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Efficient Computation of Zeros of the Moving Average Operator with Real Matricial Coefficients

by

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Abstract

This paper proposes a computationally efficient procedure for evaluating the zeros of the vector moving average operator whose coefficients are matrices. A feature of the proposed method is that it is not necessary to explicitly construct the determinantal equation from a given moving average operator in order to evaluate its zeros. Application of the procedure is illustrated and numerical examples are provided to demonstrate the relative computational accuracy of the suggested method.

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1 Introduction

The evaluation of zeros of polynomial equations is one of the most frequently occuring problems in many branches of mathematics, statistics, econometrics, engineering and natural sciences, amongst others. Many algorithms and computer programs have been developed in the last three decades for evaluating the zeros of scalar polynomial (e.g. Dunaway (1974); Jenkins and Trauch (1970); and Grant and Hitchins (1975)). These algorithms have been widely used in many applications and the problem of evaluating the zeros of low to moderate order scalar polynomials could now be assumed to have been somewhat solved. Parallel developments in the case of polynomial operators with matricial coefficients seems to have attracted rather less attention. Yet the need to evaluate the zeros of this class of polynomials is ever present, and infact such evaluation plays an indispensible role in understanding and estimating the multivariable linear systems, where there are many inputs and outputs at each time point, as has been widely considered in both the engineering and statistical literature, see, for example, Wolovich (1974), Kailath (1980), Hannan and Deistler (1988), Lütkepohl (1991), and more recently, Reinsel (1993), amongst others.

In order to provide some motivation for considering the problem addressed in this paper, we will consider the class of vector autoregressive moving average (VARMA) processes of the form

$$\sum_{j=0}^{p} \mathbf{A}(j) y(t-j) = \sum_{j=0}^{q} \mathbf{M}(j) \epsilon(t-j), \ t = 0, \pm 1, \cdots$$
(1.1)

where y(t) is a v-dimensional, observable vector process and $\epsilon(t)$ is a $v \times 1$ unobservable white noise process. The parameters $\mathbf{A}(j)$ and $\mathbf{M}(j), j = 0, \dots, p$ are coefficient matrices of order v. Interpreting z as a unit lag operator, that is, zy(t) = y(t-1), (1.1) can be succintly written as $\mathbf{A}(z)y(t) = \mathbf{M}(z)\epsilon(t)$ where the autoregressive operator, $\mathbf{A}(z) = \sum_{j=0}^{p} \mathbf{A}(j)z^{j}$, and the moving average operator, $\mathbf{M}(z) = \sum_{j=0}^{p} \mathbf{M}(j)z^{j}$, are assumed to be left coprime, and satisfy

$$det \mathbf{A}(z) \neq 0; \quad det \mathbf{M}(z) \neq 0, \ |z| \le 1$$
(1.2)

with det denoting the determinant of the indicated matrix. To increase the generality sought for model (1.1), we will further assume that the matrix pair $\{\mathbf{A}(z) : \mathbf{M}(z)\}$

are in echelon form. Writing $A_{ij}(z)$ for the (i, j)th element of $\mathbf{A}(z)$, $i, j = 1, \dots, v$ and similarly $M_{ij}(z)$ for $\mathbf{M}(z)$, the echelon form carries with it the requirement that $\{\mathbf{A}(z): \mathbf{M}(z)\}$ has the following unique structure:

$$A_{ii}(z) = 1 + \sum_{k=1}^{n_i} a_{ii}(k) z^k, \qquad M_{ii}(z) = 1 + \sum_{k=1}^{n_i} m_{ii}(k) z^k, \quad i = 1, \cdots, v;$$

$$A_{ij}(z) = \sum_{k=n_i - n_{ij}+1}^{n_i} a_{ij}(k) z^k \text{ and } M_{ij}(z) = \sum_{k=0}^{n_i} m_{ij}(k) z^k, \quad i \neq j = 1, \dots, v$$
(1.3)

wherein

$$n_{ij} = \begin{cases} \min(n_i + 1, n_j), & i > j \\ \min(n_i, n_j), & i \le j, \end{cases} \quad i, j = 1, \cdots, v.$$

Using property (1.3) it is readily verified that $[\mathbf{A}(z) : \mathbf{M}(z)]$ has row degrees $n_i, i = 1, \dots, v$, and that $\mathbf{A}(0) = \mathbf{M}(0)$ is lower triangular with units down the leading diagonal. The integers $n_i, i = 1, \dots, v$ are known as Kronecker (or structural) indices and in (1.1) $p = q = max(n_i)$. Property (1.3) intimates that particular elements of the matrices $[\mathbf{A}(j) : \mathbf{M}(j)], j = 1, \dots, p$ are restricted to be zero or one. It suffices now to note that the number of coefficients not restricted to be zero or one in $\mathbf{A}(z)$ is

$$\sum_{i=1}^{v} \sum_{j=1}^{v} n_{ij} = n + \sum_{i < j} \{\min(n_i, n_j) + \min(n_i, n_j + 1)\}, \quad n = \sum_{i=1}^{v} n_i$$

and in $\mathbf{M}(z)$ there are an additional $\sum_{i=1}^{v} \sum_{j=1}^{v} n_i = nv$ parameters giving a total of

$$n(v+1) + \sum_{i < j=1} \{\min(n_i, n_j) + \min(n_i, n_j+1)\}$$
(1.4)

freely varying parameters. For an illuminating description of the property of echelon canonical form described in this section, see, for example, Lütkepohl and Poskitt (1996) and Salau (1997) where different aspects of the concept are laid bare.

There are several advantages for employing model (1.1) over other identified VARMA representations to analyse data, and for a lucid discussion of these, see, for example, Lütkepohl and Claesson (1997). The standard methods for efficiently estimating model of the form (1.1) requires that the moving average operator, $\mathbf{M}(z)$, be invertible (cf. (1.2)). In practice, however, random fluctuations do cause the estimate of $\mathbf{M}(z)$ to occur outside the invertibility region. This problem appears to trouble all commonly employed algorithms for estimating model (1.1) as has been attested to in the literature (e.g., Hannan and Deistler(op. cit.), Salau (1995), Poskitt and Salau (1995)) where the only exception could be a procedure due to Walker (1962) which is difficult to generalize to the vector case. Thus, constant checks have to be made to ensure that $\det \hat{\mathbf{M}}(z) \neq 0$, $|z| \leq 1$ (cf. (1.2)) where the circumflex has been used to indicate the estimate of the indicated quantity. This check can be achieved with a Schur-Cohn procedure as suggested by Hannan and Rissanen (1982) and if the test fails the zeros of $\det \hat{\mathbf{M}}(z)$ will have to be evaluated with a view to identifying those that lead to invertibility for possible reflection on the unit circle. The question of which method to use to evaluate these zeros therefore bears serious consideration. This is because the accuracy with which the zeros are evaluated will impinge on the numerical accuracy of the reflection scheme.

A number of procedures have indeed been suggested for evaluating the zeros of a polynomial matrix operator, $\mathbf{M}(z)$. To a very large extent, these methods rely heavily on the explicit construction of the determinantal equation, $det \{\mathbf{M}(z)\}$, from a given polynomial operator in order to evaluate its zeros (e.g. Robinson (1983), chap. 4). For moderate to large values of v and/or p the degree of the resulting determinantal equation could be anything other than a small number and the difficulty in obtaining the zeros of this equation with good precision could also increase rapidly with the polynomial order. This problem has been alluded to elsewhere in the literature, see, for example, Hilderbrand (1974). It may be, perhaps, in recognition of this inherent problem that an alternative procedure is introduced in the numerical algorithm group (NAG) library (G13DXF). This routine is restrictive in the sense that it works on the assumption that $\mathbf{M}(0)$ is an identity matrix, and in the context of echelon canonical form, specified by (1.3), this condition cannot always be guaranteed. What is therefore needed is a general procedure that can be applied to all possible VARMA representations but gives good numerical accuracy.

The purpose of this paper therefore is to propose a more general, computationally efficient and numerically accurate procedure for the evaluation of zeros of a moving average operator. A feature of the proposed method is that it is not necessary to explicitly construct the determinantal equation in order to evaluate the zeros of a given moving average operator. The rest of this paper is organized as follows. In Section 2 we give a brief exposition of the technique that is currently in widespread use, and also presents the proposed procedure. This is followed in Section 3 by an algorithm for implementing the proposed procedure. A numerical example is also provided to illustrate the different aspects of the algorithm. Section 4 is devoted to the examination of various but different examples, contrasting the performance of the proposed procedure with that of the existing method. The paper concludes, in Section 5, with some summary remarks.

2 The Methods

In this section, the existing and proposed methods for evaluating the zeros of a given polynomial operator are introduced. For ease of presentation we have organized our discussion in this section into two subsections dealing with the conventional method and the proposed procedure, respectively.

2.1 The Conventional Approach

The standard method used in evaluating the zeros of a polynomial matrix operator is to explicitly construct its determinantal equation and then uses it to evaluate the zeros. Most of the available techniques for evaluating the zeros of a polynomial matrix operator rely on this approach. For ease of comparison, it may prove useful to highlight its main features. To illustrate the general principle, we consider the following example. Suppose we have a bivariate VARMA process in echelon form defined by the specification $n_1 = 4$ and $n_2 = 2$ for which its moving average component is of the form

$$\mathbf{M}(z) = \begin{pmatrix} 1.0 & * \\ 2.0 & 1.0 \end{pmatrix} + \begin{pmatrix} 3.0 & 0.0 \\ 7.0 & -2.5 \end{pmatrix} z + \begin{pmatrix} 3.0 & 1.0 \\ 2.0 & 1.0 \end{pmatrix} z^{2} + \begin{pmatrix} 7.0 & -2.5 \\ * & * \end{pmatrix} z^{3} + \begin{pmatrix} 2.0 & 1.0 \\ * & * \end{pmatrix} z^{4}.$$
(2.1)

Herein, an asterisk has been used to denote an element that is restricted to be naught by virtue of the structure of echelon canonical form, a notational convention that we will persist with in expressions of this type. Now observe that equation (2.1) can be compactly written as a matrix with polynomial entries (i.e. polynomial matrix):

$$\mathbf{M}(z) = \begin{pmatrix} 2z^4 + 7z^3 + 3z^2 + 3z + 1 & z^4 - 2.5z^3 + z^2 \\ 2z^2 + 7z + 2 & z^2 - 2.5z + 1 \end{pmatrix}.$$
 (2.2)

This example typifies one of the basic properties of the echelon form and also nicely illustrates that the freely varying coefficients can introduce additional naught and other constraints into the operator over and above those given by (1.4). In (2.1), one of the 13 freely varying parameters in $\mathbf{M}(z)$ is naught for example.

Let us denote the elements of $\mathbf{M}(z)$ in (2.2) by $\begin{pmatrix} m_{11}(z) & m_{12}(z) \\ m_{21}(z) & m_{22}(z) \end{pmatrix}$. Then the determinant of $\mathbf{M}(z)$ can now be readily obtained in the same way as for a matrix with scalar elements as

$$det \mathbf{M}(z) = m_{11}(z) \cdot m_{22}(z) - m_{12}(z) \cdot m_{21}(z)$$
$$= z^4 + 0.5z^3 - 5.5z^2 + 0.5z + 1.$$

In general, for v > 2, the determinant, using standard results in matrix algebra, is obtained as $\det \mathbf{M}(z) = m_{1j}(z) \cdot cof\{m_{1j}(z)\} + \cdots + m_{vj}(z) \cdot cof\{m_{vj}(z)\}$ for any $j \in \{1, \dots, v\}$ where $cof\{m_{ij}(z)\} = (-1)^{i+j}minor\{m_{ij}(z)\}, i, j = 1, \dots, v$, denotes the cofactor of $m_{ij}(z)$, see, for example, Lütkepohl (1996) and the related references therein for detailed exposition. The evaluation of zeros of $\det \mathbf{M}(z)$ can now be readily accomplished using any of the standard finite algorithms (e.g. GAUSS, NAG Library, IMSL, MATLAB, amongst others).

From the foregoing discussion, it is evident that in moderate to large multivariable systems the construction of the determinantal equation and the subsequent evaluation of zeros could impose a huge computational burden, and moreover the accuracy of zeros could likely suffer from pertubation arising from rounding errors. The natural question which now arises centred on how to overcome these problems, and this is considered next.

2.2 The Proposed Procedure

We describe in this section a convenient procedure for evaluating the zeroes of a $v \times v$ non-singular matrix $\mathbf{M}(z)$. To make the proposed procedure somewhat more general we will for now refer to polynomial matrix operator and will drop explicit reference to the qualification moving average operator in our exposition except where the context makes the use of the latter more natural. We commence the discussion by first repeating some facts from the algebra of polynomial matrices (see, e.g. Barnett, 1984; Gohberg et. al, 1982; MacDuffee, 1956) which are exploited in the subsequent developments. Now let $\mathbf{\Psi}(z) = \sum_{j=0}^{\ell} \mathbf{\Psi}(j)z^{j}$ be a $v \times v$ polynomial matrix with real coefficients and of full rank. If there exists a polynomial matrix $\mathbf{U}(z)$ such that $\mathbf{H}(z) = \mathbf{U}(z)\mathbf{\Psi}(z)$ then $\mathbf{U}(z)$ is called a right multiple of $\mathbf{\Psi}(z)$. A polynomial matrix $\mathbf{U}(z)$ is called unimodular if $\det \mathbf{U}(z)$ is a constant not equal to zero and if and only if $\mathbf{U}(z)^{-1}$ exists, and is a polynomial matrix. Thus, $\mathbf{U}(z)^{-1} = (\det \mathbf{U}(z))^{-1} \cdot adj \mathbf{U}(z) = c^{-1} \cdot adj \mathbf{U}(z)$ for some non-zero constant c since $\mathbf{U}(z)$ is unimodular where adj denotes the adjoint (adjugate) of the indicated matrix. Thus, we state the following result.

Theorem 1: [Hermite Normal Form]

Every non-singular matrix $v \times v$ polynomial matrix $\Psi(z)$ of full rank v can be transformed by left multiplication by a unimodular matrix $\mathbf{U}(z)$ to a unique matrix $\mathbf{H}(z)$, called the Hermite normal form, with the following properties:

- (i) $\mathbf{H}(z)$ is lower triangular;
- (ii) $h_{ii}(z)$ $(i = 1, \dots, v)$ are monic polynomials (i.e. the highest degree coefficient of $h_{ii}(z)$ equals unity).
- (iii) $\delta\{h_{ji}(z)\} < \delta\{h_{ii}(z)\}, j \neq i$, where δ denotes the degree of the polynomial indicated;
- (iv) $\delta\{\det \mathbf{H}(z)\} = \delta\{\det \Psi(z)\}$ and the zeros of $\det\{\Psi(z)\}$ and $\det\{\mathbf{H}(z)\}$ coincide.

Proof: By performing a series of elementary row operations (involving interchanging of any two rows, adding to any row a polynomial multiple of any other row or multiplying any row by a nonzero real number and/or a combination of these) on $\Psi(z)$, properties (i) - (iii) can be readily established. Property (iv) follows from the fact that by construction $\mathbf{H}(z) = \mathbf{U}(z)\Psi(z)$ and hence $det\{\mathbf{H}(z)\} = c \cdot det\{\Psi(z)\}$ where c is a non-zero constant since $\mathbf{U}(z)$ is unimodular. The result is now immediate, see Birkhoff and Mclane (1977, Chapter 3, Theorem 4).

Some properties of the matrix $\mathbf{H}(z)$ that are worth observing are that

$$h_{ii}(z) = z^{n_i} + \sum_{r=0}^{n_i - 1} h_{ii}(r) z^r, \ i = 1, \cdots, v;$$

$$h_{ji}(z) = \sum_{r=0}^{n_{ji}} h_{ij}(r) z^r, \ 0 < i < j = 2, \cdots, v; \ n_{ji} < n$$

and if $h_{ii}(z)$ is unity all other entries in the ith column are zero.

It is also clear from Theorem 1(iv) that the zeros of $\det \Psi(z)$ can be evaluated by examining the diagonal elements of $\mathbf{H}(z)$ and this does not involve explicit construction of the determinantal equation for although $\det \Psi(z) = \prod_{i=1}^{v} h_{ii}(z)$ where $\delta\{h_{ii}(z)\} \leq \ell v$ and ℓ , as before, is the degree of the polynomial matrix $\Psi(z)$. It is, perhaps, worth pointing out that the reduction of a polynomial matrix to its Hermite normal form is a fairly efficient numerical procedure which is effected via the Euclidean division algorithm.

3 The Algorithm

In this section we present an algorithm for implementing the procedure introduced in the previous section. Now suppose we have a non-singular matrix $\Psi(z)$ of order v with elements $\psi_{ij}(z)$ where $\psi_{ij}(z)$ is a polynomial of degree n_{ij} , i, j = $1, \dots, v$, and that the evaluation of its zeros is of interest. This goal can now be achieved in the following steps:

- Step 1: Identify the entry in the last column of $\Psi(z)$ which has the lowest degree among the non-zero entries of this column and bring it in the (v, v) position by row interchange.
- Step 2: Apply the Euclidean division algorithm, subtract from every row in that column except for the last one an appropriate polynomial multiple of the last row such that the new entries in that column have degrees strictly less than $\delta(\psi_{vv}(z))$.

- **Step 3:** Repeat step 2 until all elements in the v^{th} column, except the (v, v) element, are zero.
- **Step 4:** Ignore the v^{th} row. Repeat steps 1 and 2 for the second last column of the resulting matrix until all entries above the (v 1, v 1) entry are zero.
- Step 5: Check whether the (v, v 1) entry has degree lower than the (v 1, v 1) entry. If it does not then apply the division algorithm and the corresponding subtractions to obtain a (v, v 1) entry having lower degree than the (v 1, v 1) entry.
- Step 6: Continue the procedure for all remaining columns and ensure that the resultant matrix, say $\mathbf{H}(z)$, is lower triangular, and that the property $\delta\{h_{ji}(z)\} < \delta\{h_{ii}(z)\}, \ j \neq i = 1, \dots, v$, is preserved.
- Step 7: Check whether the $h_{ii}(z)$, $i = 1, \dots, v$ is monic. If it is not then scale all its coefficients by the coefficient of its leading term.
- **Step 8:** Evaluate the zeros of $h_{ii}(z), i = 1, \dots, v$, using standard finite routine.

Remark: It is important to emphasize that the implementation of the algorithm does not require the entire matrix $\mathbf{M}(z)$ to be stored at one go in order to obtain the required diagonal elements of $\mathbf{H}(z)$. As it can be readily verified, the Euclidean division algorithm is only applied to one column of $\mathbf{M}(z)$ at a time. This intimates that the *ith* column of $\mathbf{M}(z)$ can be stored at a time and processed to obtain $h_{ii}(z), i = 1, \dots, v$. The virtue of this mode of construction will be more appreciated when the algorithm is applied to large multivariable systems.

3.1 An Illustrative Example

In order to facilitate a clearer understanding of the algorithm presented in the previous section as well as providing some guide as to the practical implementation of the scheme, we now consider a numerical example. Suppose we have a polynomial operator given by equation (2.1). For the purpose of the present exposition, it is also most convenient to rewrite $\mathbf{M}(z)$ as a matrix with polynomial entries as in (2.2).

First, our objective is to generate matrix $\mathbf{H}(z)$ from the polynomial matrix operator $\mathbf{M}(z)$, and second to obtain the zeros of $\mathbf{M}(z)$ by examining the diagonal elements of $\mathbf{H}(z)$. The first step in the procedure is to identify the entry in the last column of matrix $\mathbf{M}(z)$ (column 2 in the present case) with the lowest degree among its non-zero entries and bring it in the (2,2) position by a row interchange. This condition has already been met. The second step of the procedure is to apply Euclidean division algorithm using the entry in position (2,2) as a divisor and subtract the polynomial multiple of the entry (2,2) from that of entry (1,1). As can be easily verified, the resulting matrix, denoted by $\tilde{\mathbf{M}}_1(z)$, is obtained as

$$\tilde{\mathbf{M}}_1(z) = \left(\begin{array}{cc} m_{11}(z) & 0\\ m_{21}(z) & m_{22}(z) \end{array}\right)$$

where $m_{11}(z) = 2z^4 + 7z^3 + 3z^2 + 3z + 1$, $m_{21}(z) = 2z^2 + 7z + 2$ and $m_{22}(z) = z^2 - 2.5z + 1$. Since $m_{12}(0) = 0$ Steps 2 and 3 of the algorithm are automatically fulfilled and we are therefore led into step 4. Repeating steps 1 and 2 for column 1 of $\tilde{\mathbf{M}}_1(z)$ reduces the matrix to the form

$$\tilde{\mathbf{M}}_2(z) = \begin{pmatrix} z^2 + 3z + 1 & 0\\ 2z^2 + 7z + 2 & z^2 - 2.5z + 1 \end{pmatrix}.$$

The application of Steps 5 and 6 to $\tilde{\mathbf{M}}_2(z)$ yields

$$\mathbf{H}(z) = \left(\begin{array}{cc} z^2 + 3z + 1 & 0 \\ z & z^2 - 2.5z + 1 \end{array}\right)$$

and the corresponding unimodular matrix, generated in an analogous manner to $\mathbf{H}(z)$ commencing with an identity matrix, is obtained as

$$\mathbf{U}(z) = \left(\begin{array}{cc} 1.0 & -z^2 \\ -2.0 & 2z^2 + 1 \end{array}\right).$$

In order to complete the step leading to the evaluation of zeros of $\mathbf{M}(z)$ via $\mathbf{H}(z)$, a check has to be carried out to ensure that each of the diagonal elements of $\mathbf{H}(z)$ is a monic polynomial, and this constitutes Step 7 of the algorithm.

As it can be readily verified, $\det \mathbf{A}(z) = \det \mathbf{H}(z) = \prod_{i=1}^{2} h_{ii}(z) = z^{4} + 0.5z^{3} - 5.5z^{2} + 0.5z + 1$ as before. Hence, the zeros of det $\mathbf{M}(z)$ are evaluated, via the

diagonal entries of $\mathbf{H}(z)$, using the NAG routine C02AGF, and are written to two decimal places as $\xi_1 = -2.62, \xi_2 = -0.38, \xi_3 = 2.0$ and $\xi_4 = 0.50$.

In choosing this example however, we are constrained to a simple, perhaps obvious, problem that can be easily verified. This is done to keep the exposition as simple as possible in order to convey the basic ideas of the procedure. Working with the diagonal elements of $\mathbf{H}(z)$ to obtain the zeros of $\mathbf{M}(z)$, as it does in this example, obviates the need to construct the determinantal equation in order to achieve the same end. Since the degree of $h_{ii}(z)$, $i = 1, \dots, v$, will always be relatively small, it is most expected that increased accuracy would obtain through the use of the proposed algorithm. This last part of the argument is heuristic and it has to be checked using various but different numerical examples. This now forms the focus of the next section.

4 Numerical Comparison Of Procedures

The theoretical discussion of the previous section indicates that improved efficiency and accuracy could result from using the proposed method to evaluate the zeros of the moving average operator but it does not, as it stands, provide a simple summary measure of its relative performance over its competitors. The purpose of this section is to provide such analysis via the use of some numerical examples. To facilitate this investigation, a suite of programs designed to perform both the implementation of the conventional approach, described in Section 2, and the algorithm introduced in Section 3 were developed. These programs, supplemented by the numerical algorithm group (NAG) routine (C02AEF), were written in FORTRAN 77 and implemented on a SUN SPARC Station 10 utilizing double precision arithmetic in all calculations. The accuracy of the programs was checked in various obvious cases.

We commence our investigation by introducing in Tables 1 and 2 the moving average operators employed, arranged according to their degrees of determinantal equation with the smallest first. Table 1 contains examples of moving average operator with unrestricted coefficients while those derived from echelon canonical form are presented in Table 2. We have chosen to present the moving average operator with unrestricted coefficients separately not only we feel they are commonly employed in most applications but because its structure is more readily assimilated. This distinction further serves to reflect the general applicability of the proposed scheme.

It is also worth mentioning that the moving average operators presented in Tables 1 and 2 have been carefully constructed to reflect the situations that are commonly encountered in practice in the statistical analysis of vector linear time series. The choice of these examples in itself is guided to some extent by our previous experience. They are so constructed such that $det \mathbf{M}(z)$ contains some repeated zeroes with real or complex values and/or a combination of these. These zeros are however chosen to be a mixture of values that are close to origin or very close to unity and of those that are clearly outside the unit circle.

The first phase of the proposed procedure is to generate the Hermite normal form, $\mathbf{H}(z)$ (cf. Theorem 1) corresponding to each of the moving average (MA) operators introduced in Tables 1 and 2, using a suite of computer programs designed for that purpose. The resulting matrices are presented in Table 3 but note that the symbol \bigcirc has been used to denote the entry that is null by construction.

Following what may be regarded as a standard practice we will measure the accuracy of the two procedures by reference to the Euclidean norm of the differences between the true and the computed zeros of $\det \mathbf{M}(z)$. This criterion measure is given as

$$C_d = \left(\sum_{j=1}^n |\xi_j - \tilde{\xi}_j|^2\right)^{\frac{1}{2}},$$

where ξ_j and $\tilde{\xi}_j$ stand for the true and computed zeros, respectively, and n denotes the number of zeros evaluated. The choice of the criterion measure is motivated by the ease of interpretation that it provides, namely, the size of C_d varies proportionally with the distance between the computed and the true zeros. This intimates that the closer is $\tilde{\xi}_j$ to ξ_j then is C_d tends to zero almost surely. Thus, the method that gives the smallest value of C_d may be considered as the most efficient method of evaluating the zeros of the moving average operator. The results obtained from this empirical investigation are now summarized in Table 4.

			×33		
	$\begin{array}{c} \begin{array}{c} 0 \\ \hline 0 \\ \hline \end{array} \end{array} $		-1.26 1.008 -1.26 -1.26 1.26 -3.76	22 22 23	
	0.0 0.14		0.025 1.35 0.9 0.26 0.05 0.0	$\begin{array}{c} -0.32\\ 0.256\\ -0.32\\ -0.32\\ 0.32\\ 0.0\end{array}$	
	+		$\begin{array}{c} 3.82 \\ 6.8 \\ - \\ 1.96 \\ 1.16 \\ 1.1 \\ 1.1 \\ 0.0 \end{array}$	$\begin{array}{c} 0.0 \\ 0.0 \\ -0.45 \\ 0.27 \\ 0.0 \\ 0.0 \end{array}$	
	$\left(\begin{array}{c} .5 \\ .0 \end{array} \right) z^3$.39 .48 .84 0.0 0.0	$\begin{array}{c} 0.64\\ 1.28\\ 1.0\\ 0.84\\ 0.0\\ 0.0\end{array}$	
	5 3 3			$\begin{array}{c} 0.05 \\ -0.1 \\ -0.6 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{array}$	
	$\left(\begin{array}{c} 2.12\\ 1.5\end{array}\right)$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	-0.27 -0.699 -0.1 0.315 0.0	-0.072 0.0 0.0 0.0 0.0 0.0 0.0	
	$z^{2} + $	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{c} 0.715\\ 0.08\\ -1.6\\ 0.732\\ 0.0\\ 0.0\end{array}$	0.35 - 0.0 0.0 0.0 0.0	
les	-1.5 2.5	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	+	+	
Valu).75 3.0	0. $-1.$ $-3.$ $-3.$ $-3.$ $-3.$	× + × × × × 7	×2	س
cients		$1.0 \\ 0.55 \\ 0.2 \\ 0.2 \\ 2.1 \\ 0.52 \\ 0.52 \\ 1.2 $	-0.6 0.0 -0.5 -2.5	$-1.88 \\ -1.504 \\ -1.88 \\ -1.88 \\ 1.88 \\ -0.64$	0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
Coeffic	$\begin{array}{c} z \\ \hline 2 \\ 0 \\ \hline \end{array}$	$\left(\begin{array}{c} -1.7\\ 0.22\\ 0.22\\ 1.32\\ 0.12\\ 1.22\\ 1.22\\ 0.4\end{array}\right)$	$\begin{array}{c} 0.5 \\ -0.96 \\ -0.3 \\ -0.3 \\ 1.3 \\ -1.65 \\ 1.0 \end{array}$	$\begin{array}{c} 0.225\\ 0.45\\ 0.45\\ 0.95\\ -0.57\\ 0.0\\ 0.0\end{array}$	$\begin{array}{c} 0.0\\ 0.0\\ 0.0\\ 0.0\\ 0.0\\ 0.0\\ 0.0\\ 0.0$
	5 1. 1.	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	-0.93 -5.08 -0.2 8.03 -0.99 0.6	-1.7 -3.4 -3.6 -3.6 -3.6 -3.6 -3.6 -0.0 0.0	$\begin{array}{c} 0.0\\ 0.0\\ -0.64\\ 0.0\\ 0.0\\ 0.0\end{array}$
	+	$\begin{array}{ccc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}$	$\begin{array}{c} 0.29 \\ -4.56 \\ 1.15 \\ 0.4 \\ 0.0 \\ 0.0 \end{array}$	$\begin{array}{c} 0.65 \\ 1.1 \\ -2.28 \\ 1.308 \\ 0.0 \\ 0.0 \end{array}$	0 0.0 0 0.0 0 0.0 0 0.0
		0 0 1 0 1 0 0 1 0 0 0 0	2.46 2.255 - 1.12 0.9 0.0	222 .472 0.15 - 0.0 0.0 0.0	0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
	$=$ $\begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$	$= \left(\begin{array}{c} 1\\0\\0\\0\\0\\0\\0\\0\end{array}\right)$	$\begin{array}{c} 1.3 \\ 1.26 \\ 1.26 \\ 0.8 \\ 1.32 \\ -0.2 \\ 0.0 \end{array}$	$\begin{array}{c} 135 & 0 \\ -08 & 1 \\ -76 & - \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{array}$	+
	$A^{(1)}_{\dagger}(z)$	$M^{(2)}_{\dagger}(z)$			
US		N	+	+	
icatio	p = 4	9	9		
Specif	v = 2	>	d,		

Table 1: Examples of Moving Average Operator with Unrestricted Coefficients

Table 2: Examples of Moving Average Operator in Echelon Canonical Form

MA Operator	Hermite Normal Form (H(z))							
$\mathbf{M}^{(1)}(z)$	$\mathbf{H}^{(1)}(z) = \begin{pmatrix} h_{11}(z) & \bigcirc \\ h_{21}(z) & h_{22}(z) \end{pmatrix}$ wherein: $h_{11}(z) = z^3 - 0.6111111111z^2 - 5.5555555555555555555555555555555555$							
$\mathbf{M}^{(2)}(z)$	$\mathbf{H}^{(2)}(z) = \begin{pmatrix} h_{11}(z) & \bigcirc & \bigcirc \\ 1.2z^3 + 2.6z^2 + 1.5z & h_{22}(z) & \bigcirc \\ 1.4z^2 + 2.8z & 0.61z^2 + 1.1z & h_{33}(z) \end{pmatrix}$ wherein: $h_{11}(z) = z^4 - 7.125z^3 + 12.0625z^2 - 7.5z + 1.5625;$ $h_{22}(z) = z^3 - 4.888888888888882z^2 + 1.33333333333333z - 8.88888888888888888;$							
	$n_{33}(z) = z + 5.6666666666666662 = 1.55555555555555555555555555555555555$							
$\mathbf{M}^{(3)}(z)$	$\mathbf{H}^{(3)}(z) = \begin{pmatrix} h_{11}(z) & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\ h_{21}(z) & h_{22}(z) & \bigcirc & \bigcirc & \bigcirc \\ h_{31}(z) & h_{32}(z) & h_{33}(z) & \bigcirc & \bigcirc \\ h_{41}(z) & h_{42}(z) & h_{43}(z) & h_{44}(z) & \bigcirc \\ h_{51}(z) & h_{52}(z) & h_{53}(z) & h_{54}(z) & h_{55}(z) \end{pmatrix}$ wherein:							
$\mathbf{W} (z)$	wherein: $h_{11}(z) = z^{5} - 10.861111111111111z^{4} + 36.95833333333333333z^{3} - 38.4027777777778z^{2} - 2.0833333333333333z^{3} - 13.88888888888889;$ $h_{21}(z) = 2.1z^{3} + 0.6z^{2} + 1.3z + 3.5;$ $h_{22}(z) = z^{4} - 7.75z^{3} + 14.625z^{2} - 3, 125z - 6.25;$ $h_{31}(z) = 0.61z^{2} - 0.22z + 1.1; h_{32}(z) = 0.52z^{3} - 0.44z^{2} + 0.29z;$ $h_{33}(z) = z^{4} - 9.5z^{3} + 32.6z^{2} - 62.5z + 62.5; h_{41}(z) = 0.43z + 1.2;$ $h_{42}(z) = 0.52z^{2} + 2.2z; h_{43}(z) = 1.3z^{3} - 3.5z^{2} + 1.25z + 0.2;$ $h_{44}(z) = z^{3} + 5z^{2} - 4z - 20; h_{51}(z) = 1.35;$ $h_{52}(z) = 2.5z + 1.05; h_{53}(z) = 0.18z^{2} + 1.32z + 0.25;$ $h_{52}(z) = 0.5z^{2} - 0.85z + 0.5; h_{53}(z) = 0.18z^{2} + 0.75z + 0.125;$							

TABLE 3	
Hermite Normal Forms corresponding to $\mathbf{M}^{(1)}(z) - \mathbf{M}^{(3)}(z)$ and \mathbf{M}	$\overset{(1)}{\dagger}(z) - \mathbf{M}^{(2)}_{\dagger}(z)$

MA Operator	Hermite Normal Form (H(z))							
$\mathbf{M}^{(1)}_{\dagger}(z)$	$\mathbf{H}_{\dagger}^{(1)}(z) = \begin{pmatrix} z^3 + 2z^2 + 4z + 8 & \bigcirc \\ 2z^2 + 0.5z & z^2 - 3z + 2 \end{pmatrix}$							
	$ \begin{pmatrix} h_{11}(z) & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & & \bigcirc & & & \\ \end{pmatrix} $							
	$ \begin{array}{c} h_{21}(z) h_{22}(z) \bigcirc \bigcirc \bigcirc \bigcirc \\ h_{31}(z) h_{32}(z) h_{33}(z) \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc $							
$\mathbf{M}^{(2)}_{\dagger}(z)$	$\mathbf{H}_{\dagger}^{(2)}(z) = \begin{bmatrix} b_{1}(z) & b_{2}(z) & b_{3}(z) & b_{4}(z) \\ h_{41}(z) & h_{42}(z) & h_{43}(z) & h_{44}(z) & \bigcirc & \bigcirc \end{bmatrix}$							
	$h_{51}(z) h_{52}(z) h_{53}(z) h_{54}(z) h_{55}(z) \bigcirc$							
	$ \begin{pmatrix} h_{61}(z) & h_{62}(z) & h_{63}(z) & h_{64}(z) & h_{65}(z) & h_{66}(z) \end{pmatrix} $							
	wherein							
	$h_{11}(z) = z^5 - 3.5z^4 - 4.5z^3 - 9z^2 - 34z + 20;$							
	$h_{21}(z) = 0.3z^4 + 2.2z^3 + 1.4z^2 + 0.22z:$							
	$h_{31}(z) = 0.22z^3 + 0.6z^2 + 1.32z;$							
	$h_{41}(z) = -1.52z^2 + 0.12z; h_{51}(z) = 1.22z; h_{61}(z) = 0.4z;$							
	$h_{22}(z) = z^4 + 6.638888889z^3 + 9.513888889z^2$							
	-7.638888889z - 13.88888889;							
	$h_{32}(z) = 1.4z^3 + 0.52z^2 + 0.2z;$							
	$h_{42}(z) = 0.3z^2 + 2.1z;$ $h_{52}(z) = 0.5z;$							
	$h_{62}(z) = 1.2z;$ $h_{33}(z) = z^4 + z^3 + z^2 + 11z + 10;$							
	$h_{43}(z) = 1.2z^3 + 0.4z^2 + 0.88z;$							
	$h_{53}(z) = 2.18z^2 - 3.5z;$ $h_{63}(z) = 2.1z;$							
	$h_{44}(z) = z^4 - 3.75z^3 + 5.375z^2 - 3.4375z + 0.78125;$							
	$h_{54}(z) = 1.4z^3 + 0.16z^2 + 0.24z; h_{64}(z) = 0.6z^2 + 2.3z;$							
	$h_{55}(z) = z^3 - 2.111111111z^2 - 0.888888888z + 2.2222222222222222222222222222222222$							
	$h_{65}(z) = z^2 + 1.4z;$							
	$h_{66}(z) = z^4 + 5.875z^3 + 3.9375z^2 - 2.5z - 1.5625.$							

TABLE 3 contd.

TABLE 4

MA Operator	n	m	C_d Values	
			$\mathbf{C}\mathbf{A}$	PP
$M^{(1)}_{\dagger}(z)$	5	1	1.33×10^{-15}	0.0
$M^{(1)}(z)$	6	2	1.27×10^{-14}	0.0
$M^{(2)}(z)$	10	4	1.25×10^{-7}	0.0
$M^{(3)}(z)$	18	7	6.71×10^{-4}	0.0
$M_{\dagger}^{(2)}(z)$	24	11	1.75×10^{-2}	0.0

Summary of Results for the Conventional and Proposed Procedures

Table 4 contains the criterion values, C_d , obtained for the two procedures, the degree of the determinantal equation (n) corresponding to the moving average operators employed in this investigation and m denotes the number of zeros of $det \mathbf{M}(z)$ that lie on or close to the unit circle. In this table, the symbols CA and PP have also been used to denote the conventional approach and the proposed procedure, respectively. The criterion value, C_d , is taken to 15 places of decimal and this enables us to capture on the essential characteristics of each of the two methods. As can be observed, the results indicate that there are marked differences in the accuracy of the two procedures. The proposed procedure seems to outperform its competitor irrespective of the location and nature of the zeros and/or the degree and dimension of the moving average operator being considered, and this reflects the general characteristics in the vast majority of the cases that we considered. As it might be anticipated, the accuracy of the conventional approach deteriorates as the degree of the determinantal equation increases. What is perhaps surprising is the fact that the degree of the determinantal equation does not have to be extremely large for the conventional method to yield unacceptably poor values of the zeros. It is our experience that when the determinantal equation contains a number of zeros that are equal in magnitude but opposite in sign, the conventional method tends to produce inaccurate results. This is in contrast to what obtains using the proposed procedure where the computed zeros are, in most cases, in close agreement with the true values.

The results in Table 4 further hinted that the bias associated with the values

of zeros obtained via the proposed procedure is much smaller than those obtained using the conventional approach. The bias associated with the proposed procedure seems to be invariant to the location of the zeros or the degree of the determinantal equation while those associated with the conventional approach increase rapidly with the degree of the determinantal equation. It is however not clear from these results what might be responsible for the observed differences in the performance of these alternative procedures. Some insight into the mechanism giving rise to this phenomenon can be gained by examining the values of zeros obtained from the two procedures individually. As an illustration we show in Table 5 the values of zeros obtained from the moving average operator, $\mathbf{M}^{(3)}(z)$, and their absolute deviations from the actual zeros for the two procedures.

The results in Table 5, written to six places of decimal, are representative and indicate that the two procedures can produce somewhat different outcomes. These results reflect what could likely happen when these procedures are used to evaluate the zeros of $det \mathbf{M}(z)$ and this is typical of situations where $det \mathbf{M}(z)$ contains a mixture of real and complex zeros that lie close to the unit circle or contains a number of repeated zeros that are equal in magnitude but opposite in sign. These features seem to have a much more pronounced effect on the performance of the conventional method than the size of the degree of the determinantal equation as it is generally believed. Closer examination of the experimental outcomes further reveals that the observed deviations from the true zeros arise from pertubation caused by rounding errors and they become magnified as the number of repeated zeros increases and/or the degree of the determinantal equation becomes large. This bias, in the context of vector ARMA estimation, is most likely to feed through to other quantities of interest such as the prediction error variance and estimates of power spectra should any attempt is made to reflect the zeros bounded by |z| = 1, through the unit circle to obtain an equivalent stable moving average operator. One would also expect similar effects to be reflected in the subsequent parameter estimates obtained via the Gauss-Newton iterative scheme or the generalized least squares procedure. Given the complexity of the problem and the absence of the knowledge of the situation we might be dealing with in practice, it therefore becomes imperative to opt for a procedure that ensures accuracy when consideration is given to the evaluation of

zeros in the estimation of vector ARMA models.

TABLE 5

Observed	Patterns	of perfor	mance	of the	Conventional	and	Proposed
		Methods	using I	$\mathbf{M}^{(3)}(z)$	Example		

j	Conventional App	Proposed	d Procedure	ξ_j	
	$\widetilde{\xi}_j$	$\mid \xi_j - ilde{\xi}_j \mid$	$\widetilde{\xi}_j$	$\mid \xi_j - \widetilde{\xi}_j \mid$	
1	-5.0	0.0	-5.0	0.0	-5.0
2	5.00018 + 0.000311i	0.000408	5.0	0.0	5.0
3	5.00018 - 0.000311i	0.000408	5.0	0.0	5.0
4	4.999641	0.000359	5.0	0.0	5.0
5	4.0	0.0	4.0	0.0	4.0
6	2.499999	0.000001	2.5	0.0	2.5
7	-2.0	0.0	-2.0	0.0	-2.0
8	2.000152	0.000152	2.0	0.0	2.0
9	1.999848	0.000152	2.0	0.0	2.0
10	1.250092	0.000092	1.25	0.0	1.25
11	1.249908	0.000092	1.25	0.0	1.25
12	1.0 + 2i	0.0	1.0 + 2i	0.0	1.0 + 2i
13	1.0 - 2i	0.0	1.0 - 2i	0.0	1.0 - 2i
14	1.111111	0.0	1.111111	0.0	1.111111
15	-0.5000031 + 0.000005i	0.008566	-0.5	0.0	-0.5
16	-0.5000031 - 0.000005i	0.008566	-0.5	0.0	-0.5
17	-0.25	0.0	-0.5	0.0	-0.5
18	-0.499994	0.000006	0.25	0.0	-0.25

5 Summary Remarks

In this paper, an alternative procedure for evaluating the zeros of the moving average operator is proposed. By means of the proposed approach, it has been possible to evaluate the zeros of large order and higher dimensional moving average operators with some degree of accuracy. The approach is more convenient in several ways. First, the evaluation of zeros does not require the explicit construction of the determinantal equation. Second, the intermediate figures obtained during calculations are always within the memory capacity of most available present day computers. This makes it easy to be implemented on microcomputers with limited memory capacity. Third, it yields good numerical accuracy irrespective of the degree and dimension of the given moving average operator. These comparative advantages suggest that there may be some practical virtues in using the proposed approach to evaluate the zeros of the moving-average type of operator.

Finally, the objective of the present study is to provide a practical solution to the problem rather than an abstract theoretical treatment of the subject-matter. It has nevertheless been impossible to avoid mathematical exposition completely so much that it has been considered best to give the main theoretical results in algorithmic form. This is done solely in the interest of clarity, however, and not to provide a pretence of rigour where no particular degree of rigour is required.

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