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**Working Paper**

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SFB 373 Discussion Paper, No. 1998,106

**Provided in Cooperation with:**

Collaborative Research Center 373: Quantification and Simulation of Economic Processes,  
Humboldt University Berlin

*Suggested Citation:* Schweizer, Martin (1998) : A minimality property of the minimal martingale measure, SFB 373 Discussion Paper, No. 1998,106, Humboldt University of Berlin, Interdisciplinary Research Project 373: Quantification and Simulation of Economic Processes, Berlin, <https://nbn-resolving.de/urn:nbn:de:kobv:11-10060779>

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# A Minimality Property of the Minimal Martingale Measure

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**Abstract:** Let  $X$  be a continuous adapted process for which there exists an equivalent local martingale measure (ELMM). The minimal martingale measure  $\hat{P}$  is the unique ELMM for  $X$  with the property that local  $P$ -martingales strongly orthogonal to the  $P$ -martingale part of  $X$  are also local  $\hat{P}$ -martingales. We prove that if  $\hat{P}$  exists, it minimizes the reverse relative entropy  $H(P|Q)$  over all ELMMs  $Q$  for  $X$ . A counterexample shows that the assumption of continuity cannot be dropped.

**Key words:** minimal martingale measure, relative entropy, equivalent martingale measures

**1991 Mathematics Subject Classification:** 60G48, 90A09

**JEL Classification Numbers:** G10

*(to appear in Statistics and Probability Letters)*

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\* Research for this paper was partially carried out within Sonderforschungsbereich 373. This paper was printed using funds made available by the Deutsche Forschungsgemeinschaft.

## 1. The result

In this section, we introduce the framework for our problem and present our main result. Let  $(\Omega, \mathcal{F}, P)$  be a probability space with a filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  satisfying the usual conditions of right-continuity and completeness, where  $T \in (0, \infty]$  is a fixed time horizon. For all unexplained terminology from stochastic analysis, we refer to Protter (1990). We consider an  $\mathbb{R}^d$ -valued  $\mathbb{F}$ -adapted process  $X = (X_t)_{0 \leq t \leq T}$  and assume that  $X$  has  $P$ -a.s. *continuous* trajectories. Intuitively,  $X$  represents the discounted price evolution of  $d$  risky assets in a financial market, and we want to exclude the possibility of having arbitrage (“money-pumps”) in this market. We therefore assume that  $X$  admits an *equivalent local martingale measure (ELMM)*, i.e., there exists a probability measure  $Q \approx P$  with  $Q = P$  on  $\mathcal{F}_0$  such that  $X$  is a local  $Q$ -martingale; see for instance Delbaen/Schachermayer (1994) for a more detailed discussion of the economic significance of such a condition. Together with the continuity of  $X$ , it implies by Theorem 2.2 of Choulli/Stricker (1996) that  $X$  is a special semimartingale satisfying the *structure condition (SC)*: In the canonical decomposition  $X = X_0 + M + A$ , the process  $M$  is an  $\mathbb{R}^d$ -valued locally square-integrable local  $P$ -martingale, and the  $\mathbb{R}^d$ -valued process  $A$  of finite variation has the form

$$A_t = \int_0^t d\langle M \rangle_s \lambda_s \quad , \quad 0 \leq t \leq T$$

for an  $\mathbb{R}^d$ -valued predictable process  $\lambda$  such that

$$K_t := \int_0^t \lambda_s^{\text{tr}} d\langle M \rangle_s \lambda_s = \sum_{i,j=1}^d \int_0^t \lambda_s^i \lambda_s^j d\langle M^i, M^j \rangle_s < \infty \quad P\text{-a.s. for all } t \in [0, T].$$

The process  $K$  is called the *mean-variance tradeoff process* of  $X$ .

Since  $X$  admits at least one ELMM, one can ask about ELMMs having some special properties. One possibility is the *minimal martingale measure*  $\widehat{P}$  introduced by Föllmer/Schweizer (1991) and generalized by Ansel/Stricker (1992, 1993). This is defined by

$$(1.1) \quad \frac{d\widehat{P}}{dP} := \widehat{Z}_T \quad \text{with } \widehat{Z} := \mathcal{E}\left(-\int \lambda dM\right),$$

where we assume that the exponential local  $P$ -martingale  $\widehat{Z}$  is strictly positive and a true  $P$ -martingale so that  $E[\widehat{Z}_T] = 1$ . If in addition  $\widehat{Z}_T \in L^2(P)$ , then Theorem (3.5) of Föllmer/Schweizer (1991) shows that every square-integrable  $P$ -martingale  $L$  strongly  $P$ -orthogonal to  $M$  is also a  $\widehat{P}$ -martingale (and strongly  $\widehat{P}$ -orthogonal to  $X$ ). Thus  $\widehat{P}$  is minimal in the sense that it preserves the martingale structure as far as possible under the constraint of turning  $X$  into a martingale. Moreover,  $\widehat{P}$  is also the natural candidate for an ELMM for  $X$  by Girsanov’s theorem.

Because the preceding description of minimality is somewhat awkward, there have been several attempts to characterize  $\widehat{P}$  in a different way. An economic characterization in a multidimensional diffusion framework has been given in Hofmann/Platen/Schweizer (1992). Föllmer/Schweizer (1991) and Schweizer (1995a) have shown that for  $X$  continuous,  $\widehat{P}$  minimizes the “free energy”  $H(Q|P) - \frac{1}{2}E_Q[K_T]$  over all ELMMs  $Q$  for  $X$  satisfying  $E_Q[K_T] < \infty$ .

Here we recall that for two probability measures  $P, Q$  and a  $\sigma$ -algebra  $\mathcal{G} \subseteq \mathcal{F}$ , the relative entropy of  $Q$  with respect to  $P$  on  $\mathcal{G}$  is

$$H_{\mathcal{G}}(Q|P) := \begin{cases} E_Q \left[ \log \frac{dQ}{dP} \Big|_{\mathcal{G}} \right] & , \text{ if } Q \ll P \text{ on } \mathcal{G} \\ +\infty & , \text{ otherwise.} \end{cases}$$

We also recall that  $H_{\mathcal{G}}(Q|P)$  is always nonnegative, increasing in  $\mathcal{G}$ , and that  $H(Q|P) := H_{\mathcal{F}}(Q|P)$  is 0 if and only if  $Q = P$ . In particular, the above characterization of  $\hat{P}$  implies that  $\hat{P}$  minimizes the relative entropy  $H(Q|P)$  over all ELMMs  $Q$  for  $X$  if  $X$  is continuous and the final value  $K_T$  of the mean-variance tradeoff process is *deterministic*. Under the same conditions,  $\hat{P}$  also minimizes  $\text{Var} \left[ \frac{dQ}{dP} \right]$  or  $\left\| \frac{dQ}{dP} \right\|_{L^2(P)}$  over all ELMMs  $Q$  for  $X$ ; see Theorem 7 of Schweizer (1995a). Miyahara (1996) has shown that  $\hat{P}$  also minimizes  $H(Q|P)$  over all ELMMs  $Q$  if  $X$  is a Markovian diffusion given by the multidimensional stochastic differential equation

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t.$$

But all these results either use a very specific structure for  $X$  or impose the very restrictive condition that  $K_T$  should be deterministic. In contrast, the main result of this paper is completely general.

**Theorem 1.** *Suppose that  $X$  is a continuous adapted process admitting at least one equivalent local martingale measure  $Q$ . If  $\hat{P}$  defined by (1.1) is a probability measure equivalent to  $P$ , then  $\hat{P}$  minimizes the reverse relative entropy  $H(P|Q)$  over all ELMMs  $Q$  for  $X$ .*

We remark that the idea of considering  $H(P|Q)$  instead of  $H(Q|P)$  first appeared in Platen/Rebolledo (1996). The assumption about  $\hat{P}$  of course just states that the minimal martingale measure  $\hat{P}$  should exist; it is thus a minimal requirement for the theorem's assertion. Theorem 1 is only true for a *continuous* process  $X$ ; we shall show by a counterexample in the next section that the conclusion fails in general if  $X$  has jumps.

The next result is a preparation for the proof of Theorem 1. It does not really need any martingale structure; we could replace  $N_{\tau}$  by any positive random variable with expectation 1. The present formulation just makes clear how we apply the lemma later on.

**Lemma 2.** *Suppose that  $N$  is a strictly positive local  $P$ -martingale with  $N_0 = 1$ . For any stopping time  $\tau$  such that the stopped process  $N^{\tau}$  is a  $P$ -martingale, we then have  $E[\log N_{\tau}] \in [-\infty, 0]$ .*

**Proof.** We cannot use Jensen's inequality because we do not know whether  $\log N_{\tau}$  is integrable. But since  $N^{\tau}$  is a strictly positive  $P$ -martingale starting from 1,  $N_{\tau}$  is strictly positive and has expectation 1. Thus we can define a probability measure  $R \approx P$  by  $\frac{dR}{dP} := N_{\tau}$ , and so we obtain

$$E_P[-\log N_{\tau}] = E_P \left[ \log \frac{dP}{dR} \right] = H(P|R) \in [0, \infty].$$

**q.e.d.**

**Proof of Theorem 1:** Let  $Q$  be any ELMM for  $X$  and denote by  $Z$  its density process with respect to  $P$ . We may also assume that  $H(P|Q) < \infty$  since there is nothing to prove

otherwise. Because  $X$  is continuous, we can write  $Z$  as  $Z = \widehat{Z}\mathcal{E}(L)$  for a local  $P$ -martingale  $L$  with  $L_0 = 0$ ; see Theorem 1 of Schweizer (1995a) or Corollary 2.3 of Choulli/Stricker (1996). Let  $(\tau_n)_{n \in \mathbb{N}}$  be a localizing sequence for  $\mathcal{E}(L)$  and  $\int \lambda dM$  and fix  $n \in \mathbb{N}$ . Then

$$\frac{dP}{dQ} \Big|_{\mathcal{F}_{\tau_n}} = \frac{1}{Z_{\tau_n}} = \frac{1}{\widehat{Z}_{\tau_n}} \frac{1}{\mathcal{E}(L)_{\tau_n}} = \frac{dP}{d\widehat{P}} \Big|_{\mathcal{F}_{\tau_n}} \frac{1}{\mathcal{E}(L)_{\tau_n}},$$

and so Lemma 2 with  $N := \mathcal{E}(L)$  implies that

$$H_{\mathcal{F}_{\tau_n}}(P|Q) = H_{\mathcal{F}_{\tau_n}}(P|\widehat{P}) - E_P[\log \mathcal{E}(L)_{\tau_n}] \geq H_{\mathcal{F}_{\tau_n}}(P|\widehat{P})$$

and therefore

$$(1.2) \quad \sup_{n \in \mathbb{N}} H_{\mathcal{F}_{\tau_n}}(P|\widehat{P}) \leq \sup_{n \in \mathbb{N}} H_{\mathcal{F}_{\tau_n}}(P|Q) \leq H(P|Q) < \infty,$$

since  $H_{\mathcal{G}}(P|Q)$  is increasing in  $\mathcal{G}$ . From Lemma 2 of Barron (1985), we thus obtain

$$\sup_{n \in \mathbb{N}} \left| \log \frac{1}{\widehat{Z}_{\tau_n}} \right| = \sup_{n \in \mathbb{N}} \left| \log \frac{dP}{d\widehat{P}} \Big|_{\mathcal{F}_{\tau_n}} \right| \in L^1(P),$$

and since  $\widehat{Z}_{\tau_n} \rightarrow \widehat{Z}_T$   $P$ -a.s. because  $\tau_n$  increases stationarily to  $T$ , the dominated convergence theorem yields

$$H(P|\widehat{P}) = E_P \left[ \log \frac{1}{\widehat{Z}_T} \right] = \lim_{n \rightarrow \infty} E_P \left[ \log \frac{1}{\widehat{Z}_{\tau_n}} \right] = \lim_{n \rightarrow \infty} H_{\mathcal{F}_{\tau_n}}(P|\widehat{P}) \leq H(P|Q)$$

by (1.2). As  $Q$  was arbitrary, the proof is complete. **q.e.d.**

**Remark.** A closer look at the above proof shows that we only need continuity of  $X$  to write the density process  $Z$  of an arbitrary ELMM as  $Z = \widehat{Z}\mathcal{E}(L)$  for some local  $P$ -martingale  $L$  null at 0. One can ask if this is also possible for a general semimartingale  $X$  satisfying the structure condition (SC), but the answer is negative. An explicit counterexample can be obtained by taking for  $X$  the sum of a Brownian motion with drift and a compensated Poisson process. Alternatively, this is a consequence of the counterexample in the next section.

## 2. The counterexample

If the process  $X$  is not continuous, the assertion of Theorem 1 is no longer true: We present here a counterexample with an ELMM  $Q^*$  such that  $H(P|Q^*) < H(P|\widehat{P})$ . It uses a bounded process in finite discrete time and basically consists of a number of elementary computations.

Fix some  $U > 1$  and consider for  $X$  a trinomial tree with time horizon 2 and parameters  $U, 1, \frac{1}{U}$ . Formally, let  $Y_1, Y_2$  be i.i.d. under  $P$  taking the values  $U, 1, \frac{1}{U}$  with probability  $\frac{1}{3}$  each. The process  $X = (X_k)_{k=0,1,2}$  is then given by  $X_0 := 1$ ,  $X_1 := Y_1$  and  $X_2 := Y_1 Y_2$ , and  $\mathbb{F}$  is the filtration generated by  $X$ . We use the notation  $\Delta X_k := X_k - X_{k-1}$  for the increments of  $X$ .

Any equivalent martingale measure (EMM)  $Q$  for  $X$  can be identified with a vector  $q \in (0, 1)^4$  via its transition probabilities

$$\begin{aligned} q_1 &:= Q[X_1 = U] & , & & q_2 &:= Q[X_2 = U | X_1 = U] \\ q_3 &:= Q[X_2 = U | X_1 = 1] & , & & q_4 &:= Q\left[X_2 = U \middle| X_1 = \frac{1}{U}\right]. \end{aligned}$$

The other transition probabilities are then determined by the martingale property of  $X$  under  $Q$  and the fact that they add to 1 at each node in the tree. An elementary computation yields

$$\begin{aligned} (2.1) \quad H(P|Q) &= E_P \left[ -\log \frac{dQ}{dP} \right] \\ &= -\frac{2}{3} \log q_1 - \frac{1}{3} \log (1 - (U+1)q_1) - \frac{1}{9} \sum_{i=2}^4 \left( 2 \log q_i + \log (1 - (U+1)q_i) \right) \\ &\quad + \log 9 - \frac{2}{3} \log U, \end{aligned}$$

and setting the gradient with respect to  $q$  equal to 0 gives an EMM  $Q^*$  with

$$q_i^* = \frac{2}{3(U+1)} \quad \text{for } i = 1, \dots, 4$$

as a candidate for the entropy-optimal EMM. Under  $Q^*$ , the random variables  $Y_1, Y_2$  are still i.i.d. and take the values  $U, 1, \frac{1}{U}$  with probability  $\frac{2}{3(U+1)}, \frac{1}{3}$  and  $\frac{2U}{3(U+1)}$ , respectively, so that  $Q^*$  is clearly equivalent to  $P$ . Inserting into (2.1) yields after some simplification

$$H(P|Q^*) = \log \frac{81}{\sqrt[3]{16}} + \frac{2}{3} \log \frac{(U+1)^2}{U}.$$

To compute the minimal EMM  $\hat{P}$  for  $X$ , we use the results of Schweizer (1995b). According to equations (2.21) and (1.2) in that paper,  $\hat{P}$  is given by the density

$$\frac{d\hat{P}}{dP} = \hat{Z}_2 = \prod_{k=1}^2 \frac{1 - \alpha_k \Delta X_k}{1 - \alpha_k \Delta A_k} = \prod_{k=1}^2 \frac{E[\Delta X_k^2 | \mathcal{F}_{k-1}] - \Delta X_k E[\Delta X_k | \mathcal{F}_{k-1}]}{E[\Delta X_k^2 | \mathcal{F}_{k-1}] - (E[\Delta X_k | \mathcal{F}_{k-1}])^2}.$$

Computing this explicitly shows that  $\hat{P}$  can be identified with the vector  $\hat{q}$  given by

$$\hat{q}_i = \frac{U+1}{2(U^2+U+1)} \quad \text{for } i = 1, \dots, 4.$$

This means that under  $\hat{P}$ ,  $Y_1$  and  $Y_2$  are again i.i.d. and take the values  $U, 1, \frac{1}{U}$  with probability  $\frac{U+1}{2(U^2+U+1)}, \frac{U^2+1}{2(U^2+U+1)}$  and  $\frac{U^2+U}{2(U^2+U+1)}$ , respectively. Inserting into (2.1) now yields

$$H(P|\hat{P}) = \log 36 - \frac{2}{3} \log \frac{U(U^2+1)(U+1)^2}{(U^2+U+1)^3}.$$

If we take for instance  $U = 2$ , we obtain

$$q_i^* = \frac{2}{9} \quad , \quad \hat{q}_i = \frac{3}{14} \quad \text{for } i = 1, \dots, 4$$

and

$$H(P|Q^*) = 4.473 < 4.475 = H(P|\hat{P}).$$

This shows that  $\hat{P}$  need not minimize the reverse relative entropy if  $X$  is not continuous so that we have indeed a counterexample. Numerical evidence suggests that  $H(P|Q^*) < H(P|\hat{P})$  for every  $U > 1$ , but we have not bothered to check this theoretically.

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