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Modelling Exchange Rates Volatility with Multivariate Long-Memory ARCH Processes

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Abstract

We consider two multivariate long-memory ARCH models, which extend the univariate long-memory ARCH models: we first consider a long-memory extension of the restricted constant conditional correlations (CCC) model introduced by Bollerslev (1990), and we propose a new unrestricted conditional covariance matrix model which models the conditional covariances as long-memory ARCH processes. We apply these two models to two daily returns on foreign exchanges (FX) rates series, the Pound-US dollar, and the Deutschmark-US dollar. The estimation results for both models show: (i) that the unrestricted model outperforms the restricted CCC model, and (ii) that all the elements of the conditional covariance matrix share the same degree of long-memory for the period April 1979-January 1997. However, this result does not hold for the floating periods March 1973-January 1997 and September 1971-January 1997. This break in the long-term structure may be caused by the European Monetary System inception in March 1979.

Keywords: Long-memory processes, heteroskedasticity, multivariate long-memory ARCH models, multivariate FIGARCH models.

JEL classification: C32, G00.

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1 Introduction

Most of the returns on speculative assets $R_t = \ln(P_t/P_{t-1})$, where P_t denotes the asset price at time t , are characterized by the absence of significant correlation between successive observations, which is a consequence of the martingale property of asset prices. However, their variations are clustered, their conditional variance is predictable, and the power transformations $|R_t|^\delta$, where δ is a positive real number, of these series exhibit a significant persistence and dependence between very distant observations called “long-range dependence” or “long-memory”.¹

Four classes of stochastic processes can mimic these properties: the long-memory ARCH processes introduced in Robinson (1991), the long-memory stochastic volatility models proposed by Harvey (1993) and Breidt, Crato and de Lima, Mandelbrot’s (1997) multifractal models, and the multi-factors models by Gallant, Hsu and Tauchen (1998).

We consider here the class of long-memory GARCH models. Although Ding and Granger (1996), and Engle and Lee (1992) proposed a mixture of ARCH and IGARCH processes, i.e., a process mixing a transitory and a permanent component, for representing the hyperbolic decay of the autocorrelations of the volatility, these features are more parsimoniously modeled by the statistical class of long-memory ARCH/GARCH processes.²

Long-memory in the class of conditional variance ARCH processes has been considered first by Robinson (1991) for testing for dynamic conditional heteroskedasticity. Later, several authors have extended several GARCH-type processes to fractional situations.³ In the same way, Ding and Granger (1996) have derived a long-memory ARCH model by using an aggregation argument.

All these models were concerned with the univariate modeling of time series. Empirical evidence on futures data,⁴ Intra-Day foreign exchange (FX) rates returns (Henry and Payne, 1997), and stock indexes (Ray and Tsay, 1997), suggest that some time series share the same long-memory component in their conditional variance. Since some time series are likely to be influenced by the same set of events, it is straightforward to consider long-memory in the conditional second moments in a multivariate framework by also modeling the covariation of the series. Furthermore, a common long-range component or ‘co-persistence’ in a matrix of conditional variances is of interest for long-term forecasts as the relative variations between these covariances are only transitory.

We first consider the extension of the univariate long-memory ARCH models in the restricted multivariate constant conditional correlations (CCC) framework proposed by Bollerslev (1990). As we observed that the volatilities and the ‘co-volatility’ of some series have a common long-memory component, see p 7, we propose a new multivariate unrestricted framework, which explicitly models the covariances as long-memory ARCH processes.

We apply these classes to the bivariate modeling of two series of FX rates returns at daily frequency for the periods September 1971-January 1997, March 1973-January 1997, and April 1979-January 1997: the Pound-US dollar, and the Deutschmark-US dollar. The effects of target

¹See Robinson (1994) and Beran (1994) for a survey on long-memory processes.

²See Robinson and Zaffaroni (1997), Ding and Granger (1996), Giraitis, Kokoszka and Leipus (1998) for a rigorous demonstration.

³See Bollerslev and Mikkelsen (1996), McCurdy and Michaud (1996), and Teyssière (1997a).

⁴See the 1996 version of Teyssière (1997b), and a previous version of this work on futures data. As data from future markets are reliable since 1982, the sample sizes were a bit short for analyzing long-memory processes and we turn to the analysis of FX data.

zone systems on the volatility have already been studied by Bollerslev (1990) and Engle and Gau (1997) among others. Our purpose is to analyze the effects of the EMS system on the long-term component of the FX rates returns volatilities.

This paper is organized as follows. Section 2 discusses the univariate and multivariate long-memory ARCH models, Section 3 presents an application to a bivariate modeling of FX rates returns. Section 4 concludes.

2 Multivariate long-memory ARCH models

2.1 The long-memory property of time series

Let $\{y_t\}$ be a stochastic process defined by the following autoregressive (AR) representation

$$(1 - \alpha(L)) y_t = \varepsilon_t \quad \varepsilon_t \text{ white noise} \quad (1)$$

where $\alpha(L)$ is a lag polynomial. We allow for long-memory in $\{y_t\}$ by considering that the order of $\alpha(L)$ is infinite, and that its coefficients α_j satisfy

$$\lim_{j \rightarrow \infty} |\alpha_j| = C j^{-(1+q)} \quad (2)$$

where C is a positive constant and q is a strictly positive real number which represents the long-memory property of the series. This implies that the coefficients α_j have a hyperbolic rate of decay. Several polynomials satisfy this property:

- The fractional q^{th} difference operator, used for defining the fractional Gaussian noise $I(q)$ process (Mandelbrot and Van Ness, 1968, Granger and Joyeux, 1980, Hosking, 1981), characterized by the real parameter q , called the degree of integration of the series. The AR representation of the fractional Gaussian noise $I(q)$ is $1 - \alpha(L) = (1 - L)^q$, with

$$\alpha_j = \frac{-\Gamma(j - q)}{\Gamma(-q)\Gamma(j + 1)}, \quad \text{with} \quad \lim_{j \rightarrow \infty} \alpha_j = \frac{-1}{\Gamma(-q)} j^{-(1+q)} \quad (3)$$

where $\Gamma(\bullet)$ denotes the Gamma function.

- The Gegenbauer polynomials, (Gray, Zhang and Woodward, 1989, Chung, 1996a) characterized by two parameters q and η , which are respectively the degree of integration of the series and the cosine of the singularity. The coefficients of these polynomials are defined by

$$\alpha_j = \sum_{k=0}^{[j/2]} \frac{(-1)^k \Gamma(-q + j - k) (2\eta)^{j-2k}}{\Gamma(-q) \Gamma(k + 1) \Gamma(j - 2k + 1)} \quad (4)$$

$$\text{with} \quad \lim_{j \rightarrow \infty} \alpha_j = \frac{\cos((j - q)\nu + (q\pi/2))}{\Gamma(-q) \sin^{-q}(\nu)} \left(\frac{2}{j}\right)^{1+q}$$

where $[\bullet]$ indicates integer part, and $\nu = \arccos(\eta)$. When $\eta = 1$, this polynomial reduces to the $I(2q)$ polynomial.

- The ratio of two Beta functions (Ding and Granger, 1996)

$$\alpha_j = \frac{B(p+j-1, q+1)}{B(p, q)}, \quad \text{with} \quad \lim_{j \rightarrow \infty} \alpha_j = \frac{q\Gamma(p+q)}{\Gamma(p)} j^{-(1+q)} \quad (5)$$

for $p, q > 0$.⁵ If $p+q = 1$, this polynomial reduces to an $I(q)$ polynomial. Since this polynomial is characterized by two parameters, it is more flexible than the polynomial of the $I(q)$ process.⁶

The Gegenbauer polynomials capture long-range dependence with persistent component at frequency ν , and are then used for modeling macroeconomic data (see Chung, 1996b). The two other polynomials, of which coefficients are always strictly positive, have been used for modeling the long range dependence of the conditional variance of some financial assets. They also satisfy

$$\sum_{j=1}^{\infty} \alpha_j = 1 \quad (6)$$

2.2 Long-memory in the conditional variance

Define a conditional heteroskedasticity model as

$$R_t = m(R_t) + \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma_t^2) \quad (7)$$

where $m(R_t)$ denotes the regression function, the conditional variance σ_t^2 depends on the information set I_t consisting of everything dated $t-1$ or earlier. Engle (1982) proposed the ARCH(p) skedastic function:

$$\sigma_t^2 = \omega + \alpha(L)\varepsilon_t^2 \quad (8)$$

where $\alpha(L)$ is a lag polynomial of order p . Robinson (1991) has considered the general case of conditional heteroskedasticity by resorting to long-memory infinite order lag polynomials, and has proposed the following long-memory ARCH:

$$\sigma_t^2 = \sigma^2 + \sum_{j=1}^{\infty} \alpha_j (\varepsilon_{t-j}^2 - \sigma^2) = \sum_{j=1}^{\infty} \alpha_j \varepsilon_{t-j}^2 \quad (9)$$

where the coefficients $\{\alpha_j\}_{j=1}^{\infty}$ of the infinite order lag polynomial $\alpha(L)$ satisfy equations (2) and (6). Thus, equation (9) defines a long-memory process in the variance, in which shocks on the error terms have a persistent effect on the conditional variance. We first consider for $\alpha(L)$ the polynomial defined by the ratio of two Beta functions. Thus, the long-memory ARCH model becomes

$$\sigma_t^2 = \sum_{j=1}^{\infty} \frac{B(p+j-1, q+1)}{B(p, q)} \varepsilon_{t-j}^2 \quad (10)$$

which is a particular case of the long-memory ARCH model derived by Ding and Granger (1996)

$$\sigma_t^2 = (1 - \mu)\sigma^2 + \mu\alpha(L)\varepsilon_t^2 \quad (11)$$

⁵If $p = 0$, $\alpha_1 = 1$, $\alpha_j = 0, \forall j > 1$

⁶Ding and Granger (1996) report examples of decay of this polynomial for several values of p and q .

where μ is a parameter $\in [0, 1]$. In our case $\mu = 1$.

This long-memory ARCH is very interesting: since the coefficients of $\alpha(L)$ are strictly positive, there is no restriction for insuring the conditional variance to be strictly positive, provided that the ratio of the Beta functions exists, i.e., $p > 0$ and $q > 0$, and at least one of the past error terms ε_t is different from zero, which are rather mild constraints. We extend this model by considering parameterized real powers for both σ_t and ε_t . Thus, the long-memory ARCH model (10) becomes:

$$\sigma_t^\delta = \sum_{j=1}^{\infty} \frac{B(p+j-1, q+1)}{B(p, q)} |\varepsilon_{t-j}|^\delta \quad (12)$$

We can obviously consider more general forms for equation (12) in which the error terms are transformed by an asymmetric function for capturing asymmetries.⁷

Long-memory in the conditional variance is also modeled by using the ARMA parameterization of some GARCH type processes, and considering fractional unit roots in the AR component of this ARMA parameterization. We then obtain the class of Fractionally Integrated GARCH processes, which bridges the gap between GARCH and Integrated GARCH processes.⁸ A FIGARCH(l, q, m) is then defined as

$$\sigma_t^2 = \frac{\omega}{1 - \beta(1)} + \left(1 - \frac{(1 - \phi(L))(1 - L)^q}{(1 - \beta(L))}\right) \varepsilon_t^2 \quad (13)$$

where $\beta(L)$ and $\phi(L)$ are lag polynomials of respectively finite order l and m , the roots of $1 - \beta(L)$ and $\phi(L)$ being outside the unit circle.⁹ This richer parameterization allows a greater flexibility than the long-memory ARCH model presented above. The counterpart of this flexibility is a more restrictive set of conditions for insuring the conditional variance to be positive. While the necessary and sufficient conditions for a FIGARCH(1, q , 0) model are not too constraining: $\beta \leq q \leq 1$ and $\omega > 0$, the sufficient conditions for a FIGARCH(1, q , 1) are more restrictive¹⁰

$$\omega > 0, \beta - q \leq \phi \leq (2 - q)/3, q(\phi - (1 - q)/2) \leq \beta(\phi - \beta + q) \quad (14)$$

This set of restrictions also applies to the FIAPARCH and FINGARCH models respectively proposed by McCurdy and Michaud (1996) and Teyssi re (1997a), who extended the APARCH model of Ding, Granger and Engle (1993) and the Engle and Ng (1993) NGARCH to fractional situations. We alternatively can use a log transformation of the conditional variance, like the FIEGARCH model proposed by Bollerslev and Mikkelsen (1996). However, as mentioned by Pagan (1996), the IEGARCH model is not identifiable by Quasi Maximum Likelihood (QML), thus if q tends to 1, the same problem is likely to occur for a FIEGARCH.

⁷See McCurdy and Michaud (1996), Teyssi re (1997a).

⁸See Bollerslev and Mikkelsen (1996), McCurdy and Michaud (1996).

⁹The link between this FIGARCH model and the previous long-memory ARCH model, with the restriction $p + q = 1$, is given by the following re-parameterization: $1 - \alpha(L) = (1 - L)^q \phi(L)/\beta(L)$, and by assuming that $\omega = (1 - \alpha(1)) \sigma^2$ is a finite parameter to be estimated. (See Robinson and Zaffaroni, 1997).

¹⁰See Bollerslev and Mikkelsen (1996). For models of higher orders, the conditions are far more complicate.

2.3 Multivariate long-memory skedastic functions

Let \mathbf{R}_t be a n -dimensional vector long-memory ARCH process

$$\mathbf{R}_t = \mathbf{m}(\mathbf{R}_t) + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim \text{i.i.d.} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_t) \quad (15)$$

where $\mathbf{m}(\mathbf{R}_t)$ denotes the vector regression function, $\boldsymbol{\varepsilon}_t$ is a n -dimensional vector of error terms with conditional covariance matrix $\boldsymbol{\Sigma}_t$. We first consider a long-memory extension of the constant conditional correlations model (CCC) proposed by Bollerslev (1990), where the conditional covariance matrix $\boldsymbol{\Sigma}_t$ has as typical element $s_{ij,t}$, with

$$\begin{aligned} s_{ii,t} &= \sigma_{ii,t}^2 = \sum_{k=1}^{\infty} \frac{B(p_i + k - 1, q_i + 1)}{B(p_i, q_i)} \varepsilon_{i,t-k}^2, \quad i = 1, \dots, n \\ s_{ij,t} &= \rho_{ij} \sigma_{ii,t} \sigma_{jj,t}, \quad i, j = 1, \dots, n \quad i \neq j \end{aligned} \quad (16)$$

where $\rho_{ij} \in (-1, 1)$ denotes the conditional correlation which is supposed to be constant. This parsimonious diagonal specification insures that the conditional covariance matrix is always positive definite, and that the multivariate long-memory ARCH process is stationary if all the univariate processes in the main diagonal are stationary.¹¹

This process can be extended by using a parameterization similar to equation (12) as follows:

$$\begin{aligned} s_{ii,t} &= \sigma_{ii,t}^2, \quad \sigma_{ii,t}^\delta = \sum_{k=1}^{\infty} \frac{B(p_i + k - 1, q_i + 1)}{B(p_i, q_i)} |\varepsilon_{i,t-k}|^\delta, \quad i = 1, \dots, n \\ s_{ij,t} &= \rho_{ij} \sigma_{ii,t} \sigma_{jj,t}, \quad i, j = 1, \dots, n \quad i \neq j \end{aligned} \quad (17)$$

We also consider the multivariate CCC FIGARCH process:

$$\begin{aligned} s_{ii,t} &= \sigma_{ii,t}^2 = \frac{\omega_i}{1 - \beta_i(1)} + \left(1 - \frac{(1 - \phi_i(L))(1 - L)^q}{1 - \beta_i(L)}\right) \varepsilon_{i,t}^2, \quad i = 1, \dots, n \\ s_{ij,t} &= \rho_{ij} \sigma_{ii,t} \sigma_{jj,t}, \quad i, j = 1, \dots, n \quad i \neq j \end{aligned} \quad (18)$$

Lastly, we propose two alternative unrestricted parameterizations for the conditional covariance matrix $\boldsymbol{\Sigma}_t$, with typical element $s_{ij,t}$:

$$s_{ij,t} = \sum_{k=1}^{\infty} \frac{B(p_{ij} + k - 1, q_{ij} + 1)}{B(p_{ij}, q_{ij})} \varepsilon_{i,t-k} \varepsilon_{j,t-k}, \quad i, j = 1, \dots, n \quad (19)$$

and

$$s_{ij,t} = \frac{\omega_{ij}}{1 - \beta_{ij}(1)} + \left(1 - \frac{(1 - \phi_{ij}(L))(1 - L)^q}{1 - \beta_{ij}(L)}\right) \varepsilon_{i,t} \varepsilon_{j,t}, \quad i, j = 1, \dots, n \quad (20)$$

In these cases, there is no analytically tractable set of conditions for insuring $\boldsymbol{\Sigma}_t$ to be positive definite, and the multivariate process to be stationary. We implement the positive definiteness constraint in the estimation procedure by resorting to numerical penalty functions.

¹¹See Bollerslev (1990), Bera and Higgins (1993), Bollerslev, Engle and Nelson (1994), Engle and Kroner (1995), or Pagan (1996), for a survey on multivariate ARCH processes.

The combination of positivity constraints by projection methods and penalty functions makes the estimation of the multivariate unrestricted FIGARCH more difficult. However, we will see later that this model is more apt to represent the conditional covariance of the bivariate series of FX rates returns.

This unrestricted long-memory ARCH process is stationary provided that Σ_t is a measurable function on the information set I_t , and $\text{trace}(\Sigma_t \Sigma_t^\top)$ is finite almost surely. The latter condition holds if the moments are bounded, i.e., $E(\text{trace}(\Sigma_t \Sigma_t^\top)^p)$ is finite for some p .¹² As our estimation results show that the two first moments of $\text{trace}(\hat{\Sigma}_t \hat{\Sigma}_t^\top)$ for the unrestricted model are close to the moments of the diagonal stationary model, we can consider that the process estimated on our data is stationary.

Since we are working with data at daily frequency, we can reasonably assume that the error terms are normally distributed.¹³ Thus, the log-likelihood function of the multivariate long-memory ARCH model is:

$$\mathcal{L}_T(\zeta) = -\frac{nT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \left(\ln |\Sigma_t| + \varepsilon_t^\top \Sigma_t^{-1} \varepsilon_t \right) \quad (21)$$

where ζ and T respectively denote the set of parameters and the sample size. The robust estimators of the variances are given by the heteroskedastic consistent sandwich covariance matrix $T^{-1} \mathcal{H}^{-1}(\hat{\zeta}) \mathcal{I}(\hat{\zeta}) \mathcal{H}^{-1}(\hat{\zeta})$ where $\mathcal{H}(\hat{\zeta})$ and $\mathcal{I}(\hat{\zeta})$ are respectively the Hessian and the outer-product-of-the-gradient matrices evaluated at the Quasi Maximum Likelihood (QML) estimates $\hat{\zeta}$.

3 Application to the bivariate modeling of exchange rates

We consider several bivariate long-memory ARCH models for two series at daily frequencies: the log of returns on the Pound-US dollar, and the Deutschmark-US dollar exchange rates, defined by $100 \ln(P_t/P_{t-1})$, where P_t denotes the exchange rate at time t . We consider here observations from the 24 August 1971, after the collapse of the Bretton-Wood fixed exchange rate system, until January 1997. A limited variation exchange rates system has been instituted in December 1971, the Smithsonian Agreement, but it lasted only 14 months, and was followed by the “floating period”. During this floating period, some European currencies were governed by several mechanisms limiting their relative variations. The “Snake in the Tunnel” mechanism allowed these currencies to vary by a maximum of $\pm 1.5\%$ against each other, the “Snake”, and by a maximum of $\pm 2.5\%$ against the dollar, “the Tunnel”. Belgium, France, Italy, Luxembourg, The Netherlands, and West Germany were the first participants. UK joined one month after and withdrew six weeks afterwards. Later, the reference to the dollar was abandoned, and then the system became the “Snake”, in which parities were several times realigned. Later, in March 1979, the European Monetary System (EMS) replaced the “Snake”. UK joined the EMS in 1990 and left two years later. A long-memory analysis should, by construction, be unaffected by this

¹²See Bollerslev, Engle and Nelson (1994).

¹³The departure to normality does not affect the QML results with data at daily frequency. However, this assumption is difficult to maintain with very high-frequency data, of which leptokurtosis can be captured with t -distributed error terms with $t < 7$ or Generalized Exponentially Distributed error terms.

series of local adjustments and breaks. However, estimation results lead us to consider periods which coincide with these events.

The most interesting period is the EMS period, April 1979- 21 January 1997 (4646 observations). We also consider two periods before the EMS, although several sub-periods with regime shifts can be considered: 24 August 1971-21 January 1997 (6630 observations), after the collapse of the Bretton-Wood system, March 1973-21 January 1997 (6233 observations), after the end of the Smithsonian Agreement.

In a first approach, we estimate non-parametrically the long-memory component in the volatility of the two series. Since the volatility of a series R_t can be represented by its absolute value transformation $|R_t|$, we can represent the ‘co-volatility’ between the series $R_{1,t}$ and $R_{2,t}$ by the quantity $\sqrt{|R_{1,t}||R_{2,t}|}$. We use Robinson’s (1995) semi-parametric discrete version of Whittle (1962) approximate ML estimator in the spectral domain. This estimator, suggested by Künsch (1987), is based on the assumption that the spectrum $f(\lambda)$ of a long-memory series near the zero frequency can be approximated as

$$\lim_{\lambda \rightarrow 0^+} f(\lambda) = G\lambda^{-2q} \quad \text{where } G \text{ is a strictly positive finite constant}$$

This approximation allows ones to avoid the consequences of a misspecification of the functional form of the spectrum in the Whittle estimator. The estimator is given by:

$$\hat{q} = \arg \min_q \left\{ \ln \left(\frac{1}{m} \sum_{j=1}^m \frac{I(\lambda_j)}{\lambda_j^{-2q}} \right) - \frac{2q}{m} \sum_{j=1}^m \ln(\lambda_j) \right\} \quad (22)$$

where $I(\lambda_j)$ is the periodogram estimated for the range of Fourier frequencies $\lambda_j = \pi j/T, j = 1, \dots, m \ll T$, the bandwidth parameter m tends to infinity with T , but more slowly: the ratio m/T tends to zero. We estimate \hat{q} for several values of the bandwidth parameter m , and we obtain:

Table 1: Estimation of the fractional degree of integration for the series of absolute returns on Pound-Dollar $|R_{1,t}|$, Deutschmark-Dollar $|R_{2,t}|$, $\sqrt{|R_{1,t}||R_{2,t}|}$ for the period April 1979 - January 1997

m	$ R_{1,t} $	$ R_{2,t} $	$\sqrt{ R_{1,t} R_{2,t} }$
$T/4$	0.238461	0.231238	0.241344
$T/8$	0.307107	0.321911	0.323034
$T/16$	0.411259	0.407288	0.439284

Thus, we observe that the two volatilities and the co-volatility have the same degree of long-memory for all the values of the bandwidth parameter m . Since the CCC model cannot account for the long-memory property of the conditional covariance, we expect that the unrestricted model, which represents the conditional covariances as long-memory ARCH processes, will have a better fit with the data.

We estimated the following bivariate long-memory ARCH:¹⁴

$$\begin{pmatrix} R_{1,t} \\ R_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix} \sim N \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s_{11,t} & s_{12,t} \\ s_{12,t} & s_{22,t} \end{pmatrix} \right] \quad (23)$$

where $R_{1,t}$ and $R_{2,t}$ respectively denote the log of returns on the Pound-Dollar and Deutschmark-Dollar. We consider the five alternative multivariate skedastic functions:

$$\textbf{(A)} \quad \begin{pmatrix} s_{11,t} \\ s_{22,t} \\ s_{12,t} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{\infty} \frac{B(p_1 + j - 1, q_1 + 1)}{B(p_1, q_1)} \varepsilon_{1,t-j}^2 \\ \sum_{j=1}^{\infty} \frac{B(p_2 + j - 1, q_2 + 1)}{B(p_2, q_2)} \varepsilon_{2,t-j}^2 \\ \rho \sqrt{s_{11,t}} \sqrt{s_{22,t}} \end{pmatrix} \quad (24)$$

$$\textbf{(B)} \quad \begin{pmatrix} \sigma_{11,t}^\delta \\ \sigma_{22,t}^\delta \\ s_{12,t} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{\infty} \frac{B(p_1 + j - 1, q_1 + 1)}{B(p_1, q_1)} |\varepsilon_{1,t-j}|^\delta \\ \sum_{j=1}^{\infty} \frac{B(p_2 + j - 1, q_2 + 1)}{B(p_2, q_2)} |\varepsilon_{2,t-j}|^\delta \\ \rho \sigma_{11,t} \sigma_{22,t} \end{pmatrix} \quad (25)$$

$$\textbf{(C)} \quad \begin{pmatrix} s_{11,t} \\ s_{22,t} \\ s_{12,t} \end{pmatrix} = \begin{pmatrix} \frac{\omega_1}{1 - \beta_1(1)} + \left(1 - \frac{(1 - \phi_1 L)(1 - L)^{q_1}}{1 - \beta_1 L} \right) \varepsilon_{1,t}^2 \\ \frac{\omega_2}{1 - \beta_2(1)} + \left(1 - \frac{(1 - \phi_2 L)(1 - L)^{q_2}}{1 - \beta_2 L} \right) \varepsilon_{2,t}^2 \\ \rho \sqrt{s_{11,t}} \sqrt{s_{22,t}} \end{pmatrix} \quad (26)$$

$$\textbf{(D)} \quad \begin{pmatrix} s_{11,t} \\ s_{22,t} \\ s_{12,t} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{\infty} \frac{B(p_1 + j - 1, q_1 + 1)}{B(p_1, q_1)} \varepsilon_{1,t-j}^2 \\ \sum_{j=1}^{\infty} \frac{B(p_2 + j - 1, q_2 + 1)}{B(p_2, q_2)} \varepsilon_{2,t-j}^2 \\ \sum_{j=1}^{\infty} \frac{B(p_3 + j - 1, q_3 + 1)}{B(p_3, q_3)} \varepsilon_{1,t-j} \varepsilon_{2,t-j} \end{pmatrix} \quad (27)$$

$$\textbf{(E)} \quad \begin{pmatrix} s_{11,t} \\ s_{22,t} \\ s_{12,t} \end{pmatrix} = \begin{pmatrix} \frac{\omega_1}{1 - \beta_1(1)} + \left(1 - \frac{(1 - \phi_1 L)(1 - L)^{q_1}}{1 - \beta_1 L} \right) \varepsilon_{1,t}^2 \\ \frac{\omega_2}{1 - \beta_2(1)} + \left(1 - \frac{(1 - \phi_2 L)(1 - L)^{q_2}}{1 - \beta_2 L} \right) \varepsilon_{2,t}^2 \\ \frac{\omega_3}{1 - \beta_3(1)} + \left(1 - \frac{(1 - \phi_3 L)(1 - L)^{q_3}}{1 - \beta_3 L} \right) \varepsilon_{1,t} \varepsilon_{2,t} \end{pmatrix} \quad (28)$$

¹⁴We truncate the expansion of the infinite order polynomial at the order 2000.

Table 2 p 14 reports the QML estimates of the constant conditional variance long-memory ARCH model **(A)** for (i) the period 1979-1997, (ii) the same period subject to the constraint that the parameters of the two conditional variances are equal, i.e., $q_1 = q_2$ and $p_1 = p_2$, (iii) for the period 1971-1997, and (iv) the period 1973-1997.

It clearly appears that for the period 1979-1997, the two series share the same degree of long-memory in the conditional variance, since the estimated values for q_1 and q_2 are very close. The constraints $q_1 = q_2$ and $p_1 = p_2$ are accepted by the likelihood ratio (LR) test, and the more parsimonious constrained model is then preferable. The estimation results for the period 1971-1997 are less appealing. The two long-memory parameters largely differ. This result completes Bollerslev's (1990) result, which showed, for a multivariate GARCH model on weekly data, a difference between these two periods in the estimated parameters explained by the inception of the EMS in March 1979. The EMS system has modified both the short-term and the long-term variations of the volatilities of exchange rates returns.¹⁵

Table 3 p 14 reports the QML estimates of the constant conditional variance long-memory ARCH model, model **(B)**, for (i) the period 1979-1997, (ii) the same period with the restrictions $p_1 = p_2$ and $q_1 = q_2$, (iii) for the period 1971-1997, and (iv) the period 1973-1997. The restriction $\delta = 2$ is rejected by the LR test. However, some parameters are instable for the two larger samples: this may be due to the misspecification of the model, of which assumption of a constant conditional correlation is too restrictive.

Table 4 p 15 displays the estimation results for the bivariate CCC FIGARCH model. The log-likelihood value is higher than the one of the long-memory ARCH model presented above, and the trade-off between the increase of the value of the likelihood function and the increase of the number of parameters leads us to choose the less parsimonious CCC FIGARCH model. As expected, the maximization procedure is much slower. As observed above, the degrees of long memory are the same for the period 1979-1997. Furthermore, the LR test accepts the hypothesis of equality of parameters β and ϕ for the two conditional variances. We will see later that the equality $\phi_1 = \phi_2$ is likely to be the consequence of the restrictive parameterization of the CCC FIGARCH.

Tables 5 and 6 p 16 display the estimation results for respectively the unrestricted models **(D)**, and **(E)**. Column (i) contains the QML estimates for the unconstrained model, column (ii) contains the estimates for the model with the restrictions $q_1 = q_2 = q_3$, column (iii) contains estimation results for the whole period 1971-1997, and column (iv) contains estimation results for the period 1973-1997. It clearly appears that the constant conditional correlation assumption for the models **(A, B, C)** is too strong since the value of the likelihood function is strongly improved. We also observe that the multivariate unrestricted FIGARCH model has a better fit with the data than the multivariate unrestricted long-memory ARCH model: the log-likelihood is strongly improved, and any parsimony criteria, e.g., the Akaike Information Criteria or the Bayes Information Criteria, favors the multivariate unrestricted FIGARCH model.¹⁶ As expected,

¹⁵We have also estimated the models **(A)** for the period 1971-1997, with two different conditional correlations, the first for the period 1971-1979, the second for the period 1979-1997. The LR test rejects the restriction that these two parameters are equal, and still rejects the restriction that the two degrees of long-memory are the same.

¹⁶The statistical properties of the AIC and BIC criteria have not been established for the class of long-memory ARCH process. Beran and Bhansali (1997) result on the BIC for ARFIMA models casts doubt on the use of the BIC for long-memory volatility models.

the estimation of the multivariate FIGARCH is more problematic; a sensible choice of starting values is necessary, but the model looks well conditioned since we did not encounter too much problems.¹⁷

More interesting, all the elements of the conditional variance matrix, i.e., the two conditional variances and the conditional covariance, display the same degree of long-memory: the correlations between the estimated parameters are very high, and the restriction $q_1 = q_2 = q_3$ is accepted by the LR test for both models **(D)** and **(E)** for the EMS period. Furthermore, the restriction $\beta_1 = \beta_2 = \beta_3$ for model **(E)** is also accepted by the LR test for the same period. In a standard GARCH framework, the polynomial $\phi(L)$ captures the local variations, while the polynomial $\beta(L)$ captures the long-term variations. As we have seen in another paper, Teyssi re (1997a), this interpretation is less obvious in a long-memory framework as the polynomial $\phi(L)$ enters in the expression of the infinite order lag polynomial.

We estimate the models **(D, E)** with the restriction $q_1 = q_2 = q_3$ for the periods 1971-1997 and 1973-1997. This restriction is accepted by the LR test. This result is due to the larger proportion of EMS observations in our sample, since this restriction is rejected respectively for the first 5500 and 3500 observations. Although, by construction, a test for structural break at a given time t for the long-memory parameter is meaningless, we can conclude from these results that this equality in the long-memory parameters for the period 1979-1997 may be caused by the EMS inception in March 1979.

4 Conclusion

We have proposed two classes of multivariate long-memory ARCH models. One of them, the constant conditional correlation (CCC) model, requires few constraints in the estimation procedure, but a strong restriction on the functional form of the conditional covariance matrix. This constraint appeared to be too strong since the long-memory CCC model is outperformed by the unrestricted long-memory conditional covariance model. We have applied these models to the bivariate modeling of the returns on two exchange rates, and have observed that for the period 1979-1997, after the EMS beginning, these two series have the same long-memory component in the conditional variance. The class of unrestricted conditional covariance models allowed us to go farther by showing that all the elements of the conditional covariance matrix share the same degree of long-memory.

This equality in the long-memory parameter for the same exchange market suggests that this parameter is linked to the features of the market. This empirical result stimulated further research for deriving a structural microeconomic model able to replicate the long-memory property of financial time-series. (See Kirman and Teyssi re, 1998).

We decided to extend this empirical analysis in another paper, Teyssi re (1998), with the new FX rate datasets at 30-minute intervals (HFDF96) recently released by Olsen & Associates. A motivation for this paper was the extension of this class of multivariate long-memory ARCH models to the framework of double long-memory time series, i.e., series with a long-memory component in both the conditional mean and the conditional variance developed in Teyssi re

¹⁷The conditional covariance matrix Σ_t was not always positive definite for all t at some steps in the line search procedure, but the BFGS algorithm converges to an optimum satisfying the positivity constraint.

(1997b), as these intra-day FX rate returns are anti-persistent.

This class of multivariate long-memory models can also be trivially extended by combining skedastic functions of different type in the conditional covariance matrix.

Another issue for future research is the extension of the concept of co-persistence in the conditional variance, developed by Bollerslev and Engle (1993), to the fractional case, although the idea of “fractional cointegration” for conditional variance processes is not yet supported by a theoretical model. (See Pagan, 1996).

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A Estimation results

Table 2: Estimation results of bivariate long-memory ARCH process, with constant conditional correlation, model (A). (Robust t -statistics between parentheses).

	(i) 1979-1997	(ii) 1979-1997	(iii) 1971-1997	(iv) 1973-1997
μ_1	-0.0032 (-0.32)	-0.0036 (-0.30)	-0.0072 (0.80)	-0.0002 (-0.03)
μ_2	-0.0003 (-0.03)	0.0000 (0.00)	-0.0174 (1.41)	-0.0081 (-0.86)
p_1	5.8472 (4.42)	6.0607 (5.66)	2.9486 (2.67)	3.8220 (3.47)
p_2	6.3221 (4.88)	—	4.1063 (2.50)	3.8918 (6.56)
q_1	0.4496 (8.11)	0.4379 (11.66)	0.5495 (3.916)	0.5173 (4.32)
q_2	0.4309 (11.47)	—	0.3475 (14.70)	0.3992 (12.30)
ρ	0.7196 (67.49)	0.7197 (67.54)	0.6192 (32.61)	0.6448 (44.30)
Log-lik	-7139.2294	-7139.7738	-9388.7127	-9033.2728

Table 3: Estimation results of bivariate long-memory ARCH process, with constant conditional correlation, model (B). (Robust t -statistics between parentheses).

	(i) 1979-1997	(ii) 1979-1997	(iii) 1971-1997	(iv) 1973-1997
μ_1	-0.0047 (0.52)	-0.0043 (-0.49)	-0.0066 (0.89)	-0.0005 (-0.07)
μ_2	-0.0018 (0.17)	-0.0012 (-0.12)	-0.0096 (1.68)	-0.0083 (-0.98)
p_1	2.5931 (3.18)	2.8592 (3.77)	2.3645 (1.69)	2.5022 (1.38)
p_2	3.1258 (3.61)	—	1.8856 (3.24)	2.4657 (4.34)
q_1	0.2385 (3.98)	0.2413 (4.60)	0.6657 (2.48)	0.4458 (1.48)
q_2	0.2427 (4.73)	—	0.2192 (4.12)	0.2961 (5.35)
δ	2.3687 (19.49)	2.3661 (20.12)	2.4004 (13.08)	2.2544 (24.81)
ρ	0.7383 (72.86)	0.7380 (72.89)	0.6628 (41.43)	0.6752 (61.96)
Log-lik	-7117.6135	-7119.7266	-9368.2204	-8998.5725

Table 4: Estimation results of bivariate CCC FIGARCH, model (C). (Robust t -statistics between parentheses).

	(i) 1979-1997	(ii) 1979-1997	(iii) 1971-1997	(iv) 1973-1997
μ_1	-0.0039 (-0.44)	-0.0036 (-0.41)	-0.0059 (-0.88)	0.0008 (0.13)
μ_2	-0.0020 (-0.20)	-0.0014 (-0.14)	-0.0154 (-1.81)	-0.0072 (-0.93)
ω_1	0.0218 (2.79)	0.0230 (3.15)	0.0081 (2.70)	0.0102 (2.68)
ω_2	0.0320 (2.71)	0.0293 (3.18)	0.0441 (1.37)	0.0310 (3.10)
β_1	0.5638 (7.27)	0.5628 (9.82)	0.7586 (9.28)	0.7330 (8.08)
β_2	0.5573 (9.18)	—	0.4016 (3.00)	0.4270 (4.75)
q_1	0.2882 (6.40)	0.2727 (6.67)	0.6692 (4.14)	0.6029 (3.63)
q_2	0.2529 (5.27)	—	0.2302 (2.65)	0.2983 (5.63)
ϕ_1	0.3687 (5.20)	0.3775 (7.56)	0.3010 (3.06)	0.2991 (3.26)
ϕ_2	0.3862 (7.33)	—	0.2895 (3.50)	0.2334 (3.45)
ρ	0.7448 (76.00)	0.7446 (75.58)	0.6734 (46.76)	0.6881 (63.71)
Log-lik	-7090.8634	-7091.9627	-9198.5869	-8937.6380

Table 5: Estimation results of bivariate long-memory ARCH process, model **(D)**. (Robust t -statistics between parentheses).

	(i) 1979-1997	(ii) 1979-1997	(iii) 1971-1997	(iv) 1973-1997
μ_1	0.0008 (0.09)	0.0008 (0.09)	-0.0078 (-0.91)	-0.0024 (-0.42)
μ_2	0.0007 (0.07)	0.0007 (0.07)	-0.0067 (-0.84)	-0.0047 (-0.71)
p_1	4.7013 (3.82)	4.6433 (4.67)	4.1442 (3.19)	3.5452 (4.49)
p_2	5.1265 (4.72)	5.1192 (5.01)	3.7340 (2.89)	3.7320 (4.91)
p_3	5.5434 (4.90)	5.4724 (5.91)	3.8172 (4.29)	4.1258 (4.98)
q_1	0.4683 (5.89)	0.4655 (6.89)	0.3817 (6.18)	0.4617 (6.31)
q_2	0.4661 (6.24)	—	0.3648 (8.35)	0.4529 (6.57)
q_3	0.4705 (5.20)	—	0.3526 (5.42)	0.4592 (5.56)
Log-lik	-6736.0071	-6736.6535	-8771.3637	-8346.8083

Table 6: Estimation results of the unrestricted bivariate FIGARCH, model **(E)**. (Robust t -statistics between parentheses).

	(i) 1979-1997	(ii) 1979-1997	(iii) 1971-1997	(iv) 1973-1997
μ_1	-0.0011 (-0.13)	-0.0009 (-1.11)	-0.0052 (-0.88)	-0.0032 (-0.59)
μ_2	-0.0008 (-0.90)	-0.0006 (-0.07)	-0.0063 (-0.83)	-0.0060 (-0.95)
ω_1	0.0108 (2.84)	0.0107 (2.97)	0.0062 (2.00)	0.0054 (1.64)
ω_2	0.0131 (2.91)	0.0129 (3.00)	0.0149 (1.10)	0.0073 (1.97)
ω_3	0.0085 (2.43)	0.0085 (2.50)	0.0031 (0.61)	0.0011 (0.48)
β_1	0.6417 (10.91)	0.6414 (12.82)	0.6895 (7.19)	0.6517 (13.72)
β_2	0.6376 (11.68)	—	0.5987 (2.58)	0.5643 (9.73)
β_3	0.6432 (11.15)	—	0.6388 (3.98)	0.6090 (12.06)
q_1	0.4235 (7.12)	0.4187 (7.75)	0.5650 (3.91)	0.4968 (8.25)
q_2	0.4179 (6.02)	—	0.4278 (1.32)	0.4039 (8.80)
q_3	0.4225 (6.88)	—	0.4741 (2.07)	0.4379 (11.74)
ϕ_1	0.3243 (8.29)	0.3296 (10.24)	0.2799 (4.35)	0.2933 (6.77)
ϕ_2	0.3089 (8.26)	0.3197 (9.75)	0.2860 (3.99)	0.2690 (6.71)
ϕ_3	0.2965 (8.93)	0.2994 (9.60)	0.2649 (3.69)	0.2690 (7.66)
Log-lik	-6683.9150	-6684.1682	-8586.5104	-8295.7384