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# Parametric versus Nonparametric Goodness of Fit Another View

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## Abstract

We consider chi-squared type tests for testing the hypothesis  $H_0$  that a density  $f$  of observations  $X_1, \dots, X_n$  lies in a parametric class of densities  $\mathcal{F}$ . We consider a version of chi-squared type test using kernel estimates for the density. The main result is, following Liero, Läuter and Konakov (1998), the derivation of the asymptotic behavior of the power of the test under Pitman and "sharp peak" type alternatives. The connection of the rate of convergence of these local alternatives, the bandwidth of the kernel estimator, the parametric estimator, the power of the test are studied.

**Keywords and Phrases:** Chi-squared test, goodness-of-fit test, density, kernel estimators, maximum likelihood estimator, local alternative, Pitman alternative, sharp peak alternative, asymptotic power.

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## 1. Introduction

Let  $X_1, X_2, \dots, X_n$  be a sample of i.i.d. random variables with density  $f$ . We wish to test whether  $f$  belongs to some parametric family  $\mathcal{F}$  of density functions

$$\mathcal{F} = \{f : f(\cdot) = f(\cdot, \theta), \theta \in \Theta \subset \mathbf{R}^k\} \quad (1)$$

against the nonparametric alternative

$$f \notin \mathcal{F}.$$

We consider the kernel estimator  $\hat{f}_n$  of the density function  $f$ ,

$$\hat{f}_n(t) = \frac{1}{n} \sum_{i=1}^n \frac{1}{h_n} K\left(\frac{t - X_i}{h_n}\right), \quad (2)$$

where the kernel function  $K(\cdot)$  is bounded, of bounded variation, has a bounded support and

$$\int K(x) dx = 1.$$

The random function  $\hat{f}_n(t)$  may be represented in the form

$$\hat{f}_n(t) = f^{(n)}(t, \theta) + \frac{1}{\sqrt{n}} \int \frac{1}{h_n} K\left(\frac{t - u}{h_n}\right) dW_n(u), \quad (3)$$

where we denote for a function  $\varphi$  (may be vector function)

$$\varphi^{(n)}(t) = \int \frac{1}{h_n} K\left(\frac{t - u}{h_n}\right) \varphi(u) du, \quad (4)$$

$$W_n(t) = \sqrt{n} (F_n(t) - F(t, \theta)).$$

Here  $F$  is the distribution function of the random variables  $X_i$  and  $F_n$  is its empirical version:

$$F_n(t) = \frac{1}{n} \sum_{j=1}^n \mathbf{1}_{]-\infty, t]}(X_j).$$

We suppose also that

$$h_n \rightarrow 0 \quad \text{and} \quad nh_n \rightarrow \infty, \quad \text{when} \quad n \rightarrow \infty.$$

It is well-known that in this case

$$\mathbf{E}_\theta \hat{f}_n(t) \rightarrow f(t, \theta),$$

i.e.  $\hat{f}_n(t)$  is asymptotically unbiased for  $f(t, \theta)$  at any point of continuity of  $f$ , and since

$$\begin{aligned} nh_n \mathbf{Var}_\theta \hat{f}_n(t) &= \mathbf{Var}_\theta \frac{1}{h_n} K \left( \frac{t - X_{n1}}{h_n} \right) = \\ &= \frac{1}{h_n} \int_0^1 K^2 \left( \frac{t-x}{h_n} \right) f(x, \theta) dx - \frac{1}{h_n} \left[ \int_0^1 K \left( \frac{t-x}{h_n} \right) f(x, \theta) dx \right]^2, \end{aligned}$$

we have

$$\lim_{n \rightarrow \infty} nh_n \mathbf{Var}_\theta \hat{f}_n(t) = f(t, \theta) \int_0^1 K^2(x) dx = kf(t, \theta),$$

i.e.  $\hat{f}_n(t)$  is mean squared consistent for  $f(t, \theta)$ . More about the properties of  $\hat{f}_n$  and its applications see, for example, in Silverman (1986), Bickel & Rosenblatt (1973), Härdle & Marron (1990), Härdle & Mammen (1993).

For a function  $\varphi(x, \theta)$  and a random point  $\theta_* = \hat{\theta}_n$  (where  $\hat{\theta}_n$  is the maximum likelihood estimator for  $\theta$ ) we introduce the notation

$$\varphi_*(x) = \varphi(x, \theta_*), \quad \nabla \varphi(x, \theta) = \left( \frac{\partial}{\partial \theta_1} \varphi(x, \theta), \dots, \frac{\partial}{\partial \theta_k} \varphi(x, \theta) \right)^T,$$

$$\nabla_* \varphi(x) = \left( \frac{\partial}{\partial \theta_1} \varphi(x, \theta_*), \dots, \frac{\partial}{\partial \theta_k} \varphi(x, \theta_*) \right)^T.$$

Further for simplicity of notation we will write  $f$  instead of  $f(\cdot, \theta)$  and  $f_*$  instead of  $f(\cdot, \hat{\theta}_n)$ . For a nonnegative function  $\pi$  we denote by  $L_\pi^2$  the  $L^2$ -space generated by the measure with density  $\pi$ . Let  $\mathbf{I}(\theta)$  be the information matrix,

$$\mathbf{I}(\theta) = \|\|I_{i,j}\|\|_{i,j=1,\dots,k} = \int \frac{\partial}{\partial \theta_i} f(x, \theta) \frac{\partial}{\partial \theta_j} f(x, \theta) f^{-1}(x, \theta) dx.$$

We denote by  $\psi_j(x, \theta)$  for  $j = 1, \dots, k$  the coordinates of the vector-function  $\psi = (\psi_1, \dots, \psi_k)^T$ ,

$$\psi(x, \theta) = \mathbf{I}^{-1/2}(\theta) f^{-1}(x, \theta) \nabla f(x, \theta).$$

So we have

$$(\psi_i(\cdot, \theta), \psi_j(\cdot, \theta))_{f^{-1}} = \delta_{i,j}$$

where

$$(\cdot, \cdot)_{f^{-1}} = (\cdot, \cdot)_{\frac{1}{f}}$$

is the inner product in the space  $L_{f^{-1}}^2$ . The same relation is true for the coordinates  $\psi_j^*(x)$  ( $j = 1, \dots, k$ ) of the vector-function  $\psi_* = \psi(x, \hat{\theta}_n)$ . For a function  $h \in L_{f_*}^2$  we put

$$[h]_L(x) = \sum_{j=1}^k (h(\cdot), \psi_j^*(\cdot))_{f_*^{-1}} \psi_j^*(x), \quad [h]^L = h - [h]_L. \quad (5)$$

It is clear that  $[h]_L$  is the projection of  $h$  in the space  $L_{f_*}^2$  on the linear space  $L = L_n$  spanned by a set  $\{\psi_j^*(x), j = 1, \dots, k\}$ .

As test statistic we suggest a function, which is measuring the normalized deviation of the kernel estimator  $\hat{f}_n(x)$  from a modified parametric estimator  $f^{(n)}(x, \hat{\theta}_n)$ . We will consider these estimators as elements of  $L_\pi^2$ . However we measure the distance between functions from  $L_\pi^2$  with the help of two seminorms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ :

$$\|h\|_1 = \|[h]_L\|_{f_*^{-1}}, \quad \|h\|_2 = \|[h]^L\|_\pi.$$

The process

$$\zeta^{(n)}(t) = \sqrt{n} \left[ \hat{f}_n(t) - f^{(n)}(t, \hat{\theta}_n) \right] \quad (6)$$

further will be called the observable empirical process. We define

$$\mathbf{T}_1^{(n)} = \|\zeta^{(n)}(\cdot)\|_{f_*^{-1}}^2 \quad \text{and} \quad \mathbf{T}_2^{(n)} = \|\zeta^{(n)}(\cdot)\|_\pi^2. \quad (7)$$

For nonnegative functions  $\pi$  and  $h$  we put

$$\mu(h) = \mathbf{k} \int h(x)\pi(x) dx, \quad \sigma^2(h) = \mathbf{k}^* \int h^2(x)\pi^2(x) dx, \quad (8)$$

where

$$\mathbf{k} = \int K^2(x) dx, \quad \mathbf{k}^* = \int (K * K)^2(x) dx.$$

As test statistic we suggest the statistic  $\mathbf{T}^{(n)}$

$$\mathbf{T}^{(n)} = \mathbf{T}_1^{(n)} + \left\{ \frac{h_n^{-1/2} [h_n \mathbf{T}_2^{(n)} - \mu(f_*)]}{\sigma(f_*)} \right\}^2. \quad (9)$$

Now we introduce the assumption under which we plan to investigate the asymptotic behavior of the statistic  $\mathbf{T}^{(n)}$ .

**A1.** For the kernel holds  $K \in L^1$  and  $\int K(x) dx = 1$ .

**A2.** The least even decreasing majorant  $K^*(x)$  of  $|K(x)|$  belongs to the  $L^1$ -space.

**A3.** The density function  $f(x, \theta)$  as function:  $\Theta \rightarrow L_{f^{-1}}^2$ , is continuously differentiable on the open kernel of  $\Theta$ . That is there exists such vector function

$$\nabla f = \left( \frac{\partial}{\partial \theta_1} f, \dots, \frac{\partial}{\partial \theta_k} f \right)$$

that

$$f(\cdot, \theta) - f(\cdot, \theta_0) = \langle \nabla f(\cdot, \theta_0), \theta - \theta_0 \rangle + \mathbf{r}_{\theta_0}(\cdot, \theta),$$

where  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbf{R}^k$ ,

$$\|\cdot\|_{f^{-1}} \rightarrow 0, \quad \text{when } \theta \rightarrow \theta_0.$$

**A4.** The Fisher information matrix  $\mathbf{I}(\theta)$  is positive definite.

## 2. The kernel smoothing

Assuming A1 and let  $K^*(x)$  be the least even decreasing majorant of  $|K(x)|$ ,

$$K^*(x) = \sup_{|t| \geq x} |K(t)|, \quad x \geq 0.$$

We consider the operators

$$[A^h f](x) = \int \frac{1}{h} K\left(\frac{x-t}{h}\right) f(t) dt = f^h(x), \quad h > 0,$$

$$[Mf](x) = \sup_{h>0} \int \frac{1}{h} K\left(\frac{x-t}{h}\right) f(t) dt.$$

Suppose that the nonnegative function  $\omega(t)$  satisfies the condition

$$\sup_I \frac{1}{|I|} \int_I \omega(x) \, dx \cdot \frac{1}{|I|} \int_I \frac{1}{\omega(x)} \, dx < \infty, \quad (10)$$

where  $I$  is an interval,  $|I|$  is the length of  $I$ .

**Theorem 2.1** (see J.B.Garnett (1981)). If  $K^* \in L^1$  and weight function  $\omega$  satisfies the condition (10), then

1.  $M$  is a bounded operator on  $L^2_\omega$ ;
2.  $A^h$  are uniformly bounded (for  $h > 0$ ) operators on  $L^2_\omega$ ;
3. if  $f \in L^2_\omega$ , then  $f^h \rightarrow f$  in the metric of the space  $L^2_\omega$ , when  $h \rightarrow 0$ .

### 3. Maximum likelihood estimator

Consider a sample  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  of i.i.d. random variables  $X_i$  with density function  $f(\cdot) = f(\cdot, \theta)$ ,  $\theta \in \Theta \subset \mathbb{R}^m$ . The maximum likelihood estimator  $\hat{\theta}_n$  is a measurable solution of the likelihood equation

$$\nabla_\theta L_n(\mathbf{X}, \theta) = \sum_{j=1}^n \nabla_\theta \log f(X_j, \theta) = \mathbf{0}. \quad (11)$$

It is well-known (see, for example, Witting & Nölle (1970), Konakov (1978), Greenwood & Nikulin (1996)), that under  $H_0$  and some smoothness conditions on function  $f(x, \theta)$  we have

$$\sqrt{n} (\hat{\theta}_n - \theta) = \frac{1}{\sqrt{n}} \sum_{j=1}^n I^{-1}(\theta) \nabla_\theta \log f(X_j, \theta) + \mathbf{r}_n, \quad (12)$$

where  $\mathbf{r}$  is a random vector such that  $\mathbf{r}_n \xrightarrow{\mathbf{P}} \mathbf{0}$  (we shall write  $\mathbf{r}_n = o_{\mathbf{P}}(\mathbf{1})$ ). Thus,

$$\sqrt{n} (\hat{\theta}_n - \theta) = I^{-1}(\theta) \int \nabla_\theta \log f(x, \theta) \, dW_n(x) + o_{\mathbf{P}}(\mathbf{1}). \quad (13)$$

It is clear that under smoothness conditions on the functions  $f(x, \theta)$

$$\int |[\nabla_\theta \log f]^h(x, \theta) - \nabla_\theta \log f(x, \theta)|^2 f(x, \theta) \, dx \rightarrow 0,$$

when  $h \rightarrow 0$ . Therefore we deduce from (13) that

$$\sqrt{n} (\hat{\theta}_n - \theta) = I^{-1}(\theta) \int [\nabla_{\theta} \log f]_n^K(x, \theta) dW_n(x) + o_{\mathbf{P}}(\mathbf{1}). \quad (14)$$

A similar representation holds under the set  $K_n$  of nonparametric local alternatives

$$K_n = \{f : f(\cdot) = f(\cdot, \theta_n) + N_n v \left( \frac{\cdot - c}{b_n} \right), n = 1, 2, \dots\}, \quad (15)$$

where

$$\theta_n = \theta + n^{-\beta} \psi,$$

$\psi$  is a given vector,  $\beta > 0$ ,  $v$  is a given function,  $N_n, b_n$  are sequences of positive numbers tending to 0,  $c$  is a constant (see e.g. Liero et al. (1998)).

#### 4. The modified empirical process

We consider the smoothed empirical process

$$\xi^{(n)}(t) = \int \frac{1}{h_n} K \left( \frac{t - x}{h_n} \right) dW_n(x), \quad (16)$$

where

$$W_n(x) = \sqrt{n} (F_n(x) - F(x, \theta)),$$

and investigate the limiting behavior of the  $L^2$ -norm of the projection  $\xi_L^{(n)}(t)$  of the process  $\xi^{(n)}(t)$  on some finite-dimensional subspace  $L$ . For a nonnegative function  $\pi$  we put, using (8),

$$\|\xi\|_{\pi}^2 = \int |\xi(x)|^2 \pi(x) dx,$$

$$\mu(f) = \mathbf{k} \int f(x) \pi(x) dx, \quad \sigma^2(f) = \mathbf{k}^* \int f^2(x) \pi^2(x) dx.$$

It is well-known that under some appropriate conditions (see Bickel and Rosenblatt (1973) and Hall (1984))

$$\mathbf{P} \left\{ \frac{h_n^{-1/2} [h_n \|\xi^{(n)}\|_{\pi}^2 - \mu(f)]}{\sigma(f)} \leq x \right\} \rightarrow \Phi(x). \quad (17)$$



Let  $\phi$  be a function such that

$$\phi \in L_\pi^2, \quad \|\phi\|_\pi = 1,$$

and  $\xi_\phi^{(n)}$  be the projection of  $\xi^{(n)}$  on the one-dimensional subspace generated by  $\phi$ ,

$$\xi_\phi^{(n)}(t) = \int \xi^{(n)}(x)\phi(x) dx \cdot \phi(t).$$

So  $\xi_\phi^{(n)} = a\phi$ , where the random coefficient  $a$  is defined by

$$a = \int \xi^{(n)}(x)\phi(x) dx = \int \phi^{(n)}(x) dW_n(x),$$

$$\phi^{(n)}(x) = \int \frac{1}{h_n} K\left(\frac{x-t}{h_n}\right) \phi(t) dt.$$

Since

$$\mathbf{E}a = 0 \quad \text{and} \quad \mathbf{Var}(a) \leq \int [\phi^{K_n}]^2(x) dx,$$

we obtain from Theorem 2.1 the following proposition.

**Proposition 4.1.** Suppose that weight function  $\pi$  satisfies the condition (10) and for some  $C = C(\theta) < \infty$

$$f(x, \theta) < C\pi(x), \quad \text{for all } x. \quad (18)$$

Then there exists such  $C_1$ , which depends only on  $C$ , the weight  $\pi$  and kernel  $K$  (and does not depend on  $\phi$ ), that

$$\mathbf{E}\|\xi_\phi^{(n)}\|_\pi^2 < C_1. \quad (19)$$

It is easily deduced from Proposition 4.1 that

**Proposition 4.2.** Suppose that the weight function  $\pi$  satisfies the condition (10) and  $L$  is a finite dimensional subspace of the space  $L_\pi^2$ . Then under condition (18) such  $C_2$  exists, which depends only on  $C$ , weight  $\pi$ , kernel  $K$  and  $\dim L$ , that

$$\mathbf{E}\|\xi_L^{(n)}\|_\pi^2 < C_2, \quad (20)$$

where  $\xi_L^{(n)}$  is the projection of  $\xi^{(n)}$  on  $L$ .

Now we denote by  $L_n$  a finite dimensional subspace,  $\dim L_n = m$ . Let  $L_n^\perp$

denote the orthogonal complement of  $L_n$  in the space  $L_\pi^2$  and  $\xi_{L_n^\perp}^{(n)}$  be the projection of  $\xi^{(n)}$  on  $L_n^\perp$ .

**Theorem 4.1.** Suppose that the weight function  $\pi$  satisfies the condition (10),  $L_n$  is a subspace of the space  $L_\pi^2$ ,  $\dim L_n$  is fixed,  $h_n \rightarrow 0$ , when  $n \rightarrow \infty$ . Then under condition (18)

$$\mathbf{P} \left\{ \frac{h_n^{-1/2} \left[ h_n \|\xi_{L_n^\perp}^{K_n}\|_\pi^2 - \mu(f) \right]}{\sigma(f)} \leq x \right\} \rightarrow \Phi(x). \quad (21)$$

*Proof.* Since

$$\|\xi_{L_n^\perp}^{(n)}\|_\pi^2 = \|\xi^{(n)}\|_\pi^2 - \|\xi_{L_n}^{(n)}\|_\pi^2,$$

and from Proposition 2 we can conclude that

$$\sup_n \mathbf{P} \left\{ \|\xi_{L_n}^{(n)}\|_\pi^2 > y \right\} \rightarrow 0, \quad \text{when } y \rightarrow \infty,$$

therefore from (17) we obtain (21).

Now we consider a case, when

$$L = L_n = \text{span}\{\psi_j^*(x), j = 1, \dots, k\}.$$

It must be remind that (see(7))

$$\mathbf{T}_2^{(n)} = \left\| \left[ \zeta_n^{(n)}(\cdot) \right]^L \right\|_\pi^2.$$

From the Theorem 4.1. we obtain the next

**Theorem 4.2.** Suppose that weight function  $\pi$  satisfies the condition (10), the density function  $f(x, \theta)$  and kernel  $K(x)$  satisfies the conditions **A1** – **A4**. Then under (18)

$$\mathbf{P} \left\{ \frac{h_n^{-1/2} \left[ h_n \|\mathbf{T}_2^{(n)} - \mu(f_*) \right]}{\sigma(f_*)} \leq x \right\} \rightarrow \Phi(x). \quad (22)$$

## 5. The observable empirical process

Really we deal with the observable empirical process

$$\begin{aligned}\zeta^{(n)}(t) &= \sqrt{n} \left[ \hat{f}_n(t) - f^{(n)}(t, \hat{\theta}_n) \right] \\ &= \sqrt{n} \left[ \hat{f}_n(t) - f^{(n)}(t, \theta) \right] + \sqrt{n} \left[ f^{(n)}(t, \theta) - f^{(n)}(t, \hat{\theta}_n) \right].\end{aligned}\quad (23)$$

It is clear (see (12), (13), (14)) that under smoothness conditions on the function  $f(x, \theta)$

$$\zeta^{(n)}(t) = \xi^{(n)} + \eta^{(n)}(t) + \mathbf{R}_n(t, \theta), \quad (24)$$

where the process  $\mathbf{R}_n(t, \theta)$  weakly converge to zero, when  $n \rightarrow \infty$ , and

$$\eta^{(n)}(t) = [\nabla_{\theta} f^{(n)}(t, \theta)]^T I^{-1}(\theta) \int [\nabla_{\theta} \log f]^{(n)}(x, \theta) dW_n(x). \quad (25)$$

Let  $P_{L_n}$  be the orthoprojector in the space  $L_{f^{-1}}^2$  on the subspace

$$L_n = \text{span}\left\{ \frac{\partial}{\partial \theta_j} f^{(n)}(x, \theta), j = 1, \dots, k \right\}.$$

We introduce the matrix  $R_n(\theta) = \|r_{i,j}\|_{i,j=1,\dots,k}$ , where

$$r_{i,j} = \int \frac{\partial}{\partial \theta_i} f^{(n)}(x, \theta) \frac{\partial}{\partial \theta_j} f^{(n)}(x, \theta) f^{-1}(x, \theta) dx, \quad i, j = 1, \dots, k.$$

and denote by  $R^{-1/2}(\theta)$  its square root. Let

$$\varphi^{(n)}(x, \theta) = \left( \varphi_1^{(n)}(x, \theta), \dots, \varphi_k^{(n)}(x, \theta) \right) = R^{-1/2}(\theta) \nabla f^{(n)}(x, \theta).$$

It is obvious that

$$\left( \varphi_i^{(n)}(\cdot, \theta), \varphi_j^{(n)}(\cdot, \theta) \right)_{f^{-1}} = \delta_{i,j},$$

where  $(\cdot, \cdot)_{f^{-1}}$  is the inner product in the space  $L_{f^{-1}}^2$ . So the system  $\{\varphi_i^{(n)}(\cdot, \theta), i = 1, \dots, k\}$  forms the orthonormal basis (in the metric of the space  $L_{f^{-1}}^2$ ) of the subspace  $L_n$ . Therefore

$$[P_{L_n} h](t) = \sum_{i=1}^k \int h(x) \varphi_i^{(n)}(x, \theta) f^{-1}(x, \theta) dx \cdot \varphi_i^{(n)}(t, \theta), \quad h \in L_{f^{-1}}^2.$$

Now we denote by  $L_n^*$  the subspace

$$L_n^* = \text{span}\left\{\frac{\partial}{\partial\theta_j}f^{(n)}(x, \hat{\theta}_n), j = 1, \dots, k\right\},$$

and put

$$[P_{L_n^*}h](t) = \sum_{i=1}^k \int h(x)\varphi_i^{(n)}(x, \hat{\theta}_n)f_*^{-1} dx \cdot \varphi_i^{(n)}(t, \hat{\theta}_n), \quad h \in L_{f^{-1}(x, \hat{\theta}_n)}^2.$$

It seems to be reasonable to expect that operators  $P_{L_n}$  and  $P_{L_n^*}$  weakly converge (on an appropriate sense) when  $n \rightarrow \infty$  to the orthoprojector  $P_L$  in the space  $L_{f^{-1}}^2$  on the subspace  $L$ ,

$$L = \text{span}\left\{\frac{\partial}{\partial\theta_j}f(x, \theta), j = 1, \dots, k\right\}.$$

For our needs it is sufficiently to prove that the asymptotic distribution of the process  $[P_{L_n^*}\zeta^{(n)}](t)$  is the same as the asymptotic distribution of the process  $[P_L\zeta^{(n)}](t)$ . It may be proved by the usual weak convergence technique. One can verify that the limiting distribution of the statistic  $\mathbf{T}_1^{(n)}$  is the  $\chi_k^2$ -distribution.

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