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# Asymptotic equivalence of discretely observed geometric Brownian motion to a Gaussian shift

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## Abstract

Financial models consider often stochastic processes satisfying certain differential equations. We show that the solution of a particular geometric Brownian motion observed in discrete time is asymptotically equivalent with a Gaussian white noise model.

## 1 Introduction

In finance, stock prices are modelled by geometric Brownian motion since the well known Black-Scholes model, classically written as follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where  $\{W_t, t \in [0, 1]\}$  is a Brownian motion, with appreciation rate  $\mu \in \mathbb{R}$  and volatility  $\sigma > 0$ . The process  $S_t$  describes the evolution of the stock price at continuous time  $t$ .

This model is further used in computing the price of derivative securities (e.g. option valuation). Thus, a good knowledge of the volatility structure, in particular, is needed in the analysis of financial analysis and forecasting. For further details of financial calculations we refer to Musiela and Rutkowski [4].

We also put an accent here on the study of the time-dependent volatility. The following geometric Brownian motion is considered

$$dS_t = \sigma(t) S_t dW_t, \tag{1}$$

where the volatility function  $\sigma$  is time-dependent,  $t \in [0, 1]$ , belongs to some set  $H$  and is bounded away from 0 ( $\sigma(t) \geq \varepsilon > 0$  for all  $t$  in  $[0, 1]$ ). Without loss of generality we assume that the initial value is  $S_0 = 1$ .

Consider  $Y_t = \log S_t$  and substitute in equation (1). The volatility is a continuous function and then a solution to this last equation is known to satisfy

$$dY_t = -\frac{\sigma^2(t)}{2} dt + \sigma(t) dW_t. \tag{2}$$

Remark that, from a practical point of view, the continuous time model (2) is obviously unrealistic. We have at our disposal discrete time observations. Therefore, we consider discrete observations of the process  $Y_t$  in (2) at times  $t_i = i/n$ , for  $i = 1, \dots, n$  and the following normalized independent increments

$$Z_i = \sqrt{n} (Y_{t_i} - Y_{t_{i-1}}) = -\frac{\sqrt{n}}{2} \int_{t_{i-1}}^{t_i} \sigma^2(u) du + \sqrt{n} \int_{t_{i-1}}^{t_i} \sigma(u) dW_u. \quad (3)$$

We show that the statistical experiment (as defined later on) induced by observations in (3) has the same asymptotic behavior as the following white noise shift model:

$$dX_t = \frac{1}{\sqrt{2}} \log \sigma(t) dt + \frac{1}{\sqrt{n}} dW_t, \quad (4)$$

with  $t \in [0, 1]$ .

The approach is in the sense of Le Cam theory of asymptotic equivalence of statistical experiments, in a nonparametric setup. We assume that the volatility belongs to a Hölder class of functions,  $H(\beta, L)$ , with  $\beta \in (1/2, 1]$  of functions that are bounded away from 0:

$$H(\beta, L) = \left\{ \sigma : [0, 1] \rightarrow [\varepsilon, \infty) \mid \varepsilon > 0, |\sigma(x) - \sigma(y)| \leq L |x - y|^\beta \right\}.$$

Note that we prove the asymptotic equivalence without indicating constructive procedures (Markov kernels) which allow to pass from one model to the other (see Nussbaum and Klemelä [6] for such results).

In this paper, we obtain the result that the geometric Brownian motion (1), observed over a regular grid is asymptotically equivalent with the white noise model drifted observations in (4), in the sense of Le Cam's deficiency measure tending to 0 with  $n \rightarrow \infty$ . In Section 2 we introduce briefly the asymptotic equivalence notion and the main tools in our proofs. Results are stated and proven in several consecutive steps in Section 3.

## 2 Notation, definitions

Let  $E_{i,n} = \{\Omega, \mathcal{A}, (P_{i,n} = P_{i,n}(\sigma) : \sigma \in H)\}$ ,  $i = 1, 2$  be two statistical experiments defined on the same probability space  $(\Omega, \mathcal{A})$  and indexed by the same parameter space.

In order to evaluate how close the two experiments are, we consider Le Cam deficiency measure or the  $\Delta$ -pseudodistance. The experiments  $E_{i,n}$  are said to be asymptotically equivalent if this distance,  $\Delta(E_{1,n}, E_{2,n})$ , tends to 0 with  $n$ . For more details about the asymptotic equivalence theory we refer to Le Cam and Yang [3], Nussbaum [5]. In the following, we describe upper bounds of this distance by means of Hellinger distance.

Assume that there exists  $\sigma_0$  in  $H$  such that the associated probability measure dominates all other probability measures of the same experiment. If the likelihood processes  $\Lambda_{i,n}(\sigma) = dP_{i,n}(\sigma)/dP_{i,n}(\sigma_0)$  are defined on the same probability space (or have such versions), then the  $\Delta$ -distance satisfies

$$\Delta(E_{1,n}, E_{2,n}) \leq \frac{1}{2} \sup_{\sigma \in H} E |\Lambda_{1,n}(\sigma) - \Lambda_{2,n}(\sigma)|.$$

This upper bound can be expressed by means of Hellinger distance as follows. The Hellinger distance between two likelihood processes is by definition

$$H^2(\Lambda_{1,n}, \Lambda_{2,n}) = \frac{1}{2} E \left( \sqrt{\Lambda_{1,n}} - \sqrt{\Lambda_{2,n}} \right)^2.$$

Then we deduce easily that

$$\Delta(E_{1,n}, E_{2,n}) \leq \sup_{\sigma \in H} H(\Lambda_{1,n}, \Lambda_{2,n}). \quad (5)$$

In particular, we use the following formulas, where  $N(\mu, \sigma^2)$  denotes the Gaussian law with mean  $\mu$  and variance  $\sigma^2$ ,

$$H^2(N(\mu_1, \sigma^2), N(\mu_2, \sigma^2)) \leq \frac{(\mu_1 - \mu_2)^2}{4\sigma^2}, \quad (6)$$

$$H^2(N(\mu, \sigma_1^2), N(\mu, \sigma_2^2)) \leq 2 \frac{(\sigma_1 - \sigma_2)^2}{\sigma_1^2 + \sigma_2^2}. \quad (7)$$

### 3 Equivalence results

Let  $P_n^Y$  and  $P_n^X$  denote the distribution of  $\{Y_i = Y_{t_i}\}_{i=1, \dots, n}$  in (2) and  $\{X_t(n)\}_{t \in [0,1]}$  in (4), respectively, and  $E_n^Y = \left\{ [0, 1]^n, \mathcal{B}_{[0,1]}^n, (P_n^Y(\sigma), \sigma \in H(\beta, L)) \right\}$ ,  $E_n^X = \left\{ C_{[0,1]}, \mathcal{B}_{C_{[0,1]}}, (P_n^X(\sigma), \sigma \in H(\beta, L)) \right\}$  the associated experiments. We prove that these experiments are equivalent in a few steps.

Consider the following random variables

$$Z_i^* = -\frac{\sigma^2(t_i)}{2\sqrt{n}} + \sigma(t_i) \xi_i, \quad (8)$$

where  $\xi_i$  are independent, having standard Gaussian law.

**Lemma 1** *If  $E_{1,n}$  is the experiment generated by  $\{Z_i\}_{i=1, \dots, n}$  in (3) and  $E_{1,n}^*$  by  $\{Z_i^*\}_{i=1, \dots, n}$  in (8), then these experiments are asymptotically equivalent, i.e.*

$$\lim_{n \rightarrow \infty} \Delta(E_{1,n}, E_{1,n}^*) = 0.$$

**Proof.** Let us remark that observations  $Z_i$  and  $Z_i^*$  in (3) and (8) are independent and have Gaussian law,

$$P_i : N \left( -\frac{h_i(\sigma^2)}{2\sqrt{n}}, h_i(\sigma^2) \right), \text{ where } h_i(\sigma^2) = n \int_{t_{i-1}}^{t_i} \sigma^2(u) du,$$

respectively,

$$P_i^* : N \left( -\frac{\sigma^2(t_i)}{2\sqrt{n}}, \sigma^2(t_i) \right).$$

In order to prove that  $\Delta(E_{1,n}, E_{1,n}^*) \rightarrow 0$ , when  $n \rightarrow \infty$ , it suffices by (5) to prove that  $H^2(P_{1,n}, P_{1,n}^*) \rightarrow 0$ , as  $n \rightarrow \infty$ , where  $P_{1,n} = \bigotimes_{i=1}^n P_i$  and  $P_{1,n}^* = \bigotimes_{i=1}^n P_i^*$ . We have

$$\begin{aligned} H^2(P_{1,n}, P_{1,n}^*) &\leq 2 \sum_{i=1}^n H^2(P_i, P_i^*) \\ &\leq 2 \sum_{i=1}^n \left[ H^2(P_i, \tilde{P}_i) + H^2(\tilde{P}_i, P_i^*) \right], \end{aligned}$$

where  $\tilde{P}_i$  denotes the law of the following independent random variables

$$\tilde{Z}_i = -\frac{h_i(\sigma^2)}{2\sqrt{n}} + \sigma(t_i) \xi_i.$$

Using formulas (6) and (7), we get

$$\begin{aligned} H^2(P_{1,n}, P_{1,n}^*) &\leq 2 \sum_{i=1}^n \left[ 2 \frac{\left( \sqrt{h_i(\sigma^2)} - \sigma(t_i) \right)^2}{h_i(\sigma^2) + \sigma^2(t_i)} + \frac{1}{4\sigma^2(t_i)} \frac{(h_i(\sigma^2) - \sigma^2(t_i))^2}{4n} \right] \\ &\leq \sum_{i=1}^n (h_i(\sigma^2) - \sigma^2(t_i))^2 \left[ \frac{1}{\varepsilon^2} + \frac{1}{8n\varepsilon^2} \right]. \end{aligned}$$

Remark that for the continuous function  $\sigma$  there exists a value  $u_i \in [t_{i-1}, t_i]$  such that  $h_i(\sigma^2) = \sigma^2(u_i)$  and then

$$\begin{aligned} |h_i(\sigma^2) - \sigma^2(t_i)| &= |\sigma^2(u_i) - \sigma^2(t_i)| \\ &\leq 2B(\beta, L) |u_i - t_i|^\beta = O(1) n^{-\beta}, \end{aligned}$$

where  $B(\beta, L)$  is an upper bound of functions  $\sigma$  in  $H(\beta, L)$ . Finally,

$$H^2(P_{1,n}, P_{1,n}^*) \leq n^{1-2\beta} (1 + o(1)),$$

which tends to 0 for all  $\beta > 1/2$ . ■

**Remark 1** In addition to the main result we notice the following facts. Similarly, let us consider the process  $M_t$  satisfying

$$dM_t = \sigma(t) dW_t$$

and discrete observations of this process at times  $t_i = i/n$ , for  $i = 1, \dots, n$ , together with

$$U_i = \sqrt{n} (M_{t_i} - M_{t_{i-1}}) = \sqrt{n} \int_{t_{i-1}}^{t_i} \sigma(u) dW_u. \quad (9)$$

Analogously to Lemma 1, it is stated in the following Lemma that the Gaussian random variables

$$U_i^* = \sigma(t_i) \xi_i \quad (10)$$

introduce an equivalent experiment to (9). Via the same steps as for the observations  $Z_i^*$  we obtain in our Theorem the equivalence of (10) to observations  $X_i$  in (4). This proves the fact that the drift part of the discrete observations  $Y_i$  in (2) is asymptotically non significant.

**Lemma 2** If  $E_{2,n}$  is the experiment associated to  $\{U_i\}_{i=1,\dots,n}$  in (9) and  $E_{2,n}^*$  to  $\{U_i^*\}_{i=1,\dots,n}$  in (10) then

$$\lim_{n \rightarrow \infty} \Delta(E_{2,n}, E_{2,n}^*) = 0.$$

**Proof.** The proof goes similarly to that of Lemma 1, by writing the laws of  $\{U_i\}_{i=1,\dots,n}$  and  $\{U_i^*\}_{i=1,\dots,n}$ ,  $P_{2,n} : \bigotimes_{i=1}^n N(0, h_i(\sigma^2))$  and  $P_{2,n}^* : \bigotimes_{i=1}^n N(0, \sigma^2(t_i))$ , respectively. By inequalities (6) and (7) on the Hellinger distance and using the Hölder property of the volatility function  $\sigma$ , we conclude immediately. ■

The following result states that the solution  $Y_t$  of the geometric Brownian motion, discretely observed and the Gaussian drifted model  $X_t$  are equivalent. Moreover, the non random expectation term in (8) is asymptotically non informative and all the information can be 'extracted' from the variance term, in model (10).

**Theorem 1** The experiments  $E_n^Y$  and  $E_n^X$  associated to  $\{Y_i\}_{i=1,\dots,n}$  in (3) and  $\{X_t(n)\}_{t \in [0,1]}$  in (4), respectively, are asymptotically equivalent, i.e.

$$\lim_{n \rightarrow \infty} \Delta(E_n^Y, E_n^X) = 0.$$

Moreover, the drift part of observations  $\{Y_i\}_{i=1,\dots,n}$  in (3) is asymptotically non significant in this model.

**Proof.** We use several times the transitivity property of the asymptotic equivalence property. It is sufficient to prove that  $E_{1,n}^*$  (see Lemma 1), respectively  $E_{2,n}^*$  (see Lemma 2) are equivalent with the same global experiment  $E_n^X$ .

Let us consider first  $E_{2,n}^* = \left\{ [0, 1]^n, \mathcal{B}_{[0,1]}^n, (P_{2,n}^*(\sigma), \sigma \in H(\beta, L)) \right\}$  and remark that  $P_{2,n}^* = \bigotimes_{i=1}^n P_{\sigma(t_i)}^2$  can be written as a product measure of distributions from an exponential family having parameters  $\sigma(t_i)$ . By following the same steps as in Grama and Nussbaum [2] (see Example 4.2, p 189), we deduce that this model is equivalent to

$$Z_i^{**} = \frac{\log \sigma(t_i)}{\sqrt{2}} + \varepsilon_i, \quad (11)$$

where  $\sigma \in H(\beta, L)$  and  $\varepsilon_i$  are i.i.d. standard normal distributed.

The proof that  $E_{1,n}^* = \left\{ [0, 1]^n, \mathcal{B}_{[0,1]}^n, (P_{1,n}^*(\sigma), \sigma \in H(\beta, L)) \right\}$  with  $P_{1,n}^*(\sigma) = \bigotimes_{i=1}^n P_{\sigma(t_i)}^1$  is asymptotically equivalent to (11) follows the same arguments slightly differently, as given below.

The final step is immediate and consists of applying Brown and Low [1] in order to deduce that the experiment induced by observations  $Z_i^{**}$  is asymptotically equivalent to  $E_n^X$  in (4). These facts finish the proof of our theorem.

The global experiment  $E_{1,n}^*$  has to be reduced to the local experiment of the same observations with  $\sigma \in H_{\sigma_0, \gamma_n}(\beta, L)$ , a shrinking neighborhood of  $\sigma_0$ . For a fixed function  $\sigma_0$  in  $H(\beta, L)$  define

$$H_{\sigma_0, \gamma_n}(\beta, L) = \{ \sigma \in H(\beta, L) : \|\sigma - \sigma_0\|_\infty \leq \gamma_n \},$$

where  $\gamma_n = (n/\log n)^{-\beta/(2\beta+1)}$ .

Observations  $Z_i^*$ , (8), have a normal density written as an exponential model as follows

$$\begin{aligned} p_i(x) &= \frac{1}{\sigma(t_i)\sqrt{2\pi}} \exp \left\{ -\frac{1}{2\sigma^2(t_i)} \left( x + \frac{\sigma^2(t_i)}{2\sqrt{n}} \right)^2 \right\} \\ &= \exp \left\{ \lambda_i \frac{x^2}{2} - \frac{x}{2\sqrt{n}} + \frac{1}{8\lambda_i n} + \frac{1}{2} \log \left( -\frac{\lambda_i}{2\pi} \right) \right\}, \end{aligned}$$

where  $\lambda_i = -1/\sigma^2(t_i)$ . We write this exponential model as

$$p_i(x) = \exp \{ \lambda_i U(x) - V_n(\lambda_i) \} \exp \left\{ -\frac{x}{2\sqrt{n}} \right\},$$

where  $U(x) = x^2/2$  and

$$\begin{aligned} V_n(\lambda_i) &= -\frac{1}{2} \log \left( -\frac{\lambda_i}{2\pi} \right) - \frac{1}{8\lambda_i n}, \\ V_n'(\lambda_i) &= -\frac{1}{2\lambda_i} + \frac{1}{8\lambda_i^2 n}, \\ V_n''(\lambda_i) &= \frac{1}{2\lambda_i^2} \left( 1 - \frac{1}{2\lambda_i n} \right) = I_n(\lambda_i). \end{aligned}$$

Via Theorem 3.2 in Grama and Nussbaum [2], the local  $E_{1,n}^*$  is equivalent to the discrete Gaussian experiment:

$$Z_i^{(1)} = \sigma(t_i) + I_n(\sigma_0(t_i))^{-1/2} \varepsilon_i, \sigma \in H_{\sigma_0, \gamma_n}(\beta, L). \quad (12)$$

Denote  $I(\lambda_i) = 1/(2\lambda_i^2)$  and consider the variables

$$Z_i^{(2)} = \sigma(t_i) + I(\sigma_0(t_i))^{-1/2} \varepsilon_i, \sigma \in H_{\sigma_0, \gamma_n}(\beta, L). \quad (13)$$

We prove that the experiments generated by  $Z^{(1)} = \{Z_i^{(1)}\}_{i=1, \dots, n}$  and  $Z^{(2)} = \{Z_i^{(2)}\}_{i=1, \dots, n}$  are asymptotically equivalent, in order to bring our problem back to the case of Grama and Nussbaum [2] and apply the variance-stabilizing transformation.

We show that the Hellinger distance tends to 0 with  $n$ , as follows

$$\begin{aligned} H^2(Z^{(1)}, Z^{(2)}) &\leq 2 \sum_{i=1}^n H^2(N(\sigma(t_i), I_n(\sigma_0(t_i))^{-1}), N(\sigma(t_i), I(\sigma_0(t_i))^{-1})) \\ &\leq 4 \sum_{i=1}^n \frac{\left(I_n(\sigma_0(t_i))^{-1/2} - I(\sigma_0(t_i))^{-1/2}\right)^2}{I_n(\sigma_0(t_i))^{-1} + I(\sigma_0(t_i))^{-1}} \\ &\leq 4 \sum_{i=1}^n \frac{\left(\left(1 - \frac{1}{2n\sigma_0(t_i)}\right)^{-1/2} - 1\right)^2}{\left(1 - \frac{1}{2n\sigma_0(t_i)}\right)^{-1} + 1}. \end{aligned}$$

If we denote  $a_i = 1/(2n\sigma_0(t_i))$  then

$$\frac{\left((1 - a_i)^{-1/2} - 1\right)^2}{(1 - a_i)^{-1} + 1} = \frac{\left(1 - (1 - a_i)^{-1/2}\right)^2}{2 - a_i} \leq a_i^2$$

and

$$H^2(Z^{(1)}, Z^{(2)}) \leq \frac{1}{n^2} \sum_{i=1}^n \frac{1}{\sigma_0^2(t_i)} \leq \frac{1}{n\varepsilon^2} \xrightarrow{n \rightarrow \infty} 0.$$

The last step is to apply a variance-stabilizing transformation  $\Gamma$ , solution of the differential equation  $\Gamma'(\lambda) = \sqrt{I(\lambda)}$ , which gives  $\Gamma(\lambda) = \log \lambda / \sqrt{2}$ . We obtain the equivalent local Gaussian experiment

$$Z_i^{(3)} = \frac{\log \sigma(t_i)}{\sqrt{2}} + \varepsilon_i, \sigma \in H_{\sigma_0, \gamma_n}(\beta, L).$$

By globalization arguments of Grama and Nussbaum [2] (Theorem 3.8) we obtain the global experiment in (11) (where  $\sigma$  in the class  $H(\beta, L)$ ). ■



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