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On sequential parameter estimation for some linear stochastic differential equations with time delay *

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Abstract

We consider the parameter estimation problem for the scalar diffusion type process described by the stochastic equation with time delay

$$dX(t) = \sum_{i=0}^m \vartheta_i X(t - r_i) dt + dW(t).$$

The asymptotic behavior of the classical maximum likelihood estimator (MLE) very depends on the true values of parameter $\vartheta = (\vartheta_0, \vartheta_1, \dots, \vartheta_m)'$.

Here we construct a sequential MLE with preassigned least square accuracy for the so-called stationary and the periodic cases of the solution $X(\cdot)$. The limit behaviour of the duration of the procedure with given accuracy is obtained.

Keywords: stochastic differential equations; time delay; maximum likelihood estimator; sequential analysis; least square accuracy.

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1 Introduction

Assume $(W(t), \mathcal{F}_t, t \geq 0)$ is a realvalued Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ and $(X(t), t \geq -r)$ satisfies the following differential equation with time delay

$$\left. \begin{aligned} dX(t) &= \sum_{i=0}^m \vartheta_i X(t - r_i) dt + dW(t), \quad t \geq 0, \\ X(s) &= X_0(s), \quad s \in [-r, 0]. \end{aligned} \right\} \quad (1)$$

The parameters $r_i, \vartheta_i, i = 0, \dots, m$ are real numbers with $0 = r_0 < r_1 < \dots < r_m =: r$, if $m \geq 1$ and $r_0 = r = 0$ if $m = 0$. The initial process $(X_0(s), s \in [-r, 0])$ is supposed to be cadlag and all $X_0(s), s \in [-r, 0]$ are assumed to be \mathcal{F}_0 -measurable. Moreover assume that

$$E \int_{-1}^0 X_0^2(s) ds < \infty.$$

The equation (1) is a special case of so-called affine stochastic differential equation studied in detail e.g. in [Mo/Sch] and [Mo]. In particular it holds, that (1) has a uniquely determined solution $(X(t), t \geq -r)$ having the representation

$$\left. \begin{aligned} X(t) &= \sum_{j=0}^m \vartheta_j \int_{-r_j}^0 x_0(t - s - r_j) X_0(s) ds + \\ &\quad + x_0(t) X_0(0) + \int_0^t x_0(t - s) dW(s), \quad t > 0 \\ X(t) &= X_0(t), \quad t \in [-r, 0] \end{aligned} \right\} \quad (2)$$

and satisfying $E \int_0^T X^2(s) ds < \infty$ for every T with $0 < T < \infty$. Here the function $x_0(\cdot)$ denotes the fundamental solution of the corresponding to (1) linear deterministic equation

$$x_0(t) = 1 + \sum_{j=0}^m \int_0^t \vartheta_j x_0(t - r_j), \quad t \geq 0, \quad (3)$$

$$x_0(s) = 0, \quad s \in [-r, 0); \quad x_0(0) = 1.$$

(see [Ha/Ve] for details on (3)).

Fix a subject Θ of R^{m+1} and assume the vector $\vartheta = (\vartheta_0, \vartheta_1, \dots, \vartheta_m)' \in \Theta$ is unknown and has to be estimated based on the observation $(X(t))$. The delay times r_i are supposed to be known.

The measures $P_\vartheta, \vartheta \in R^{m+1}$ generated by the solutions of (1) form an exponential family in the sense of [Ku/So]. Thus, one possibility to estimate ϑ is to use the maximum likelihood method. The corresponding log-likelihood-function is given by

$$\ell_t(\vartheta) = \vartheta' \Phi(t) - \frac{1}{2} \vartheta' G(t) \vartheta \quad , \quad \vartheta \in \Theta, \quad t > 0, \quad (4)$$

where

$$\Phi(t) = \left(\int_0^t X(s - r_i) dX(s), i = 0, \dots, m \right)',$$

and

$$G(t) = \left(\int_0^t X(s - r_i) X(s - r_j) ds, i, j = 0, \dots, m \right)$$

denotes the Fisher information matrix (for details see [Gu/Ku] and [Ku/So]). Another method is provided by sequential estimation. Sequential estimation of one-dimensional parameters in exponential families of processes have been studied e.g. in [Li/Sh] and [Nov], see also [Ku/So] (1997), Chapter 10. The more-dimensional parameter case cannot be treated in the same way. Indeed, the construction of the stopping time for the observation in these papers very uses the one-dimensionality of the Fisher information. For processes arising from linear stochastic differential equations without time delay having more-dimensional parameters, sequential methods have been developed in [Ko/Pe] (1985), (1987), (1992).

Here we shall extend these results to equations of the type (1). We shall construct for every $\varepsilon > 0$ a sequential procedure ϑ^* to estimate ϑ with ε -accuracy in the square mean sense, i.e. with $E[\vartheta^* - \vartheta]^2 \leq \varepsilon$.

The method used below is a two step construction of a random time, where the first step uses the trace of the Fisher information matrix and follows the line of the one-dimensional case mentioned above.

A generalization of the sequential estimators, constructed in the sequel, to differential equations of the type (1) but based on noisy observations, will be presented in a subsequent paper.

2 Results

Consider the process $(X(t), t \geq -r)$ described by equation (1) above.

Throughout this paper we suppose that the following assumption holds.

Assumption (A) : For every $\vartheta \in \Theta$ there exist a (deterministic scalar) positive increasing function $\varphi(\cdot)$ on $[0, \infty)$ with $\lim_{T \rightarrow \infty} \varphi(T) = \infty$ and a possibly random $(m + 1) \times (m + 1)$ -matrix function $I_\infty(T)$, $T \in [0, \infty)$, being continuous periodic with period $\Delta \geq 0$ ($\Delta = 0$ means $I_\infty(T) \equiv I_\infty(0)$ and positive definite for every T). Moreover, it holds

$$\lim_{T \rightarrow \infty} \left| \frac{G(T)}{\varphi(T)} - I_\infty(T) \right| = 0 \quad a.s. \quad (5)$$

The assumption (A) is satisfied under further restrictions on Θ only. For example, if $m = 1$ then it holds exactly in the following two cases.

Consider the set Λ of all complex roots of the so-called characteristic equation

$$\lambda - \vartheta_0 - \vartheta_1 e^{-\lambda r} = 0$$

and put $v_0 = v_0(\vartheta) = \max\{Re\lambda | \lambda \in \Lambda\}$. It can be easily shown that $v_0 < \infty$. Then (A) holds for $\Theta = \{\vartheta \in R^2 \mid v_0(\vartheta) < 0 \text{ or } [v_0(\vartheta) > 0 \text{ and } v_0(\vartheta) \notin \Lambda]\}$, see [Gu/Ku] (1998) for details. If $v_0 < 0$ then the equation (1) admits a stationary solution and every solution tends to it in distribution, moreover we have $\Delta = 0$, we call this case the "stationary case". If $v_0 > 0$ and $v_0 \notin \Lambda$ the equality (5) is valid with some $\Delta > 0$. We denote this case as the "periodic" one.

A similar picture appears in the classical multidimensional linear equation

$$dX(t) = AX(t)dt + dW(t), \quad t \geq 0, \quad X(0) = X_0$$

with the Fisher information matrix

$$\Gamma(T) = \int_0^T X(t)X'(t)dt.$$

Here $W(\cdot)$ is a d -dimensional standard Wiener process and A a given $d \times d$ matrix. Let λ_{\max} and λ_{\min} be eigenvalues of A having the maximal and minimal absolute value under all of eigenvalues of e^A , respectively. It is well known that the limiting matrix $\lim_{T \rightarrow \infty} T^{-1}\Gamma(T)$ exist and is a positive definite deterministic matrix in the stable case ($Re\lambda_{\max} < 0$) and $\Gamma(T)$ increase exponentially in the unstable case ($Re\lambda_{\min} > 0$). Note that for stable case the sequential parameter estimation problem of matrix A was considered in [Ko/Pe] (1985), for the scalar model in [Nov] and [Li/Sh], for unstable case in [Ko/Pe] (1987) and in mixed case ($Re\lambda_{\max} > 0$, $Re\lambda_{\min} < 0$ and $\lambda + \mu \neq 0$ for all eigenvalues λ, μ of A) in [Ko/Pe] (1992).

The sequential estimation problem for the matrix A in the stable case by noisy observations was studied in [Va/Ko] (1987) and [Va/Ko] (1990).

Let us return to the study of (1) and let Assumption (A) be true.

To estimate ϑ with preassigned accuracy $\varepsilon > 0$ we shall start with the maximum likelihood estimator of ϑ for the given length T of observation defined by the equality

$$\hat{\vartheta}(T) = G^{-1}(T)\Phi(T), \quad T > 0. \quad (6)$$

From (1) and (6) we find the deviation of the estimator $\hat{\vartheta}(T)$ from ϑ :

$$\hat{\vartheta}(T) - \vartheta = G^{-1}(T)\zeta(T) \quad (7)$$

with

$$\zeta(T) = \int_0^T Z(t)dW(t), \quad Z(t) = (X(t), X(t - r_1), \dots, X(t - r_m))'.$$

Now we make a time substitution which enables us to control the second moments of the noise ζ .

Fix an arbitrary increasing sequence $(c_n)_{n \geq 1}$ of reals tending to infinity. Let us define the sequence of (\mathcal{F}_t) -stopping times $(\tau_\varepsilon(n), n \geq 1)$ as follows

$$\tau_\varepsilon(n) = \inf\{T > 0 : trG(T) = \varepsilon^{-1}c_n\}, \quad (8)$$

These moments are finite a.s. due to the condition (5).

One can easily verify that for any $\varepsilon > 0$ the sequence $(\zeta(\tau_\varepsilon(n)), n \geq 1)$ satisfies the equalities

$$E_{\vartheta} \|\zeta(\tau_\varepsilon(n))\|^2 = \varepsilon^{-1} c_n, \quad n \geq 1. \quad (9)$$

(Throughout this paper $\|\cdot\|$ denotes the Euclidian norm.)

The equalities (9) suggest that the estimation of the parameter ϑ should be performed at the moments $\tau_\varepsilon(n)$:

$$\vartheta_n(\varepsilon) = \hat{\vartheta}(\tau_\varepsilon(n)), \quad n \geq 1. \quad (10)$$

According to (7) in order to obtain the estimates with fixed least square deviation now one should control the behaviour of the sequence of random matrices $(G^{-1}(\tau_\varepsilon(n)), n \geq 1)$. It can be achieved by conducting the observations up to the moment $\tau_\varepsilon(n)$ with a specially chosen number n . Let

$$\nu_\varepsilon = \inf\{N \geq 1 : S_N(\varepsilon) \geq \varrho\}, \quad (11)$$

where $S_N(\varepsilon) = \sum_{n=1}^N \beta_n^2(\varepsilon)$, $\beta_n^2(\varepsilon) = (\varepsilon \cdot c_n^{-1})^2 \cdot \|G^{-1}(\tau_\varepsilon(n))\|^{-2}$, $\varrho = \sum_{n \geq 1} 1/c_n$.

The sequential plan $(T(\varepsilon), \vartheta_\varepsilon^*)$ of estimation of the vector ϑ will be defined by

$$T(\varepsilon) = \tau_\varepsilon(\nu_\varepsilon), \quad \vartheta_\varepsilon^* = S_{\nu_\varepsilon}^{-1} \sum_{n=1}^{\nu_\varepsilon} \beta_n^2(\varepsilon) \cdot \vartheta_n(\varepsilon). \quad (12)$$

Obviously, ν_ε is a $(\mathcal{F}_{\tau_n(\varepsilon)})$ -stopping time, and therefore, by construction, $T(\varepsilon)$ turns out to be an (\mathcal{F}_t) -stopping time.

In such a way the sequential estimate ϑ_ε^* is a random weighted mean of the maximum likelihood estimates, calculated at the stopping times $\tau_\varepsilon(n), n \geq 1$.

The following theorem summarizes the main result.

Theorem 1. *Assume that Assumption A holds. Then for any $\varepsilon > 0$ and any $\vartheta \in \Theta$ the sequential estimation plan (12) of ϑ possesses the properties:*

$$1^\circ. T(\varepsilon) < \infty \quad P_\vartheta - a.s.,$$

$$2^\circ. E_\vartheta \|\vartheta_\varepsilon^* - \vartheta\|^2 \leq \varepsilon$$

and the following inequalities hold $P_\vartheta - a.s.$

$$3^\circ. 0 < \underline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \cdot \varphi(T(\varepsilon)) \leq \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \cdot \varphi(T(\varepsilon)) < \infty.$$

Proof. 1° . Let us verify the finiteness of $T(\varepsilon) = \tau_\varepsilon(\nu_\varepsilon)$. While the moments $\tau_\varepsilon(n)$ are finite for all $n \geq 1$, it suffices to establish the finiteness of the moment ν_ε . Making

use of the definition (9) of $\tau_\varepsilon(n)$ and the condition (5) we have

$$\lim_{n \rightarrow \infty} \left| \frac{\varepsilon^{-1} \cdot c_n}{\varphi(\tau_\varepsilon(n))} - \text{tr} I_\infty(\tau_\varepsilon(n)) \right| = 0 \quad a.s. \quad (13)$$

and as follows by the definition of $\beta_n^2(\varepsilon)$

$$\lim_{n \rightarrow \infty} |\beta_n^2(\varepsilon) - \beta^2(\tau_\varepsilon(n))| = 0 \quad a.s., \quad (14)$$

where

$$\beta^2(u) = [\text{tr} I_\infty(u) \cdot \|I_\infty^{-1}(u)\|]^{-2}. \quad (15)$$

Note that by the conditions on the matrix function $I_\infty(u)$ we have

$$\inf_{u \in R^1} \beta^2(u) > 0.$$

Then $\sum_{n=1}^{\infty} \beta_n^2(\varepsilon) = \infty$ a.s. and for all $\varepsilon > 0$ the moments ν_ε and $T(\varepsilon)$ are finite a.s.

2°. Now we estimate the mean square deviation of ϑ_ε^* . From (7), (9), (12) and by definitions of ν_ε , β_n and ϱ it follows that

$$\begin{aligned} E_\vartheta \|\vartheta_\varepsilon^* - \vartheta\|^2 &= E_\vartheta S_{\nu_\varepsilon}^{-2} \left\| \sum_{n=1}^{\nu_\varepsilon} \beta_n^2(\varepsilon) (\vartheta_n(\varepsilon) - \vartheta) \right\|^2 \leq \\ &\leq E_\vartheta S_{\nu_\varepsilon}^{-1} \sum_{n \geq 1} \beta_n^2(\varepsilon) \|\vartheta_n(\varepsilon) - \vartheta\|^2 \leq \varrho^{-1} \sum_{n \geq 1} E_\vartheta \beta_n^2(\varepsilon) \cdot \\ &\cdot \|G^{-1}(\tau_\varepsilon(n))\|^2 \cdot \|\zeta(\tau_\varepsilon(n))\|^2 = \varepsilon^2 \varrho^{-1} \sum_{n \geq 1} \frac{1}{c_n^2} E_\vartheta \|\zeta(\tau_\varepsilon(n))\|^2 = \\ &= \varepsilon \varrho^{-1} \sum_{n \geq 1} \frac{1}{c_n} = \varepsilon. \end{aligned}$$

For the first inequality we used the Cauchy-Bunjakovsky inequality.

3°. In order to establish the limiting relationships for $T(\varepsilon)$ we note that as in (14) for all $n \geq 1$ it holds

$$\lim_{\varepsilon \rightarrow 0} |\beta_n^2(\varepsilon) - \beta^2(\tau_\varepsilon(n))| = 0 \quad a.s. \quad (16)$$

According to (16) and by the definition of the moment ν_ε for small but positive ε we have the inequalities

$$\nu' \leq \nu_\varepsilon \leq \nu'' \quad a.s. \quad (17)$$

with

$$\begin{aligned} \nu' &= \inf\{N \geq 1 : N > \varrho[\sup_{u \in [0, \Delta]} \beta^2(u)]^{-1}\} - 1, \\ \nu'' &= \inf\{N \geq 1 : N > \varrho[\inf_{u \in [0, \Delta]} \beta^2(u)]^{-1}\}. \end{aligned}$$

Similar (13) we can obtain

$$\lim_{\varepsilon \rightarrow 0} \left| \frac{\varepsilon^{-1} c_{\nu\varepsilon}}{\varphi(T(\varepsilon))} - \text{tr} I_\infty(T(\varepsilon)) \right| = 0 \quad \text{a.s.} \quad (18)$$

From (17) and (18) follows the assertion 3° of the Theorem 1

$$0 < \mu' \leq \lim_{\varepsilon \rightarrow 0} \varepsilon \cdot \varphi(T(\varepsilon)) \leq \mu'' < \infty,$$

where

$$\mu' = c_{\nu'} \cdot \left[\sup_{u \in [0, \Delta)} I_\infty(u) \right]^{-1}, \quad \mu'' = c_{\nu''} \cdot \left[\inf_{u \in [0, \Delta)} I_\infty(u) \right]^{-1}. \quad (19)$$

Theorem 1 is proved.

3 Example

Consider system (1) with $m = 1$, $r_1 = 1$

$$\left. \begin{aligned} dX(t) &= \vartheta_0 X(t) dt + \vartheta_1 X(t-1) dt + dW(t), \quad t \geq 0, \\ X(s) &= X_0(s), \quad s \in [-1, 0]. \end{aligned} \right\} \quad (20)$$

Assume for reasons of citation, that X_0 is continuous.

The sequential plan $(T(\varepsilon), \vartheta_\varepsilon^*)$ of estimation $\vartheta = (\vartheta_0, \vartheta_1)'$ will be defined as (12) with the Fisher information matrix

$$G(T) = \begin{pmatrix} \int_0^T X^2(t) dt & \int_0^T X(t) X(t-1) dt \\ 0 & 0 \\ \int_0^T X(t) X(t-1) dt & \int_0^T X^2(t-1) dt \\ 0 & 0 \end{pmatrix} \quad (21)$$

[Ku/So].

We can reformulated Theorem 1 for this cases as follows.

Theorem 2. *Let the parameters ϑ_0 and ϑ_1 in (20) such that we have the stationary or periodic case (for the notation see chapter one). Then the sequential plan (12) of estimation $\vartheta = (\vartheta_0, \vartheta_1)'$ possesses the properties:*

1°. $T(\varepsilon) < \infty \quad P_\vartheta - \text{a.s.},$

2°. $E_\vartheta \|\vartheta_\varepsilon^* - \vartheta\|^2 \leq \varepsilon.$

3°. *Besides the following limit inequalities*

$$0 < \underline{\lim}_{\varepsilon \rightarrow 0} \gamma(\varepsilon) T(\varepsilon) \leq \overline{\lim}_{\varepsilon \rightarrow 0} \gamma(\varepsilon) T(\varepsilon) < \infty \quad P_\vartheta - \text{a.s.} \quad (22)$$

are fulfilled, where $\gamma(\varepsilon) = \varepsilon$ in the stationary case and $\gamma(\varepsilon) = (\ln\varepsilon^{-1})^{-1}$ in the periodic case. Moreover, in the periodic case the limiting inequality

$$\lim_{\varepsilon \rightarrow 0} |T(\varepsilon) - \frac{1}{2v_0} \ln\varepsilon^{-1}| < \infty \quad a.s. \quad (23)$$

holds.

Proof of 1° – 2°: According to Theorem 1 the assertions 1° and 2° of Theorem 2 will be proved if the matrix $G(T)$ (21) satisfies the condition (5).

Now we establish the auxiliary equalities

$$\lim_{T \rightarrow \infty} T^{-1}G(T) = I_\infty \quad a.s. \quad (24)$$

for the stationary case and

$$\lim_{T \rightarrow \infty} |e^{-2v_0 T}G(T) - I_\infty(T)| = 0 \quad a.s. \quad (25)$$

for the periodic case, $v_0 > 0$.

Here

$$I_\infty = \begin{pmatrix} \int_0^\infty x_0^2(t)dt & \int_0^\infty x_0(t)x_0(t+1)dt \\ 0 & 0 \\ \int_0^\infty x_0(t)x_0(t+1)dt & \int_0^\infty x_0^2(t)dt \\ 0 & 0 \end{pmatrix}$$

and $I_\infty(T)$ is a periodic matrix

$$I_\infty(T) = \begin{pmatrix} g_{11}(T) & g_{12}(T) \\ g_{12}(T) & g_{22}(T) \end{pmatrix},$$

$$g_{ij}(T) = \int_0^\infty e^{-2v_0 t} U_i(T-t)U_j(T-t)dt, \quad i, j = 0, 2,$$

$$U_i(t) = \phi_i(t)X_0(0) + b \int_{-1}^0 \phi_i(t-s-1)e^{-v_0(s+1)}X_0(s)ds + \int_0^\infty \phi_i(t-s)e^{-v_0 s}dW(s),$$

$$\phi_i(t) = A_i \cdot \cos(\xi_0 t) + B_i \cdot \sin(\xi_0 t), \quad i = 0, 2,$$

$$A_0 = \frac{2(v_0 - a + 1)}{(v_0 - a + 1)^2 + \xi_0^2}, \quad B_0 = \frac{2\xi_0}{(v_0 - a + 1)^2 + \xi_0^2},$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = e^{-v_0} \begin{pmatrix} \cos \xi_0 & -\sin \xi_0 \\ \sin \xi_0 & \cos \xi_0 \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix},$$

$$\xi_o = \arg\{Im\lambda \mid \lambda \in \Lambda, Re\lambda = v_0, Im\lambda > 0\}.$$

Taking into account the representation

$$X(t) = x_0(t)X_0(0) + b \int_{-1}^0 x_0(t-s-1)X_0(s)ds + \int_0^t x_0(t-s)dW(s) \quad (26)$$

for the solution $(X(t), t \geq -1)$ of (21) [Gu/Ku], [Ku/So] and the fact that in the stationary case

$$\int_0^\infty x_0^2(t)dt < \infty,$$

we can see that

$$\lim_{t \rightarrow \infty} |X(t) - Z(t)| = 0 \quad a.s.,$$

where $Z(t) = \int_{-\infty}^t x_0(t-s)dW(s)$ is a stationary process with the correlation matrix I_∞ , which is ergodic [Gu/Ku], [Ku/So]. Then the equality (24) hold. In the periodic case according to [Gu/Ku]

$$x_0(t) = \phi_0(t)e^{v_0 t} + o(e^{\gamma t})$$

and

$$x_0(t-1) = \phi_2(t)e^{v_0 t} + o(e^{\gamma t})$$

for some γ with $\gamma < v_0$. Similar to Lemma 4.8 in [Gu/Ku] we can prove the equality

$$\lim_{t \rightarrow \infty} |e^{-\theta_0 t} X(t) - U_0(t)| = 0 \quad a.s.$$

From here we have

$$\begin{aligned} & \lim_{t \rightarrow \infty} |e^{-2v_0 T} \int_0^T X^2(t)dt - \int_0^\infty e^{-2v_0 t} U^2(T-t)dt| = \\ & = \lim_{T \rightarrow \infty} | \int_0^T e^{-2v_0(T-t)} [e^{-2v_0 t} X^2(t) - U^2(t)]dt + \int_0^T e^{-2v_0(T-t)} U^2(t)dt - \\ & - \int_0^\infty e^{-2v_0 t} U^2(T-t)dt | = \lim_{T \rightarrow \infty} \int_T^\infty e^{-2v_0 t} U^2(T-t)dt = 0 \quad a.s. \end{aligned}$$

The other equations in (25) may be proved analogously. Note that according to [Gu/Ku] $I_\infty(u) > 0$ for $u \in [0, \Delta)$ and the matrix function $I_\infty(u)$ is continuous on R^1 . It follows $I_\infty(u) > 0$ for $u \in [0, \Delta]$. Then (24), (25) and the conditions (5) for the matrix $G(T)$ defined by (21) are established.

3°. In order to obtain the exact limiting relationships for $T(\varepsilon)$ in the stationary case it suffices to note that by the definition of stopping times $\tau_\varepsilon(n)$ and (24) we get for all $n \geq 1$

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \varepsilon \cdot \tau_\varepsilon(n) &= c_n \cdot (tr I_\infty)^{-1} > 0 \text{ a.s.}, \\ \lim_{\varepsilon \rightarrow 0} \varepsilon \cdot G(\tau_\varepsilon(n)) &= c_n \cdot (tr I_\infty)^{-1} \cdot I_\infty > 0 \text{ a.s.}\end{aligned}\tag{27}$$

and as follows

$$\lim_{\varepsilon \rightarrow 0} \beta_n^2(\varepsilon) = (tr I_\infty \cdot \|I_\infty^{-1}\|)^{-2} > 0 \text{ a.s.}\tag{28}$$

Take into account that in this case $\varphi(T) = T$, from (8), (11), (24) and (28) we have

$$\mu_1 \leq \underline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \cdot T(\varepsilon) \leq \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \cdot T(\varepsilon) \leq \mu_2\tag{29}$$

with

$$\mu_1 = c_{\nu-1} \cdot (tr I_\infty)^{-1}, \mu_2 = c_\nu (tr I_\infty)^{-1},\tag{30}$$

$$\nu = \inf\{N \geq 1 : N > \varrho(tr I_\infty \cdot \|I_\infty^{-1}\|)^2\}.$$

Then the inequalities (22) for the stationary case hold.

Now we establish the assertion 3° of Theorem 2 for the periodic case.

By the definition (8) and according to (25) we have

$$\lim_{\varepsilon \rightarrow 0} |\varepsilon^{-1} c_{\nu_\varepsilon} \cdot e^{-2\nu_0 T(\varepsilon)} - tr I_\infty(T(\varepsilon))| = 0 \text{ a.s.}\tag{31}$$

Since $\inf_u tr I_\infty(u) > 0$ we can rewrite (31) in the form

$$\lim_{\varepsilon \rightarrow 0} \left[T(\varepsilon) - \frac{1}{2\nu_0} \ln \varepsilon^{-1} - \frac{1}{2\nu_0} \ln c_{\nu_\varepsilon} + \frac{1}{2\nu_0} \ln tr I_\infty(T(\varepsilon)) \right] = 0 \text{ a.s.}$$

From here and (17) we can obtain the relationships

$$\lim_{\varepsilon \rightarrow 0} (\ln \varepsilon^{-1})^{-1} \cdot T(\varepsilon) = \frac{1}{2\nu_0} \text{ a.s.}$$

and

$$\tilde{\mu}_1 \leq \underline{\lim}_{\varepsilon \rightarrow 0} \left[T(\varepsilon) - \frac{1}{2\nu_0} \ln \varepsilon^{-1} \right] \leq \overline{\lim}_{\varepsilon \rightarrow 0} \left[T(\varepsilon) - \frac{1}{2\nu_0} \ln \varepsilon^{-1} \right] \leq \tilde{\mu}_2$$

with

$$\begin{aligned}\tilde{\mu}_1 &= \frac{1}{2\nu_0} \ln c_{\nu'} \left(\sup_{u \in [0, \Delta]} tr I_\infty(u) \right)^{-1}, \\ \tilde{\mu}_2 &= \frac{1}{2\nu_0} \ln c_{\nu''} \left(\inf_{u \in [0, \Delta]} tr I_\infty(u) \right)^{-1}.\end{aligned}$$

The assertion 3^o of Theorem 2 is established. Theorem 2 is proved.

From Theorem 2 it follows that the duration $T(\varepsilon)$ of the sequential estimation has a nonrandom lower and upper bounds $\gamma^{-1}(\varepsilon)\tilde{\mu}_1$ and $\gamma^{-1}(\varepsilon)\tilde{\mu}_2$ respectively asymptotically. These bounds have the same increasing rate with $\varepsilon \rightarrow 0$. From assertions 2 and 3 of Theorem 2 follows that the convergence rate of the mean square deviation of the sequential estimator ϑ_ε^* corresponds with the rate of convergence of the MLE in stationary and periodic cases [Gu/Ku].

According to the inequalities (29) the duration of observations $T(\varepsilon)$ in stationary case is approximately not great than $\varepsilon^{-1}\mu_2$ with μ_2 defined by (30) when ε is small. Note that in this case one can obtain the following limiting equalities

$$\lim_{\varepsilon \rightarrow 0} \nu_\varepsilon = \nu \text{ a.s.} \quad (32)$$

and

$$\lim_{\varepsilon \rightarrow 0} \varepsilon T(\varepsilon) = \mu_2 \text{ a.s.}$$

Here ν is defined by (30). To obtain (32) we change the definition of ν_ε a little bit. Replace the magnitudes $\beta_n^{-2}(\varepsilon)$ in the definition of ν_ε in (11) by the nearest integer from above and choose (c_n) in such a way that the constant ϱ in (11) is irrational. In this case, the limit $\lim_{\varepsilon \rightarrow 0} S_N(\varepsilon)$ is strictly greater than ϱ , and this implies (32).

From (32) it follows that by small ε the moments $\nu_\varepsilon = \nu$ a.s. and by the property (28) it is obvious that the sequential estimate ϑ_ε^* may be represented in stationary case as the mean of finite numbers ν of maximum likelihood estimates $\hat{\vartheta}$ which are calculated at the moments $\tau_\varepsilon(n)$:

$$\vartheta_\varepsilon^* \sim \frac{1}{\nu} \sum_{n=1}^{\nu} \hat{\vartheta}(\tau_\varepsilon(n)). \quad (33)$$

The number ν may be asymptotically estimated with help of the property (24) and by the definition (30) of the moment ν .

It should be pointed out also that by known bound α for $\inf_{u \in [0, \Delta]} \beta^2(u) \geq \alpha > 0$ with $\beta^2(u)$ defined by (15), according to (18) we obtain

$$\nu_\varepsilon \leq \inf\{N \geq 1 : N > \varrho\alpha^{-1}\} = 1$$

by small ε if the sequence (c_n) is such that $\varrho < \alpha$. Then for the sequential estimate ϑ_ε^* defined by (12) for small ε we have

$$\vartheta_\varepsilon^* = \hat{\vartheta}(\tau_1(\varepsilon)) \text{ a.s.}$$

Remark. From Theorem 2 we can see that the sequential estimators ϑ_ε^* converge to the true value ϑ in mean square as $\varepsilon \rightarrow 0$ in stationary and periodic cases. Moreover, for any sequence $(\varepsilon_n, n \geq 1)$ of positive integers such that $\sum_{n \geq 1} \varepsilon_n < \infty$, we can define

the sequence of estimators $(\tilde{\vartheta}_n, n \geq 1)$, $\tilde{\vartheta}_n = \vartheta_{\varepsilon_n}^*$, $n \geq 1$. Then the sequence $(\tilde{\vartheta}_n)$ of estimators for ϑ is strongly consistent. It follows from the assertion 2 of Theorem 2 and the Borel - Cantelli lemma.

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