

Huberman, Gur

Working Paper

## Arbitrage pricing theory

Staff Report, No. 216

**Provided in Cooperation with:**

Federal Reserve Bank of New York

*Suggested Citation:* Huberman, Gur (2005) : Arbitrage pricing theory, Staff Report, No. 216, Federal Reserve Bank of New York, New York, NY

This Version is available at:

<https://hdl.handle.net/10419/60653>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

Federal Reserve Bank of New York  
Staff Reports

Arbitrage Pricing Theory

Gur Huberman  
Zhenyu Wang

Staff Report no. 216  
August 2005

This paper presents preliminary findings and is being distributed to economists and other interested readers solely to stimulate discussion and elicit comments. The views expressed in the paper are those of the authors and are not necessarily reflective of views at the Federal Reserve Bank of New York or the Federal Reserve System. Any errors or omissions are the responsibility of the authors.

## **Arbitrage Pricing Theory**

Gur Huberman and Zhenyu Wang

*Federal Reserve Bank of New York Staff Reports*, no. 216

August 2005

JEL classification: G12

### **Abstract**

Focusing on capital asset returns governed by a factor structure, the Arbitrage Pricing Theory (APT) is a one-period model, in which preclusion of arbitrage over static portfolios of these assets leads to a linear relation between the expected return and its covariance with the factors. The APT, however, does not preclude arbitrage over dynamic portfolios. Consequently, applying the model to evaluate managed portfolios is contradictory to the no-arbitrage spirit of the model. An empirical test of the APT entails a procedure to identify features of the underlying factor structure rather than merely a collection of mean-variance efficient factor portfolios that satisfies the linear relation.

Key words: arbitrage, asset pricing model, factor model

---

Huberman: Columbia University Graduate School of Business (e-mail: gh16@columbia.edu).  
Wang: Federal Reserve Bank of New York and University of Texas at Austin McCombs School of Business (e-mail: zhenyu.wang@ny.frb.org). This review of the arbitrage pricing theory was written for the forthcoming second edition of *The New Palgrave Dictionary of Economics*, edited by Lawrence Blume and Steven Durlauf (London: Palgrave Macmillan). The views expressed in this paper are those of the authors and do not necessarily reflect the position of the Federal Reserve Bank of New York or the Federal Reserve System.

# 1 Introduction

The Arbitrage Pricing Theory (APT) was developed primarily by Ross (1976a, 1976b). It is a one-period model in which every investor believes that the stochastic properties of returns of capital assets are consistent with a factor structure. Ross argues that if equilibrium prices offer no arbitrage opportunities over static portfolios of the assets, then the expected returns on the assets are approximately linearly related to the factor loadings. (The factor loadings, or betas, are proportional to the returns' covariances with the factors.) The result is stated in Section 2.

Ross' (1976a) heuristic argument for the theory is based on the preclusion of arbitrage. This intuition is sketched in Section 3. Ross' formal proof shows that the linear pricing relation is a necessary condition for equilibrium in a market where agents maximize certain types of utility. The subsequent work, which is surveyed below, derives either from the assumption of the preclusion of arbitrage or the equilibrium of utility-maximization. A linear relation between the expected returns and the betas is tantamount to an identification of the stochastic discount factor (SDF). Sections 4 and 5, respectively, review this literature.

The APT is a substitute for the Capital Asset Pricing Model (CAPM) in that both assert a linear relation between assets' expected returns and their covariance with other random variables. (In the CAPM, the covariance is with the market portfolio's return.) The covariance is interpreted as a measure of risk that investors cannot avoid by diversification. The slope coefficient in the linear relation between the expected returns and the covariance is interpreted as a risk premium. Such a relation is closely tied to mean-variance efficiency, which is reviewed in Section 6.

Section 6 also points out that an empirical test of the APT entails a procedure to identify at least some features of the underlying factor structure. Merely stating that some collection of portfolios (or even a single portfolio) is mean-variance efficient relative to the mean-variance frontier spanned by the existing assets does not constitute a test of the APT, because one can always find a mean-variance efficient portfolio. Consequently, as a test of the APT it is not sufficient to merely show that a set of factor portfolios satisfies the linear relation between the expected return and its covariance with the factors portfolios.

A sketch of the empirical approaches to the APT is offered in Section 7 whereas Section 8 describes various procedures to identify the underlying factors. The large number of factors proposed in the literature and the variety of statistical or ad hoc procedures to find them indicate that a definitive insight on the topic is still missing.

Finally, Section 9 surveys the applications of the APT, the most prominent being the evaluation of the performance of money managers who actively change their portfolios.

Unfortunately, the APT does not necessarily preclude arbitrage opportunities over dynamic portfolios of the existing assets. Therefore, the applications of the APT in the evaluation of managed portfolios are contradictory at least in spirit to the APT, which obtains price restrictions by assuming the absence of arbitrage.

## 2 A Formal Statement

The APT assumes that investors believe that the  $n \times 1$  vector,  $r$ , of the single-period random returns on capital assets satisfies the factor model

$$r = \mu + \beta f + e , \tag{1}$$

where  $e$  is an  $n \times 1$  vector of random variables,  $f$  is a  $k \times 1$  vector of random variables (factors),  $\mu$  is an  $n \times 1$  vector and  $\beta$  is an  $n \times k$  matrix. With no loss of generality, normalize (1) to make  $E[f] = 0$  and  $E[e] = 0$ , where  $E[\cdot]$  denotes expectation and  $0$  denotes the matrix of zeros with the required dimension. The factor model (1) implies  $E[r] = \mu$ .

The mathematical proof of the APT requires restrictions on  $\beta$  and the covariance matrix  $\Omega = E[ee']$ . An additional customary assumption is that  $E[e|f] = 0$ , but this assumption is not necessary in some of the APT's developments.

The number of assets,  $n$ , is assumed to be much larger than the number of factors,  $k$ . In some models,  $n$  is infinity or approaches infinity. In this case, representation (1) applies to a sequence of capital markets; the first  $n$  assets in the  $(n + 1)$ st market are the same as the assets in the  $n$ th market and the first  $n$  rows of the matrix  $\beta$  in the  $(n + 1)$ st market constitute the matrix  $\beta$  in the  $n$ th market.

The APT asserts the existence of a constant  $a$  such that, for each  $n$ , the inequality

$$(\mu - X\lambda)Z^{-1}(\mu - X\lambda) \leq a \tag{2}$$

holds for a  $(k+1) \times 1$  vector  $\lambda$ , and an  $n \times n$  positive definite matrix  $Z$ . Here,  $X = (\iota, \beta)$ , in which  $\iota$  is an  $n \times 1$  vector of ones. Let  $\lambda_0$  be the first component of  $\lambda$  and  $\lambda_1$  consists of the rest of the components. If some portfolio of the assets is risk-free, then  $\lambda_0$  is the return on the risk-free portfolio. The positive definite matrix  $Z$  is often the covariance matrix  $E[ee']$ . *Exact* arbitrage pricing obtains if (2) is replaced by

$$\mu = X\lambda = \iota\lambda_0 + \beta\lambda_1 . \tag{3}$$

The vector  $\lambda_1$  is referred to as the risk premium, and the matrix  $\beta$  is referred to as the beta or loading on factor risk.

The interpretation of (2) is that each component of  $\mu$  depends approximately linearly on the corresponding row of  $\beta$ . This linear relation is the same across assets.

The approximation is better, the smaller the constant  $a$ ; if  $a = 0$ , the linear relation is exact and (3) obtains.

### 3 Intuition

The intuition behind the model draws from the intuition behind Arrow-Debreu security pricing. A set of  $k$  fundamental securities spans all possible future states of nature in an Arrow-Debreu model. Each asset's payoff can be described as the payoff on a portfolio of the fundamental  $k$  assets. In other words, an asset's payoff is a weighted average of the fundamental assets' payoffs. If market clearing prices allow no arbitrage opportunities, then the current price of each asset must equal the weighted average of the current prices of the fundamental assets.

The Arrow-Debreu intuition can be couched in terms of returns and expected returns rather than payoffs and prices. If the unexpected part of each asset's return is a linear combination of the unexpected parts of the returns on the  $k$  fundamental securities, then the expected return of each asset is the same linear combination of the expected returns on the  $k$  fundamental assets.

To see how the Arrow-Debreu intuition leads from the factor structure (1) to exact arbitrage pricing (3), set the idiosyncratic term  $e$  on the right-hand side of (1) equal to zero. Translate the  $k$  factors on the right-hand side of (1) into the  $k$  fundamental securities in the Arrow-Debreu model. Then (3) follows immediately.

The presence of the idiosyncratic term  $e$  in the factor structure (1) makes the model more general and realistic. It also makes the relation between (1) and (3) more tenuous. Indeed, 'no arbitrage' arguments typically prove the weaker (2). Moreover, they require a weaker definition of arbitrage (and therefore a stronger definition of no arbitrage) in order to get from (1) to (2).

The proofs of (2) augment the Arrow-Debreu intuition with a version of the law of large numbers. That law is used to argue that the average effect of the idiosyncratic terms is negligible. In this argument, the independence among the components of  $e$  is used. Indeed, the more one assumes about the (absence of) contemporaneous correlations among the component of  $e$ , the tighter the bound on the deviation from exact APT.

### 4 No-Arbitrage Models

Huberman (1982) formalizes Ross' (1976a) heuristic argument. A portfolio  $v$  is an  $n \times 1$  vector. The cost of the portfolio  $v$  is  $v'v$ , the income from it is  $v'r$ , and its return is  $v'r/v'v$  (if its cost is not zero). Huberman defines arbitrage as the existence of zero-cost

portfolios such that a subsequence  $\{w\}$  satisfies

$$\lim_{n \rightarrow \infty} E[w'r] = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \text{var}[w'r] = 0, \quad (4)$$

where  $\text{var}[\cdot]$  denotes variance. The first requirement in (4) is that the expected income associated with  $w$  becomes large as the number of assets increases. The second requirement in (4) is that the risk (as measured by the income's variance) vanishes as the number of assets increases. Accordingly, a sequence of capital markets offers no arbitrage if there is no subsequence  $\{w\}$  of zero-cost portfolios that satisfy (4).

Huberman shows that if the factor model (1) holds and if the covariance matrix  $E[ee']$  is diagonal for all  $n$  and uniformly bounded, then the absence of arbitrage implies (2) with  $Z = I$  and a finite bound  $a$ . The idea of his proof is as follows. Consider the orthogonal projection of the vector  $\mu$  on the linear space spanned by the columns of  $X$ :

$$\mu = X\hat{\lambda} + \alpha, \quad (5)$$

where  $\alpha'X = 0$  and  $\hat{\lambda}$  is a  $k \times 1$  vector. The projection implies

$$\alpha'\alpha = \min_{\lambda} (\mu - X\lambda)'(\mu - X\lambda). \quad (6)$$

A violation of (2) is the existence of a subsequence of  $\{\alpha'\alpha\}$  that approaches infinity. The vector  $\alpha$  is often referred to as a pricing error and it can be used to construct arbitrage. For any scalar  $h$ , the portfolio  $w = h\alpha$  has zero cost because the first column of  $X$  is  $\iota$ . The factor model (1) and the projection (5) imply  $E[w'r] = h(\alpha'\alpha)$  and  $\text{var}[w'r] = h^2(\alpha'E[ee']\alpha)$ . If  $\sigma^2$  is the upper bound of the diagonal elements of  $E[ee']$ , then  $\text{var}[w'r] \leq h^2(\alpha'\alpha)\sigma^2$ . If  $h$  is chosen to be  $(\alpha'\alpha)^{-2/3}$ , then  $E[w'r] = (\alpha'\alpha)^{1/3}$  and  $\text{var}[w'r] \leq (\alpha'\alpha)^{-1/3}\sigma^2$ , which imply that (4) is satisfied by a subsequence of the zero-cost portfolios  $\{(\alpha'\alpha)^{-2/3}\alpha\}$ .

Using the no-arbitrage argument, the exact APT can be proven to hold in the limit for well-diversified portfolios. A portfolio  $w$  is well diversified if  $w'\iota = 1$  and  $\text{var}[w'e] = 0$ , i.e., if the portfolio's return contains only factor variance. A sequence of portfolios,  $\{w\}$ , is well diversified if  $w'\iota = 1$  and  $\lim_{n \rightarrow \infty} \text{var}[w'e] = 0$ . Suppose there are  $m$  sequences of well-diversified portfolios and  $m$  is a fixed number larger than  $k + 1$ . For each  $n$ , let  $W$  be an  $n \times m$  matrix, in which each column is one of the well-diversified portfolios. The exact APT holds in the limit for the well-diversified portfolios if and only if there exists a sequence of  $k \times 1$  vectors,  $\{\lambda\}$ , such that

$$\lim_{n \rightarrow \infty} (W'\mu - \tilde{X}\lambda)'(W'\mu - \tilde{X}\lambda) = 0, \quad (7)$$

where  $\tilde{X} = (j, W'\beta)$  and  $j$  is an  $m \times 1$  vector of ones. The projection of  $W'\mu$  on the columns of  $\tilde{X}$  gives  $W'\mu = \tilde{X}\hat{\lambda} + \alpha$ , in which  $\alpha'\tilde{X} = 0$ . If equation (7) does not hold,

a subsequence of  $\alpha$  satisfies  $\alpha' \alpha > \delta$  for some positive constant  $\delta$ . This sequence of  $\alpha$  can be used to construct arbitrage as follows. For any scalar  $h$ , define a portfolio as  $v = hW\alpha$ , which is then costless because  $v'i = h\alpha'W'i = h\alpha'j = 0$ . It follows from  $\alpha' \tilde{X} = 0$  that  $E[v'r] = h\alpha'\alpha$  and  $\text{var}[v'r] = h^2\alpha'W'E[ee']W\alpha$ . If  $h$  is chosen to be  $(\alpha'W'E[ee']W\alpha)^{-1/3}$ , then  $\text{var}[v'r] = h^{-1}$ . Since  $\{W\}$  is well-diversified and  $E[ee']$  is diagonal and uniformly bounded, it follows that  $\lim_{n \rightarrow \infty} h = \infty$ . This implies that portfolio sequence  $\{v\}$  is arbitrage because it satisfies (4).

Ingersoll (1984) generalizes Huberman's result, showing that the factor model, uniform boundedness of the elements of  $\beta$  and no arbitrage imply (2) with  $Z = E[ee']$ , which is not necessarily diagonal. A variant of Ingersoll's argument is as follows. Write the positive definite matrix  $Z$  as the product  $Z = UU'$ , where  $U$  is an  $n \times n$  nonsingular matrix. Then, consider the orthogonal projection of the vector  $U^{-1}\mu$  on the column space of  $U^{-1}X$ :

$$U^{-1}\mu = U^{-1}X\lambda + \alpha, \quad (8)$$

where  $\alpha'U^{-1}X = 0$ . The rest of the argument is similar to those presented earlier.

Chamberlain and Rothschild (1983) employ Hilbert space techniques to study capital markets with (possibly infinitely) many assets. The preclusion of arbitrage implies the continuity of the cost functional in the Hilbert space. Let  $L$  equal the maximum eigenvalue of the limit covariance matrix  $E[ee']$  and  $d$  equal the supremum of all the ratios of expectation to standard deviation of the incomes on all costless portfolios with a nonzero weight on at least one asset. Chamberlain and Rothschild demonstrate that (2) holds with  $a = Ld^2$  and  $Z = I$  if asset prices allow no arbitrage.

With two additional assumptions, Chamberlain (1983) provides explicit lower and upper bounds on the left-hand side of (2). He further shows that exact arbitrage pricing obtains if and only if there is a well-diversified portfolio on the mean-variance frontier. The first of his additional assumptions is that all the factors can be represented as limits of traded assets. The second additional assumption is that the variances of incomes on any sequence of portfolios that are well diversified in the limit and that are uncorrelated with the factors converge to zero.

## 5 Utility-Based Arguments

In utility-based arguments, investors are assumed to solve the following problem:

$$\max_{c_0, c_T, w} E[u(c_0, c_T)] \quad \text{subject to} \quad c_0 \leq b - w'i \text{ and } c_T \leq w'r, \quad (9)$$

where  $b$  is the initial wealth, and  $u(c_0, c_T)$  is a utility function of initial and terminal consumption  $c_0$  and  $c_T$ . The utility function is assumed to increase with initial and



with terminal consumption. The first order condition is

$$E[rM] = \iota , \tag{10}$$

where  $M = (\partial u / \partial c_T) / (\partial u / \partial c_0)$ . The random variable  $M$  satisfying (10) is referred to as the stochastic discount factor (SDF) by Hansen and Jagannathan (1991, 1997). Substitution of the factor model (1) into the first order condition gives

$$\mu = \iota \lambda_0 + \beta \lambda_1 + \alpha , \tag{11}$$

where  $\lambda_0 = 1/E[M]$ ,  $\lambda_1 = -E[fM]/E[M]$  and  $\alpha = -E[eM]/E[M]$ . It follows from (11) that

$$(\mu - X\lambda)'(\mu - X\lambda) = \alpha'\alpha , \tag{12}$$

where  $X = (\iota, \beta)$  and  $\lambda = (\lambda_0, \lambda_1)'$ .

Clearly, the APT (2) holds for  $Z = I$  and  $a$  if  $\alpha'\alpha$  is uniformly bounded by  $a$ . Ross (1976a) is the first to set up an economy in which  $\alpha'\alpha$  is uniformly bounded. The exact APT (3) holds if and only if

$$E[eM] = 0 . \tag{13}$$

If the SDF is a linear function of the factors, then equation (13) holds. Conversely, if equation (13) holds, there exists an SDF, which is a linear function of factors, such that equation (10) is satisfied. However, the SDF does not have to be a linear function of factors for the purpose of obtaining the exact APT. A nonlinear function,  $M = g(f)$ , of factors for the SDF would also imply (13) under the assumption  $E[e|f] = 0$ .

Connor (1984) shows that if the market portfolio is well diversified, then every investor holds a well-diversified portfolio (i.e. a  $k+1$  fund separation obtains; the funds are associated with the factors and with the risk-free asset, which Connor assumes to exist). With this, the first order condition of any investor implies exact arbitrage pricing in a competitive equilibrium.

Connor and Korajczyk (1986) extend Connor's previous work to a model with investors who have better information about returns than most other investors. The former class of investors is sufficiently small, so the pricing result remains intact and it is used to derive a test of the superiority of information of the allegedly better informed investors.

Connor and Korajczyk (1985) extend Connor's single-period model to a multi-period model. They assume that the capital assets are the same in all periods, that each period's cash payoffs from these assets obey a factor structure, and that competitive equilibrium prices are set as if the economy had a representative investor who maximizes

exponential utility. They show that exact arbitrage pricing obtains with time-varying risk premium (but, similar to Stambaugh (1983), with constant factor loadings.)

Chen and Ingersoll (1983) argue that if a well-diversified portfolio exists and it is the optimal portfolio of some utility-maximizing investor, then the first order condition of that investor implies exact arbitrage pricing.

Dybvig (1983) and Grinblatt and Titman (1983) consider the case of finite assets and provide explicit bounds on the deviations from exact arbitrage pricing. These bounds are functions of the per capita asset supplies, individual bounds on absolute risk aversion, variance of the idiosyncratic risk, and the interest rate. To derive his bound, Dybvig assumes that the support of the distribution of the idiosyncratic term  $e$  is bounded below, that each investor's coefficient of absolute risk aversion is non-increasing and that the competitive equilibrium allocation is unconstrained Pareto optimal. To derive their bound, Grinblatt and Titman require a bound on a quantity related to investors' coefficients of absolute risk aversion and the existence of  $k$  independent, costless and well diversified portfolios.

## 6 Mean-Variance Efficiency

The APT was developed as a generalization of the CAPM, which asserts that the expectations of assets' returns are linearly related to their covariances (or betas, which in turn are proportional to the covariances) with the market portfolio's return. Equivalently, the CAPM says that the market portfolio is mean-variance efficient in the investment universe containing all possible assets. If the factors in (1) can be identified with traded assets then exact arbitrage pricing (3) says that a portfolio of these factors is mean-variance efficient in the investment universe consisting of the assets  $r$ .

Huberman and Kandel (1985b), Jobson and Korkie (1982, 1985) and Jobson (1982) note the relation between the APT and mean-variance efficiency. They propose likelihood-ratio tests of the joint hypothesis that a given set of random variables are factors in model (1) and that exact arbitrage pricing (3) obtains. Kan and Zhou (2001) point out a crucial typographical error in Huberman and Kandel (1985b). Peñaranda and Sentana (2004) study the close relation between the Huberman and Kandel's spanning approach and the celebrated volatility bounds in Hansen and Jagannathan (1991).

Even when the factors are not traded assets, (3) is a statement about mean-variance efficiency: Grinblatt and Titman (1987) assume that the factor structure (1) holds and that a risk-free asset is available. They identify  $k$  traded assets such that a portfolio of them is mean-variance efficient if and only if (3) holds. Huberman, Kandel and Stambaugh (1986) extend the work of Grinblatt and Titman by characterizing the sets of  $k$  traded assets with that property and show that these assets can be described as

portfolios if and only if the global minimum variance portfolio has nonzero systematic risk. To find these sets of assets, one must know the matrices  $\beta\beta'$  and  $E[ee']$ . If the latter matrix is diagonal, factor analysis produces an estimate of it, as well as an estimate of  $\beta\beta'$ .

The interpretation of (3) as a statement about mean-variance efficiency contributes to the debate about the testability of the APT. (Shanken (1982, 1985) and Dybvig and Ross (1985), however, discuss the APT's testability without mentioning that (3) is a statement about mean-variance efficiency.) The theory's silence about the factors' identities renders any test of the APT a joint test of the pricing relation and the correctness of the factors. As a mean-variance efficient portfolio always exists, one can always find 'factors' with respect to which (3) holds. In fact, any single portfolio on the frontier can serve as a 'factor.'

Thus, finding portfolios which are mean-variance efficient — or failure to find them — neither supports nor contradicts the APT. It is the factor structure (1) which, combined with (3), provides refutable hypotheses about assets' returns. The factor structure (1) imposes restrictions which, combined with (3), provide refutable hypotheses about assets' returns. The factor structure suggests looking for factors with two properties: (i) their time-series movements explain a substantial fraction of the time series movements of the returns on the priced assets, and (ii) the unexplained parts of the time series movements of the returns on the priced assets are approximately uncorrelated across the priced assets.

## 7 Empirical Tests

Empirical work inspired by the APT typically ignores (2) and instead studies exact arbitrage pricing (3). This type of work usually consists of two steps: an estimation of factors (or at least of the matrix  $\beta$ ) and then a check to see whether exact arbitrage pricing holds. In the first step, researchers typically use the following regression model to estimate the parameters in the factor model:

$$r_t = \alpha + \beta f_t + e_t, \quad (14)$$

where  $r_t$ ,  $f_t$  and  $e_t$  are the realization of the variables in period  $t$ . The factors observed in empirical studies often have a nonzero mean, denoted by  $\delta$ . Let  $T$  be the total number of periods and  $\Sigma$ , the summation over  $t = 1, \dots, T$ . The ordinary least-square (OLS) estimates are

$$\hat{\mu} = \frac{1}{T} \sum r_t \quad \text{and} \quad \hat{\delta} = \frac{1}{T} \sum f_t \quad (15)$$

$$\hat{\beta} = \left( \sum (r_t - \hat{\mu})(f_t - \hat{\delta})' \right) \left( \sum (f_t - \hat{\delta})(f_t - \hat{\delta})' \right)^{-1} \quad (16)$$

$$\hat{\alpha} = \hat{\mu} - \hat{\beta}\hat{\delta} \quad (17)$$

$$\hat{\Omega} = \frac{1}{T} \sum \hat{e}_t \hat{e}_t' \quad \text{where} \quad \hat{e}_t = r_t - \hat{\alpha} - \hat{\beta}f_t . \quad (18)$$

These are also maximum-likelihood estimators if the returns and factors are independent across time and have a multivariate normal distribution.

In the second step, researchers may use the exact pricing (3) and (14) to obtain the following restricted version of the regression model,

$$r_t = \iota\lambda_0 + \beta(f_t + \lambda_1) + e_t . \quad (19)$$

Under the assumption that returns and factors follow identical and independent normal distributions, the maximum-likelihood estimators are

$$\bar{\beta} = \left( \sum (r_t - \iota\bar{\lambda}_0)(f_t + \bar{\lambda}_1)' \right) \left( \sum (f_t + \bar{\lambda}_1)(f_t + \bar{\lambda}_1)' \right)^{-1} \quad (20)$$

$$\bar{\Omega} = \frac{1}{T} \sum \bar{e}_t \bar{e}_t' \quad \text{where} \quad \bar{e}_t = r_t - \iota\bar{\lambda}_0 - \bar{\beta}(f_t + \bar{\lambda}_1) \quad (21)$$

$$\bar{\lambda} = (\bar{X}'\bar{\Omega}^{-1}\bar{X})^{-1}\bar{X}'\bar{\Omega}^{-1}(\hat{\mu} - \bar{\beta}\hat{\delta}) \quad \text{where} \quad \bar{X} = (\iota, \bar{\beta}) . \quad (22)$$

These estimators need to be solved simultaneously from the above three equations. Notice that  $\bar{\beta}$  and  $\bar{\Omega}$  are the OLS estimators in (19) for a given  $\bar{\lambda}$ . The last equation shows that  $\bar{\lambda}$  is the generalized least-square estimator in the cross-sectional regression of  $\hat{\mu} - \bar{\beta}\hat{\delta}$  on  $\bar{X}$  with  $\bar{\Omega}$  being the weighting matrix. To test the restriction imposed by the exact APT, researchers use the likelihood-ratio statistic,

$$\text{LR} = T(\log |\bar{\Omega}| - \log |\hat{\Omega}|) , \quad (23)$$

which follows a  $\chi^2$  distribution with  $n - k - 1$  degrees of freedom when the number of observations,  $T$ , is very large. When factors are payoffs of traded assets or a risk-free asset exists, the exact APT imposes more restrictions. For these cases, Campbell, Lo and MacKinley (1997 chapter 6) provide an overview. If the observations of returns and factors do not follow independent normal distribution, similar tests can be carried out using the generalized method of moments (GMM). Jagannathan and Wang (2002) and Jagannathan, Skoulakis and Wang (2002) provide an overview of the application of the GMM for testing asset pricing models including the APT.

Interest is sometimes focused only on whether a set of specified factors are priced or on whether their loadings help explain the cross section of expected asset returns. For this purpose, most researchers study the cross-sectional regression model

$$\hat{\mu} = \hat{X}\lambda + v \quad \text{or} \quad \hat{\mu} = \iota\lambda_0 + \hat{\beta}\lambda_1 + v , \quad (24)$$

where  $\hat{X} = (\iota, \hat{\beta})$  and  $v$  is an  $n \times 1$  vector of errors for this equation. The OLS estimator of  $\lambda$  in this regression is tested to see if it is different from zero or not. To test this

specification, asset characteristics  $z$ , such as firm size, that are correlated with mean asset returns are added to the regression:

$$\hat{\mu} = \iota\lambda_0 + \hat{\beta}\lambda_1 + z\lambda_2 + v . \quad (25)$$

A significant  $\lambda_1$  and insignificant  $\lambda_2$  are viewed as evidence in support of the specified factors being part of the exact APT. Black, Jensen and Scholes (1972) and Fama and MacBeth (1973) pioneered this cross-sectional approach to test the CAPM. Chen, Roll and Ross (1986) used it to test the exact APT. Shanken (1992) and Jagannathan and Wang (1998) developed the statistical foundations of the cross-sectional tests. The cross-sectional approach is now a popular tool for analyzing risk premiums on the loadings of proposed factors.

## 8 Specification of Factors

The tests outlined above are joint tests that the matrix  $\beta$  is correctly estimated and that exact arbitrage pricing holds. Estimation of the factor loading matrix  $\beta$  entails at least an implicit identification of the factors. The three approaches listed below have been used to identify factors.

The first consists of an algorithmic analysis of the estimated covariance matrix of asset returns. For instance, Roll and Ross (1980), Chen (1983) and Lehman and Modest (1985a, 1985b) use factor analysis and Chamberlain and Rothschild (1983) and Connor and Korajczyk (1985, 1986) recommend using principal component analysis.

The second approach is one in which a researcher starts at the estimated covariance matrix of asset returns and uses his judgment to choose factors and subsequently estimate the matrix  $\beta$ . Huberman and Kandel (1985a) note that the correlations of stock returns of firms of different sizes increases with a similarity in size. Therefore, they choose an index of small firms, one of medium size firms and one of large firms to serve as factors. In a similar vein, Fama and French (1993) use the spread between the stock returns of small and large firms as one of their factors. Echoing the findings of Rosenberg, Reid, and Lanstein (1984), Chan, Hamao and Lakonishok (1991) and Fama and French (1992) observe that expected stock returns and their correlations are also related to the ratio of book-to-market equity. Based on these observations, Fama and French (1993) add the spread between stock returns of value and growth firms as another factor.

The third approach is purely judgmental in that it is one in which the researcher primarily uses his intuition to pick factors and then estimates the factor loadings and checks if they explain the cross-sectional variations in estimated expected returns (i.e., he checks (3)). Chan, Chen and Hsieh (1985) and Chen, Roll and Ross (1986) select

financial and macroeconomic variables to serve as factors. They include the following variables: the return on an equity index, the spread of short and long-term interest rates, a measure of the private sector's default premium, the inflation rate, the growth rates of industrial production and the aggregate consumption. Based on economic intuition, researchers continue to add new factors, which are too many to enumerate here.

The first two approaches are implemented to conform to the factor structure underlying the APT: the first approach by the algorithmic design and the second because researchers check that the factors they use indeed leave the unexplained parts of asset returns almost uncorrelated. The third approach is implemented without regard for the factor structure. Its attempt to relate the assets' expected returns to the covariance of the assets' returns with other variables is more in the spirit of Merton's (1973) intertemporal CAPM than in the spirit of the APT.

The empirical work cited above examines the extent to which the exact APT (with whatever factors are chosen) explains the cross-sectional variation in assets' mean returns better than the CAPM. It also examines the extent to which other variables — usually those that include various firm characteristics — have marginal explanatory power beyond the factor loadings to explain the cross-section of assets' mean returns. The results usually suggest that the APT is a useful model in comparison with the CAPM. (Otherwise, they would probably have gone unpublished.) However, the results are mixed when the alternative is firm characteristics. Researchers who introduce factors tend to report results supporting the APT with their factors and test portfolios. Nevertheless, different tests and construction of portfolios often reject the proposed APT. For example, Fama and French (1993) demonstrate that exact APT using their factors holds for portfolios constructed by sorting stocks on firm size and book-to-market ratio, whereas Daniel and Titman (1997) demonstrate that the same APT does not hold for portfolios that are constructed by sorting stocks further on the estimated loadings with respect to the Fama and French's factors.

The APT often seems to describe the data better than competing models. It is wise to recall, however, that the purported empirical success of the APT may well be due to the weakness of the tests employed. Some of the questions that come to mind are: Which factors capture the data best? What is the economic interpretation of the factors? What are the relations among the factors that different researchers have reported? As any test of the APT is a joint test that the factors are correctly identified and that the linear pricing relation holds, a host of competing theories exist side by side under the APT's umbrella. Each fails to reject the APT but has its own factor identification procedure. The number of factors, as well as the methods of factor construction, is exploding. The multiplicity of competing factor models indicates igno-

rance of the true factor structure of asset returns and suggests a rich and challenging research agenda.

## 9 Applications

The APT lends itself to various practical applications due to its simplicity and flexibility. The three areas of applications critically reviewed here are: asset allocation, the computation of the cost of capital, and the performance evaluation of managed funds.

The application of the APT in asset allocation is motivated by the link between the factor structure (1) and mean-variance efficiency. Since the structure with  $k$  factors implies the existence of  $k$  assets that span the efficient frontier, an investor can construct a mean-variance efficient portfolio with only  $k$  assets. The task is especially straightforward when the  $k$  factors are the payoffs of traded securities. When  $k$  is a small number, the model reduces the dimension of the optimization problem. The use of the APT in the construction of an optimal portfolio is equivalent to imposing the restriction of the APT in the estimation of the mean and covariance matrix involved in the mean-variance analysis. Such a restriction increases the reliability of the estimates because it reduces the number of unknown parameters.

If the factor structure specified in the APT is incorrect, however, the optimal portfolio constructed from the APT will not be mean-variance efficient. This uncertainty calls for adjusting, rather than restricting, the estimates of mean and covariance matrix by the APT. The degree of this adjustment should depend on investors' prior belief in the model. Pastor and Stambaugh (2000) introduce the Bayesian approach to achieve this adjustment. Wang (2005) further shows that the Bayesian estimation of the return distribution results in a weighted average of the distribution restricted by the APT and the unrestricted distribution matched to the historical data.

The proliferation of APT-based models challenges an investor engaging in asset allocation. In fact, Wang (2005) argues that investors averse to model uncertainty may choose an asset allocation that is not mean-variance efficient for any probability distributions estimated from the prior beliefs in the model.

Being an asset pricing model, the APT should lend itself to the calculation of the cost of capital. Elton, Gruber and Mei (1994) and Bower and Schink (1994) used the APT to derive the cost of capital for electric utilities for the New York State Utility Commission. Elton, Gruber and Mei specify the factors as unanticipated changes in the term structure of interest rates, the level of interest rates, the inflation rate, the GDP growth rate, changes in foreign exchange rates, and a composite measure they devise to measure changes in other macro factors. In the meantime, Bower and Schink use the factors suggested by Fama and French (1993) to calculate the cost of capital

for the Utility Commission. However, the Commission did not adopt any of the above-mentioned multi-factor models but used the CAPM instead (see DiValentino (1994).)

Other attempts to apply the APT to compute the cost of capital include Bower, Bower and Logue (1984), Goldenberg and Robin (1991) who use the APT to study the cost of capital for utility stocks, and Antoniou, Garrett and Priestley (1998) who use the APT to calculate the cost of equity capital when examining the impact of the European exchange rate mechanism. Different studies use different factors and consequently obtain different results, a reflection of the main drawback of the APT — the theory does not specify what factors to use. According to Green, Lopez and Wang (2003), this drawback is one of the main reasons that the U.S. Federal Reserve Board has decided not to use the APT to formulate the imputed cost of equity capital for priced services at Federal Reserve Banks.

The application of asset pricing models to the evaluation of money managers was pioneered by Jensen (1967). When using the APT to evaluate money managers, the managed funds' returns are regressed on the factors, and the intercepts are compared with the returns on benchmark securities such as Treasury bills. Examples of this application of the APT include Busse (1999), Carhart (1997), Chan, Chen, Lakonishok (2002), Cai, Chan and Yamada (1997), Elton, Gruber and Blake (1996), Mitchell and Pulvino (2001), and Pastor and Stambaugh (2002).

The APT is a one-period model that delivers arbitrage-free pricing of existing assets (and portfolios of these assets), given the factor structure of their returns. Applying it to price derivatives on existing assets or to price trading strategies is problematic, because its stochastic discount factor is a random variable which may be negative. Negativity of the SDF in an environment which permits derivatives leads to a pricing contradiction, or arbitrage. Consider, for instance, the price of an option that pays its holder whenever the SDF is negative. Being a limited liability security, such an option should have a positive price, but applying the SDF to its payoff pattern delivers a negative price. (The observation that the stochastic discount factor of the CAPM may be negative is in to Dybvig and Ingersoll (1982) who also studied some of the implications of this observation.)

Trading and derivatives on existing assets are closely related. Famously, Black and Scholes (1972) show that dynamic trading of existing securities can replicate the payoffs of options on these existing securities. Therefore, one should be careful in interpreting APT-based excess returns of actively managed funds because such funds trade rather than hold on to the same portfolios. Examples of interpretations of asset management techniques as derivative securities include Merton (1981) who argues that market-timing strategy is an option, Fung and Hsieh (2001) who show that hedge funds using trend-following strategies behave like a look-back straddle, and Mitchell and Pulvino



(2001) who demonstrate that merger arbitrage funds behave like an uncovered put.

Motivated by the challenge of evaluating dynamic trading strategies, Glosten and Jagannathan (1994) suggest replacing the linear factor models with the Black-Scholes model. Wang and Zhang (2005) study the problem extensively and develop an econometric methodology to identify the problem in factor-based asset pricing models. They show that the APT with many factors is likely to have large pricing errors over actively managed funds, because empirically these models deliver SDFs which allow for arbitrage over derivative-like payoffs.

It is ironic that some of the applications of the APT require extensions of the basic model which violate its basic tenet — that assets are priced as if markets offer no arbitrage opportunities.

## Bibliography

- Antoniou, A., Garrett, I., Priestley, R., 1998. Calculating the equity cost of capital using the APT: the impact of the ERM. *Journal of International Money and Finance* 14, 949–965.
- Black, F., Jensen, M., Scholes, M., 1972. The capital-asset pricing model: some empirical tests. *Studies in the Theory of Capital Markets*.
- Bower, D., Bower, R., Logue, D., 1984. Arbitrage pricing and utility stock returns. *Journal of Finance* 39, 1041–1054.
- Bower, R., Schink, G., 1994. Application of the Fama-French model to utility stocks. *Financial Markets, Institutions and Instruments* 3, 74–96.
- Busse, J., 1999. Volatility timing in mutual funds: evidence from daily returns, *Review of Financial Studies* 12, 1009–1041.
- Cai, J., Chan, K., Yamada, T., 1997. The performance of Japanese mutual funds, *Review of Financial Studies* 10, 237–273.
- Campbell, J., Lo, A., MacKinley, C., 1997. *The Econometrics of Financial Markets*, Princeton University Press.
- Carhart, M., 1997. On persistence in mutual fund performance, *Journal of Finance* 52, 57–82.
- Chamberlain, G., 1983. Funds, factors and diversification in arbitrage pricing models. *Econometrica* 51, 1305–23.
- Chamberlain, G., Rothschild, M., 1983. Arbitrage, factor structure, and mean variance analysis on large asset markets. *Econometrica* 51, 1281–304.
- Chan, L., Chen, H., Lakonishok, J., 2002. On mutual fund investment styles. *Review of Financial Studies* 15, 1407–1437.
- Chan, K., Chen, N., Hsieh, D., 1985. An exploratory investigation of the firm size effect. *Journal of Financial Economics* 14, 451–71.
- Chan, L., Hamano, Y., Lakonishok, J., 1991. Fundamentals and stock returns in Japan. *Journal of Finance* 46, 1739–64
- Chen, N., 1983. Some empirical tests of the theory of arbitrage pricing. *Journal of Finance* 38, 1393–414.
- Chen, N., Ingersoll, J., 1983. Exact pricing in linear factor models with infinitely many assets: a note. *Journal of Finance* 38, 985–8.
- Chen, N., Roll, R., Ross, S., 1986. Economic forces and the stock markets. *Journal of Business* 59, 383–403.
- Connor, G., 1984. A unified beta pricing theory. *Journal of Economic Theory* 34, 13–31.
- Connor, G., Korajczyk, R., 1985. Risk and return in an equilibrium APT: theory and

- tests. Banking Research Center Working Paper 129, Northwestern University.
- Connor, G., Korajczyk, R., 1986. Performance measurement with the arbitrage pricing theory: a framework for analysis. *Journal of Financial Economics* 15, 373–94.
- Daniel, K., Titman, S., 1997. Evidence on the Characteristics of Cross Sectional Variation in Stock Returns. *Journal of Finance* 52, 1–33
- DiValentino, L., 1994. Preface. *Financial Markets, Institutions and Instruments*, 3, 6–8.
- Dybvig, P., 1983. An explicit bound on deviations from APT pricing in a finite economy. *Journal of Financial Economics* 12, 483–96.
- Dybvig, P., Ross, S., 1985. Yes, the APT is testable. *Journal of Finance* 40, 1173–88.
- Elton, E., Gruber, M., Blake, C., 1996. Survivorship bias and mutual fund performance. *Review of Financial Studies* 9, 1097–1120.
- Elton, E., Gruber, M., Mei, J., 1994. Cost of capital using arbitrage pricing theory: A case study of nine New York utilities. *Financial Markets, Institutions and Instruments* 3, 46–73.
- Fama, E., MacBeth, J., 1973. Risk, return and equilibrium: empirical tests. *Journal of Political Economy* 71, 607–636.
- Fama, E., French, K., 1992. The cross-section of expected stock returns. *Journal of Finance* 47, 427–486.
- Fama, E., French, K., 1993. Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics* 33, 3–56.
- Fung, W., Hsieh, D., 2001. The risks in hedge fund strategies: theory and evidence from trend followers. *Review of Financial Studies* 14, 313–341.
- Glosten, L., Jagannathan, R., 1994. A contingent claim approach to performance evaluation, *Journal of Empirical Finance* 1, 133–160.
- Goldenberg, G., Robin, A., 1991. The arbitrage pricing theory and cost-of-capital estimation: The case of electric utilities. *Journal of Financial Research* 14, 181–196.
- Green, E., Lopez, J., Wang, Z., 2003. Formulating the imputed cost of equity for priced services at Federal Reserve Banks. *Economic Policy Review* 9, 55–8.
- Grinblatt, M., Titman, S., 1983. Factor pricing in a finite economy. *Journal of Financial Economics* 12, 495–507.
- Grinblatt, M., Titman, S., 1987. The relation between mean-variance efficiency and arbitrage pricing. *Journal of Business* 60, 97–112 .
- Hansen, L., Jagannathan, R., 1991. Implications of security market data for models of dynamic economies. *Journal of Political Economy* 99, 225–262
- Huberman, G., 1982. A simple approach to arbitrage pricing. *Journal of Economic Theory* 28, 183–91.

- Huberman, G., Kandel, S., 1985a. A size based stock returns model. Center for Research in Security Prices Working Paper 148, University of Chicago.
- Huberman, G., Kandel S., 1985b. Likelihood ratio tests of asset pricing and mutual fund separation. Center for Research in Security Prices Working Paper 149, University of Chicago.
- Huberman, G., Kandel, S., Stambaugh, R., 1987. Mimicking portfolios and exact arbitrage pricing. *Journal of Finance* 42, 1–9.
- Ingersoll, J., 1984. Some results in the theory of arbitrage pricing. *Journal of Finance* 39, 1021–1039.
- Jagannathan, R., Wang, Z., 1998. An asymptotic theory for estimating beta-pricing models using cross-sectional regression. *Journal of Finance* 53, 1285–1309.
- Jagannathan, R., Wang, Z., 2002. Empirical evaluation of asset pricing models: a comparison of the SDF and beta methods. *Journal of Finance* 57, 2337–2367.
- Jagannathan, R., Skoulakis, G., Wang, Z., 2002. Generalized method of moments: applications in finance. *Journal of Business and Economic Statistics* 20, 470–481.
- Jensen, M., 1968. The performance of mutual funds in the period 1945–1964. *Journal of Finance* 23, 389–416.
- Jobson, J., 1982. A multivariate linear regression test of the arbitrage pricing theory. *Journal of Finance* 37, 1037–42.
- Jobson, J., Korkie, B., 1982. Potential performance and tests of portfolio efficiency. *Journal of Financial Economics* 10, 433–66.
- Jobson, J., Korkie, B., 1985. Some tests of linear asset pricing with multivariate normality. *Canadian Journal of Administrative Sciences* 2, 114–38.
- Kan, R., Guofu, Z., 2001. Tests of mean-variance spanning. Working paper, Washington University in St. Louis.
- Lehman, B., Modest, D., 1985a. The empirical foundations of the arbitrage pricing theory I: the empirical tests. Department of Economics Working Paper 291, Columbia University.
- Lehman, B., Modest, D., 1985b. The empirical foundations of the arbitrage pricing theory II: the optimal construction of basis portfolios. Department of Economics Working Paper 292, Columbia University.
- Merton, R., 1973. An intertemporal capital asset pricing model. *Econometrica* 41, 867–87.
- Merton, R., 1981. On market timing and investment performance I: any equilibrium theory of values for markets forecasts. *Journal of Business* 54, 363–406.
- Mitchell, M., Todd Pulvino, T., 2001. Characteristics of risk and return in risk arbitrage, *Journal of Finance* 56, 2135–2175.
- Pastor, L., Stambaugh, R., 2000. Comparing asset pricing models: an investment

- perspective. *Journal of Financial Economics* 56, 335–381.
- Pastor, L., Stambaugh, R., 2002. Mutual fund performance and seemingly unrelated assets. *Journal of Financial Economics* 63, 315–349.
- Peñaranda, F., Sentana, E., 2004. Spanning tests in return and stochastic discount factor mean-variance frontiers: a unifying approach. Working paper 0410, CEMFI, Madrid.
- Roll, R., Ross, S., 1980. An empirical investigation of the arbitrage pricing theory. *Journal of Finance* 35, 1073–103.
- Rosenberg, B., Reid, K., Lanstein, R. 1984. Persuasive evidence of market inefficiency. *Journal of Portfolio Management* 11, 9–17.
- Ross, S., 1976a. The arbitrage theory of capital asset pricing. *Journal of Economic Theory* 13, 341–60.
- Ross, S., 1976b. Risk, return and arbitrage. *Risk Return in Finance* ed. I. Friend and J. Bicksler, Cambridge, Mass.: Ballinger.
- Shanken, J., 1982. The arbitrage pricing theory: is it testable?. *Journal of Finance* 37, 1129–240.
- Shanken, J., 1985. A multi-beta CAPM or equilibrium APT?: a reply. *Journal of Finance* 40, 1189–96.
- Shanken, J., 1992. On the estimation of beta-Pricing models. *Review of Financial Studies* 5, 1–33.
- Stambaugh, R., 1983. Arbitrage pricing with information. *Journal of Financial Economics* 12, 357–69.
- Wang, Z., 2005. A shrinkage approach to model uncertainty and asset allocation. *Review of Financial Studies* 18, 673–705.
- Wang, Z., Zhang, X., 2005. Empirical evaluation of asset pricing models: arbitrage and pricing errors over contingent claims. Working Paper, Federal Reserve Bank of New York and Cornell University.