties of the variogram estimator of fractal dimension, methods of Constantine and Hall (1994) may be employed to prove that u times the sum over a of $v_a \Delta_{W_a}/w_a$ equals $o_p(t_\xi)$. (Here it is critical that m diverge to infinity.) If B is sufficiently large then u times the sum over a of the $O_p(\cdot)$ remainder at (A.4) also equals $o_p(t_\xi)$, and similar methods may be applied to terms represented by Q_3 in the Taylor expansion. (The high-order contributions to bias of \hat{H} include terms of order ξ^α , but since we assumed $\alpha > \frac{1}{2}$ then this equals $o(t_\xi)$.) Arguing thus we may ultimately show that $(R/S)_n = C\delta^{\alpha/2}(w^0)^{-1}\{1+S_\xi(n)+o_p(t_\xi)\}$. Hence, $\log(R/S)_n$ equals a quantity which does not depend on n and which goes into the intercept term in the regression, plus $(\alpha/2)\log n + S_\xi(n) + o_p(t_\xi)$. The result asserted in section 2.3 follows from this property.