

Klemelä, Jussi

**Working Paper**

## Asymptotic minimax risk in the uniform norm for the white noise model on the sphere

SFB 373 Discussion Paper, No. 1998,21

**Provided in Cooperation with:**

Collaborative Research Center 373: Quantification and Simulation of Economic Processes, Humboldt University Berlin

*Suggested Citation:* Klemelä, Jussi (1998) : Asymptotic minimax risk in the uniform norm for the white noise model on the sphere, SFB 373 Discussion Paper, No. 1998,21, Humboldt University of Berlin, Interdisciplinary Research Project 373: Quantification and Simulation of Economic Processes, Berlin, <https://nbn-resolving.de/urn:nbn:de:kobv:11-10056691>

This Version is available at:

<https://hdl.handle.net/10419/61271>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

# Asymptotic Minimax Risk in the Uniform Norm for the White Noise Model on the Sphere \*

Jussi Klemelä  
Rolf Nevanlinna Institute  
University of Helsinki

## Abstract

Estimation of the signal function defined on the unit sphere of the Euclidean space is considered. Gaussian continuous time white noise model is supposed. Uniform norm is chosen as a loss function and exact asymptotic minimax risk is derived extending the result of Korostelev (1993). The exact asymptotic minimax risk is given also for the  $L_2$ -loss applying the result of Pinsker (1982).

**Mathematics Subject Classifications:** 62G07, 62C20, 62H11

**Key Words and Phrases:** asymptotic minimax risk, spherical data, uniform norm, white noise model.

## 1 Introduction

Let  $S_d = \{x \in \mathbf{R}^{d+1} : \|x\| = 1\}$ . Consider the observation of the form

$$dX(x) = f(x)dx + \epsilon dW(x), \quad x \in S_d$$

where  $\epsilon > 0$  is the noise size and  $dW(x)$  is the Gaussian white noise in  $S_d$ , that is for measurable sets  $B, B_1, B_2 \subset S_d$ ,

---

\*This research was supported by Sonderforschungsbereich 373 “Quantifikation und Simulation Ökonomischer Prozesse”.

- (i)  $\mathcal{L} \left( \int_B dW(x) \right) = N(0, \mu(B))$  where  $\mu$  is the Lebesgue measure of  $S_d$ ,
- (ii) if  $B_1 \cap B_2 = \emptyset$ , then  $\int_{B_1} dW(x)$  and  $\int_{B_2} dW(x)$  are independent.

The Lebesgue measure of the sphere can be defined with the help of the Lebesgue measure of  $\mathbf{R}^{d+1}$ , see Rudin (1986, page 175). It is a Borel measure, invariant with respect to rotations and  $\mu(S_d) = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$ .

Let  $\Lambda(\beta, L)$  be the class of Hölder functions

$$\Lambda(\beta, L) = \{f : M \rightarrow \mathbf{R} \mid |f(x) - f(y)| \leq L\delta(x, y)^\beta, x, y \in S_d\}$$

where  $0 < \beta < 1$  and  $\delta$  is the Riemannian metric of  $S_d$ , that is,  $\delta(x, y) = \arccos(x'y)$ . We define the discrepancy of an arbitrary estimator  $\hat{f}_\epsilon(x)$  and the true signal function  $f(x)$  by the sup norm

$$\|\hat{f}_\epsilon - f\|_\infty = \sup_{x \in S_d} |\hat{f}_\epsilon(x) - f(x)|.$$

Let  $w(u)$ ,  $u \geq 0$ , be a loss function, i. e. a continuous bounded monotone function of  $u$ ,  $w(0) = 0$ . Introduce the minimax risk

$$r_\epsilon = r_\epsilon(w, \beta, L) = \inf_{\hat{f}_\epsilon} \sup_{f \in \Lambda(\beta, L)} E_{\epsilon, f} w \left( \psi_\epsilon^{-1} \left\| \hat{f}_\epsilon - f \right\|_\infty \right)$$

where  $\psi_\epsilon = (\epsilon^2 \log \epsilon^{-2})^{\beta/(2\beta+d)}$ .

**Theorem 1** Let  $A_{\beta, d} = \lim_{\kappa \rightarrow \infty} \|\kappa^{d/2} K_{\kappa, \eta}\|_2^{-2\beta/(2\beta+d)}$  where  $K_{\kappa, \eta} : S_d \rightarrow \mathbf{R}$ ,

$$K_{\kappa, \eta}(x) = (1 - (\kappa\delta(x, \eta))^\beta)_+ \tag{1}$$

with  $\kappa > 0$  and  $\eta \in S_d$ . Let

$$C_0 = L^{d/(2\beta+d)} A_{\beta, d} \left( \frac{2d}{2\beta+d} \right)^{\beta/(2\beta+d)}.$$

Then

$$\lim_{\epsilon \rightarrow 0} r_\epsilon = w(C_0).$$

The constant  $A_{\beta,d}$  can be calculated explicitly,

$$\begin{aligned}
\|\kappa^{d/2} K_{\kappa,\eta}\|_2^2 &= \kappa^d \mu_{d-1}(S_{d-1}) \int_0^\pi (1 - (\kappa\theta)^\beta)_+^2 \sin^{d-1} \theta \, d\theta \\
&= \kappa^d \mu_{d-1}(S_{d-1}) \int_0^{\kappa\pi} (1 - t^\beta)_+^2 \sin^{d-1}(\kappa^{-1}t) \kappa^{-1} dt \\
&\sim \mu_{d-1}(S_{d-1}) \int_0^1 (1 - t^\beta)^2 t^{d-1} dt \\
&= \mu_{d-1}(S_{d-1}) \frac{2\beta^2}{(2\beta + d)(\beta + d)d}
\end{aligned} \tag{2}$$

where  $\mu_{d-1}(S_{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ ,  $d \geq 2$  and we use the convention  $\mu_0(S_0) = 2$ . Thus,

$$\lim_{\epsilon \rightarrow 0} r_\epsilon = w \left( L^{d/(2\beta+d)} \left( \frac{(\beta + d)d^2}{\mu_{d-1}(S_{d-1})\beta^2} \right)^{\beta/(2\beta+d)} \right).$$

Korostelev (1993) proved corresponding result for estimating regression function defined on the unit interval from  $n$  samples  $X_i = f(i/n) + \epsilon \xi_i$ ,  $\xi_i \stackrel{i.i.d.}{\sim} N(0, 1)$ . Donoho (1994) presented results for the continuous time white noise model on the real line, and gave also evaluation of the asymptotic minimax risk for estimating derivatives of the signal function.

Previously for instance Hall, Watson, and Cabrera (1987) have studied kernel density estimation on the sphere and Hendriks (1990) has studied Fourier series density estimates on Riemannian manifolds. However, exact minimax constants have not been previously given in the case of spherical data.

The proof of Theorem 1 is given in sections 2 and 3. The upper bound is proved in Section 2 using ideas presented in Korostelev (1993) and the lower bound is proved in Section 3 using ideas presented in Donoho (1994). In Section 4 exact asymptotic minimax risk is given when estimating signal function belonging to the spherical Sobolev ball and the  $L_2$ -loss is used. This extends the result in Pinsker (1982).

## 2 The Upper Bound

Let us define the estimator

$$\hat{f}(x) = \sum_{i=1}^N \hat{a}_{Ni} I_{A_{Ni}}(x)$$

where

$$\hat{a}_{Ni} = c(\kappa) \int_{S_d} K_{\kappa, \eta(N, i)}(x) X(dx),$$

$K_{\kappa, \eta}$  was defined in (1),  $c(\kappa)^{-1} = \int_{S_d} K_{\kappa, \eta} d\mu$ ,

$$\kappa = \left( \frac{C_0 \psi_\epsilon}{L} \right)^{-1/\beta},$$

$$N = \lceil \chi^{-1} (\epsilon^2 \log \epsilon^{-2})^{-d/(2\beta+d)} \rceil = \lceil \chi^{-1} \psi_\epsilon^{-d/\beta} \rceil$$

where  $\lceil b \rceil$  is the smallest integer  $\geq b$ ,

$$\chi = \frac{1}{\mu(S_d)} \left( \frac{\gamma C_0 d}{L(2\beta + d)} \right)^{d/\beta}$$

where  $\gamma > 0$  is arbitrary,  $\eta(N, i) \in S_d$ ,  $i = 1, \dots, N$ , are asymptotically equidistant on  $S_d$ , and  $A_{Ni}$ ,  $i = 1, \dots, N$ , is a partition of  $S_d$  such that  $A_{Ni}$  is the set of those  $x \in S_d$  for which  $\eta_{N, i}$  is the closest member of the set  $\{\eta_{N, 1}, \dots, \eta_{N, N}\}$  (ties can be broken arbitrarily). Thus

$$A_{Ni} \subset \{x \in S_d : i = \operatorname{argmin}_{i=1, \dots, N} \delta(\eta(N, i), x)\}.$$

Let us start with the variation of the estimator.

**Lemma 2**

$$\lim_{\epsilon \rightarrow 0} P \left( \psi_\epsilon^{-1} \|\hat{f} - E\hat{f}\|_\infty > (1 + \gamma) C_0 \frac{2\beta}{2\beta + d} \right) = 0.$$

*Proof.* Let us denote  $Z_\epsilon(x) = \hat{f}(x) - E\hat{f}(x)$ . Then

$$\begin{aligned} D_\epsilon^2 &\stackrel{def}{=} \operatorname{Var}(\psi_\epsilon^{-1} Z_\epsilon(\eta(N, 1))) = \psi_\epsilon^{-2} \operatorname{Var}(\hat{a}_{N1}) \\ &= \psi_\epsilon^{-2} \epsilon^2 c(\kappa)^2 \int_{S_d} K_{\kappa, \eta(N, 1)}^2(x) d\mu(x) \\ &\sim \psi_\epsilon^{-2} \epsilon^2 \kappa^d \mu_{d-1}(S_{d-1})^{-1} \frac{2d(\beta + d)}{2\beta + d} \end{aligned}$$

as is seen from Equation (2) and from

$$c(\kappa)^{-1} = \int_{S_d} (1 - (\kappa \delta(x, \eta))^\beta)_+ d\mu(x)$$

$$\begin{aligned}
&= \mu_{d-1}(S_{d-1}) \int_0^\pi (1 - (\kappa\theta)^\beta)_+ \sin^{d-1} \theta \, d\theta \\
&= \mu_{d-1}(S_{d-1}) \int_0^{\kappa\pi} (1 - t^\beta)_+ \sin^{d-1}(\kappa^{-1}t) \kappa^{-1} dt \\
&\sim \kappa^{-d} \mu_{d-1}(S_{d-1}) \int_0^1 (1 - t^\beta) t^{d-1} dt \\
&= \kappa^{-d} \mu_{d-1}(S_{d-1}) \frac{\beta}{(\beta + d)d}
\end{aligned} \tag{3}$$

Let us denote  $y = (1 + \gamma)C_0 2\beta / (2\beta + d)$ . Then it can be checked that

$$\frac{y^2}{D_\epsilon^2} = 2d(1 + \gamma)^2 \log(\epsilon^{-2/(2\beta+d)})$$

and for sufficiently small  $\epsilon$ ,

$$\begin{aligned}
P(\|\psi_\epsilon^{-1} Z_\epsilon\|_\infty > y) &\leq P\left(\max_{i=1, \dots, N} \psi_\epsilon^{-1} |Z_\epsilon(\eta(N, i))| > y\right) \\
&\leq NP\left(D_\epsilon^{-1} \psi_\epsilon^{-1} |Z_\epsilon(\eta(N, i))| > \frac{y}{D_\epsilon}\right) \\
&\leq N \exp\left\{-\frac{1}{2} \frac{y^2}{D_\epsilon^2}\right\} \\
&\leq N \exp\left\{-\frac{d(1 + \gamma)^2 \log \epsilon^{-2}}{2\beta + d}\right\} \\
&\leq \chi^{-1} (\log \epsilon^{-2})^{-d/(2\beta+d)} (\epsilon^2)^{d((1+\gamma)^2-1)/(2\beta+d)},
\end{aligned}$$

which approaches to 0 as  $\epsilon \rightarrow 0$ . □

Next, let us consider the bias of the estimator.

**Lemma 3**

$$\limsup_{\epsilon \rightarrow 0} \sup_{f \in \Lambda(\beta, L)} \psi_\epsilon^{-1} \|f - E\hat{f}\|_\infty \leq (1 + \gamma)C_0 \frac{d}{2\beta + d}.$$

*Proof.* We have that

$$\begin{aligned}
&\|E\hat{f} - f\|_\infty \\
&\leq \max_{i=1, \dots, N} |E\hat{f}(\eta(N, i)) - f(\eta(N, i))| + L \max_i \min_{j \neq i} \delta(\eta(N, i), \eta(N, j))^\beta
\end{aligned}$$

Now

$$\limsup_{\epsilon \rightarrow 0} \max_i \min_{j \neq i} \delta(\eta(N, i), \eta(N, j)) \leq \limsup_{\epsilon \rightarrow 0} \left(\frac{\mu(S_d)}{N}\right)^{1/d}$$

and thus

$$\begin{aligned} & \limsup_{\epsilon \rightarrow 0} \psi_\epsilon^{-1} \max_i \min_{j \neq i} \delta(\eta(N, i), \eta(N, j))^\beta \\ & \leq \limsup_{\epsilon \rightarrow 0} \psi_\epsilon^{-1} \left( \frac{\mu(S_d)}{N} \right)^{\beta/d} \leq (\mu(S_d)\chi)^{\beta/d} \leq \frac{\gamma C_0 d}{L(2\beta + d)}. \end{aligned}$$

On the other hand, for  $i = 1, \dots, N$ ,

$$\begin{aligned} & |E\hat{f}(\eta(N, i)) - f(\eta(N, i))| \\ & = |E\hat{a}_{N,i} - f(\eta(N, i))| \\ & = \left| c(\kappa) \int_{S_d} K_{\kappa, \eta(N, i)}(x) f(x) d\mu(x) - f(\eta(N, i)) \right| \\ & \leq c(\kappa) \int_{S_d} K_{\kappa, \eta(N, i)}(x) |f(x) - f(\eta(N, i))| d\mu(x) \\ & \leq c(\kappa) L \int_{S_d} (1 - (\kappa\delta(\eta(N, i), x))^\beta)_+ \delta(\eta(N, i), x)^\beta d\mu(x) \\ & = c(\kappa) L \mu_{d-1}(S_{d-1}) \int_0^\pi (1 - (\kappa\theta)^\beta)_+ \theta^\beta \sin^{d-1} \theta d\theta \\ & = c(\kappa) L \mu_{d-1}(S_{d-1}) \int_0^{\kappa\pi} (1 - t^\beta)_+ (\kappa^{-1}t)^\beta \sin^{d-1}(\kappa^{-1}t) \kappa^{-1} dt \\ & \sim c(\kappa) L \kappa^{-\beta-d} \mu_{d-1}(S_{d-1}) \int_0^1 (1 - t^\beta) t^{\beta+d-1} dt \\ & = c(\kappa) L \kappa^{-\beta-d} \mu_{d-1}(S_{d-1}) \frac{\beta}{(2\beta + d)(\beta + d)} \\ & \sim \kappa^{-\beta} L \frac{d}{2\beta + d} = \psi_\epsilon C_0 \frac{d}{2\beta + d} \end{aligned}$$

where we used Equation (3). □

The upper bound follows from lemma 2 and lemma 3 by an similar argument as was given in Korostelev (1993).

### 3 The Lower Bound

We will give the proof for the lower bound. Let  $J_{\kappa, \eta} : S_d \rightarrow \mathbf{R}$ ,

$$J_{\kappa, \eta}(x) = L\kappa^{-\beta} K_{\kappa, \eta}(x) = L\kappa^{-\beta} (1 - (\kappa\delta(x, \eta))^\beta)_+$$

where  $\kappa > 0$ ,  $\eta \in S_d$ . Let  $N = N(\kappa)$  be the greatest integer such that there exists such  $\eta(N, i) \in S_d$ ,  $i = 1, \dots, N$ , that the functions  $J_{\kappa, \eta(N, i)}$  have disjoint supports. Now we have that

$$\liminf_{\kappa \rightarrow \infty} \kappa^{-d} N(\kappa) \geq \text{Const.} \quad (4)$$

Let

$$\mathcal{C}(\kappa, \{\eta(N, i)\}) = \left\{ \sum_{i=1}^N \theta_i J_{\kappa, \eta(N, i)} : |\theta_i| \leq 1, i = 1, \dots, N \right\}.$$

Now  $\mathcal{C}(\kappa, \{\eta(N, i)\}) \subset \Sigma(\beta, L)$ . The complete class of estimators for estimating  $f \in \mathcal{C}(\kappa, \{\eta(N, i)\})$  consists of all procedures of the form

$$\hat{f}_\epsilon = \sum_{i=1}^N \hat{\theta}_i J_{\kappa, \eta(N, i)} \quad (5)$$

where  $\hat{\theta}_i = \delta_i(y_1, \dots, y_N)$ ,  $i = 1, \dots, N$ , for measurable functions  $\delta_i$  and

$$y_i = \int_{S_d} \xi_i(x) X(dx), \quad i = 1, \dots, N$$

with  $\xi_i = J_{\kappa, \eta(N, i)} / \|J_{\kappa, \eta(N, i)}\|_2^2$ .

When  $\hat{f}_\epsilon$  is of the form (5) and  $f \in \mathcal{C}(\kappa, \{\eta(N, i)\})$  then

$$\begin{aligned} \left\| \hat{f}_\epsilon - f \right\|_\infty &\geq \max_{i=1, \dots, N} \left| \hat{f}_\epsilon(\eta(N, i)) - f(\eta(N, i)) \right| \\ &= |J_{\kappa, \eta(N, 1)}(\eta(N, 1))| \left\| \hat{\theta} - \theta \right\|_{l_N^\infty} \\ &= L\kappa^{-\beta} \left\| \hat{\theta} - \theta \right\|_{l_N^\infty}. \end{aligned}$$

Thus

$$\begin{aligned} r_\epsilon &\geq \inf_{\hat{f}_\epsilon} \sup_{f \in \mathcal{C}(\kappa, \{\eta(N, i)\})} E_{\epsilon, f} w \left( \psi_\epsilon^{-1} \left\| \hat{f}_\epsilon - f \right\|_\infty \right) \\ &\geq \inf_{\hat{\theta}} \sup_{|\theta_i| \leq 1} E_{G(\theta, \sigma_N^2 I_N)} w \left( \psi_\epsilon^{-1} L\kappa^{-\beta} \left\| \hat{\theta} - \theta \right\|_{l_N^\infty} \right) \end{aligned}$$

where  $G(\theta, \sigma_n^2 I_N)$  is the normal distribution with the expectation vector  $\theta$  and the covariance matrix  $\sigma_N^2 I_N$ , where  $I_N$  is the  $N \times N$  identity matrix and  $\sigma_N^2 = \text{Var}(y_1) = \epsilon^2 / \|J_{\kappa, \eta(N, 1)}\|_2^2$ .



By Korostelev's lemma (Korostelev 1993, Donoho 1994), if

$$\sigma_N^{-1} \leq \sqrt{2 - \gamma} \sqrt{\log N}, \quad (6)$$

for some  $0 < \gamma < 2$ , then

$$\inf_{\hat{\theta}} \sup_{|\theta_i| \leq 1} E_{G(\theta, \sigma_N^2 I_N)} w \left( \left\| \hat{\theta} - \theta \right\|_{l_N^\infty} \right) \longrightarrow w(1), \text{ when } N \longrightarrow \infty.$$

Let  $0 < \gamma < 2$ ,

$$C'_0 = L^{d/(2\beta+d)} A_{\beta,d} \left( \frac{(2-\gamma)d}{2\beta+d} \right)^{\beta/(2\beta+d)},$$

and

$$\kappa = \left( \frac{C'_0 \psi_\epsilon}{L} \right)^{-1/\beta} = \left( L^{d/\beta} A_{\beta,d}^{(2\beta+d)/\beta} \frac{(2-\gamma)d}{2\beta+d} \epsilon^2 \log \epsilon^{-2} \right)^{-1/(2\beta+d)}.$$

Now

$$\begin{aligned} \sigma_N^{-1} &= \epsilon^{-1} \left\| J_{\kappa, \eta(N,1)} \right\|_2 \\ &= \epsilon^{-1} \left\| L \kappa^{-\beta} K_{\kappa, \eta(N,1)} \right\|_2 \\ &= \epsilon^{-1} L \kappa^{-\beta-d/2} \left\| \kappa^{d/2} K_{\kappa, \eta(N,1)} \right\|_2 \\ &\sim \epsilon^{-1} L \kappa^{-\beta-d/2} A_{\beta,d}^{-(2\beta+d)/(2\beta)} \\ &= \left( \frac{(2-\gamma)d}{2\beta+d} \log \epsilon^{-2} \right)^{1/2} \\ &\leq (2-\gamma)^{1/2} \left( \log (\epsilon^2 \log \epsilon^{-2})^{-d/(2\beta+d)} \right)^{1/2} \\ &= (2-\gamma)^{1/2} \sqrt{\log(\text{Const.} \times \kappa^d)}. \end{aligned}$$

Thus (6) is satisfied ultimately as  $\epsilon \rightarrow 0$  by Equation (4).

The lower bound follows by Korostelev's lemma because

$$\psi_\epsilon^{-1} L \kappa^{-\beta} \sim C'_0 \sim L^{d/(2\beta+d)} A_{\beta,d} \left( \frac{(2-\gamma)d}{2\beta+d} \right)^{\beta/(2\beta+d)}$$

and  $0 < \gamma < 2$  was chosen arbitrarily.  $\square$

## 4 Estimation with the $L_2$ -loss

Let us also consider estimation with the  $L_2$ -loss assuming that the signal function belongs to the Sobolev ball. We will give exact asymptotic minimax constant applying the result of Pinsker (1982) to the spherical case.

Let  $H_s(S_d)$ ,  $s > 0$ , be the Sobolev space of  $S_d$ . Then  $f \in L_2(S_d, \mu)$  is in  $H_s(S_d)$  if and only if

$$\|f\|_s^2 \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \sigma_k^s \sum_{f_j \in E_k} \langle f, f_j \rangle_{L_2(S_d, \mu)}^2 < \infty, \quad (7)$$

where  $(\sigma_k)_{k=0}^{\infty}$  is the set of different eigenvalues of the Laplace-Beltrami operator  $\Delta$  in increasing order,  $(E_k)_{k=0}^{\infty}$  is the set of eigenspaces of  $\Delta$  ( $E_k$  corresponds to  $\sigma_k$  for each  $k$ ), and  $(f_j)_{j=0}^{\infty}$  is an orthonormal basis of  $L_2(S_d, \mu)$  consisting of real eigenfunctions of  $\Delta$  (Giné, 1975).

Let  $\Sigma(s, M) = \{f \in L_2(S_d, \mu) : \|f\|_s^2 \leq M\}$ , where  $s > 0$  and  $\|\cdot\|_s$  is defined in (7). Define

$$R_\epsilon(\Sigma(s, M)) = \inf_{\hat{f}} \sup_{f \in \Sigma(s, M)} E_{\epsilon, f} \left\| \hat{f} - f \right\|_{L_2(S_d, \mu)}^2.$$

Now we can state the following theorem.

**Theorem 4**

$$R_\epsilon(\Sigma(s, M)) \sim \epsilon^{4s/(2s+d)} M^{d/(2s+d)} P_{s,d}, \quad \epsilon \rightarrow 0,$$

where

$$P_{s,d} = \left( \frac{2s}{(s+d)(d-1)!} \right)^{2s/(2s+d)} \frac{(2s+d)^{d/(2s+d)}}{d}.$$

The proof of the theorem follows from Pinsker (1980, Theorem 1). In fact, consider the observations

$$y_{k,i} = \theta_{k,i} + \epsilon \xi_{k,i}, \quad k = 1, 2, \dots, \quad i = 1, \dots, \tau_k,$$

where  $\tau_k \geq 1$  are integers,  $\xi_{k,i}$  are i. i. d.  $N(0, 1)$ ,  $\epsilon > 0$  is the noise size, and the sequence  $\theta = (\theta_{k,i})$  is in  $l_2$ . Consider the parameter space

$$\Theta = \left\{ \theta \in l_2 : \sum_{k=1}^{\infty} a_k \sum_{i=1}^{\tau_k} \theta_{k,i}^2 \leq M \right\}$$

where  $a_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Define

$$R_\epsilon^*(\Theta) = \inf_{\hat{\theta}} \sup_{\theta \in \Theta} E_{\epsilon, \theta} \left\| \hat{\theta} - \theta \right\|_{l_2}^2.$$

From Pinsker (1980, Theorem 1) it follows that

$$R_\epsilon^*(\Theta) \sim \omega^2 \quad (\epsilon \rightarrow 0)$$

where

$$\omega^2 = \epsilon^2 \sum_{\mu_\epsilon a_k < 1} \tau_k (1 - (\mu_\epsilon a_k)^{1/2}) \quad (8)$$

and  $\mu_\epsilon$  is the unique solution of

$$\epsilon^2 \sum_{\mu_\epsilon a_k < 1} \tau_k a_k ((\mu_\epsilon a_k)^{-1/2} - 1) = M. \quad (9)$$

If one chooses  $a_k = \sigma_k^s$  and  $\tau_k = \dim E_k^{(d)}$ , where  $\sigma_k$  are the eigenvalues and  $E_k^{(d)}$  are the eigenspaces of the Laplace-Beltrami operator, respectively, then it holds that

$$R_\epsilon(\Sigma(s, M)) = R_\epsilon^*(\Theta).$$

For the circle  $S_1$  it is known that when  $k \geq 1$ ,  $\sigma_k = k^2$ ,  $\dim E_k = 2$ , and orthonormal basis of  $E_k$  is  $\{\pi^{-1/2} \cos k\theta, \pi^{-1/2} \sin k\theta\}$ . For the sphere  $S_2$  it is known that  $\sigma_k = k(k+1)$  and  $\dim E_k = 2k+1$ , when  $k \geq 1$ . For the  $d$ -dimensional sphere  $S_d$  it is known that  $\sigma_k = k(k+d-1)$  and

$$\begin{aligned} \dim E_k^{(d)} &= \text{card} \{(l_2, \dots, l_d) \in \mathcal{Z}_+^{d-1} : k \geq l_2 \geq l_3 \geq \dots \geq l_{d-1} \geq |l_d|\} \\ &= \frac{(2k+d-1)(k+d-2)!}{k!(d-1)!} \end{aligned}$$

(Takeuchi, 1994, pages 216, 223). It can be seen that

$$\dim E_k^{(d)} = \sum_{i=0}^k \dim E_i^{(d-1)} = \sum_{i=0}^{d-1} \alpha_i k^i,$$

where  $\alpha_i$  are integers and  $\alpha_{d-1} = 2/(d-1)!$ .

Thus Theorem 4 follows by applying (8) if we choose  $a_k = [k(k+d-1)]^s$  and  $\tau_k = \sum_{i=0}^{d-1} \alpha_i k^i$ , where  $\alpha_i$  are integers and  $\alpha_{d-1} = 2/(d-1)!$ .

## References

- [1] Giné, E. M. (1975). Invariant tests for uniformity on compact Riemannian manifolds based on Sobolev norms. *Ann. Statist.* **3** 1243–1266.

- [2] Donoho, D. L. (1994). Asymptotic minimax risk for sup-norm loss: Solution via optimal recovery. *Probab. Theory Relat. Fields* **99** 145–170
- [3] Hall, P., Watson, G. S., and Cabrera, J. (1987). Kernel density estimation with spherical data. *Biometrika* **74** 751–62.
- [4] Hendriks, H. (1990). Nonparametric estimation of a probability density on a Riemannian manifold using Fourier expansions. *Ann. Statist.* **18** 832–849.
- [5] Korostelev, A. P. (1993). Exact asymptotically minimax estimator for nonparametric regression in uniform norm. *Theory Probab. Appl.* **38** 775–782.
- [6] Pinsker, M. S. (1980). Optimal filtering of square integrable signals in Gaussian white noise. *Problems Inform. Transmission* **16** 52–68.
- [7] Takeuchi, M. (1994). *Modern Spherical Functions*. Translations of Mathematical Monographs Vol. 135. American Mathematical Society, Providence, Rhode Island.