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# Dynamic Managerial Compensation: On the Optimality of Seniority-based Schemes\*

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## Abstract

We characterize the firm's optimal contract for a manager who faces costly effort decisions and whose ability to generate profits for the firm changes stochastically over time. The optimal contract is obtained as the solution to a dynamic mechanism design problem with hidden actions and persistent private shocks to the manager's productivity. When the manager is risk-neutral, the optimal contract often entails a simple pay package that is linear in the firm's cash flows. Furthermore, the power of incentives (i.e., the sensitivity of pay to performance) typically increases over time, thus providing a possible justification for the practice of putting more stocks and options in the packages of managers with a longer tenure in the firm. Building on the insights from the risk-neutral case, we then explore the properties of optimal contracts for risk-averse managers. We find that risk-aversion reduces, and in some cases can even revert, the profitability of seniority-based compensation schemes whose power of incentives increases over time.

*JEL classification:* D82

*Keywords:* dynamic mechanism design, adverse selection, moral hazard, incentives, optimal pay scheme, risk-aversion, stochastic process.

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# 1 Introduction

The contracts that the most successful firms offer to their top employees are designed taking into account that managerial ability to generate profits is bound to change over time. Shocks to managerial productivity are expected to originate from variations in the business environment that are to a large extent anticipated at the time of contracting but whose ultimate effect on managerial productivity is the managers' private information.

The questions that this paper addresses are the following: (i) How should firms respond to such shocks to managerial productivity? In particular should they induce their managers to work harder when their productivity increases or rather when they experience a negative productivity shock? and (ii) What type of compensation schemes induce the managers to respond to the shocks the way it is optimal for the firm, when both managerial effort and the shocks to managerial productivity are the managers' private information?

We assume that the firm perfectly understands the value of not renegeing on its promises and thus commits to the compensation scheme that it offers to its managers at the time they are hired. We then use a mechanism design approach to solve for the dynamic contract that maximizes the expected sum of the firm's cash flows, net of managerial compensation.

Our first result provides a characterization of a class of contracts that permit the firm to sustain *any* implementable effort policy at minimum cost. These compensation schemes are the dynamic analog of those proposed by Laffont and Tirole (1986) for a static setting. They specify a *bonus* for each period that is paid conditional upon the firm's cash flows exceeding a critical target that typically depends on past cash-flows, as well as on possible messages sent by the manager over time. The role of these messages is to permit the firm to adjust its compensation scheme in response to variations in managerial productivity.

Our second and third results identify conditions for a given effort policy to be implementable, as well as for the possibility of implementing a given effort policy with a *linear* or a *pseudo-linear* compensation scheme. By a linear scheme, we mean one whose compensation is linear in performance, i.e., in cash flows; by a pseudo-linear scheme, we mean one that gives the manager an utility

that is linear in performance. While such schemes are less high-powered than the bonus schemes described above, there are interesting cases where the optimal effort policy for the firm can indeed be implemented at minimum cost with such schemes (e.g., when the manager is risk neutral and the shocks to managerial productivity are expected to follow an ARIMA process).

Equipped with the aforementioned results, we then investigate the properties of profit-maximizing contracts, and in particular the optimality of *seniority-based* compensation schemes. The latter are schemes that provide managers with a longer tenure in the firm with more high-powered incentives, thus inducing them to exert higher effort. To this purpose, we start by considering the case where the manager is risk-neutral.<sup>1</sup> We show that, for many processes of interest, the power of incentives under the optimal contract indeed increases over time, thus providing a possible explanation for the frequent practice of putting more stocks and options in the package of managers with a longer tenure in the firm (see, e.g., Lippert and Porter, 1997, but also Gibbons and Murphy, 1991). Contrary to other explanations proposed in the literature (e.g., declining disutility of effort or career concerns<sup>2</sup>), the optimality of seniority-based schemes is not driven by variations in the managers' preferences, or by variations in their outside option. It results from an optimal allocation of informational rents over time. In other words, it originates in the firm's desire to minimize the managers' compensation while preserving their incentives for both effort and information revelation.

The driving force behind this result is the assumption that the effect of a manager's initial productivity  $\theta_1$  on his future productivity  $\theta_t$  (formally captured by the impulse response of  $\theta_1$  on  $\theta_t$ , as explained in detail below) is expected to decline over time. For example, when the manager's private information evolves according to an AR(1) process, this assumption is satisfied as long as the coefficient of linear dependence of  $\theta_t$  on  $\theta_{t-1}$  is less than one. The property of declining impulse responses implies that, to minimize a manager's information rent, as computed at the time of hiring, the firm finds it optimal to induce low effort in the early periods of the relationship and higher effort

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<sup>1</sup>Because the firm contracts with the manager at the time the latter is already privately informed about his initial productivity, interesting dynamics emerge even without risk aversion, and even without imposing limited liability on the manager side. Private information at the contracting stage in fact implies that the firm cannot "sell out" its business to the manager, for there does not exist a uniform price that extracts all surplus from all types of the manager.

<sup>2</sup>For a detailed analysis of career concerns incentives, see e.g., Dewatripont, Jewitt and Tirole (1999).

in the subsequent periods. The reason is that the surplus that a manager who is highly productive at the contracting stage can obtain by mimicking a less productive type increases in the effort that the firm asks the latter to exert, as shown first in Laffont and Tirole 1986. Indeed, this surplus originates in the possibility of generating, on average, the same cash flows as the less productive type by working less, thus economizing on the disutility of effort. It follows that, by reducing the effort of those types who are less productive at the contracting stage, the firm reduces the rent that it must provide to the most productive types. The benefit of reducing the effort of the least productive types is however higher when done in the early stages of the relationship, when the effect of the initial productivity is still pronounced, than in the distant future, when such an effect is fable. It follows that effort (and by implication the power of incentives in the compensation scheme that sustains it) typically increases with tenure and gradually approaches the first-best level in the long run.<sup>3</sup>

We also show that, when the manager is risk neutral, the optimal effort policy can often be sustained by paying the manager according to a simple (state-contingent) linear scheme according to which, in each period, the firm pays the manager a fixed salary plus a bonus that is linear in the firm's cash flows (or, equivalently, in the firm's stock price, assuming that the latter also depends on managerial effort). When the manager's productivity evolves according to an ARIMA process (more generally, according to any process where the impulse responses exhibit a certain separability with respect to the initial productivity), the slope of the linear scheme increases deterministically over time, i.e., it depends only on the manager's initial productivity and on the number of periods he has been working for the firm. More generally, though, the optimal contract requires that the manager be given the possibility of proposing changes to his pay over time in response to the shocks to his productivity.<sup>4</sup>

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<sup>3</sup>This property of declining distortions is quite common in the literature on dynamic contracting with adverse selection (see, e.g., Baron and Besanko, 1984, Besanko, 1986, Courty and Li 2000, Battaglini, 2005, among others). As explained in Pavan, Segal, and Toikka (2009), the precise statistical property that is responsible for this result is the assumption of declining impulse responses of  $\theta_1$  on  $\theta_t$ , as opposed to other properties of the stochastic process such as the degree of correlation between  $\theta_1$  and  $\theta_t$ , or the ability of  $\theta_1$  to forecast  $\theta_t$ , as measured by the volatility of the forecast error of  $\theta_t$  given  $\theta_1$ . One of the key contributions of the current paper is to study the extent to which such predictions are robust to the possibility that the agent is risk averse.

<sup>4</sup>The idea that a manager must be given the possibility of proposing changes to his reward package is consistent with the recent empirical literature on managerial compensation where it is found that this practice has become frequent in

Building on the insights from the risk-neutral case, we then explore the properties of optimal compensation schemes for risk-averse managers. We find that, other things equal, risk-aversion reduces (and in some cases can even revert) the profitability of seniority-based compensation schemes whose power of incentives increases, on average, over time. The reason is that these schemes entail a high sensitivity of compensation to performance precisely in those periods in which the manager faces high uncertainty about his ability to generate cash flows for the firm. Indeed, while a manager's productivity at the time he is hired is a fairly good predictor of his productivity in the near future, it is a fairly poor predictor of his productivity in the distant future. Increasing the sensitivity of compensation to performance over time thus means exposing the manager to a great deal of risk. Whether risk-averse managers with a longer tenure in the firm receive more or less high-powered incentives than younger ones then depends on the interaction between their degree of risk-aversion and the dynamics of the impulse responses of the process governing the evolution of managerial productivity (more below).

We also find that, contrary to the risk-neutral case, the effort that the firm asks a manager to exert after the first period need not be monotone in his productivity, nor need it converge to the first-best level in the long run. Contrary to the risk neutral case, the characterization of the optimal effort policy under risk aversion cannot be derived in closed forms. Part of the contribution of the paper is also in showing how the optimal effort policy can be obtained through a recursive approach that leads to a program whose solution can be obtained numerically. Although a closed form solution is in general not available, the numerical solutions when applied to simple examples confirm that the driving forces behind the dynamics of the optimal contract are the ones discussed above.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 characterizes a compensation scheme that sustains all implementable effort policy at minimum cost. Section 4 provides conditions for an effort policy to be implementable, and for the possibility of implementing it at minimum cost with linear and pseudo-linear schemes. Section 5 characterizes the properties of optimal contracts and discusses the effect of risk aversion on the optimality of seniority-based

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the last decade (see, among others, Kuhnen and Zwiebel, 2008, and Bebchuck and Fried, 2004).

schemes. Section 6 discusses the related literature. Section 7 concludes with a few final remarks. All proofs are in the Appendix at the end of the manuscript.

## 2 The Model

### 2.1 The environment

**Players, actions and information.** The firm’s shareholders (hereafter referred to as the principal) hire a manager (the agent) to work on a project over  $T$  periods, where  $T$  can be either finite or infinite. In each period  $t \in \mathbb{N}_+$  the agent receives some private information  $\theta_t \in \Theta_t \subset \mathbb{R}$  about his ability to generate profits for the firm (his type), and then chooses effort  $e_t \in E = \mathbb{R}$ .<sup>5</sup>

**Payoffs.** The principal’s payoff is the discounted sum of the firm’s cash flows, net of the agent’s compensation:

$$U^P(\pi^T, c^T) = \sum_{t=1}^T \delta^{t-1} [\pi_t - c_t].$$

where  $\delta < 1$  is the common discount factor,

$$\pi_t = \theta_t + e_t$$

is the period- $t$  cash flow,  $c_t$  is the period- $t$  compensation to the agent,  $\pi^T \equiv (\pi_t)_{t=1}^T$ , and  $c^T \equiv (c_t)_{t=1}^T$ .<sup>6</sup> The function  $U^P$  also corresponds to the principal’s Bernoulli utility function used to evaluate lotteries over  $(\pi^T, c^T)$ . Both  $\theta^T$  and  $e^T$  are the agent’s private information. On the contrary, the stream of cash flows  $\pi^T$  is verifiable, which implies that the agent can be rewarded as a function of the firm’s cash flows. By choosing effort  $e_t$  in period  $t$ , the agent suffers a disutility  $\psi(e_t)$ . The agent’s preferences over (lotteries over) streams of consumption levels  $c^T$  and streams of effort choices

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<sup>5</sup>That effort can take negative values should not raise concerns: effort simply stands for the effect of the manager’s activity on the firm’s performance, which can be either positive or negative.

<sup>6</sup>Note that, because  $\theta_t$  is *not* restricted to be independent of past shocks  $\theta^{t-1} \equiv (\theta_1, \dots, \theta_{t-1})$ , there is no loss of generality in assuming that  $\pi_t$  depends only on  $\theta_t$ , as opposed to the entire history  $\theta^t = (\theta_1, \dots, \theta_t)$ . To see this, suppose that  $\pi_t = f_t(\theta^t) + h_t(e^t)$  for some functions  $f_t : \mathbb{R}^t \rightarrow \mathbb{R}$  and  $h_t : \mathbb{R}^t \rightarrow \mathbb{R}$ . It then suffices to change variables and simply let  $\theta_t^{new} = f_t(\theta^t)$ . The assumption that effort in period  $t$  affects the cash flows only in period  $t$  is more restrictive, although common in the literature. This assumption is not essential for our results. We refer the reader to the earlier version of the paper (Garret and Pavan, 2009) for the analysis of the case where  $\pi_t$  is a function of the entire history  $e^t$  of past effort choices.

$e^T \equiv (e_t)_{t=1}^T$  are described by an expected utility function with Bernoulli utility given by<sup>7</sup>

$$U^A(c^T, e^T) = \mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t(c_t) \right) - \sum_{t=1}^T \delta^{t-1} \psi(e_t) \quad (1)$$

where  $\mathcal{V} : \mathbb{R} \rightarrow \mathbb{R}$  and all  $v_t : \mathbb{R} \rightarrow \mathbb{R}$  functions are strictly increasing, weakly concave, surjective (i.e., onto) and differentiable. This representation covers as special cases the situation where the agent has time-additive-separable preferences for consumption smoothing ( $v_t$  strictly concave,  $\mathcal{V}$  linear) as well as the case where the agent is risk-averse but cares only about his total compensation ( $\mathcal{V}$  strictly concave,  $v_t$  linear). The case of risk neutrality corresponds to both  $v$  and  $\mathcal{V}$  linear. As standard, the aforementioned specification presumes time consistency. In what follows, we will thus assume that, at each history  $h_t$ , the agent maximizes the expectation of  $U^A(c^T, e^T)$ , where the expectation is taken with respect to whatever information is available to the agent at history  $h_t$ .

**Stochastic process.** In each period  $t$ ,  $\theta_t$  is drawn from a cumulative distribution function  $F_t(\cdot|\theta^{t-1})$ , where  $\theta^{t-1} \equiv (\theta_1, \dots, \theta_{t-1}) \in \Theta^{t-1} \equiv \times_{s=1}^{t-1} \Theta_s$  is the history of past productivity levels. We assume that, for any  $t$  any  $\theta^{t-1}$ ,  $F_t(\cdot|\theta^{t-1})$  is absolutely continuous over  $\mathbb{R}$  and strictly increasing with density  $f_t(\theta_t|\theta^{t-1}) > 0$  over a connected set<sup>8</sup>  $Supp[F_t(\cdot|\theta^{t-1})] \subset \Theta_t$ , and then denote by  $\underline{\theta}_t(\theta^{t-1}) \equiv \inf\{Supp[F_t(\cdot|\theta^{t-1})]\}$  and  $\bar{\theta}_t(\theta^{t-1}) \equiv \sup\{Supp[F_t(\cdot|\theta^{t-1})]\}$  the infimum and supremum of the support. We also assume that, for each  $t$ ,  $F_t(\theta_t|\cdot)$  is differentiable in  $\theta^{t-1}$ .<sup>9</sup> Hereafter, we identify the process governing the evolution of the agent's productivity with the collection of kernels  $F \equiv \langle F_t \rangle_{t=1}^T$ . For each  $t$ , we then let

$$R^t \equiv \left\{ \theta^t \in \Theta^t : \theta_1 \in \Theta_1 \text{ and } \theta_l \in Supp[F_l(\cdot|\theta^{l-1})], \text{ all } l = 2, \dots, t \right\}$$

denote the set of productivity histories that are consistent with the process  $F$ . We then define the impulse responses corresponding to the process  $F$  as follows. For any  $t$  any  $\theta^t \in \Theta^t$ , we let  $J_t^t(\theta^t) \equiv 1$ ,

<sup>7</sup>As is common in the literature, we equate the agent's period- $t$  consumption  $c_t$  with the compensation from the principal. In other words, we assume that the agent cannot secretly save.

<sup>8</sup>The notation  $Supp[F_t(\cdot|\theta^{t-1})]$  stands for the support of the distribution  $F_t(\cdot|\theta^{t-1})$ , defined to be the smallest compact subset of  $\mathbb{R}$  whose complement has probability zero. Note that the kernels  $F_t(\cdot|\theta^{t-1})$  are defined for all possible histories  $\theta^{t-1} \in \Theta^{t-1}$  including those that have zero measure from an ex ante perspective.

<sup>9</sup>Throughout, if  $\theta^{t-1}$  is such that  $\theta_s = \underline{\theta}_s$  for some  $s < t$ , then  $\partial F_t(\theta_t|\theta^{t-1})/\partial \theta_s$  denotes the right-hand derivative of  $F_t$  with respect to  $\theta_s$ . Likewise, if  $\theta_s = \bar{\theta}_s$ , then  $\partial F_t(\theta_t|\theta^{t-1})/\partial \theta_s$  denotes the left-hand derivative.



whereas, for any  $t$  any  $\tau > t$ , any  $\theta^\tau$  such that  $\theta^t \in \Theta^t$  and  $\theta_s \in \text{Supp}[F_s(\cdot|\theta^{s-1})]$  any  $s > t$ , we let

$$J_t^\tau(\theta^\tau) \equiv \sum_{\substack{K \in \mathbb{N}, l \in \mathbb{N}^{K+1}: \\ t=l_0 < \dots < l_K = \tau}} \prod_{k=1}^K I_{l_{k-1}}^{l_k}(\theta^{l_k}),$$

with

$$I_l^m(\theta^m) \equiv -\frac{\partial F_m(\theta_m|\theta^{m-1})/\partial \theta_l}{f_m(\theta_m|\theta^{m-1})}.$$

These  $J$  functions are the nonlinear analogs of the familiar constant linear impulse responses for autoregressive processes. For example, when productivity evolves according to an AR(1) process, i.e., when  $\theta_t = \gamma\theta_{t-1} + \varepsilon_t$ , when  $\gamma$  is a scalar and  $(\tilde{\varepsilon}_s)_{s=1}^T$  is a sequence of independent random variables, the impulse response of  $\theta_t$  on  $\theta_\tau$ ,  $\tau > t$ , is simply  $J_t^\tau = \gamma^{\tau-t}$ . More generally, the impulse response  $J_t^\tau(\theta^\tau)$  of  $\theta_t$  on  $\theta_\tau$  captures the total effect of an infinitesimal variation of  $\theta_t$  on  $\theta_\tau$ , taking into account all effects on intermediate types  $(\theta_{t+1}, \dots, \theta_{\tau-1})$ . In the special case of a Markov process, because each  $I_l^m(\theta^m)$  is equal to zero for all  $l < m - 1$  and depends on  $\theta^m$  only through  $(\theta_m, \theta_{m-1})$ , the impulse response  $J_t^\tau(\theta^\tau)$  reduces to a function of  $(\theta_t, \dots, \theta_\tau)$  and can be conveniently written as  $J_t^\tau(\theta_t, \theta_{t+1}, \dots, \theta_\tau) = \prod_{k=t+1}^\tau I_{k-1}^k(\theta_k, \theta_{k-1})$ , with each  $I_{k-1}^k$  given by

$$I_{k-1}^k(\theta_k, \theta_{k-1}) = \frac{-\partial F_k(\theta_k|\theta_{k-1})/\partial \theta_{k-1}}{f_k(\theta_k|\theta_{k-1})}.$$

As anticipated in the Introduction, these impulse response functions play a key role in determining the dynamics of the profit-maximizing contract. Throughout, we will assume that the process  $F$  satisfies the property of “*first-order stochastic dominance in types*”: for all  $t \geq 2$ ,  $\theta^{t-1} > \theta^{t-1'}$  implies  $F_t(\theta_t|\theta^{t-1}) \leq F_t(\theta_t|\theta^{t-1'})$  for all  $\theta_t$ . This assumption in turn implies that  $J_t^\tau(\theta^\tau) \geq 0$  for any  $t$  and  $\tau$ ,  $\tau \geq t$ , any  $\theta^\tau \in R^\tau$ .

**Technical assumptions.** To validate a certain dynamic envelope theorem (see Pavan, Segal, and Toikka 2009 for details), and guarantee interior solutions, we will also make the following technical assumptions. We will assume that the sets  $\Theta_t$  and the functions  $J_t^\tau(\cdot)$  are uniformly bounded, in the following sense: there exist nonnegative scalars  $B_1, B_2 < +\infty$  such that  $|\theta_t| < B_1$  and  $J_t^\tau(\theta^\tau) < B_2$ , all  $t$  and  $\tau$ ,  $\tau \geq t$ , all  $\theta^\tau \in R^\tau$ .<sup>10</sup> Lastly, we will impose the following conditions on the disutility

<sup>10</sup>Throughout, given any function  $g : \mathbb{R}^T \rightarrow \mathbb{R}$  the notation  $\mathbb{E}^{\tilde{\theta}^T|\theta^t}[g(\tilde{\theta}^T)]$  will denote the expected value of the

function  $\psi$ . Firstly,  $\psi(e) = 0$  for all  $e \leq 0$ . Secondly,  $\psi$  is continuously differentiable over  $\mathbb{R}$ . Thirdly, there exists a scalar  $\bar{e} > 0$  such that (i)  $\psi'(\bar{e}) > 1$ , and (ii)  $\psi$  is thrice continuously differentiable over  $(0, \bar{e})$  with  $\psi''(e) > 0$  and  $\psi'''(e) \geq 0$  for all  $e \in (0, \bar{e})$ . Finally, there exist scalars  $C > 0$  and  $L > 1$  such that  $\psi(e) = Le - C$  for all  $e > \bar{e}$ . These conditions are satisfied, for example, when  $\bar{e} > 1$ ,  $\psi(e) = (1/2)e^2$  for all  $e \in (0, \bar{e})$ , and  $\psi(e) = \bar{e}e - \bar{e}^2/2$  for all  $e > \bar{e}$ . As mentioned above, these conditions guarantee that the agent's payoff satisfies certain equi-Lipschitz continuity conditions which in turn permit one to conveniently express the value function through a differentiable envelope formula.

## 2.2 The mechanism design problem

The principal's problem consists of choosing a mechanism detailing for each period  $t$  a recommendation for the agent's effort along with a level of consumption where both effort and consumption possibly depend on the history of observed cash flows  $\pi^t$  and on messages sent by the agent over time.

By the Revelation Principle, we restrict attention to direct revelation mechanisms for which a *truthful* and *obedient* strategy is optimal for each type  $\theta_1$  of the agent. Hereafter, we refer to any such mechanism as *incentive compatible*.<sup>11</sup> Letting  $\Pi^t = \times_{\tau=1}^t \Pi_\tau$ , with  $\Pi_\tau = \{\pi_\tau \in \mathbb{R} : \pi_\tau = \theta_\tau + e, \theta_\tau \in \Theta_\tau, e \in E\}$  denoting the set of possible period- $t$  feasible cash flows, a direct mechanism  $\Omega = \langle \xi_t, s_t \rangle_{t=1}^T$  thus consists of a collection of functions  $\xi_t : \Theta^t \times \Pi^{t-1} \rightarrow E$  and  $s_t : \Theta^t \times \Pi^t \rightarrow \mathbb{R}$  such that  $\xi_t(\theta^t, \pi^{t-1})$  is the recommended level of effort for period  $t$  given the agent's reports  $\theta^t$  and the observed history of cash flows  $\pi^{t-1}$ , while  $s_t(\theta^t, \pi^{t-1}, \pi_t)$  is the principal's payment (equivalently, the agent's consumption) at the end of period  $t$  given the reports  $\theta^t$  and the cash flows  $\pi^t = (\pi^{t-1}, \pi_t)$ . Note that  $s_t(\theta^t, \pi^{t-1}, \pi_t)$  depends on both past and current cash flows. With a slight abuse of notation,

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function  $g(\tilde{\theta}^T)$  given the unique probability measure over  $\Theta^T$  that corresponds to the kernels  $F$  starting from history  $\theta^t$ .

<sup>11</sup>A truthful and obedient strategy prescribes that the agent truthfully reports his private information and follows the principal's effort recommendation at each history. Such a strategy is said to be optimal for type  $\theta_1$  if it maximizes type  $\theta_1$ 's expected payoff among all possible (measurable) strategies. By the principle of optimality, in any incentive-compatible mechanism, with  $F$ -probability one, truthful and obedient behavior must remain optimal also at future histories that are consistent with truthful and obedient behavior in the past.

for any  $\theta^t \in R^t$ , we then denote by  $e_t(\theta^t) \equiv \xi(\theta^t, \pi^t(\theta^t))$  the equilibrium effort choice for period  $t$  and by  $c_t(\theta^t) = s_t(\theta^t, \pi^t(\theta^t))$  the equilibrium consumption level for period  $t$ , given  $\theta^t$ , where  $\pi^t(\theta^t) = (\pi_s(\theta^s))_{s=1}^t$  with  $\pi_s(\theta^s)$  defined recursively by  $\pi_s(\theta^s) = \theta_s + e_s(\theta^s)$  all  $s \leq t$ .

The timing of play in each period  $t$  is the following:

- At the beginning of period  $t$ , the agent learns  $\theta_t \in \Theta_t$ ;
- The agent then sends a report  $\hat{\theta}_t \in \Theta_t$ ;
- The mechanism reacts by prescribing an effort choice  $\xi_t(\hat{\theta}_t, \pi^{t-1})$  and a contingent reward scheme  $s_t(\hat{\theta}_t, \pi^{t-1}, \cdot) : \Pi_t \rightarrow \mathbb{R}$ ;
- The agent privately chooses effort  $e_t$ , the cash flows  $\pi_t = \theta_t + e_t$  are then observed, and the compensation  $c_t = s_t(\theta^t, \pi^t)$  is paid and consumed.

The principal offers the mechanism  $\Omega$  at date 1, after the agent has observed the first realization  $\theta_1$  of the process  $F$ .<sup>12</sup> If the agent refuses to participate in the mechanism  $\Omega$ , both the agent and the principal receive their outside options, which we assume to be equal to zero. If, instead, the agent accepts  $\Omega$ , he then stays in the relationship in all subsequent periods.<sup>13</sup> In the terminology of mechanism design, we refer to a mechanism for which participation is optimal for each type  $\theta_1$  as *individually rational*.

**Definition 1** *We say that the compensation scheme  $s \equiv \langle s_t(\cdot) \rangle_{t=1}^{t=T}$  implements the effort policy  $e \equiv \langle e_t(\cdot) \rangle_{t=1}^{t=T}$  if any mechanism  $\Omega = \langle \xi_t, s_t \rangle_{t=1}^T$  in which the compensation scheme is  $s$  and the recommendation policy  $\xi$  satisfies  $\xi_t(\theta^t, \pi^{t-1}) = e_t(\theta^t)$  for any  $(\theta^t, \pi^{t-1})$  such that  $\theta^t \in R^t$  and  $\pi_s = \theta_s + e_s(\theta^s)$  all  $s \leq t-1$ , is incentive-compatible and individually-rational for the agent. We say that the policy  $e$  is implementable if there exists a compensation scheme that implements it.*

<sup>12</sup>As explained above, this assumption implies that the principal cannot simply sell out the firm to the agent even in the absence of limited liability constraints and in the presence of risk neutrality.

<sup>13</sup>That participation must be guaranteed only in period one is clearly not restrictive when the principal can ask the agent to post bonds. Below, we will discuss situations/implementations where, even in the absence of bonding, participation can be guaranteed after any history consistent with truthful and obedient play in past periods.

### 3 Cost-minimizing compensation schemes

We start by characterizing a compensation scheme that implements all implementable effort policies at minimal cost for the principal.

**Proposition 1** *Suppose that the effort policy  $e$  is implementable. Let  $c^{opt}(\cdot; e) \equiv \left\langle c_t^{opt}(\cdot; e) \right\rangle_{t=1}^{t=T}$  denote the ( $R$ -measurable) solution to the following problem:*

$$\begin{aligned} & \min_c \mathbb{E}^{\tilde{\theta}^T} \left[ \sum_{t=1}^T \delta^{t-1} c_t(\tilde{\theta}^t) \right] \\ & \text{s.t. } \mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t(c_t(\theta^t)) \right) = W(\theta^T) \text{ for } F\text{-almost all } \theta^T \end{aligned} \quad (2)$$

where, for each  $\theta^T \in R^T$ ,

$$W(\theta^T) \equiv \sum_{t=1}^T \delta^{t-1} \psi(e_t(\theta^t)) + \int_{\theta_1}^{\theta_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(e_t(\tilde{\theta}^t)) \right] ds + \sum_{t=2}^T \delta^{t-1} H_t(\theta^t)$$

with

$$\begin{aligned} H_t(\theta^t) & \equiv \int_{\theta_t}^{\theta_t} \mathbb{E}^{\tilde{\theta}^T | \theta^{t-1}, s} \left[ \sum_{\tau=t}^T \delta^{\tau-t} J_\tau^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds \\ & \quad - \mathbb{E}^{\tilde{\theta}_t | \theta^{t-1}} \left[ \int_{\tilde{\theta}_t}^{\tilde{\theta}_t} \mathbb{E}^{\tilde{\theta}^T | \theta^{t-1}, s} \left[ \sum_{\tau=t}^T \delta^{\tau-t} J_\tau^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds \right] \end{aligned}$$

for all  $t \geq 2$ , all  $\theta^t \in R^t$ . Consider the compensation scheme  $s$  defined as follows: in each period  $t$ , given the reports  $\theta^t \in \Theta^t$  and the observed cash flows  $\pi^t \in \mathbb{R}^t$ , the principal pays the agent a compensation  $s_t(\theta^t, \pi^t) = c_t^{opt}(\theta^t; e)$  if  $\theta^t \in R^t$  and  $\pi_t \geq \pi_t(\theta^t) \equiv \theta_t + e_t(\theta^t)$  and charges the agent a sufficiently low penalty otherwise. The compensation scheme  $s$  described above implements the policy  $e$  at minimum cost for the principal.

By definition, if a mechanism  $\Omega$  exists which implements the policy  $e$ , then at any period  $t$  and for  $F$ -almost all truthful histories<sup>14</sup>  $\theta^t$ , the agent's expected payoff in  $\Omega$  under a truthful and obedient strategy must coincide with the value function  $V^\Omega(\theta^t)$  defined to be the supremum of the agent's expected payoff in  $\Omega$ , given the truthful history  $\theta^t$ , among all possible strategies. The proof

<sup>14</sup>A truthful history is one that is reached by reporting truthfully and following the principal's effort recommendations in each previous period. Without loss, these histories can be denoted by the realized sequence of productivities  $\theta^t$ .

in the Appendix then uses the envelope theorem for dynamic stochastic problems of Pavan, Segal and Toikka (2009) to establish that, after any truthful history  $\theta^{t-1}$ ,  $V^\Omega(\theta^{t-1}, \cdot)$  must be Lipschitz continuous in  $\theta_t$  and satisfy the following envelope condition

$$\frac{\partial V^\Omega(\theta^t)}{\partial \theta_t} = \mathbb{E}^{\tilde{\theta}^T | \theta^t} \left[ \sum_{\tau=t}^T \delta^{\tau-1} J_\tau^\top(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] \quad (3)$$

for  $F$ -almost every truthful history  $\theta^t$ . Using the fact that, in any incentive-compatible mechanism  $\Omega$ , at  $F$ -almost every truthful history  $\theta^t$

$$V^\Omega(\theta^t) = \mathbb{E}^{\tilde{\theta}_{t+1} | \theta^t} \left[ V^\Omega(\theta^t, \tilde{\theta}_{t+1}) \right]$$

and iterating across periods then permits us to establish that, for  $F$ -almost all histories  $\theta^T$ , the agent's utility from the compensation he receives from the principal must satisfy

$$\mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t(c_t(\theta^t)) \right) = W(\theta^T) + V^\Omega(\underline{\theta}_1) \quad (4)$$

with  $V^\Omega(\underline{\theta}_1)$  nonnegative so as to guarantee participation by all period-1 types. This means that the utility that the agent derives from his compensation is *essentially* uniquely determined by the effort policy  $e$ , up to a nonnegative constant.

Given the property above, if the principal replaces the original mechanism  $\Omega$  implementing the policy  $e$  with one where the compensation is as in the proposition then (i) the agent continues to have the right incentives to report truthfully and follow the principal's effort recommendations (this step follows from replication arguments similar to those that establish the Revelation Principle, adapted to the environment considered here) and (ii) the cost to the principal is no greater than under the original scheme  $\Omega$ , thus establishing the result.

The value of Proposition 1 is twofold: it characterizes the cost to the principal of sustaining any implementable policy  $e$ , and it shows that all implementable effort policies can be sustained by offering the agent a compensation scheme that, in each period  $t$ , pays a *bonus* conditional upon the period- $t$  cash flows exceeding a target that depends on the history of reports about the manager's productivity.

A necessary condition for  $c^{opt}(\cdot; e)$  to solve program (2) in the Proposition is that it satisfies the inverse Euler equation in Corollary 1 below. (The proof in the Appendix adapts arguments similar to those in Rogerson (1985) to the present environment.<sup>15</sup>)

**Corollary 1** *Given any policy  $e$ , the compensation  $c^{opt}(\cdot; e)$  solves program (2) in Proposition 1 only if, with  $F$ -probability one, the following condition holds for any two adjacent periods  $1 \leq t, t+1 \leq T$ :*

$$\frac{1}{v'_t(c_t(\theta^t))} = \mathbb{E}^{\tilde{\theta}_{t+1}|\theta^t} \left[ \frac{\delta}{v'_{t+1}(c_{t+1}(\theta^t, \tilde{\theta}_{t+1}))} \right]. \quad (5)$$

## 4 Implementable policies

Building on the result in the previous section we now provide conditions for a given effort policy to be implementable. We start with the following definition.

**Definition 2** *We say that the recommendation policy  $\xi$  is an extension of the effort policy  $e$  from  $R$  to  $\Theta$  if (i) for any  $t$ ,  $\xi_t(\theta^t, \pi^{t-1})$  is independent of past cash flows  $\pi^{t-1}$  (and hence denoted by  $\xi_t(\theta^t)$ ), and (ii) for any  $t$  any  $\theta^t \in R^t$ ,  $\xi_t(\theta^t) = e_t(\theta^t)$ .*

An extension is thus a recommendation policy that is insensitive to the observed cash flows and that recommends the equilibrium effort whenever the agent, with his reports, does not reveal any departure from truthtelling in the past. We then have the following result.

**Proposition 2** *Suppose that there exists an extension  $\xi$  of the policy  $e$  that satisfies the following single-crossing conditions*

$$\mathbb{E}^{\tilde{\theta}^T|\theta^t} \left[ \begin{array}{l} \psi'(\xi_t(\hat{\theta}^{t-1}, \theta_t)) - \psi'(\xi_t(\hat{\theta}^{t-1}, \hat{\theta}_t) + \hat{\theta}_t - \theta_t) \\ + \sum_{\tau=t+1}^T \delta^{\tau-t} J_\tau^r(\theta^t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau) \\ \times [\psi'(\xi_\tau(\hat{\theta}^{t-1}, \theta_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)) - \psi'(\xi_\tau(\hat{\theta}^{t-1}, \hat{\theta}_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau))] \end{array} \right] [\theta_t - \hat{\theta}_t] \geq 0. \quad (6)$$

for any  $t$ , any  $\theta^t \in R^t$ , any  $\hat{\theta}^t \in \Theta^t$ . In addition, assume that one of the following conditions holds: (i) the process  $F$  is Markov; (ii) each  $\xi_t(\theta^t)$  depends on  $\theta^t$  only through  $\theta_1$ . In each of these cases, the policy  $e$  is implementable.

<sup>15</sup>Note that Condition (5) holds irrespective of whether the agent's preferences are time-additively-separable.

Note that the single-crossing conditions in the propositions are trivially satisfied for example when, in each period, the effort that the firm recommends is nondecreasing in the history of reported productivity. More generally, these conditions are weaker monotonicity conditions requiring only that, in the continuation game starting with an arbitrary history  $(\theta^t, \hat{\theta}^{t-1}, e^{t-1})$ , effort increases, on average, with the agent's report about his current productivity.

To see the idea behind the result in the proposition, consider first the case where the process is Markov. In this case, whether or not the agent has been truthful in the past is irrelevant for the incentives he faces in the continuation game that starts after any history  $(\theta^t, \hat{\theta}^{t-1}, e^{t-1})$ . Suppose, then, that the principal offers a scheme of the type described in Proposition 1; that is, in each period  $t$ , given the reports  $\hat{\theta}^t \in \Theta^t$ , the principal pays a bonus  $c_t(\hat{\theta}^t)$  if the current cash flow exceeds the target  $\pi_t = \hat{\theta}_t + \xi_t(\hat{\theta}^t)$  and charges the agent a sufficiently low penalty otherwise. Faced with this scheme, the effort that the agent exerts in each period  $t$  is determined entirely by the history of reports  $\hat{\theta}^t$  made over time, and by the agent's current type  $\theta_t$  and is given by  $\hat{e}_t = \xi_t(\hat{\theta}^t) + \hat{\theta}_t - \theta_t$ , all  $t$ . Then let  $U(\theta^t; \hat{\theta}^{t-1}, \hat{\theta}_t)$  denote the agent's expected payoff in the continuation game that starts in period  $t$  with history  $(\theta^t, \hat{\theta}^{t-1})$  when, in period  $t$ , the agent sends the report  $\hat{\theta}_t$ , he then chooses effort  $\hat{e}_t = \xi_t(\hat{\theta}^t) + \hat{\theta}_t - \theta_t$ , and then starting from period  $t + 1$  onward he follows a truthful and obedient strategy. The proof in the Appendix verifies that, when the single-crossing conditions in the proposition are satisfied, there exists a compensation scheme of the type described above such that

$$\left[ \frac{dU(\theta^t; \hat{\theta}^{t-1}, \theta_t)}{d\theta_t} - \frac{\partial U(\theta^t; \hat{\theta}^{t-1}, \hat{\theta}_t)}{\partial \theta_t} \right] [\theta_t - \hat{\theta}_t] \geq 0,$$

all  $\theta^t \in R^t$ ,  $\hat{\theta}^t \in \Theta^t$ . As is well known, this endogenous single-crossing condition on derivatives implies that  $U(\theta^t; \hat{\theta}^{t-1}, \theta_t) \geq U(\theta^t; \hat{\theta}^{t-1}, \hat{\theta}_t)$  all  $\theta^t \in R^t$ ,  $\hat{\theta}^t \in \Theta^t$ , meaning that no one-shot deviation from the truthful and obedient strategy is ever profitable for the agent. Together with continuity at infinity, this implies that, under the proposed scheme, following a truthful and obedient strategy is optimal for each type  $\theta_1$ , thus establishing the implementability of the proposed effort policy.<sup>16</sup>

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<sup>16</sup>Note that the scheme described above, contrary to that in Proposition 1, does not constrain the agent to report only feasible sequences of types  $\theta^T \in R$ . Allowing the agent to report also types that are inconsistent with the process  $F$  is just a trick that permits us to establish the suboptimality of one-stage deviations from the truthful and obedient

Next consider the case where the process is possibly non-Markov, but where the effort policy depends only on  $\theta_1$ . The proof in the Appendix shows that one can construct compensation schemes in which the agent is asked to report only his initial type and such that the utility that he derives from the payments he receives over time is linear in the cash flows  $\pi_t$ ,  $t > 1$ . The idea here is the following. First, because the policy depends only on  $\theta_1$ , the agent does not need to report anything to the principal after the first period. This eliminates the complication stemming from the fact that, with non-Markov processes, verifying the agent's incentives for truthtelling is typically hard due to the fact that his continuation payoff may depend on the entire history of past types. Second, by using a scheme that makes the agent's payoff, gross of his disutility of effort, linear in the cash flows  $\pi_t$ ,  $t > 1$ , the principal can control the agent's incentives for effort without the need to set bonuses that depend on past types (i.e., without communicating with the agent after the first period). It suffices to set the sensitivity of the the payoff to each cash flow  $\pi_t$ ,  $t > 1$ , equal to  $\psi'(\xi_t(\theta_1))$ . It is then easy to see that, conditional on having reported truthfully in the first period, and irrespective of the effort exerted in the preceding periods, at any period  $t > 1$ , the agent finds it optimal to choose a level of effort  $e_t = \xi_t(\theta_1)$  for which the marginal disutility is equal to the marginal increase in compensation. The only difficulty for the principal is then to induce the agent to report truthfully in the first period. When the effort policy satisfies the single-crossing condition in the proposition<sup>17</sup>, this can be achieved by committing to pay, in addition to the compensation described above, a bonus in period one conditional upon the period-1 cash flow exceeding a target  $\pi_1 = \theta_1 + \xi_1(\theta_1)$  that depends only on the first period announcement. It is important to recognize that, while the schemes described above do facilitate the verification of the implementability of a policy, they are not essential for its sustainability: any effort policy that can be sustained with a scheme of the type described above can also be sustained with a bonus scheme of the type described in Proposition 1, at a weakly lower cost for the principal.

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strategy after *any* history, including those that entailed a deviation in the past. Together with continuity a infinity, this in turn establishes the optimality of the truthful and obedient strategy, and hence the implementability of the desired effort policy.

<sup>17</sup>Note that, when the policy depends only on  $\theta_1$ , the only relevant single-crossing condition is the one for  $t = 1$ .



Lastly, note that the case where the effort policy is independent of the agent's reports after the first period is of particular interest. As we will shown below, the optimal policy takes this form for example when the agent is risk neutral and the process for the agent's productivity is separable in the first component, in the following sense.

**Definition 3** *The process  $F$  is separable in the first component (SFC) if each impulse response function  $J_1^t(\theta^t)$  depends on  $\theta^t$  only through  $\theta_1$ , which in turn is the case when the process  $F$  admits the following representation*

$$\theta_t = \kappa_t(\theta_1) + \phi_t(\varepsilon_2, \dots, \varepsilon_t)$$

for some functions  $\kappa_t : \Theta_1 \rightarrow \mathbb{R}$  and  $\phi_t : \mathcal{E}^t \rightarrow \mathbb{R}$ , with each shock  $\varepsilon_t$  drawn independently from a distribution  $G_t$  with support  $\mathcal{E}_t \subset \mathbb{R}$ .

Note that the family of processes that satisfy the SFC property includes all ARIMA processes: in this case, each impulse response  $J_1^t$  is simply a scalar.

#### 4.1 Implementability (at minimum cost) with linear and pseudo-linear schemes

Before turning to the properties of optimal policies, we conclude this section with a result concerning the possibility of implementing certain effort policies with linear and pseudo-linear schemes. By the former, we mean compensation schemes in which the total compensation that the agent receives over time is linear in the cash flows. By the latter, we mean compensation schemes in which the *utility* that the agent derives from his compensation is linear in the cash flows.<sup>18</sup> The reason for looking at these schemes is twofold. Many compensation packages used in the real world based on a time-varying combination of stocks and options are known to give rise to linear sensitivities of compensation to firm's performance, as measured for example by cash flows and stock prices. Second, such schemes provide a desirable form of robustness. They guarantee that the agent finds it optimal to report truthfully and choose the desired level of effort even if he is concerned about the possibility that the

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<sup>18</sup>The discussion after Proposition 2 mentioned already some of the advantages of linear and pseudo-linear schemes. However, there we restricted attention to policies that depend only on  $\theta_1$ . Furthermore, we did not address the question of what conditions guarantee that such schemes implement the desired effort policies at minimal cost for the principal.

firm's cash flow are not only a function of his productivity and effort but also of noise that is beyond his control. In other words, such schemes continue to implement the desired effort policy even if the agent believes the cash flows to be given by  $\pi_t = \theta_t + e_t + \eta_t$  where  $\eta_t$  is transitory noise, independent of the process  $F$  and of the agent's effort decisions.

Motivated by the aforementioned considerations, we then have the following result.

**Proposition 3** *Let  $\xi$  be any extension of the policy  $e$  satisfying the following single-crossing conditions*

$$\mathbb{E}^{\hat{\theta}^T | \theta^t} \left[ \begin{array}{l} \sum_{\tau=t}^T \delta^{\tau-t} J_t^T(\theta^t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau) \\ \times [\psi'(\xi_\tau(\hat{\theta}^{t-1}, \theta_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)) - \psi'(\xi_\tau(\hat{\theta}^{t-1}, \hat{\theta}_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau))] \end{array} \right] [\theta_t - \hat{\theta}_t] \geq 0 \quad (7)$$

for any  $t$ , any  $\theta^t \in R^t$ , any  $\hat{\theta}^t \in \Theta^t$ . In addition, assume that one of the following conditions holds: (i) the process  $F$  is Markov; (ii) each  $\xi_t(\theta^t)$  depends on  $\theta^t$  only through  $\theta_1$ . In each of these cases, the policy  $e$  can be implemented at minimal cost for the principal with an appropriately designed pseudo-linear scheme.

First note that the single crossing conditions in the proposition are stronger than those in Proposition 2. These schemes thus permit the principal to implement fewer effort policies than the corresponding bonus schemes of Proposition 2. Nonetheless, there are interesting cases where the optimal effort policies can indeed be implemented with such schemes, as will be shown below.

Next note that these schemes are designed so as to achieve multiple goals at once. First, they undo the agent's risk-aversion by making his payoff, gross of the cost of effort, linear in the cash-flows  $\pi_t$ ,  $t \geq 1$ . By setting the sensitivity to each cash flow equal to  $\alpha_t(\theta^t) = \psi'(\xi_t(\theta^t))$  such linearity guarantees that, conditional on having reported truthfully, and irrespective of past effort choices, the agent finds it optimal to exert the right level of effort, thus taking care of the moral-hazard part of the problem. Second, these schemes are designed so as to induce truth-telling. This is obtained through a careful design of the part of the compensation that is independent of the realized cash flow but which depends on the reported productivity. This takes care of the adverse selection part of the problem. Lastly, these schemes are designed so that the compensation the agent receives over time is

the one that minimizes the cost to the principal. This is obtained by distributing the compensation over time according to Rogerson inverse Euler condition, as defined in Corollary 1, thus taking care of the optimal consumption smoothing part of the problem.

To illustrate, consider for simplicity the case where  $\mathcal{V}$  is linear but where each  $v_t$  is possibly strictly concave, thus maintaining a preference for consumption smoothing, while making the payoff time-additively separable. Let  $\xi$  be any extension of the desired effort policy  $e$  from  $R$  to  $\Theta$ . Let  $c^{opt}(\cdot; \xi)$  be the cost-minimizing allocation of consumption over time, as defined in the program of Proposition 1, with all policies and corresponding constraints naturally extended from  $R$  to  $\Theta$ . Then consider the following pseudo-linear payment scheme  $s^{PL}$ . In each period  $t$ , and for all  $(\theta^t, \pi^t)$ , the principal pays the agent a compensation

$$s_t^{PL}(\theta^t, \pi^t) = v_t^{-1} \left( v_t \left( c_t^{opt}(\theta^t; \xi) \right) + \psi'(\xi_t(\theta^t))[\pi_t - \theta_t - \xi_t(\theta^t)] \right). \quad (8)$$

It is easy to see that, under such a scheme, in each period  $t$ , conditional on having reported truthfully, the agent has the right incentives for choosing the desired level of effort  $\xi_t(\theta^t)$ . It is also easy to see that, on path, the compensation that the agent receives over time is given by  $s_t^{PL}(\theta^t, \pi^t) = c_t^{opt}(\theta^t; \xi)$  which is exactly the compensation that minimizes the cost to the principal among all those that are consistent with the agent's incentive compatibility, as indicated in Proposition 1. As shown in the proof in the Appendix, that the effort policy satisfies the single crossing conditions in the proposition in turn guarantees that, when either the process is Markov, or the policy depends only on  $\theta_1$ , (i) the agent has the right incentives to report all his types truthfully, and (ii) each type  $\theta_1$  has the incentive to participate, thus establishing implementability of the desired effort policy.

Next, consider the special case where the agent is risk neutral (both  $\mathcal{V}$  and each  $v_t$  linear). It is worthwhile noticing that, while in this case the particular way the compensation is distributed over time is irrelevant, certain choices have the advantage of guaranteeing that, even if the agent had the option to leave the relationship after the first period, he would never find it optimal to do so. To see this, consider for example the case where  $F$  is Markov,  $R = \Theta$ , and  $T$  is finite. By committing

to pay the agent in each period a compensation

$$s_t(\theta^t, \pi^t) = S_t(\theta^t) + \alpha_t(\theta^t)\pi_t \quad (9)$$

with

$$S_t(\theta^t) \equiv \psi(\xi_t(\theta^t)) - \alpha_t(\theta^t) [\theta_t + \xi_t(\theta^t)] + H_t(\theta^t)$$

if  $t > 1$  and

$$S_1(\theta_1) \equiv \psi(\xi_1(\theta_1)) - \alpha_1(\theta_1) [\theta_1 + \xi_1(\theta_1)] + \int_{\theta_1}^{\theta_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(\xi_t(\tilde{\theta}^t)) \right] ds$$

if  $t = 1$ , the principal guarantees that, after each period  $t$ , the agent's continuation payoff under a truthful and obedient strategy from period  $t$  onwards is equal to

$$\int_{\theta_t}^{\theta_t} \mathbb{E}^{\tilde{\theta}_{>t}^T | s} \left[ \sum_{\tau=t}^T \delta^{\tau-1} J_t^\tau(s, \tilde{\theta}_{>t}^\tau) \psi'(\xi_\tau(\hat{\theta}^{t-1}, s, \tilde{\theta}_{>t}^\tau)) \right] ds$$

irrespective of whether or not the agent has been truthful and obedient in the past. Because the latter is clearly positive, this implies that the agent never finds it optimal to leave the firm.

When the agent is risk-averse, the pseudo-linear schemes give the agent a payoff which is linear in the cash flows but typically entail a non-linear compensation. However, when the single crossing conditions of Proposition 3 hold, it may be possible to implement the desired effort policy with a payment scheme where the compensation itself is linear in the cash flows. To illustrate, consider again the case where  $\mathcal{V}$  is linear but where each  $v_t$  is strictly concave and let  $w_t \equiv v_t^{-1}$ . Then let  $s^L$  be the linear scheme defined for all  $t$ , all  $(\theta^t, \pi^t)$ , by

$$s_t^L(\theta^t, \pi^t) \equiv c_t^{opt}(\theta^t; \xi) + w_t' \left( v_t \left( c_t^{opt}(\theta^t; \xi) \right) \right) \psi'(\xi_t(\theta^t)) [\pi_t - \theta_t - \xi_t(\theta^t)]$$

By the convexity of each  $w_t$  and Jensen's inequality, we then have that

$$s_t^L(\theta^t, \pi^t) \leq s_t^{PL}(\theta^t, \pi^t),$$

with equality if and only if  $\pi_t = \theta_t + \xi_t(\theta^t)$ , where  $s^{PL}$  is the pseudo-linear scheme defined in (8). Faced with the above linear scheme, the agent's payoff under a truthful and obedient strategy is thus the same as when he faces the corresponding pseudo-linear scheme, whereas his payoff under any

other plan is weakly lower. This means that, whenever the desired effort policy is implementable at minimal cost with a pseudo-linear schemes it is also implementable at minimal cost with a linear scheme. Given the popularity of linear schemes in the real world and in the managerial compensation literature, the above result is somewhat reassuring.

## 5 Profit-maximizing policies

### 5.1 Risk-neutrality

Equipped with the results in the previous section, we now characterize the policies that maximize the firm's expected cash flows, net of the cost of managerial compensation. We start by characterizing optimal policies for risk-neutral agents. Recall that, by definition, in any mechanism that is incentive-compatible for the agent, the payoff that each type  $\theta_1$  expects under a truthful and obedient strategy must coincide with the value function. Applying (3) to  $t = 1$ , the latter in turn must satisfy

$$V^\Omega(\theta_1) = \int_{\underline{\theta}_1}^{\theta_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(s, \tilde{\theta}_{>1}^t) \psi'(e_t(s, \tilde{\theta}_{>1}^t)) \right] ds + V^\Omega(\underline{\theta}_1) \quad (10)$$

for all  $\theta_1$ . Integrating (10) by parts, we then have that the expected surplus that the principal must leave to the agent, as computed at the time of hiring, is given by

$$\mathbb{E} \left[ V^\Omega(\tilde{\theta}_1) \right] = \mathbb{E}^{\tilde{\theta}^T} \left[ \eta(\tilde{\theta}_1) \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(e_t(\tilde{\theta}^t)) \right] + V^\Omega(\underline{\theta}_1)$$

where  $\eta(\theta_1) \equiv [1 - F_1(\theta_1)]/f_1(\theta_1)$  denotes the inverse hazard rate of the first-period distribution.

Using the fact that the principal's expected payoff is equal to

$$\mathbb{E}[U^P] = \mathbb{E} \left[ \sum_{t=1}^T \delta^{t-1} \{ \tilde{\theta}_t + e_t(\tilde{\theta}^t) - \psi(e_t(\tilde{\theta}^t)) \} \right] - \mathbb{E} \left[ V^\Omega(\tilde{\theta}_1) \right]$$

we then have that

$$\mathbb{E}[U^P] = \mathbb{E} \left[ \sum_{t=1}^T \delta^{t-1} \left\{ \tilde{\theta}_t + e_t(\tilde{\theta}^t) - \psi(e_t(\tilde{\theta}^t)) - \eta(\tilde{\theta}_1) J_1^t(\tilde{\theta}^t) \psi'(e_t(\tilde{\theta}^t)) \right\} \right] - V^\Omega(\underline{\theta}_1). \quad (11)$$

Given that  $J_1^t(\theta^t) \geq 0$  for all  $t$  all  $\theta^t$  under FOSD, we then have that the effort policy that maximizes (11) can be obtained by pointwise maximization of  $\mathbb{E}[U^P]$  and is given in the next proposition.

**Proposition 4** *Suppose that the agent is risk neutral. (i) Let  $e^*$  be the effort policy implicitly defined, for all  $t$  all  $\theta^t \in R^t$ , by<sup>19</sup>*

$$\psi'(e_t^*(\theta^t)) = 1 - \eta(\theta_1)J_1^t(\theta^t) \psi''(e_t^*(\theta^t)). \quad (12)$$

*Suppose that the policy  $e^*$  is implementable. Then, under any optimal contract for the principal,  $e^*$  is sustained in each period at  $F$ -almost all histories.*

*(ii) The following conditions guarantee that the policy  $e^*$  is implementable: (a) the process  $F$  is either Markov, or separable in the first component (SFC); and (b) each function  $\eta(\theta_1)J_1^t(\theta^t)$  is nonincreasing. Under these conditions, the policy  $e^*$  can also be implemented by a linear schemes.<sup>20</sup> Furthermore, by committing to pay the agent in each period  $t$  according to (9) the principal guarantees that the agent never finds it optimal to leave the relationship.*

First, consider part (i). It is easy to see that the policy  $e^*$  maximizes (11). That, when it is implementable,  $e^*$  is sustained under any optimal contract for the principal then follows from the fact that the principal's payoff in any mechanism that is incentive-compatible for the agent is given by (11), together with the fact that, from Proposition 1, there exists a compensation scheme  $s^*$  that implements  $e^*$  and that gives the lowest period-1 type  $\underline{\theta}_1$  an expected payoff equal to his outside option, in which case  $V^\Omega(\underline{\theta}_1) = 0$ .

Next consider part (ii). The result follows from Propositions 2 and 3. When each function  $\eta(\theta_1)J_1^t(\theta^t)$  is nonincreasing, then each function  $e_t^*(\theta^t)$  is nondecreasing. In the Appendix, we show that, when this is the case, there always exists an extension  $\xi$  of  $e^*$  from  $R$  to  $\Theta$  that satisfies not only the single-crossing conditions (6) of Proposition 2 but also the more stringent conditions (7) of Proposition 3. The results in those propositions then imply that when  $F$  is Markov, or it is separable in the first component (SFC)—in which case each  $e_t^*(\theta^t)$  and hence also each corresponding  $\xi_t(\theta^t)$  depends on  $\theta^t$  only through  $\theta_1$ —then the policy  $e^*$  can be implemented both by the bonus schemes of Proposition 1 and by the linear schemes described in (9). That, in this case, participation can

<sup>19</sup>The formula in (12) presumes that the effort  $e^*(\theta^t)$  implicitly defined by (12) is strictly positive, which is the case if and only if  $\psi''(0) < 1/[\eta(\theta_1)J_1^t(\theta^t)]$ . When this is not the case then  $e_t^*(\theta^t) = 0$ .

<sup>20</sup>The extension  $\xi$  of the policy  $e^*$  from  $R$  to  $\Theta$  is given in the Appendix.

be guaranteed not only in the first period but also in any subsequent period then follows from what discussed after Proposition 3.

Note that the assumption that each function  $\eta(\theta_1)J_1^t(\theta^t)$  is nondecreasing in  $\theta^t$  is trivially satisfied, for example, when the density of the first period distribution  $F_1$  is log-concave and the process  $F$  is ARIMA. As anticipated above, in this case, the impulse response functions  $J_1^t$  are scalars and the dynamics of effort under the optimal contract is entirely deterministic. The implementation of the optimal contract is then particularly simple. In period one, the principal offers the agent a menu of contracts, indexed by  $\theta_1$ . Each contract specifies for each period  $t$  a fixed payment  $S_t(\theta_1)$ , given by (9), along with a variable pay  $\alpha_t(\theta_1)\pi_t$  that is linear in the observed cash flow  $\pi_t$ . Both the fixed component  $S_t(\theta_1)$  and the variable component  $\alpha_t(\theta_1)$  vary over time, but in the case of a process that is separable in the first component, do not depend on the productivity shocks experienced after the first period.

A few more observations are in order. First, note that, when the optimal effort policy is the one in Proposition 4, then in each period  $t$ , the agent's effort  $e_t^*(\theta^t)$  is downward distorted with respect to the first-best level, which is implicitly defined by  $\psi'(e^{FB}) = 1$  for all  $t$  all  $\theta^t$ . This property originates in the principal's desire to limit the surplus (equivalently, the rent) that she must leave to the agent to induce him to truthfully reveal his productivity level. By offering to those types who are less productive at the hiring stage (for which  $\eta(\theta_1)$  is high) a contract with low-powered incentives, the principal makes it less attractive for the most productive types to mimic. As anticipated in the Introduction, this is because the surplus that a more productive type obtains by mimicking a less productive one is the disutility of effort that he can save by generating the same cash flows as the less productive type by working less. This means that the lower the effort that the least productive types are induced to exert, the lower the surplus that the principal must leave to the most productive ones.

Second, note that the dynamics of effort under the policy  $e^*$  is entirely driven by the dynamics of the impulse responses  $J_1^t(\theta^t)$ . The reason is that these functions capture the extent to which the agent's initial private information  $\theta_1$  has persistent effects on the surplus that the principal must

leave to the most productive types to induce them to reveal their private information. To illustrate, consider the following examples.

**Example 1** Suppose that  $\theta_t$  evolves according to an AR(1) process

$$\theta_t = \gamma\theta_{t-1} + \varepsilon_t$$

for some  $\gamma \in (0, 1)$ . Then  $J_1^t(\theta^t) = \gamma^{t-1}$  for all  $t$  all  $\theta^t$ . It follows that  $e_t^*$  increases over time and, for any  $\theta_1$ ,

$$\lim_{t \rightarrow \infty} e_t^*(\theta_1) = e^{FB}.$$

**Example 2** Assume that each  $\theta_t$  is i.i.d., so that  $J_1^t(\theta^t) = 0$  for all  $t \geq 2$  all  $\theta^t$ . Then effort is distorted only in the first period, i.e.  $e_1^*(\theta_1) < e_1^{FB}$  and  $e_t^*(\theta^t) = e^{FB}$  for all  $t \geq 2$  all  $\theta^t$ .

**Example 3** Suppose that  $\theta_t$  follows a random walk, i.e.

$$\theta_t = \theta_{t-1} + \varepsilon_t$$

Then  $e_t^*(\theta^t)$  is constant over time and depends only on  $\theta_1$ .

The property that effort increases over time and gradually converges to the first-best level under the AR(1) process of Example 1 is actually more general; it also applies for example to any ARIMA processes for which  $J_1^t$  decreases with  $t$  with  $\lim_{t \rightarrow \infty} J_1^t = 0$ , where  $J_1^t$  are nonnegative scalars that depend on the various parameters of the ARIMA process.

Note that, as the examples make clear, the statistical property of the process  $F$  that is responsible for the dynamics of effort is the dynamics of the impulse responses  $J_1^t(\theta^t)$  and not the dynamics of the correlation between  $\theta_1$  and  $\theta_t$ , or the extent to which  $\theta_1$  is a good predictor of future types  $\theta_t$ , as measured by the variance of the forecast error of  $\theta_t$  given  $\theta_1$ . As one can see for example by looking at the random walk case of Example 3,  $Corr(\theta_1, \theta_t)$  decreases over time and  $Var([\theta_t - \mathbb{E}(\theta_t|\theta_1)]^2)$  increases over time. Nonetheless,  $e_t^*(\theta^t) - e^{FB}$  is constant over time due to the fact that the impulse responses  $J_1^t$  of  $\theta_1$  on  $\theta_t$  are constant in the random walk case.



Also note that, when productivity evolves according to a random walk, then because effort is constant over time, the optimal mechanism can be implemented by offering in period one the same menu of linear contracts that the principal would offer in a static relationship, and then committing to use the same compensation scheme that the agent selects in period one also in each subsequent period. Each linear scheme (indexed by  $\theta_1$ ) has a fixed compensation of

$$S(\theta_1) \equiv \psi(e^*(\theta_1)) + \int_{\underline{\theta}_1}^{\theta_1} \psi'(e^*(s)) ds - \alpha(\theta_1)[\theta_1 + e^*(\theta_1)]$$

together with a piece-rate  $\alpha(\theta_1)\pi_1$  with  $\alpha(\theta_1) = \psi'(e^*(\theta_1))$  that is constant over time. These contracts are reminiscent of those derived in Laffont and Tirole (1986) in a static setting. Contrary to the static case, the entire linear scheme  $S(\theta_1) + \alpha(\theta_1)\pi_t$  — as opposed to the point  $S(\theta_1) + \alpha(\theta_1)[\theta_1 + e^*(\theta_1)]$  — is now used over time. This is a direct consequence of the fact that the firm's performance  $\pi_t$  now changes stochastically over time in response to the shocks affecting the agent's productivity. Also note that while the optimal mechanism can be implemented by using in each period the static optimal contract for period one, this does not mean that the dynamic optimal mechanism coincides with a sequence of static optimal contracts, as in Baron and Besanko (1984). Because the agent's type  $\theta_t$  (and its distribution) changes over time, the sequence of static optimal contracts entails a different choice of effort for each period. What the result implies is that, despite the lack of stationarity, it is optimal for the principal to commit to use the same scheme selected in period one also in each subsequent period, thus inducing the same effort as if the agent's type were constant over time. Out of curiosity, also note that the optimal reward scheme (and the corresponding effort dynamics) when  $\theta_t$  follows a random walk coincides with the one that the principal would offer in an environment in which the shocks have only a transitory (as opposed to permanent) effect on the agent's productivity. Indeed, it is easy to verify that the dynamics of effort under the optimal contract is the same when  $\theta_t = \theta_1 + \sum_{s=2}^t \varepsilon_s$  as when  $\theta_t = \theta_1 + \varepsilon_t$ . This property holds more generally for any process  $F$  that is separable in the first component, i.e., such that  $\theta_t = \kappa_t(\theta_1) + \phi_t(\varepsilon_2, \dots, \varepsilon_t)$  for some functions  $\kappa$  and  $\phi$ : the dynamics of effort is then completely independent of the functions  $\phi_t$  that determine the effect of shocks experienced after period one on the agent's productivity.

**Seniority.** We now turn to the key property discussed in the Introduction that the power of incentives increases with tenure (i.e., with the length of the employment relationship). What the examples above have in common is the property that the effect of the agent’s initial productivity on his future productivity, as captured by the impulse response  $J_1^t$  is either constant (in the random walk-case) or declines over time (gradually, in the AR(1) case, and with a single jump to zero at  $t = 2$  in the case of independent types). More generally, all these examples share the property of “declining impulse responses” defined as follows.

**Definition 4** *The process  $F$  satisfies the property of “declining impulse responses” if, for any  $s > t \geq 1$ , any  $(\theta^t, \theta_{>t}^s)$ ,  $\theta_s \geq \theta_t$  implies that  $J_1^s(\theta^t, \theta_{>t}^s) \leq J_1^t(\theta^t)$ .*

Inspecting the formula for the principal’s expected payoff as given by (11), together with standard comparative statics arguments, then immediately reveals that, when this property holds, then as long as the agent’s productivity does not fall, the effort that he is asked to exert under the optimal policy does not fall either. Declining impulse responses thus provide a condition under which the agent’s effort increases monotonically over time.

An alternative definition of the diminishing effect of the agent’s initial productivity on his future productivity can be obtained by taking an ex-ante view and assuming that

$$\frac{\partial \mathbb{E}^{\tilde{\theta}^T | \theta_1}[\tilde{\theta}_t]}{\partial \theta_1} = \mathbb{E}^{\tilde{\theta}^t | \theta_1} \left[ J_1^t(\tilde{\theta}^t) \right]$$

decreases with  $t$ . In case the agent’s disutility of effort is quadratic, one can indeed verify that, under this condition, the agent’s expected effort under the optimal policy,  $\mathbb{E}^{\tilde{\theta}_t | \theta_1}[e_t^*(\tilde{\theta}^t)]$  (as well as the expected slope of the incentive scheme that sustains it) increases over time. This property, however, does not necessarily extend to more general disutility functions, for it relies on a certain convexity property of the principal’s payoff as given by (11).

What the aforementioned definitions of declining impulse responses both capture is the idea that the effect of a manager’s initial productivity on his subsequent productivity declines over time, a property that seems appropriate for many situations of interest. This property implies that, as a manager’s tenure in the firm grows, the firm finds it optimal to induce him to exert higher effort.

This requires a higher sensitivity of his compensation to performance, which can be obtained, for example, by putting more stocks and options in the compensation package.

Interestingly, note that the reason why, in our theory, the power of incentives increases over time has nothing to do with variations in the manager's preferences or in his outside option (as, for example, in the career-concerns or in the learning-by-doing literature). It is entirely driven by the fact that, when hired, the manager is expected to possess private information about his ability to generate profits for the firm, along with the fact that the effects of such initial private information on future productivity is expected to decline over time. Under these assumptions, to reduce the incentives of those managers who are most productive at the hiring stage from mimicking the less productive ones, it is more effective for the firm to distort effort in the latter's contracts more in the first few periods, when the effect of the initial productivity is still pronounced, than in the distant future when such an effect is fable. Our theory thus complements the aforementioned two theories by offering an alternative justification for the optimality of seniority-based schemes whose power of incentives increases over time.

## 5.2 Risk aversion

We now investigate how the optimality of seniority-based schemes may be affected by the possibility that the agent is risk-averse. For simplicity, we assume here that  $T$  is finite (in which case discounting can be dropped) and start by considering the case where the agent does not have preferences for consumption smoothing, so that  $\mathcal{V}$  is concave, but each  $v_t$  is linear. Using the result in Proposition 1, the principal's expected payoff under any cost-minimizing compensation scheme can be expressed as

$$\begin{aligned} \mathbb{E}[U^P] &= \mathbb{E} \left[ \sum_{t=1}^T (\tilde{\theta}_t + e_t(\tilde{\theta}^t)) \right] \\ &\quad - \mathbb{E} \left[ \mathcal{V}^{-1} \left( \sum_{t=1}^T \psi(e_t(\tilde{\theta}^t)) + \int_{\underline{\theta}_1}^{\tilde{\theta}_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{t=1}^T J_1^t(\tilde{\theta}^t) \psi'(e_t(\tilde{\theta}^t)) \right] ds + \sum_{t=2}^T H_t(\tilde{\theta}^t) \right) \right] \end{aligned} \tag{13}$$

where

$$\begin{aligned}
H_t(\theta^t) &\equiv \int_{\underline{\theta}_t(\theta^{t-1})}^{\theta_t} \mathbb{E}^{\tilde{\theta}^T | \theta^{t-1}, s} \left[ \sum_{\tau=t}^T J_t^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds \\
&\quad - \mathbb{E}^{\tilde{\theta}_t | \theta^{t-1}} \left[ \int_{\underline{\theta}_t}^{\tilde{\theta}_t} \mathbb{E}^{\tilde{\theta}^T | \theta^{t-1}, s} \left[ \sum_{\tau=t}^T J_t^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds \right]
\end{aligned}$$

The expression in (13) is the analogue of the *dynamic virtual surplus* formula in (11) for the case of a risk-averse agent. We now show how the formula in (13) can be used to characterize the profit-maximizing effort policy. We start by considering policies that depend on  $\theta_1$  only, and denote these policies by  $\hat{e}$ . We return to policies that depend on the full history  $\theta^T$  at the end of the section. To facilitate the characterization of the necessary conditions, we consider an example in which  $\theta_t$  follows an ARIMA process (in which case the  $J_1^t$  functions are scalars) and where the inverse of the agent's utility function over consumption is quadratic.

**Example 4** Suppose that  $T < \infty$  and that the process  $F$  is ARIMA with  $\theta_1$  drawn from an absolutely continuous distribution  $F_1$  strictly increasing on an interval  $\Theta_1 = [\underline{\theta}_1, \bar{\theta}_1] \subset \mathbb{R}$ , and with each shock  $\varepsilon_t$  drawn from an absolutely continuous distribution  $G_t$  strictly increasing on a compact interval  $\mathcal{E}_t \subset \mathbb{R}$ , independently from  $\theta_1$  and from any other shock. Assume that the agent's utility function over consumption is given by  $\mathcal{V}(c) = \frac{1}{\alpha} \sqrt{2\alpha c + \beta^2} - \frac{\beta}{\alpha}$  with  $\alpha, \beta > 0$  and note that this function is chosen so that  $\mathcal{V}^{-1}(u) = \frac{\alpha}{2} u^2 + \beta u$ . Let  $K > 1/\beta$  and assume that the disutility of effort  $\psi$  is such that  $\log \psi'$  is strictly concave on  $(0, \bar{e})$  and that for any  $e \geq \bar{e}$ ,  $\psi(e) = \psi(\bar{e}) + (e - \bar{e})K$ . The essentially unique<sup>21</sup> policy  $\hat{e}^*$  that maximizes the principal's expected payoff, as given by (13), among those that depend on  $\theta_1$  only is such that, for any  $t$ ,  $F$ -almost any  $\theta_1$ ,

$$\begin{aligned}
&\psi'(\hat{e}_t^*(\theta_1)) \left[ \alpha \left( \sum_{s=1}^T \psi(\hat{e}_s^*(\theta_1)) + \int_{\underline{\theta}_1}^{\theta_1} \sum_{s=1}^T J_1^s \psi'(\hat{e}_s^*(q)) dq \right) + \beta \right] \\
&\geq 1 - \frac{\psi''(\hat{e}_t^*(\theta_1)) J_1^t}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} \left[ \alpha \left( \sum_{s=1}^T \psi(\hat{e}_s^*(q)) + \int_{\underline{\theta}_1}^q \sum_{s=1}^T J_1^s \psi'(\hat{e}_s^*(r)) dr \right) + \beta \right] f_1(q) dq \\
&\quad - \alpha \sum_{s=2}^t \left[ \left( \sum_{\tau=s}^T J_s^\tau \psi'(\hat{e}_\tau^*(\theta_1)) \right) J_s^t \psi''(\hat{e}_t^*(\theta_1)) \right] \text{Var}(\varepsilon_s),
\end{aligned} \tag{14}$$

with the inequality holding as equality if  $\hat{e}_t^*(\theta_1) > 0$ .

<sup>21</sup>The qualifier ‘‘essentially’’ is due to the fact that the optimal policy is determined only  $F$ -almost everywhere.

Recall that the choice of the optimal effort policy trades off two concerns: (1) limiting the agent's intertemporal informational rent (as perceived at the time of hiring) and (2) insuring the agent against the risk associated with variations in his compensation that are necessary to guarantee incentive-compatibility. To see this more clearly, assume that  $T = 2$ . When applied to  $t = 1$  (and assuming an interior solution), Condition (14) becomes

$$\begin{aligned} & \psi'(\hat{e}_1^*(\theta_1)) \left[ \alpha \left( \sum_{s=1}^2 \psi(\hat{e}_s^*(\theta_1)) + \int_{\underline{\theta}_1}^{\theta_1} \sum_{s=1}^2 J_1^s \psi'(\hat{e}_s^*(q)) dq \right) + \beta \right] \\ = & 1 - \frac{\psi''(\hat{e}_1^*(\theta_1))}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} \left[ \alpha \left( \sum_{s=1}^2 \psi(\hat{e}_s^*(q)) + \int_{\underline{\theta}_1}^q \sum_{s=1}^2 J_1^s \psi'(\hat{e}_s^*(r)) dr \right) + \beta \right] f_1(q) dq. \end{aligned} \quad (15)$$

Because  $\theta_1$  is known at the time of contracting, the agent does not face any risk concerning his period-one productivity. The optimal choice of effort for period one is thus determined uniquely by the desire to limit the agent's informational rent. It is easy to see that Condition (15) reduces to Condition (12) for the risk-neutral case when  $\alpha = 0$  and  $\beta = 1$ . Also note that, as in the risk-neutral case, the policy  $\hat{e}_1^*(\theta_1)$  is increasing in  $\theta_1$ . This simply follows from the fact that the (measure of the) set of types  $(\theta_1, \bar{\theta}_1)$  to whom the principal must give a higher rent when he increases  $\hat{e}_1^*(\theta_1)$  is decreasing in  $\theta_1$ . However, contrary to the risk-neutral case, distortions do not vanish "at the top". In fact, when applied to  $\bar{\theta}_1$ , Condition (15) becomes

$$\psi'(\hat{e}_1^*(\bar{\theta}_1)) \left[ \alpha \left( \sum_{s=1}^2 \psi(\hat{e}_s^*(\bar{\theta}_1)) + \int_{\underline{\theta}_1}^{\bar{\theta}_1} \sum_{s=1}^2 J_1^s \psi'(\hat{e}_s^*(q)) dq \right) + \beta \right] = 1, \quad (16)$$

whereas efficiency (under risk aversion) requires that in each period the agent exerts a constant level of effort  $e^{FB}$  implicitly given by

$$\psi'(e^{FB}) = \mathcal{V}'(\mathcal{V}^{-1}(T\psi(e^{FB}))) \quad (17)$$

which, when applied to the example here, becomes

$$\psi'(e^{FB}) [\alpha (2\psi(e^{FB})) + \beta] = 1.$$

The reason is that, with risk aversion, the compensation that the principal must pay to type  $\bar{\theta}_1$  to discourage him from mimicking a lower type reduces  $\bar{\theta}_1$ 's marginal utility of money; this in turn

makes effort more costly to sustain which explains why the principal finds it optimal to distort (downwards) type  $\bar{\theta}_1$ 's effort (see also Battaglini and Coate, 2008, for a similar result in a two-type model).

Next, consider the effort policy for  $t = 2$ . When applied to  $t = 2$ , Condition (14) becomes (assuming again an interior solution):

$$\begin{aligned} & \psi'(\hat{e}_2^*(\theta_1)) \left[ \alpha \left( \sum_{s=1}^2 \psi(\hat{e}_s^*(\theta_1)) + \int_{\theta_1}^{\theta_1} \sum_{s=1}^2 J_1^s \psi'(\hat{e}_s^*(q)) dq \right) + \beta \right] \\ = & 1 - \frac{\psi''(\hat{e}_2^*(\theta_1)) J_1^2}{f(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} \left[ \alpha \left( \sum_{s=1}^2 \psi(\hat{e}_s^*(q)) + \int_{\theta_1}^q \sum_{s=1}^2 J_1^s \psi'(\hat{e}_s^*(r)) dr \right) + \beta \right] f(q) dq \\ & - \alpha \psi'(\hat{e}_2^*(\theta_1)) \psi''(\hat{e}_2^*(\theta_1)) \text{Var}(\varepsilon_2). \end{aligned} \quad (18)$$

The key difference between (18) and (15) is the last term on the right-hand side of (18). This term captures the principal's concern about exposing the agent to the risk associated with the uncertainty that the agent faces about his second period's productivity. Other things equal, this term contributes to reducing effort, as anticipated in the Introduction.

To further appreciate the distinctions/similarities between the risk-neutral and the risk-averse case, take the specification of the example, and suppose that  $\theta_t = \gamma\theta_{t-1} + \varepsilon_t$ ,  $\gamma \in (0, 1]$ , with  $\theta_1$  uniformly distributed over  $[0, 1]$ . In addition, assume that  $\beta = 1$ , and that  $\psi(e) = \frac{e^2}{4}$  for all  $e \in (0, \bar{e})$ , with  $\bar{e} > 4$  and  $K = \frac{\bar{e}}{2}$ . Then note that, under risk-neutrality ( $\alpha = 0$ ,  $\beta = 1$ ), the (fully)<sup>22</sup> optimal policies are given by  $\hat{e}_1^*(\theta_1) = 2 - (1 - \theta_1)$  and  $\hat{e}_2^*(\theta_1) = 2 - (1 - \theta_1)\gamma$ , with  $e_1^{FB} = e_2^{FB} = 2$ . As discussed above, to minimize the agent's informational rent, the principal optimally distort both  $e_1$  and  $e_2$  downward. Furthermore, because the effect of the agent's initial type on his future types, as captured by the impulse responses  $J_1^t = \gamma^{t-1}$ , declines over time, it is optimal to distort more in the early stages of the relationship than in the later ones. Furthermore, the smaller the impulse responses, the smaller the distortion from the first-best effort: in the example,  $\hat{e}_2^*(\theta_1) = \hat{e}_1^*(\theta_1)$  [i.e., no seniority] when  $\gamma = 1$  [random walk case] and  $\hat{e}_2^*(\theta_1) = e^{FB}$  when  $\gamma = 0$  [ $\theta_1$  and  $\theta_2$  independent].

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<sup>22</sup>Recall, from Proposition 4, that under risk-neutrality the optimal policy depends on  $\theta^2$  only via  $\theta_1$  when  $F$  is ARIMA.

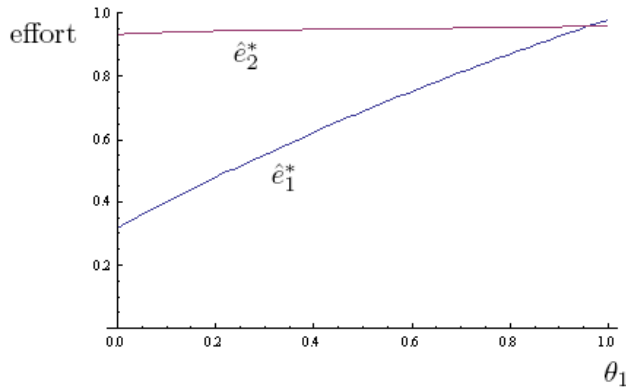


Figure 1: Optimal (shock-independent) effort policies: AR1

Now, to see how risk aversion affects the choice of effort, Figure 1 depicts the optimal  $\theta_1$ -contingent policies for the aforementioned specification with  $\alpha = \beta = 1$ ,  $\gamma = \frac{1}{2}$ , and  $\varepsilon_2$  uniformly distributed over  $[0, 1]$ .<sup>23</sup>

While a form of seniority continues to hold ( $\hat{e}_2^*(\theta_1)$  is higher than  $\hat{e}_1^*(\theta_1)$  for most period-one types), risk aversion tends to depress  $e_2$ , thus reducing the optimality of reward schemes that pay managers with a longer tenure with more high powered incentives. Furthermore, now there exist values of  $\theta_1$  for which  $\hat{e}_2^*(\theta_1) < \hat{e}_1^*(\theta_1)$ . To see the reason for this, note that, when evaluated at  $\bar{\theta}_1$ , equations (15) and (18) are symmetric except for the term  $-\alpha\psi''(\hat{e}_2(\bar{\theta}_1))\psi'(\hat{e}_2(\bar{\theta}_1))Var(\varepsilon_2)$ . As discussed above, this term captures the additional cost associated with a high second-period effort. This cost stems from the fact that a higher effort requires a higher sensitivity of compensation to performance and hence a higher volatility of the agent's compensation. To better appreciate where this term comes from, recall, from Proposition 1, that incentive-compatibility requires that the total compensation that the agent receives in each state  $(\theta_1, \theta_2)$  — equivalently,  $(\theta_1, \varepsilon_2)$  — be given by

$$\nu^{-1} \left( \sum_{t=1}^2 \psi(\hat{e}_t(\theta_1)) + \psi'(\hat{e}_2(\theta_1))[\varepsilon_2 - \mathbb{E}[\tilde{\varepsilon}_2]] + \int_{\theta_1}^{\theta_1} \left[ \sum_{t=1}^2 J_1^t \psi'(\hat{e}_t(s)) \right] ds \right). \quad (19)$$

It is then immediate that reducing  $\hat{e}_2(\theta_1)$  permits the principal to reduce the agent's exposure to the risk generated by  $\varepsilon_2$ . For high values of  $\theta_1$ , this new effect dominates the rent-extraction effect discussed above for the case of risk neutrality, thus resulting in  $\hat{e}_2^*(\theta_1) < \hat{e}_1^*(\theta_1)$ .

<sup>23</sup>We approximated the solution using a sixth-order polynomial.

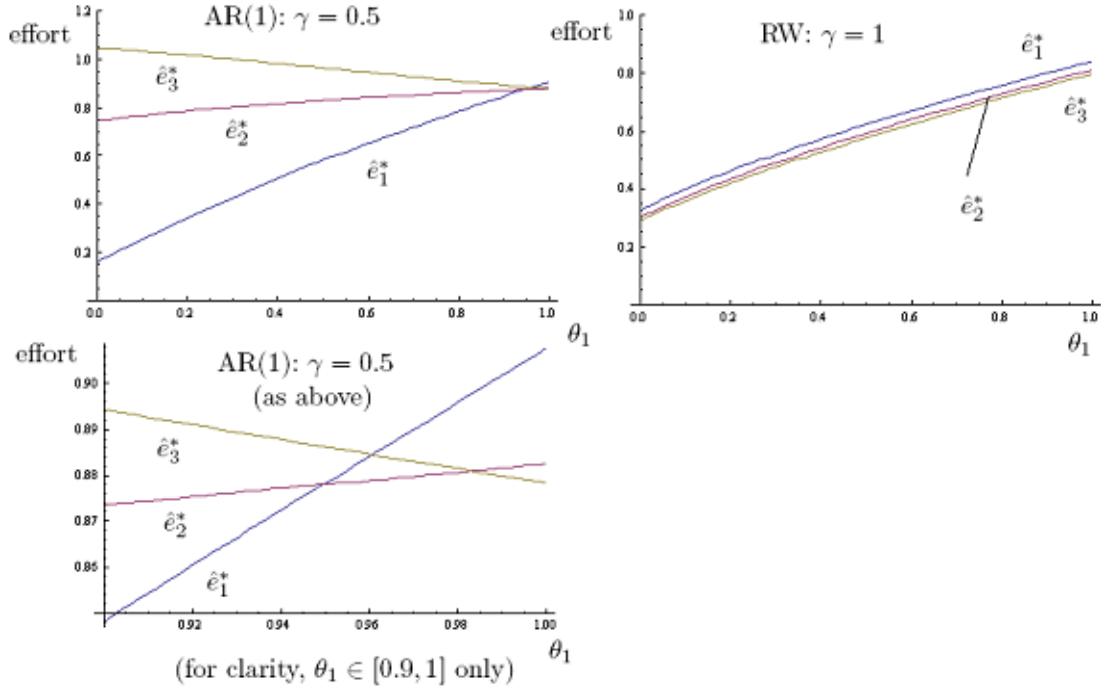


Figure 2: Optimal (shock-independent) effort policies: AR1 and random walk

When the effect of  $\theta_1$  on  $\theta_2$  is small (i.e., for low values of  $\gamma$ ), this new effect mitigates but does not overturn the optimality of compensation schemes that provide managers with a longer tenure with more high powered incentives. When instead the effect of  $\theta_1$  on  $\theta_2$  is strong (i.e., for high values of  $\gamma$ ) this new effect can completely reverse the optimality of such schemes. In the limit, when the shocks to the agent's productivity become fully persistent ( $\theta_t$  follows a random walk, i.e.  $\gamma = 1$ ), one can easily see from (15) and (18) that  $\hat{e}_2^*(\theta_1) < \hat{e}_1^*(\theta_1)$  for all  $\theta_1$ .

The aforementioned properties extend to  $T > 2$ . Figure 2 depicts the optimal policies for the same specification considered above but now letting  $T = 3$ . The left-hand side is for the case of an AR(1) process with  $\gamma = 1/2$ , while the right-hand side is for the case of a random-walk ( $\gamma = 1$ ).

As the figure shows, when  $\gamma = 1$ , effort decreases over time, for all  $\theta_1$ . As anticipated in the Introduction, this reflects the fact that reducing effort in period  $t$  is more effective in reducing the agent's exposure to risk than reducing effort in period  $s < t$ . When instead  $\gamma = 1/2$  then, on average, effort is higher in the later periods than in early ones, although the opposite is true for high values of  $\theta_1$ , as in the  $T = 2$  case. Furthermore, effort in later periods can now be decreasing in



$\theta_1$ . This follows from the fact that, when  $t$  is high, reducing the agent’s effort in period  $t$  has little effect on the agent’s informational rent—which continues to be given by (10) as in the risk-neutral case—while it has a strong effect on the agent’s exposure to risk (the opposite is true when  $t$  is small). Now, distorting effort in any period  $t$  to reduce the agent’s rent is always relatively more effective for low types than for high ones. The reason is the same as in a static setting: When the period-1 distribution is log-concave, the measure of types whose rent is reduced when reducing type  $\theta_1$ ’s effort is decreasing in  $\theta_1$ . This explains why the optimal effort policy is increasing in  $\theta_1$  in the early stages of the relationship. Together with the fact that a reduction in effort in period  $s$  is an (imperfect) substitute for a reduction of effort in period  $t > s$  when it comes to reducing the agent’s total exposure to risk, this implies that the optimal effort policy must eventually become decreasing in type once  $t$  has grown large enough.

Another way the principal could mitigate the effect of the agent’s uncertainty about his future productivity on his compensation is by conditioning the effort policy in periods  $t > 1$  on the shocks to productivity experienced after the first period. To gauge the effect of this additional flexibility, consider again the same specification assumed above. While a complete analytical characterization of the fully-optimal policy escapes us because of the complexity of the optimization problem, we could approximate the optimal policy with 6th-degree polynomials. The result for the case where  $T = 2$  is depicted in Figure 3, where we considered the same parametrization as in Figure 1, but now allowed the second-period effort to depend on the shock  $\varepsilon_2$  [equivalently on  $\theta_2 = \gamma\theta_1 + \varepsilon_2$ ]. Again, when the effect of  $\theta_1$  on  $\theta_2$  is not too high (in the example,  $\lambda = 1/2$ ), the optimality of seniority-based schemes that provide more high powered incentives over time is maintained:  $\mathbb{E}e_2^*(\theta_1, \tilde{\varepsilon}_2)$  is on average higher than  $e_1^*(\theta_1)$ , with the inequality reversed only for sufficiently high values of  $\theta_1$  [here  $\mathbb{E}e_2^*(\theta_1, \tilde{\varepsilon}_2)$  is a short-cut for  $\mathbb{E}^{\tilde{\theta}_2|\theta_1}[e_2^*(\theta_1, \tilde{\theta}_2)]$ ].

Also note that second-period effort is typically decreasing in the shock  $\varepsilon_2$  [equivalently in  $\theta_2$ , for given  $\theta_1$ ], as illustrated in Figure 4. This negative correlation permits the principal to further reduce the agent’s exposure to risk, as one can see directly from (19).

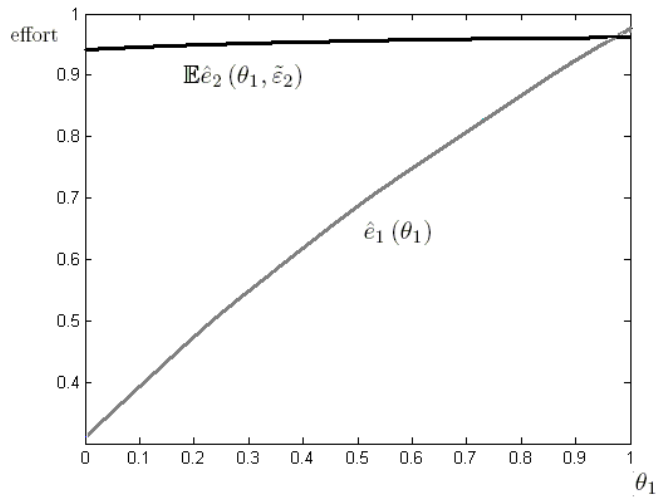


Figure 3: (Fully optimal) first-period and average second-period effort policies

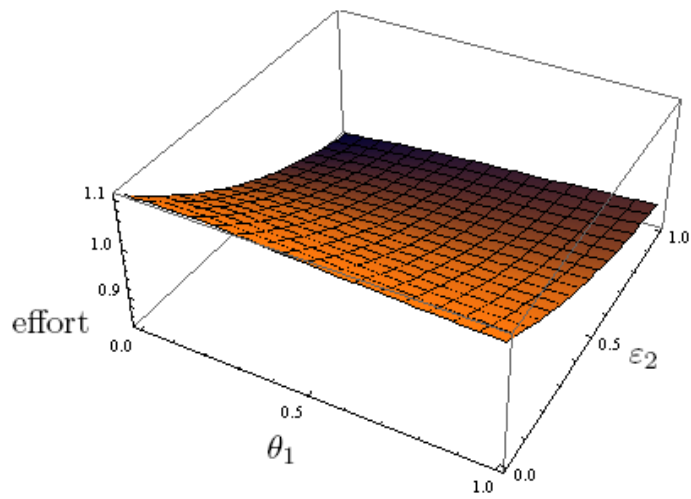


Figure 4: (Fully-optimal) second-period effort

The fact that the optimal policy  $e_2^*(\theta_1, \varepsilon_2)$  is typically decreasing in  $\varepsilon_2$  also implies that, with risk aversion, it becomes difficult to sustain the optimal effort policy with the pseudo-linear schemes of Proposition 3. Indeed, as one can see from the single crossing-conditions (7), these schemes typically require that effort be nondecreasing in type in the last period. In contrast, the bonus schemes of Proposition 1 simply require the cash flows (as opposed to effort) to be nondecreasing, as one can see from (6). We thus expect firms to rely more on bonus schemes and less on linear and pseudo-linear schemes when they value reducing their managers' exposure to risk.

**Consumption smoothing.**

We now turn to the case where not only is the agent risk averse, he also has preferences for intertemporal consumption smoothing. For simplicity, and without any serious loss, we assume that  $v_t = v$  all  $t$ , with  $v$  strictly concave, while  $\mathcal{V}$  is linear. Because the agent's payoff now depends on the timing the payments are made, the principal must now distribute the payments optimally over time, adding an additional dimension to the problem. One way to arrive to the optimal policy  $e^*$  is the following. Using Proposition 1 along with Corollary 1, one determines the cost for the principal of sustaining any effort policy  $e$ . Once this is accomplished, one then chooses the policy  $e^*$  that maximizes the principal's expected cash flows, net of the compensation to the agent.

An alternative route, which is often more parsimonious, is the following. One treats the utility of consumption as an additional control and then maximizes the principal's expected payoff with respect to effort and the agent's utility of consumption, subject to the latter satisfying the necessary condition for incentive-compatibility

$$\sum_{t=1}^T v(c_t(\theta^t)) = W(\theta^T) \text{ for } F\text{-almost all } \theta^T \tag{20}$$

where  $W$  is the function defined in Proposition 1. This alternative approach often facilitates the computation, for it does not require computing the cost-minimizing compensation scheme for each possible effort policy. To illustrate, suppose for simplicity that  $T = 2$  and that  $\delta = 1$ . Denote the utility of consumption in each period by  $u_1(\theta_1) = v(c_1(\theta_1))$  and  $u_2(\theta_1, \theta_2) = v(c_2(\theta_1, \theta_2))$ . Then,

for any  $\theta^T = (\theta_1, \theta_2)$ , the necessary condition (20) can be rewritten as

$$u_1(\theta_1) + u_2(\theta_1, \theta_2) = \sum_{t=1}^2 \psi(e_t(\theta^t)) + \int_{\underline{\theta}_1}^{\theta_1} \mathbb{E}^{\tilde{\theta}_2|s} \left[ \sum_{t=1}^2 J_1^t(\tilde{\theta}^t) \psi'(e_t(\tilde{\theta}^t)) \right] ds \quad (21)$$

$$+ \int_{\underline{\theta}_2}^{\theta_2} \psi'(e_2(\theta_1, s)) ds - \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[ \int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(e_2(\theta_1, s)) ds \right].$$

Evaluating (21) at  $(\theta_1, \underline{\theta}_2)$  allows us to express each function  $u_1(\theta_1)$  and  $u_2(\theta_1, \theta_2)$  as functions of the effort policy  $e$  and the auxiliary function  $u_2(\cdot, \underline{\theta}_2)$ . Specifically, for all  $(\theta_1, \theta_2)$ ,

$$u_1(\theta_1) = -u_2(\theta_1, \underline{\theta}_2) + \psi(e_1(\theta_1)) + \psi(e_2(\theta_1, \underline{\theta}_2)) \quad (22)$$

$$+ \int_{\underline{\theta}_1}^{\theta_1} \psi'(e_1(s)) ds + \int_{\underline{\theta}_1}^{\theta_1} \mathbb{E}^{\tilde{\theta}_2|s} \left[ J_1^2(s, \tilde{\theta}_2) \psi'(e_2(s, \tilde{\theta}_2)) \right] ds$$

$$- \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[ \int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(e_2(\theta_1, s)) ds \right],$$

and

$$u_2(\theta_1, \theta_2) = u_2(\theta_1, \underline{\theta}_2) + \psi(e_2(\theta_1, \theta_2)) - \psi(e_2(\theta_1, \underline{\theta}_2)) \quad (23)$$

$$+ \int_{\underline{\theta}_2}^{\theta_2} \psi'(e_2(\theta_1, s)) ds$$

where Condition (23) follows from applying (3) to  $T = 2$  with the above utility specification. The optimal mechanism can then be obtained by maximizing the principal's expected payoff

$$\mathbb{E} \left[ \tilde{\theta}_1 + \tilde{\theta}_2 + e_1(\tilde{\theta}_1) + e_2(\tilde{\theta}_1, \tilde{\theta}_2) - v^{-1}(u_1(\tilde{\theta}_1)) - v^{-1}(u_2(\tilde{\theta}_1, \tilde{\theta}_2)) \right],$$

with respect to the functions  $e_1(\cdot)$ ,  $e_2(\cdot, \cdot)$ , and  $u_2(\cdot, \underline{\theta}_2)$ , and then using the fact that the utilities of consumptions  $u_1$  and  $u_2$  are given by (22) and (23).

As an illustration, consider the same specification as in Example 4, i.e. let  $v(c) = \frac{1}{\alpha} \sqrt{2\alpha c + \beta^2} - \frac{\beta}{\alpha}$  for  $\alpha, \beta > 0$ , and assume that  $J_1^2(\theta_1, \theta_2) = \frac{1}{2}$  (an equivalent choice was made for Figures 3 and 4). Using again sixth-order polynomials, we can then compute numerically the optimal effort policies. These policies have the same qualitative features as the ones obtained above for the case without consumption smoothing, i.e., for the case where  $\mathcal{V}$  is strictly concave and  $v$  linear. In particular, both  $e_1(\theta_1)$  and  $\mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[ e_2(\theta_1, \tilde{\theta}_2) \right]$  are increasing in  $\theta_1$ , and  $e_2(\theta_1, \theta_2)$  is decreasing in  $\theta_2$ , as in Figures 3

and 4. We thus expect the key trade-offs identified above to extend to the case of risk-aversion with preferences for consumption smoothing. The only key difference is that the firm must now optimally distribute the payments to the agent over time in accordance to the inverse Euler condition of Corollary 1.

## 6 Related literature

The literature on managerial compensation is clearly too large to be successfully summarized within the context of this paper. We refer the reader to Prendergast (1999) for an excellent overview and to Edmans and Gabaix (2009) for a survey of some recent developments.

Particularly related to our paper is the empirical literature on the use of seniority-based compensation schemes. This literature finds mixed evidence as to the effect of tenure on performance-based pay. While some papers suggest that managers with a longer tenure tend to have weaker incentives and explain this by the fact that the board of directors tends to become captured by the CEOs over time (e.g., Hill and Phan, 1991), others point to the contrary (e.g., Lippert and Porter, 1997, but also Gibbons and Murphy, 1991). These differences often originate in the choices about which incentive measures are relevant (e.g., whether to consider stock options). At the theoretical level, our paper contributes to this literature by offering a novel explanation for the optimality (or suboptimality) of seniority-based schemes which, to the best of our knowledge, was not noticed before.

The paper is also related to the literature on “dynamic moral hazard” and to its application to managerial compensation. Seminal works in this literature include Lambert (1983), Rogerson (1985) and Spear and Srivastava (1987). These works provide qualitative insights about the optimal policy, but do not provide a full characterization. This has been possible only in restricted settings: Phelan and Townsend (1991) characterize optimal policies numerically in a discrete-time model, while Sannikov (2008) uses a continuous-time setting with Brownian shocks to characterize the optimal policy as the solution to a differential equation. In contrast to these results, Holmstrom and Milgrom (1987) show that the optimal contract has a simple structure when (a) the agent does not value the timing of payments, (b) noise follows a Brownian motion and (c) the agent’s utility is exponential

and defined over consumption net of the disutility of effort. Under these assumptions, the optimal contract takes the form of a simple linear aggregator of aggregate profits. Contrary to this literature, in the current paper, we assumed that, in each period, the agent observes the shock to his productivity before choosing effort. In this respect, the paper is closely related to Laffont and Tirole (1986) who first proposed this alternative timing. This alternative approach permits one to use techniques from the mechanism design literature to solve for the optimal contract. This approach has been recently applied to dynamic managerial compensation by Edmans and Gabaix (2010). Relative to their work, our contribution is twofold: (i) We provide conditions for the implementability of contingent effort policies that permit the manager to change effort in response to the shocks to his productivity and (ii) we characterize the contracts that permit the firm to respond to such contingencies. We then used such a characterization to determine the profit-maximizing effort policy.<sup>24</sup> Allowing for general processes and characterizing the optimal effort policies is instrumental to our results about the dynamics of the power of incentives, the optimality of linear and pseudo-linear schemes, and the effect of the agent's risk aversion on the dynamics of the power of incentives. In this respect, the paper is also related to our work on managerial turnover in a changing world (Garrett and Pavan, 2010). In that paper we assumed that all agents are risk-neutral and focused on the dynamics of retention decisions. In this paper, in contrast, we abstracted from retention (i.e., assumed a single agent) and focused instead on the effect of risk-aversion on the dynamics of effort and on the optimality of seniority-based compensation schemes.

Related is also the literature on the optimal use of financial instruments in dynamic principal-agent relationships. For instance, DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), Sannikov (2007),<sup>25</sup> and Biais et al. (2010) study optimal financial contracts for a manager who privately observes the dynamics of cash-flows and can divert funds from investors for private con-

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<sup>24</sup>Two other important differences are that (i) we allow for the possibility that the agent is privately informed at the time he signs the contract, and (ii) we consider the possibility that the agent has preferences for consumption smoothing. Such possibilities are not only realistic, they have important implications for the dynamics of effort and of the sustaining compensation scheme.

<sup>25</sup>As in our work, and contrary to the other papers cited here, Sannikov (2007) allows the agent to possess private information prior to signing the contract. Assuming the agent's initial type can be either "bad" or "good", he characterizes the optimal separating menu where only good types are funded.

sumption. In these papers, it is typically optimal to induce the highest possible effort (which is equivalent to no stealing/no saving); the instrument which is then used to create incentives is the probability of terminating the project. One of the key findings is that the optimal contract can often be implemented using long-term debt, a credit line, and equity. The equity component represents a linear component to the compensation scheme which is used to make the agent indifferent as to whether or not to divert funds for private use. Since the agent's cost of diverting funds is constant over time and output realizations, so is the equity share. In contrast, we provide an explanation for why and how this share typically changes over time. While these papers suppose that cash-flows are i.i.d., Tchisty (2006) explores the consequences of correlation and shows that the optimal contract can be implemented using a credit line with an interest rate that increases with the balance. DeMarzo and Sannikov (2008) consider an environment in which both the investors and the agent learn about the firm's true productivity (which evolves according to a Brownian motion). In this last paper, as in ours, the agent's private information is correlated over time.

From a methodological standpoint, in this paper we applied recent results from the dynamic mechanism design literature to arrive to the characterization of optimal contracts. In particular, the approach here builds on the techniques developed by Pavan, Segal, and Toikka (2009). That paper provides a general treatment of incentive compatibility in dynamic settings. It extends previous work by Besanko (1985), Courty and Li (2003), and Battaglini (2005), among others, by allowing for significantly more general payoffs and stochastic processes and by identifying the role of impulse responses as the key driving force for the dynamics of optimal contracts.<sup>26</sup> An important dimension in which the current paper makes progress is the characterization of optimal mechanisms for risk-averse agents with correlated information<sup>27</sup>. In this respect, the paper is also related to the literature on optimal dynamic taxation (also known as Mirrleesian taxation, or new public finance). Battaglini and Coate (2008) consider a discrete-time-two-type model with Markov transitions and show continuity of the optimal mechanism with respect to the agent's degree of risk aversion.<sup>28</sup> Zhang (2009) considers

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<sup>26</sup>We refer the reader to that paper for a more extensive review of the dynamic mechanism design literature.

<sup>27</sup>For static models with risk aversion, see also Salanie (1990), and Laffont and Rochet (1998).

<sup>28</sup>Their characterization of the optimal scheme with risk-aversion is partial, but nonetheless it shows that the optimal scheme converges to the one for risk-neutrality as the coefficient of risk-aversion vanishes.

a model with finitely many types, but where contracting occurs in continuous-time and where the arrival rate of the transitions between types follows a Poisson process. For most of the analysis, he restricts attention to two types and finds that many of the results derived for the i.i.d. case (e.g., Albanesi and Sleet, 2006) carry over to the environment with persistent types. In particular, the celebrated “immiserization result” according to which consumption converges to its lower bound, extends to a setting with correlated types. One qualitative difference with respect to the i.i.d. case is that the size of the “wedges”, i.e. the distortions due to the agent’s private information, is significantly larger when types are persistent, a result which is consistent with the findings in the current paper. Consistent with Battaglini and Coate (2006), he also finds that, contrary to the risk-neutral case, distortions do not vanish as soon as the agent becomes a high type. More recently Fahri and Werning (2010) use techniques similar to those developed in Pavan, Segal, and Toikka (2009) and in the current paper to provide insights on the dynamics of taxes in a setting with a continuum of types drawn from an AR(1) process. Our results appear broadly consistent with the aforementioned findings; however, by considering more general processes and payoffs, we shed light on properties of the optimal contract not identified before (in particular on the effect of risk-aversion on the optimality of seniority-based schemes).

## 7 Conclusions

We investigated the properties of optimal compensation schemes in an environment in which managerial ability to generate profits is expected to change over time and is the managers’ private information. We showed how optimal contracts can be obtained as a solution to a dynamic mechanism design problem. We then used the solution (i) to shed light on the properties of compensation schemes that sustain the desired effort policy at minimal cost for the firm, (ii) to identify conditions for a given effort policy to be implementable, and (iii) to derive the properties of profit-maximizing effort policies.

When the manager is risk-neutral, we showed that it is typically optimal for the firm to offer a reward package whose power of incentives increases over time, thus inducing the manager to exert,



on average, more effort as his tenure in the firm grows. This result hinges on the assumption that the effect of the manager's productivity at the time he is hired on his productivity in the subsequent periods declines over time, a property that appears reasonable for many cases of interest. Under this assumption, reducing the power of incentives in the contracts of those types who are least productive at the initial contracting stage so as to discourage more productive types to mimic is more effective when done in the early periods, when the effect of the initial private information is still pronounced than in the later periods when such an effect has vanished.

Building on the results for the risk neutral case, we then showed that risk aversion reduces, and in some cases can even revert, the profitability of such seniority-based schemes. The reason is that contracts whose power of incentives increases, on average, over time require a high sensitivity of pay to performance precisely in those periods in which the uncertainty the manager faces about his productivity (and hence about his pay), as perceived at the time he is hired, is the highest, thus exposing him to a lot of risk.

While the results described above refer to contracts for managerial compensation, many of the insights apply more generally to other dynamic contracting problems. In particular, we expect our results to offer valuable insights also for the design of optimal taxes in environments with persistent shocks to workers' productivity, a problem that is at the frontier of the new public finance literature.

## 8 Appendix

**Proof of Proposition 1.** The proof proceeds in three steps and is based on two lemmas. Lemma 1 establishes that, in any incentive-compatible mechanism implementing the policy  $e$ , the agent's value function at each history must satisfy a certain envelope condition. The second lemma uses this envelope condition iteratively to show that for  $F$ -almost all sequences  $\theta^T$ , the utility that the agent derives from the compensation he receives from the principal must be equal to  $W(\theta^T) + K$ , where  $W$  is uniquely pinned down by the policy  $e$  and where  $K$  is a constant that must be nonnegative if the mechanism is individually rational for the agent. The last step verifies that, if a mechanism  $\Omega$  exists

that implements the policy  $e$ , then the one whose compensation is as defined in the proposition also implements the same policy at a (weakly) lower cost for the principal, thus establishing the result.

*Step 1.* Given any mechanism  $\Omega = \langle \xi, s \rangle$ , and any history  $h_t$ , let  $V^\Omega(h_t)$  denote the agent's value function at history  $h_t$ . This function is defined to be the supremum of the agent's expected payoff over all possible reporting and effort strategies. We say that the mechanism  $\Omega$  remains incentive compatible at history  $h_t$  if  $V^\Omega(h_t)$  coincides with the agent's expected payoff under a truthful and obedient strategy from  $t$  onwards. We then have the following result.

**Lemma 1** *Fix the history  $h_{t-1} = (\theta^{t-1}, \hat{\theta}^{t-1}, e^{t-1})$ .<sup>29</sup> If the mechanism  $\Omega$  remains incentive compatible at  $F_t(\cdot|\theta^{t-1})$ -almost all histories  $h_t = (h_{t-1}, \theta_t)$ , then  $V^\Omega(h_{t-1}, \cdot)$  is Lipschitz continuous in  $\theta_t$  over  $\text{Supp}[F_t(\cdot|\theta^{t-1})]$  and at each  $\theta_t \in \text{Supp}[F_t(\cdot|\theta^{t-1})]$  at which  $V^\Omega(h_{t-1}, \cdot)$  is differentiable and  $\Omega$  is incentive compatible*

$$\frac{dV^\Omega(h_{t-1}, \theta_t)}{d\theta_t} = \mathbb{E}^{\tilde{\theta}^T | \theta^t} \left[ \sum_{\tau=t}^T \delta^{\tau-1} J_t^\tau(\tilde{\theta}^\tau) \psi'(\xi_\tau((\hat{\theta}^{t-1}, \tilde{\theta}_{\geq t}^\tau), \tilde{\pi}^{\tau-1}(h_{t-1}, \tilde{\theta}_{\geq t}^{\tau-1}))) \right]$$

where  $\pi^{\tau-1}(h_{t-1}, \theta_{\geq t}^{\tau-1}) = (\pi_s)_{s=1}^{\tau-1}$  with  $\pi_s = \theta_s + e_s$  for all  $s < t$  and  $\pi_s = \theta_s + \xi_s(\hat{\theta}^{t-1}, \theta_{\geq t}^s, \pi^{s-1})$  all  $s \geq t$ .

**Proof of the lemma.** Consider the following fictitious environment. At any period  $s \geq t$ , the agent can misreport his private information  $\theta_s$  but is then “forced” to choose effort so as to perfectly “hide” his lies. That is, at any period  $s \geq t$ , and for any given sequence of reports  $(\hat{\theta}^s)$ , the agent must exert effort  $e_s$  so that the observed cash flow  $\pi_s = \theta_s + e_s$  equals the one expected by the principal when the agent's period- $s$  type is  $\hat{\theta}_s$  and he follows the principal's recommended effort choice  $\xi_s(\hat{\theta}^s, \pi^{s-1})$ . This is to say that, at any period  $s \geq t$ , given the public history  $(\hat{\theta}^s, \pi^{s-1})$  and the true productivity  $\theta_s$  the agent must choose effort

$$e_s(\theta_s; \hat{\theta}^s, \pi^{s-1}) = \hat{\theta}_s + \xi_s(\hat{\theta}^s, \pi^{s-1}) - \theta_s. \quad (24)$$

Now fix the period- $(t-1)$  history  $h_{t-1}$ . Given the reports  $\hat{\theta}_{\geq t}^T$ , let  $(\hat{c}_{\geq t}^T, \hat{\pi}_{\geq t}^T)$  be the stream of payments and cash flows in the continuation game that starts at period  $t$  with history  $(h_{t-1}, \theta_t)$  when

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<sup>29</sup>The first element of  $h_{t-1}$  denotes the true productivity history, whereas the term  $\hat{\theta}^{t-1}$  denotes the history of reports. The last term  $e^{t-1}$  denotes the history of effort choices.

in the continuation game the agent sends the reports  $(\hat{\theta}_{\geq t}^T)$  and then follows the behavior described above (i.e., chooses effort according to (24)). For any  $\hat{\theta}_{\geq t}^T$  and any sequence of true types  $(\theta_{\geq t}^T)$ , the agent's payoff in this fictitious environment is given by

$$U^A(\theta_{\geq t}^T; \hat{\theta}_{\geq t}^T, h_{t-1}) = \sum_{s=t}^T \delta^{s-1} [\hat{c}_s - \psi(\hat{\theta}_s + \xi_s(\hat{\theta}^s, \hat{\pi}^{s-1}) - \theta_s)] + X(h_{t-1})$$

where  $X(h_{t-1})$  is a function of the past history  $h_{t-1}$ . The assumptions that  $\psi$  is continuously differentiable with derivative bounded uniformly over  $E$  implies that  $U^A$  is totally differentiable in  $\theta_{\geq t}^s$ , any  $s \geq t$ , and equi-Lipschitz continuous in  $\theta_{\geq t}^T$  in the norm:

$$\|\theta_{\geq t}^T\| \equiv \sum_{s \geq t} \delta^{s-t} |\theta_s|.$$

Together with the assumption that the impulse responses are uniformly bounded, this means that the value function  $V^\Omega(h_{t-1}, \cdot)$  is Lipschitz continuous in  $\theta_t$  over  $\Theta_t$  and hence also over  $Supp[F_t(\cdot|\theta^{t-1})]$ . This result follows from Pavan, Segal, and Toikka (2009).

Now suppose that, given  $h_{t-1}$ , the mechanism  $\Omega$  remains incentive compatible at  $F_t(\cdot|\theta^{t-1})$ -almost any history  $h_t = (h_{t-1}, \theta_t)$  in the unrestricted game where the agent is free to choose any effort he wants at any point in time. It is then necessarily incentive compatible also in this fictitious game where he is forced to choose effort according to (24). The results in Pavan, Segal, and Toikka then imply that at any  $\theta_t \in Supp[F_t(\cdot|\theta^{t-1})]$ , at which  $V^\Omega(h_{t-1}, \cdot)$  is differentiable and  $\Omega$  is incentive compatible

$$\frac{dV^\Omega(h_{t-1}, \theta_t)}{d\theta_t} = \mathbb{E}^{\tilde{\theta}^T|\theta^t} \left[ \sum_{\tau=t}^T \delta^{\tau-1} J_\tau^T(\tilde{\theta}^\tau) \frac{\partial U^A(\tilde{\theta}_{\geq t}^T; \tilde{\theta}_{\geq t}^T, h_{t-1})}{\partial \theta_\tau} \right]$$

where  $\partial U^A(\theta_{\geq t}^T; \theta_{\geq t}^T, h_{t-1})/\partial \theta_\tau$  denotes the partial derivative of  $U^A(\theta_{\geq t}^T; \theta_{\geq t}^T, h_{t-1})$  with respect to the true type  $\theta_\tau$  under truthtelling. The result then follows from the fact that

$$\frac{\partial U^A(\theta_{\geq t}^T; \theta_{\geq t}^T, h_{t-1})}{\partial \theta_\tau} = \psi'(\xi_\tau((\hat{\theta}^{t-1}, \theta_{\geq t}^\tau), \pi^{\tau-1}(h_{t-1}, \theta_{\geq t}^{\tau-1}))).$$

■

*Step 2.* Using the previous lemma inductively, then leads to the following result.

**Lemma 2** *Let  $\Omega$  be any incentive-compatible and individually rational mechanism implementing the policy  $e$ . Then for  $F$ -almost all  $\theta^T$*

$$\mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t(c_t(\theta^t)) \right) = W(\theta^T) + K$$

with  $K \geq 0$ .

**Proof of the lemma.** Using the result in the previous lemma and the fact that any mechanism  $\Omega$  that is individually-rational and incentive-compatible for all  $\theta_1$  must remain incentive-compatible at  $F$ -almost all truthful histories<sup>30</sup>  $\theta^t \in R^t$ , any  $t$ , we then have that, for  $F$ -almost all truthful histories  $\theta^t \in R^t$

$$V^\Omega(\theta^t) = V^\Omega(\theta^{t-1}, \underline{\theta}_t(\theta^{t-1})) + \int_{\underline{\theta}_t(\theta^{t-1})}^{\theta_t} \mathbb{E}^{\tilde{\theta}^T | \theta^{t-1}, s} \left[ \sum_{\tau=t}^T \delta^{\tau-1} J_\tau^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds. \quad (25)$$

Furthermore, the fact that  $\Omega$  is incentive compatible for all  $\theta_1$ , implies that, for any  $t$ ,  $F$ -almost all truthful histories  $\theta^{t-1} \in R^t$ ,

$$V^\Omega(\theta^{t-1}) = \mathbb{E}^{\tilde{\theta}_t | \theta^{t-1}} \left[ V^\Omega(\theta^{t-1}, \tilde{\theta}_t) \right].$$

Combining, we have that

$$V^\Omega(\theta^{t-1}, \underline{\theta}_t(\theta^{t-1})) = V^\Omega(\theta^{t-1}) - \mathbb{E}^{\tilde{\theta}_t | \theta^{t-1}} \left[ \int_{\underline{\theta}_t(\theta^{t-1})}^{\tilde{\theta}_t} \mathbb{E}^{\tilde{\theta}^T | \theta^{t-1}, s} \left[ \sum_{\tau=t}^T \delta^{\tau-1} J_\tau^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds \right].$$

Applying (25) to  $V^\Omega(\theta^{t-1})$  we then have that

$$\begin{aligned} V^\Omega(\theta^{t-1}, \underline{\theta}_t(\theta^{t-1})) &= V^\Omega(\theta^{t-2}, \underline{\theta}_{t-1}(\theta^{t-2})) \\ &+ \int_{\underline{\theta}_{t-1}(\theta^{t-2})}^{\theta_{t-1}} \mathbb{E}^{\tilde{\theta}^T | \theta^{t-2}, s} \left[ \sum_{\tau=t-1}^T \delta^{\tau-1} J_{t-1}^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds \\ &- \mathbb{E}^{\tilde{\theta}_t | \theta^{t-1}} \left[ \int_{\underline{\theta}_t(\theta^{t-1})}^{\tilde{\theta}_t} \mathbb{E}^{\tilde{\theta}^T | \theta^{t-1}, s} \left[ \sum_{\tau=t}^T \delta^{\tau-1} J_t^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds \right]. \end{aligned} \quad (26)$$

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<sup>30</sup>As explained also in the main text, a truthful history  $h_t = (\theta^t, \hat{\theta}^{t-1}, e^{t-1})$  is one that is reached by reporting truthfully and following the principal's effort recommendations in each previous period. Because in any such history  $\hat{\theta}_s = \theta_s$  and  $e_s = \xi_s(\theta^s, \pi^{s-1}(\theta^{s-1}))$  all  $s < t$ , without risk of confusion, these histories can be conveniently denoted by the realized sequence of productivities  $\theta^t$ .

Combining (25) with (26) we then have that

$$V^\Omega(\theta^t) = V^\Omega(\theta^{t-2}, \underline{\theta}_{t-1}(\theta^{t-2})) + \int_{\underline{\theta}_{t-1}(\theta^{t-2})}^{\theta_{t-1}} \mathbb{E}^{\tilde{\theta}^T | \theta^{t-2}, s} \left[ \sum_{\tau=t-1}^T \delta^{\tau-1} J_{t-1}^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds + \delta^{t-1} H_t(\theta^t).$$

Applying the same steps inductively to each  $V^\Omega(\theta^{s-1}, \underline{\theta}_s(\theta^{s-1}))$ ,  $s \leq t$ , then leads to the conclusion that

$$V^\Omega(\theta^t) = V^\Omega(\underline{\theta}_1) + \int_{\underline{\theta}_1}^{\theta_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{\tau=1}^T \delta^{\tau-1} J_1^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] ds + \sum_{s=2}^t \delta^{s-1} H_s(\theta^s). \quad (27)$$

Using the fact that the agent's payoff under a truthful and obedient strategy must coincide with the value function at  $F$ -almost all histories and, in the case  $T = \infty$ , continuity at infinity, then establishes the result, with  $K = V^\Omega(\underline{\theta}_1)$ . The fact that the mechanism is individually-rational for all  $\theta_1$  then implies that  $K \geq 0$ . ■

*Step 3.* Now it is easy to see that, if a mechanism  $\Omega$  exists that implements the policy  $e$ , then  $e$  can also be implemented by a mechanism  $\Omega'$  such that  $s'_t(\theta^t, \pi^t) = c_t(\theta^t)$  if  $\theta^t \in R^t$  and  $\pi_t \geq \pi_t(\theta^t) \equiv \theta_t + e_t(\theta^t)$ , and  $s'_t(\theta^t, \pi^t) = -L_t$  with  $L_t > 0$  large enough, otherwise [here  $c_t(\theta^t) = s_t(\theta^t, \pi^t(\theta^t))$  is the equilibrium compensation under the original mechanism  $\Omega$ ]. The proof follows from essentially the same replication arguments that establish the Revelation Principle.

Having established that, in any mechanism  $\Omega$  that implements the policy  $e$ , the utility the agent derives from his compensation must satisfy

$$\mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t(c_t(\theta^t)) \right) = W(\theta^T) \text{ for } F\text{-almost all } \theta^T,$$

by the definition of  $c^{opt}(\cdot; e)$ , it is then easy to see that any mechanism  $\Omega^{opt}$  where the compensation scheme is such that  $s_t(\theta^t, \pi^t) = c_t^{opt}(\theta^t; e)$  if  $\theta^t \in R^t$  and  $\pi_t \geq \pi_t(\theta^t) \equiv \theta_t + e_t(\theta^t)$ , and  $s_t(\theta^t, \pi^t) = -L_t$  otherwise, implements  $e$  at minimum cost. That it induces participation by all period-1 types follows from the fact that the payoff that each type  $\theta_1$  expects under a truthful and obedient strategy is given by

$$\int_{\underline{\theta}_1}^{\theta_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(e_t(\tilde{\theta}^t)) \right] ds$$

which is nonnegative under the assumption of first-order-stochastic dominance in types. ■

**Proof of Corollary 1.** The proof is by contradiction. Suppose  $c^{opt}(\cdot; e)$  solves program (2) in Proposition 1 and assume that there exists a period  $t$  and an  $F$ -positive probability set  $Q \subset R^t$  such that, for  $\theta^t \in Q$ ,

$$\frac{1}{v'_t(c_t^{opt}(\theta^t; e))} > \mathbb{E}^{\tilde{\theta}_{t+1}|\theta^t} \left[ \frac{\delta}{v'_{t+1}(c_{t+1}^{opt}(\theta^t, \tilde{\theta}_{t+1}; e))} \right].$$

The argument for the case where the inequality is reversed is symmetric. Then consider the following alternative policy  $c^\#(\cdot; e)$ . For any  $\tau \neq t, t+1$ , any  $\theta^\tau \in R^\tau$ ,  $c^\#(\theta^\tau; e) = c_\tau^{opt}(\theta^\tau; e)$ ; in period  $t$ ,

$$c_t^\#(\theta^t; e) = \begin{cases} c_t^{opt}(\theta^t; e) & \text{if } \theta^t \notin Q \\ v_t^{-1} \left( v_t(c_t^{opt}(\theta^t; e)) + k \right) & \text{if } \theta^t \in Q, \end{cases}$$

while in period  $t+1$ ,

$$c_{t+1}^\#(\theta^{t+1}; e) = \begin{cases} c_{t+1}^{opt}(\theta^{t+1}; e) & \text{if } \theta^{t+1} \notin Q \\ v_{t+1}^{-1} \left( v_{t+1}(c_{t+1}^{opt}(\theta^{t+1}; e)) - k/\delta \right) & \text{if } \theta^{t+1} \in Q. \end{cases}$$

Clearly, this scheme also satisfies the constraints in program (2). The difference in terms of ex-ante expected cost to the principal under this scheme and under the original scheme  $c^{opt}(\cdot; e)$  is given by

$$\Delta(k) \equiv F(\tilde{\theta}^t \in Q) \mathbb{E}^{\{\tilde{\theta}^t: \tilde{\theta}^t \in Q\}} \left[ \begin{aligned} & \delta^{t-1} [v^{-1}(v(c_t^{opt}(\tilde{\theta}^t; e)) + k) - c_t^{opt}(\tilde{\theta}^t; e)] \\ & + \delta^t \mathbb{E}^{\tilde{\theta}_{t+1}|\tilde{\theta}^t} [v^{-1}(v(c_{t+1}^{opt}(\tilde{\theta}^{t+1}; e)) - \frac{k}{\delta}) - c_{t+1}^{opt}(\tilde{\theta}^{t+1}; e)] \end{aligned} \right]$$

where  $F(\tilde{\theta}^t \in Q)$  denotes the ex-ante probability that  $\tilde{\theta}^t \in Q$  and  $\mathbb{E}^{\{\tilde{\theta}^t: \tilde{\theta}^t \in Q\}}[\cdot]$  denotes the conditional expectation of  $[\cdot]$  over  $R^t$ , given the sigma-algebra generated by the event that  $\tilde{\theta}^t \in Q$ .

Clearly,  $\Delta(0) = 0$  and

$$\frac{\partial \Delta(0)}{\partial k} = F(\tilde{\theta}^t \in Q) \mathbb{E}^{\{\tilde{\theta}^t: \tilde{\theta}^t \in Q\}} \delta^{t-1} \left[ \frac{1}{v'_t(c_t^{opt}(\tilde{\theta}^t; e))} + \mathbb{E}^{\tilde{\theta}_{t+1}|\tilde{\theta}^t} \left( \frac{\delta}{v'_{t+1}(c_{t+1}^{opt}(\tilde{\theta}^{t+1}; e))} \right) \right] > 0.$$

The principal can then reduce her expected payment to the agent by switching to the compensation  $c^\#(\cdot; e)$  with  $k < 0$  arbitrarily small, contradicting the assumption that  $c^{opt}(\cdot; e)$  solves program (2).

■

**Proof of Proposition 2.** We start with (i). Consider the following compensation scheme  $s$ . For any  $t$  any  $(\theta^t, \pi^t) \in \Theta^t \times \Pi^t$ ,  $s_t(\theta^t, \pi^t) = c_t(\theta^t)$  if  $\pi_t \geq \theta_t + \xi_t(\theta^t)$  and  $s_t(\theta^t, \pi^t) = -L_t$  otherwise, with

the functions  $c_t$  satisfying for all  $\theta^T \in \Theta^T$ ,

$$\begin{aligned} \mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t(c_t(\theta^t)) \right) &= \sum_{t=1}^T \delta^{t-1} \psi(\xi_t(\theta^t)) \\ &+ \int_{\theta_1}^{\theta_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(\xi_t(\tilde{\theta}^t)) \right] ds \\ &+ \sum_{t=2}^T \delta^{t-1} H_t(\theta^t) \end{aligned} \quad (28)$$

where for each  $\theta^t \in \Theta^{t31}$

$$\begin{aligned} H_t(\theta^t) &\equiv \int_{\theta_t}^{\theta_t} \mathbb{E}^{\tilde{\theta}_{>t}^T | s} \left[ \sum_{\tau=t}^T \delta^{\tau-t} J_t^\tau(s, \tilde{\theta}_{>t}^\tau) \psi'(\xi_\tau(\theta^{t-1}, s, \tilde{\theta}_{>t}^\tau)) \right] ds \\ &- \mathbb{E}^{\tilde{\theta}_t | \theta_{t-1}} \left[ \int_{\theta_t}^{\tilde{\theta}_t} \mathbb{E}^{\tilde{\theta}_{>t}^T | s} \left[ \sum_{\tau=t}^T \delta^{\tau-t} J_t^\tau(s, \tilde{\theta}_{>t}^\tau) \psi'(\xi_\tau(\theta^{t-1}, s, \tilde{\theta}_{>t}^\tau)) \right] ds \right]. \end{aligned}$$

Under the proposed scheme, given the announcement  $\hat{\theta}^s$  and the true type profile  $\theta^s$ , the effort the agent chooses in each period  $s$  is given by

$$c_s(\theta_s; \hat{\theta}^s) = \xi_s(\hat{\theta}^s) + \hat{\theta}_s - \theta_s \quad (29)$$

Hence, we can conveniently denote histories by  $(\theta^t, \hat{\theta}^t)$ . Now let

$$\begin{aligned} U(\theta^t; \hat{\theta}^t) &\equiv \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta_t} \left[ \mathcal{V} \left( \sum_{s=1}^t \delta^{s-1} v_s(c_s(\hat{\theta}^s)) + \sum_{s=t+1}^T \delta^{s-1} v_s(c_s(\hat{\theta}^t, \tilde{\theta}_{>t}^s)) \right) \right] \\ &- \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta_t} \left[ \sum_{s=1}^t \delta^{s-1} \psi(e_s(\theta_s; \hat{\theta}^s)) + \sum_{s=t+1}^T \delta^{s-1} \psi(\xi_s(\hat{\theta}^t, \tilde{\theta}_{>t}^s)) \right] \end{aligned}$$

denote the expected payoff when, following  $(\theta^t, \hat{\theta}^{t-1})$ , in period  $t$  the agent sends the message  $\hat{\theta}_t$ , he then chooses effort  $e_t(\theta_t; \hat{\theta}^t) = \xi_t(\hat{\theta}^t) + \hat{\theta}_t - \theta_t$  and then from period  $t+1$  onwards follows a truthful

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<sup>31</sup>Note that the expressions for the expectations of the future realizations of the process use the property that the process is Markov.

and obedient strategy. Under the proposed scheme,

$$\begin{aligned}
U(\theta^t; \hat{\theta}^t) &= \sum_{s=1}^t \delta^{s-1} \left[ \psi(\xi_t(\hat{\theta}^t)) - \psi(e_s(\theta_s; \hat{\theta}^s)) \right] + \int_{\theta_1}^{\hat{\theta}_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(\xi_t(\tilde{\theta}^t)) \right] ds \\
&\quad + \sum_{s=2}^t \delta^{s-1} H_s(\hat{\theta}^s) \\
&\quad + \mathbb{E}^{\tilde{\theta}_{t+1} | \theta_t} \left[ \int_{\tilde{\theta}_{t+1}}^{\tilde{\theta}_{t+1}} \mathbb{E}^{\tilde{\theta}_{>t+1}^T | s} \left[ \sum_{\tau=t+1}^T \delta^{\tau-1} J_{t+1}^\tau(s, \tilde{\theta}_{>t+1}^\tau) \psi'(\xi_\tau(\hat{\theta}^t, s, \tilde{\theta}_{>t+1}^\tau)) \right] ds \right] \\
&\quad - \mathbb{E}^{\tilde{\theta}_{t+1} | \hat{\theta}_t} \left[ \int_{\tilde{\theta}_{t+1}}^{\tilde{\theta}_{t+1}} \mathbb{E}^{\tilde{\theta}_{>t+1}^T | s} \left[ \sum_{\tau=t+1}^T \delta^{\tau-1} J_{t+1}^\tau(s, \tilde{\theta}_{>t+1}^\tau) \psi'(\xi_\tau(\hat{\theta}^t, s, \tilde{\theta}_{>t+1}^\tau)) \right] ds \right].
\end{aligned}$$

Integrating by parts and using (29) one can verify that

$$\frac{\partial U(\theta^t; \hat{\theta}^t)}{\partial \theta_t} = \delta^{t-1} \psi'(\xi_t(\hat{\theta}^t)) + \hat{\theta}_t - \theta_t + \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta_t} \left[ \sum_{s=t+1}^T \delta^{s-1} J_t^s(\theta_t, \tilde{\theta}_{>t}^s) \psi'(\xi_s(\hat{\theta}^t, \tilde{\theta}_{>t}^s)) \right].$$

One can also verify that

$$\frac{dU(\theta^t; \hat{\theta}^{t-1}, \theta_t)}{d\theta_t} = \delta^{t-1} \psi'(\xi_t(\hat{\theta}^{t-1}, \theta_t)) + \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta_t} \left[ \sum_{s=t+1}^T \delta^{s-1} J_t^s(\theta_t, \tilde{\theta}_{>t}^s) \psi'(\xi_s(\hat{\theta}^{t-1}, \theta_t, \tilde{\theta}_{>t}^s)) \right]$$

The single crossing conditions in the proposition imply that

$$\left[ \frac{dU(\theta^t; \hat{\theta}^{t-1}, \theta_t)}{d\theta_t} - \frac{\partial U(\theta^t; \hat{\theta}^{t-1}, \hat{\theta}_t)}{\partial \theta_t} \right] [\theta_t - \hat{\theta}_t] \geq 0$$

all  $\theta^t \in R^t$ ,  $\hat{\theta}^t \in \Theta^t$ , which in turn implies that  $U(\theta^t; \hat{\theta}^{t-1}, \theta_t) \geq U(\theta^t; \hat{\theta}^{t-1}, \hat{\theta}_t)$  all  $\theta^t \in R^t$ ,  $\hat{\theta}^t \in \Theta^t$ .

Because these conditions apply to all  $t$ , all histories  $(\theta^t, \hat{\theta}^t) \in R^t \times \Theta^t$ , we then conclude that, after any history  $(\theta^t, \hat{\theta}^{t-1})$ , a single deviation in period  $t$  from the truthful and obedient strategy in the continuation game that starts in period  $t$  with history  $(\theta^t, \hat{\theta}^{t-1})$  is never optimal for the agent.

Together with continuity at infinity, this establishes that, when the principal offers the mechanism  $\Omega = \langle \xi, s \rangle$  with recommendation policy  $\xi$  satisfying the single-crossing conditions in the proposition and with compensation scheme as defined above, then each type  $\theta_1$  finds it optimal to participate and follow a truthful and obedient strategy at all periods. This establishes that  $e$  is implementable.

Next, consider case (ii) and note that in this case the single-crossing conditions (6) are trivially satisfied at any  $t > 1$ . Let  $w_t \equiv v_t^{-1}$  and  $\mathcal{W} \equiv \mathcal{V}^{-1}$ . Let  $s$  be the compensation scheme defined as



follows. For  $t = 1$ ,

$$s_1(\theta_1, \pi_1) = w_1 \left( \mathcal{W} \left( \begin{array}{c} \sum_{t=1}^T \delta^{t-1} \psi(\xi_t(\theta_1)) + \int_{\theta_1}^{\hat{\theta}_1} \mathbb{E}^{\tilde{\theta}_{>1}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(s, \tilde{\theta}_{>1}^t) \psi'(\xi_t(s)) \right] ds \\ - \mathbb{E}^{\tilde{\theta}_{>1}^T | \theta_1} \left[ \sum_{t=2}^T \delta^{t-1} \psi'(\xi_t(\theta_1)) (\tilde{\theta}_t + \xi_t(\theta_1)) \right] \end{array} \right) \right)$$

if  $\pi_1 \geq \theta_1 + \xi_1(\theta_1)$  and  $s_1(\theta_1, \pi_1) = -L_1$  otherwise. For any  $t > 1$ ,

$$s_t(\theta^t, \pi^t) = w_t \left( \begin{array}{c} -\delta^{1-t} \sum_{\tau=1}^{t-1} \delta^{\tau-t} v_\tau(s_\tau(\theta^\tau, \pi^\tau)) \\ + \delta^{1-t} \mathcal{W} \left( \begin{array}{c} \sum_{t=1}^T \delta^{t-1} \psi(\xi_t(\theta_1)) + \int_{\theta_1}^{\hat{\theta}_1} \mathbb{E}^{\tilde{\theta}_{>1}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(s, \tilde{\theta}_{>1}^t) \psi'(\xi_t(s)) \right] ds \\ - \mathbb{E}^{\tilde{\theta}_{>1}^T | \theta_1} \left[ \sum_{t=2}^T \delta^{t-1} \psi'(\xi_t(\theta_1)) (\tilde{\theta}_t + \xi_t(\theta_1)) \right] \\ + \sum_{\tau=2}^t \delta^{\tau-1} \psi'(\xi_\tau(\theta_1)) \pi_\tau \end{array} \right) \end{array} \right)$$

if  $\pi_1 \geq \xi_1(\theta_1) + \theta_1$ , and  $s_t(\theta^t, \pi^t) = -L_t$  otherwise. Note that the above scheme is constructed so that, at each period  $t$ , if the agent were to receive no further payments from the principal in the future, then upon meeting the first-period target  $\pi_1 = \xi_1(\hat{\theta}_1) + \hat{\theta}_1$ , his utility from past and current payments would depend only on the first period report  $\hat{\theta}_1$  and be linear in the cash flows  $(\pi_s)_{s=2}^t$  generated between period 2 and period  $t$ . To see that such a scheme induces the desired effort choices, suppose first that  $T$  is finite. By construction, given any history of reports  $\hat{\theta}^T$  and any history of profit realizations  $\pi^T$ , with  $\pi_1 \geq \xi_1(\hat{\theta}_1) + \hat{\theta}_1$ , the agent's total utility from money is given by

$$\begin{aligned} \mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t(s_t(\hat{\theta}^t, \pi^t)) \right) &= \sum_{t=1}^T \delta^{t-1} \psi(\xi_t(\hat{\theta}_1)) + \int_{\theta_1}^{\hat{\theta}_1} \mathbb{E}^{\tilde{\theta}_{>1}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(s, \tilde{\theta}_{>1}^t) \psi'(\xi_t(s)) \right] ds \\ &\quad + \sum_{t=2}^T \delta^{t-1} \psi'(\xi_t(\hat{\theta}_1)) (\pi_t - \mathbb{E}^{\tilde{\theta}_{>1}^T | \hat{\theta}_1} [\tilde{\theta}_t] - \xi_t(\hat{\theta}_1)). \end{aligned}$$

Next, consider the case where  $T = \infty$ . By continuity of  $\mathcal{V}$ ,

$$\begin{aligned} \mathcal{V} \left( \sum_{t=1}^{\infty} \delta^{t-1} v_t(s_t(\hat{\theta}^t, \pi^t)) \right) &= \lim_{Z \rightarrow \infty} \mathcal{V} \left( \sum_{t=1}^Z \delta^{t-1} v_t(s_t(\hat{\theta}^t, \pi^t)) \right) \\ &= \sum_{t=1}^{\infty} \delta^{t-1} \psi(\xi_t(\hat{\theta}_1)) + \int_{\theta_1}^{\hat{\theta}_1} \mathbb{E}^{\tilde{\theta}_{>1}^T | s} \left[ \sum_{t=1}^{\infty} \delta^{t-1} J_1^t(s, \tilde{\theta}_{>1}^t) \psi'(\xi_t(s)) \right] ds \\ &\quad + \sum_{t=2}^{\infty} \delta^{t-1} \psi'(\xi_t(\hat{\theta}_1)) (\pi_t - \mathbb{E}^{\tilde{\theta}_{>1}^T | \hat{\theta}_1} [\tilde{\theta}_t] - \xi_t(\hat{\theta}_1)). \end{aligned}$$

It is then easy to see that, under the proposed scheme, the agent's decision problem is time-separable after  $t = 1$ . After having reported  $\hat{\theta}_1$  in period one, and having delivered a cash flow  $\pi_1 \geq \xi_1(\hat{\theta}_1) + \hat{\theta}_1$ , at any period  $t > 1$ , and for any possible history  $(\theta^t, (\hat{\theta}_1, \hat{\theta}_{>1}^t), e^{t-1})$ , it is optimal

for the agent to choose a level of effort  $e_t = \xi_t(\hat{\theta}_1)$ . To establish the result it then suffices to verify that each type  $\theta_1$  finds it optimal to report truthfully in period one. The payoff that each type  $\theta_1$  obtains by reporting  $\hat{\theta}_1$  and then choosing optimally a level of effort  $e_1(\theta_1; \hat{\theta}_1) = \xi_1(\hat{\theta}_1) + \hat{\theta}_1 - \theta_1$  for  $t = 1$  and  $e_t = \xi_t(\hat{\theta}_1)$  for any  $t > 1$  is

$$\begin{aligned} U(\theta_1; \hat{\theta}_1) &= \psi(\xi_1(\hat{\theta}_1)) - \psi(\xi_1(\hat{\theta}_1) + \hat{\theta}_1 - \theta_1) \\ &\quad + \int_{\theta_1}^{\hat{\theta}_1} \mathbb{E}^{\tilde{\theta}_{>1}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(s, \tilde{\theta}_{>1}^t) \psi'(\xi_t(s)) \right] ds \\ &\quad - \mathbb{E}^{\tilde{\theta}_{>1}^T | \hat{\theta}_1} \left[ \sum_{t=2}^T \delta^{t-1} \psi'(\xi_t(\hat{\theta}_1)) (\tilde{\theta}_t + \xi_t(\hat{\theta}_1)) \right] + \mathbb{E}^{\tilde{\theta}_{>1}^T | \theta_1} \left[ \sum_{t=2}^T \delta^{t-1} \psi'(\xi_t(\hat{\theta}_1)) (\tilde{\theta}_t + \xi_t(\hat{\theta}_1)) \right] \end{aligned}$$

Once again, integrating by parts, one can verify that the single-crossing condition (6) in the proposition implies that

$$\left[ \frac{dU(\theta_1; \theta_1)}{d\theta_1} - \frac{\partial U(\theta_1; \hat{\theta}_1)}{\partial \theta_1} \right] [\theta_1 - \hat{\theta}_1] \geq 0$$

which in turn implies that  $U(\theta_1; \theta_1) \geq U(\theta_1; \hat{\theta}_1)$  all  $\theta_1, \hat{\theta}_1 \in \Theta_1$ , thus establishing the result. ■

### Proof of Proposition 3.

We start with the following definition. For any given extension  $\xi$  of any given policy  $e$ , let  $c^{opt}(\cdot; \xi) \equiv \left\langle c_t^{opt}(\cdot; \xi) \right\rangle_{t=1}^{t=T}$  denote any arbitrary ( $\Theta$ -measurable) extension of the compensation  $c^{opt}(\cdot; e)$  defined in Proposition 1 such that

$$\mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t(c_t^{opt}(\theta^t)) \right) = W^\#(\theta^T) \quad \text{all } \theta^T \in \Theta^T \quad (30)$$

where

$$\begin{aligned} W^\#(\theta^T) &\equiv \sum_{t=1}^T \delta^{t-1} \psi(\xi_t(\theta^t)) \\ &\quad + \int_{\theta_1}^{\theta_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(\xi_t(\tilde{\theta}^t)) \right] ds + \sum_{t=2}^T \delta^{t-1} H_t^\#(\theta^t) \end{aligned}$$

with

$$\begin{aligned} H_t^\#(\theta^t) &\equiv \int_{\underline{\theta}_t(\theta^{t-1})}^{\theta_t} \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta^{t-1}, s} \left[ \sum_{\tau=t}^T \delta^{\tau-t} J_\tau^\#(\theta^{t-1}, s, \tilde{\theta}_{>t}^\tau) \psi'(\xi_\tau(\hat{\theta}^{t-1}, s, \tilde{\theta}_{>t}^\tau)) \right] ds \\ &\quad - \mathbb{E}^{\tilde{\theta}_t | \theta_{t-1}} \left[ \int_{\underline{\theta}_t}^{\tilde{\theta}_t} \mathbb{E}^{\tilde{\theta}_{>t}^T | s} \left[ \sum_{\tau=t}^T \delta^{\tau-t} J_\tau^\#(\theta^{t-1}, s, \tilde{\theta}_{>t}^\tau) \psi'(\xi_\tau(\hat{\theta}^{t-1}, s, \tilde{\theta}_{>t}^\tau)) \right] ds \right] \end{aligned}$$

In other terms,  $c^{opt}(\cdot; \xi)$  is any ( $\Theta$ -measurable) compensation  $c \equiv \langle c_t(\cdot) \rangle_{t=1}^{t=T}$  that minimizes  $\mathbb{E} \left[ \sum_{t=1}^T \delta^{t-1} c_t(\hat{\theta}^t) \right]$  among those that satisfy the constraints given by (30).

Next, consider the following pseudo-linear scheme. For  $t = 1$ ,

$$s_1(\theta_1, \pi_1) = w_1 \left( \mathcal{W} \left( \mathcal{V} \left( v_1 \left( c_1^{opt}(\theta_1; \xi) \right) \right) + \psi'(\xi_1(\theta_1)) [\pi_1 - \theta_1 - \xi_1(\theta_1)] \right) \right)$$

while for any  $t > 1$  any  $(\theta^t, \pi^t)$

$$s_t(\theta^t, \pi^t) = w_t \left( \frac{1}{\delta^{t-1}} \mathcal{W} \left( \mathcal{V} \left( \sum_{\tau=1}^t \delta^{\tau-1} v_t \left( c_\tau^{opt}(\theta^\tau; \xi) \right) \right) + \sum_{\tau=1}^t \delta^{\tau-1} \psi'(\xi_\tau(\theta^\tau)) [\pi_\tau - \theta_\tau - \xi_\tau(\theta^\tau)] \right) - \frac{1}{\delta^{t-1}} \sum_{\tau=1}^{t-1} \delta^{\tau-1} v_t(s_\tau(\theta^\tau, \pi^\tau)) \right)$$

where recall that  $w_t \equiv v_t^{-1}$  and  $\mathcal{W} \equiv \mathcal{V}^{-1}$ . Note that this scheme is constructed so that, when the agent reports his type truthfully and chooses effort obediently, the payments he receives over time coincide with the cost-minimizing compensations given by  $c^{opt}(\cdot; \xi)$ . From Proposition 1, it then follows that, if this scheme implements the policy  $e$ , it then necessarily implements it at minimal cost for the principal. In what follows, it thus suffices to show that, under such a scheme, a truthful and obedient strategy is optimal for the agent. To this purpose, note that this scheme is also chosen so that, given any sequence of reports and cash flows  $(\theta^T, \pi^T)$ , the utility that the agent derives from the entire stream of payments is given by

$$\mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t(s_t(\theta^t, \pi^t)) \right) = W^\#(\theta^T) + \sum_{t=1}^T \delta^{t-1} \psi'(\xi_t(\theta^t)) (\pi_t - \theta_t - \xi_t(\theta^t)).$$

The latter is linear in each cash-flow  $\pi_t$  with coefficient of linear dependence  $\alpha_t(\theta^t) = \psi'(\xi_t(\theta^t))$ . As mentioned already in the proof of Proposition 2, such linearity guarantees that the decision problem for the agent is time-separable: in each period  $t \geq 1$ , after having reported  $\hat{\theta}^t$  and irrespective of the history  $(\theta^t, e^{t-1})$ , the agent finds it optimal to choose a level of effort  $e_t = \xi_t(\hat{\theta}^t)$ . In other words, such linearity takes care of the moral-hazard part of the problem. To establish the result, it then suffices to verify that, in each period  $t$ , the agent finds it optimal to report his type  $\theta_t$  truthfully. To

establish this, let

$$\begin{aligned}
& U(\theta^t; \hat{\theta}^t, e^{t-1}) \\
& \equiv \\
& \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta^t} \left[ \mathcal{V} \left( \begin{aligned} & \sum_{\tau=1}^{t-1} \delta^{\tau-1} v_\tau \left( s_\tau \left( \hat{\theta}^\tau, (\theta_z + e_z)_{z=1}^\tau \right) \right) + \\ & + \delta^{t-1} v_t \left( s_t \left( \hat{\theta}^t, \left( (\theta_z + e_z)_{z=1}^{t-1}, \theta_t + \xi_t(\hat{\theta}^t) \right) \right) \right) + \\ & \sum_{\tau=t+1}^T \delta^{\tau-1} v_\tau \left( s_\tau \left( \hat{\theta}^t, \tilde{\theta}_{>t}^\tau, \left( (\theta_z + e_z)_{z=1}^{t-1}, \theta_t + \xi_t(\hat{\theta}^t), (\tilde{\theta}_z + \xi_z(\hat{\theta}^t, \tilde{\theta}_{>t}^z))_{z=t+1}^\tau \right) \right) \right) \end{aligned} \right) \right] \\
& \quad - \sum_{\tau=1}^{t-1} \delta^{\tau-1} \psi(e_\tau) + \delta^{t-1} \psi(\xi_t(\hat{\theta}^t)) - \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta^t} \left[ \sum_{\tau=t+1}^T \psi(\xi_\tau(\hat{\theta}^t, \tilde{\theta}_{>t}^\tau)) \right]
\end{aligned}$$

denote the expected payoff when, under the proposed scheme, following the history  $(\theta^t, \hat{\theta}^{t-1}, e^{t-1})$ , in period  $t$  the agent sends the message  $\hat{\theta}_t$ , he then chooses optimally effort  $e_t = \xi_t(\hat{\theta}^t)$  and then from period  $t+1$  onwards he follows a truthful and obedient strategy. To establish that truthful reporting in each period is optimal for the agent, we consider the two cases in the Proposition separately.

**Case (i): Markov process.** Using the definitions of  $s$  and  $c^{opt}(\cdot; \xi)$  and the fact that the process is Markov, we have that in this case

$$\begin{aligned}
U(\theta^t; \hat{\theta}^t, e^{t-1}) &= \int_{\theta_1}^{\hat{\theta}_1} \mathbb{E}^{\tilde{\theta}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(\xi_t(\tilde{\theta}^t)) \right] ds \\
&+ \sum_{\tau=1}^{t-1} \delta^{\tau-1} \left\{ \psi(\xi_\tau(\hat{\theta}^\tau)) - \psi(e_\tau) + \psi'(\xi_\tau(\hat{\theta}^\tau)) \left[ \theta_\tau + e_\tau - \hat{\theta}_\tau - \xi_\tau(\hat{\theta}^\tau) \right] \right\} \\
&+ \delta^{t-1} \psi'(\xi_t(\hat{\theta}^t)) \left[ \theta_t - \hat{\theta}_t \right] \\
&+ \sum_{\tau=2}^t \delta^{\tau-1} H_\tau^\#(\hat{\theta}^\tau) + \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta^t} \left[ \sum_{s=t+1}^T \delta^{s-1} H_s^\#(\hat{\theta}^t, \tilde{\theta}_{>t}^s) \right]
\end{aligned} \tag{31}$$

from which we obtain that

$$\frac{\partial U(\theta^t; \hat{\theta}^t, e^{t-1})}{\partial \theta_t} = \delta^{t-1} \psi'(\xi_t(\hat{\theta}^t)) + \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta^t} \left[ \sum_{s=t+1}^T \delta^{s-1} J_t^s(\theta_t, \tilde{\theta}_{>t}^s) \psi'(\xi_s(\hat{\theta}^t, \tilde{\theta}_{>t}^s)) \right]$$

and

$$\frac{dU(\theta^t; \hat{\theta}^{t-1}, \theta_t, e^{t-1})}{d\theta_t} = \delta^{t-1} \psi'(\xi_t(\hat{\theta}^{t-1}, \theta_t)) + \mathbb{E}^{\tilde{\theta}_{>t}^T | \theta^t} \left[ \sum_{s=t+1}^T \delta^{s-1} J_t^s(\theta_t, \tilde{\theta}_{>t}^s) \psi'(\xi_s(\hat{\theta}^{t-1}, \theta_t, \tilde{\theta}_{>t}^s)) \right]$$

The single crossing conditions in the proposition imply that

$$\left[ \frac{dU(\theta^t; \hat{\theta}^{t-1}, \theta_t)}{d\theta_t} - \frac{\partial U(\theta^t; \hat{\theta}^{t-1}, \hat{\theta}_t)}{\partial \theta_t} \right] [\theta_t - \hat{\theta}_t] \geq 0$$

all  $\theta^t \in R^t$ ,  $\hat{\theta}^t \in \Theta^t$ , which in turn implies that  $U(\theta^t; \hat{\theta}^{t-1}, \theta_t) \geq U(\theta^t; \hat{\theta}^{t-1}, \hat{\theta}_t)$  all  $\theta^t \in R^t$ ,  $\hat{\theta}^t \in \Theta^t$ . As argued in the proof of Proposition 2, these conditions guarantee that, after any history  $(\theta^t, \hat{\theta}^{t-1}, e^{t-1})$ , a single deviation in period  $t$  from the truthful and obedient strategy in the continuation game that starts in period  $t$  with history  $(\theta^t, \hat{\theta}^{t-1}, e^{t-1})$  is never optimal for the agent. Together with continuity at infinity, this implies that the agent finds it optimal to report truthfully at all histories. The above scheme  $s$  thus implements the policy  $e$ .

**Case (ii). Policies depending on  $\theta_1$  only.** Using the fact that

$$\mathbb{E}^{\tilde{\theta}^T | \theta^{t-1}, \theta_t} [J_t^T(\tilde{\theta}^\tau)] = \frac{\partial}{\partial \theta_t} \left[ \mathbb{E}^{\tilde{\theta}^T | \theta^{t-1}, \theta_t} [\tilde{\theta}_\tau] \right]$$

all  $t, \tau, \tau > t$ , all  $\theta^t$ , one can verify that

$$\begin{aligned} \mathcal{V} \left( \sum_{t=1}^T \delta^{t-1} v_t \left( s_t(\hat{\theta}^t, \pi^t) \right) \right) &= \sum_{t=1}^T \delta^{t-1} \psi(\xi_t(\hat{\theta}_1)) + \int_{\underline{\theta}_1}^{\hat{\theta}_1} \mathbb{E}^{\tilde{\theta}_{>1}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(s, \tilde{\theta}_{>1}^t) \psi'(\xi_t(s)) \right] ds \\ &\quad + \sum_{t=2}^T \delta^{t-1} \psi'(\xi_t(\hat{\theta}_1)) (\pi_t - \mathbb{E}^{\tilde{\theta}_{>1}^T | \hat{\theta}_1} [\tilde{\theta}_t] - \xi_t(\hat{\theta}_1)). \end{aligned}$$

It is therefore easy to see that, under the proposed scheme, after having reported  $\hat{\theta}_1$  in period one, at any period  $t \geq 1$ , and for any possible history  $(\theta^t, \hat{\theta}^t, e^{t-1})$ , it is optimal for the agent to choose a level of effort  $e_t = \xi_t(\hat{\theta}_1)$ . It is also easy to see that the payoff that each type  $\theta_1$  obtains by reporting  $\hat{\theta}_1$  in period one and then choosing effort optimally at any subsequent information set is equal to

$$\begin{aligned} U(\theta_1; \hat{\theta}_1) &= \int_{\underline{\theta}_1}^{\hat{\theta}_1} \mathbb{E}^{\tilde{\theta}_{>1}^T | s} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(s, \tilde{\theta}_{>1}^t) \psi'(\xi_t(s)) \right] ds \\ &\quad + \sum_{t=1}^T \delta^{t-1} \psi'(\xi_t(\hat{\theta}_1)) \left[ \mathbb{E}^{\tilde{\theta}_t | \theta_1} [\tilde{\theta}_t] - \mathbb{E}^{\tilde{\theta}_t | \hat{\theta}_1} [\tilde{\theta}_t] \right] \end{aligned}$$

from which we obtain that

$$\frac{\partial U(\theta_1; \hat{\theta}_1)}{\partial \theta_1} = \mathbb{E}^{\tilde{\theta}_{>1}^T | \theta_1} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(\theta_1, \tilde{\theta}_{>1}^t) \psi'(\xi_t(\hat{\theta}_1)) \right]$$

and

$$\frac{dU(\theta_1; \theta_1)}{d\theta_1} = \mathbb{E}^{\tilde{\theta}_{>1}^T | \theta_1} \left[ \sum_{t=1}^T \delta^{t-1} J_1^t(\theta_1, \tilde{\theta}_{>1}^t) \psi'(\xi_t(\theta_1)) \right]$$

Once again, the single-crossing condition (6) in the proposition implies that, for all  $\theta_1, \hat{\theta}_1 \in \Theta_1$ ,

$$\left[ \frac{dU(\theta_1; \theta_1)}{d\theta_1} - \frac{\partial U(\theta_1; \hat{\theta}_1)}{\partial \theta_1} \right] [\theta_1 - \hat{\theta}_1] \geq 0.$$

thus implying that  $U(\theta_1; \theta_1) \geq U(\theta_1; \hat{\theta}_1)$  all  $\theta_1, \hat{\theta}_1 \in \Theta_1$ , which establishes the result. ■

**Proof of Proposition 4. Part (i).** The assumptions that  $\psi$  is continuously differentiable with  $\psi(e) = 0$  for all  $e < 0$ ,  $\psi''(e) > 0$  and  $\psi'''(e) \geq 0$  for all  $e \in [0, \bar{e}]$ ,  $\psi'(e) = K$  for all  $e > \bar{e}$ , together with  $J_1^t(\theta^t) \geq 0$  for all  $t$  all  $\theta^t$ , imply that, for all  $t$  all  $\theta^t \in R^t$ , the principal's payoff, as given by (11), is strictly increasing in  $e_t$  for all  $e_t < e_t^*(\theta^t)$ , and strictly decreasing in  $e_t$  for all  $e_t > e_t^*(\theta^t)$ , where  $e_t^*(\theta^t)$  is implicitly given by (12) when  $\psi_+'''(0) < 1/[\eta(\theta_1)J_1^t(\theta^t)]$  and by  $e_t^*(\theta^t) = 0$  otherwise. It is then immediate that when the policy  $e^*$  is implementable, then, under any optimal contract for the principal,  $e^*$  is implemented in each period  $t$  at  $F$ -almost all histories. This follows from the fact that the principal's payoff in any mechanism that is incentive-compatible and individually-rational for the agent is given by (11) together with the fact the policy  $e^*$  maximizes (11) and the fact that, if the policy  $e^*$  is implementable, then from Proposition 1 there exists a compensation scheme  $s^*$  that implements  $e^*$  and that gives the lowest period-1 type  $\underline{\theta}_1$  an expected payoff equal to his outside option, in which case  $V^\Omega(\underline{\theta}_1) = 0$ .

**Part (ii).** The result follows from Propositions 2 and 3. Consider the following extension of the policy  $e^*$  from  $R$  to  $\Theta$ . For any  $t \geq 2$ , any  $2 \leq s \leq t$ , let  $\varphi_s : \Theta^t \rightarrow \Theta^t$  be the function defined, for all  $\theta^t \in \Theta^t$ , by

$$\varphi_s(\theta^t) \equiv \begin{cases} \theta^t & \text{if } \theta_s \in \text{Supp}[F_s(\cdot | \theta^{s-1})] \\ (\theta^{s-1}, \min\{\text{Supp}F_s(\cdot | \theta^{s-1})\}, \theta_{s+1}, \dots, \theta_t) & \text{if } \theta_s < \min\{\text{Supp}[F_s(\cdot | \theta^{s-1})]\} \\ (\theta^{s-1}, \max\{\text{Supp}[F_s(\cdot | \theta^{s-1})]\}, \theta_{s+1}, \dots, \theta_t) & \text{if } \theta_s > \max\{\text{Supp}[F_s(\cdot | \theta^{s-1})]\} \end{cases} .$$

For all  $\theta_1 \in \Theta_1$ , then let  $\lambda^1(\theta_1) \equiv \theta_1$ , while for any  $t \geq 2$ , let  $\lambda^t : \Theta^t \rightarrow R^t$  be the function defined, for all  $\theta^t \in \Theta^t$ , by  $\lambda^t(\theta^t) \equiv \varphi_t \circ \varphi_{t-1} \circ \dots \circ \varphi_2(\theta^t)$ . Note that the function  $\lambda^t$  maps each vector of reports  $\theta^t \in \Theta^t \setminus R^t$  which reveals that the agent has been untruthful in previous periods into a

vector of reports  $\hat{\theta}^t = (\lambda_s^t(\theta^t))_{s=1}^t$  that is consistent with truth-telling in all previous periods. This is obtained by replacing recursively any report  $\theta_s$  that, given the past reports  $\hat{\theta}^{s-1} = (\lambda_l^t(\theta^t))_{l=1}^{s-1}$  is smaller than any feasible type with  $\hat{\theta}_s = \min\{\text{Supp}[F_s(\cdot|\hat{\theta}^{s-1})]\}$ , and, likewise, by replacing any report  $\theta_s$  that is higher than any feasible type with  $\hat{\theta}_s = \max\{\text{Supp}[F_s(\cdot|\hat{\theta}^{s-1})]\}$ .

Now, let  $\xi$  be the recommendation policy given by  $\xi_t(\theta^t) = e_t^*(\lambda^t(\theta^t))$  all  $t$ , all  $\theta^t \in \Theta^t$ . Clearly, when  $\Theta^t = R^t$  all  $t$ , the policy  $\xi$  is simply the policy which recommends the equilibrium effort at all histories irrespective of whether or not the agent has been truthful in the past. When, instead,  $R^t \subset \Theta^t$  for some  $t$ , the policy  $\xi$  is obtained by replacing the reports that indicate a departure from truth-telling in past periods with the reports  $\lambda^t(\theta^t)$  and then recommending the equilibrium effort  $e_t^*(\lambda^t(\theta^t))$  for the type history  $\lambda^t(\theta^t)$ .

Assumption (b) in the proposition guarantees that each function  $e_t^*(\theta^t)$  is nondecreasing. By construction, each function  $\xi_t(\theta^t)$  is also nondecreasing. We conclude that the extension  $\xi$  constructed above satisfies not only the single-crossing conditions (6) of Proposition 2 but also the stronger conditions (7) of Proposition 3. From the results in those propositions, we then conclude that when, in addition,  $F$  is Markov or each function  $J_1^T(\theta^t)$  depends only on  $\theta^t$  only through  $\theta_1$  (in which case so does each function  $\xi_t$ ), then the policy  $e^*$  can be implemented at minimum costs by both the bonus schemes of Proposition 1 or by the linear schemes of Proposition 3. ■

**Proof of Example 4.** Using (13), the principal's expected payoff in any incentive-compatible mechanism  $\hat{\Omega}$  implementing a policy  $\hat{e}$  that is contingent on  $\theta_1$  only (i.e. such that there exists a collection of functions  $\hat{e}_t : \Theta_1 \rightarrow \mathbb{R}$ ,  $t = 2, \dots, T$ , such that  $e_t(\theta_1, \varepsilon^t) = \hat{e}_t(\theta_1)$  for all  $\varepsilon^t$ ) is given by  $\mathbb{E}[Z(\tilde{\theta}_1)]$ , where, for any  $\theta_1 \in \Theta_1$ ,

$$\begin{aligned} Z(\theta_1) \equiv & \mathbb{E} \left[ \sum_{s=1}^T \left( J_1^s \theta_1 + \sum_{\tau=2}^s J_\tau^s \tilde{\varepsilon}_\tau \right) \right] + \sum_{s=1}^T \hat{e}_s(\theta_1) \\ & - \frac{\alpha}{2} \mathbb{E} \left[ \left( \begin{aligned} & \sum_{s=1}^T \psi(\hat{e}_s(\theta_1)) + V^{\hat{\Omega}}(\underline{\theta}_1) \\ & + \int_{\underline{\theta}_1}^{\theta_1} \sum_{s=1}^T J_1^s \psi'(\hat{e}_s(z)) dz + \sum_{s=2}^T \hat{H}_s(\theta_1, \tilde{\varepsilon}^s) \end{aligned} \right)^2 \right] \\ & - \beta \mathbb{E} \left[ \left( \begin{aligned} & \sum_{s=1}^T \psi(\hat{e}_s(\theta_1)) + V^{\hat{\Omega}}(\underline{\theta}_1) \\ & + \int_{\underline{\theta}_1}^{\theta_1} \sum_{s=1}^T J_1^s \psi'(\hat{e}_s(\tilde{\theta}_1)) d\tilde{\theta}_1 + \sum_{s=2}^T \hat{H}_s(\theta_1, \tilde{\varepsilon}^s) \end{aligned} \right) \right], \end{aligned}$$

where, for  $s = 2, \dots, T$ ,

$$\hat{H}_s(\theta_1, \varepsilon^s) = [\varepsilon_s - \mathbb{E}[\tilde{\varepsilon}_s]] \sum_{\tau=s}^T J_\tau^s \psi'(\hat{e}_\tau(\theta_1)).$$

For each  $s = 2, \dots, T$ , let  $\sigma_s^2 \equiv \text{Var}(\tilde{\varepsilon}_s)$ . Then

$$\begin{aligned} Z(\theta_1) &= \sum_{s=1}^T \left( J_1^s \theta_1 + \sum_{\tau=2}^s J_\tau^s \mathbb{E}[\tilde{\varepsilon}_\tau] \right) + \sum_{s=1}^T \hat{e}_s(\theta_1) \\ &\quad - \frac{\alpha}{2} \left( \sum_{s=1}^T \psi(\hat{e}_s(\theta_1)) + V^{\hat{\Omega}}(\theta_1) + \int_{\theta_1}^{\theta_1} \sum_{s=1}^T J_1^s \psi'(\hat{e}_s(z)) dz \right)^2 \\ &\quad - \frac{\alpha}{2} \sum_{s=2}^T \sigma_s^2 \left[ \sum_{\tau=s}^T J_\tau^s \psi'(\hat{e}_\tau(\theta_1)) \right]^2 \\ &\quad - \beta \left( \sum_{s=1}^T \psi(\hat{e}_s(\theta_1)) + V^{\hat{\Omega}}(\theta_1) + \int_{\theta_1}^{\theta_1} \sum_{s=1}^T J_1^s \psi'(\hat{e}_s(z)) dz \right). \end{aligned}$$

The principal's relaxed program then consists of choosing a vector of effort functions  $\hat{e}_t : \Theta_1 \rightarrow \mathbb{R}$ ,  $t = 1, \dots, T$ , along with a scalar  $V^{\hat{\Omega}}(\theta_1) \geq 0$ , so as to maximize  $\mathbb{E}[Z(\tilde{\theta}_1)]$ . It is immediate that, at the optimum,  $V^{\hat{\Omega}}(\theta_1) = 0$ . Furthermore, given that  $\psi'(\bar{e}) > 1/\beta$ , it is also immediate that any policy  $\hat{e} = (\hat{e}_t(\cdot))_{t=1}^T$  that maximizes  $\mathbb{E}[Z(\tilde{\theta}_1)]$  must have the property that, for any  $t$ ,  $\hat{e}_t(\theta_1) \in [0, \bar{e}]$  for almost every  $\theta_1 \in \Theta_1$ .

Now let  $g : [0, K] \rightarrow \mathbb{R}$  be the function defined by  $g(0) = 0$ ,  $g(y) = \psi'^{-1}(y)$ , all  $y \in (0, K)$ , and  $g(K) = \bar{e}$ . For any  $t = 1, \dots, T$ , any  $\theta_1 \in \Theta_1$ , then let  $u_t(\theta_1) \equiv \psi'(\hat{e}_t(\theta_1))$  and  $x_t(\theta_1) \equiv \int_{\theta_1}^{\theta_1} u_t(z) dz$ . Omitting the first term, which does not depend on the effort policy, the principal's relaxed problem thus consists in choosing functions  $u : \Theta_1 \rightarrow [0, K]^T$  and  $x : \Theta_1 \rightarrow \mathbb{R}_+^T$  that maximize

$$\int_{\theta_1}^{\bar{\theta}_1} L(\theta_1, u(\theta_1), x(\theta_1)) d\theta_1$$

where, for any  $(\theta_1, u, x) \in \Theta_1 \times [0, K]^T \times \mathbb{R}_+^T$ ,

$$L(\theta_1, u, x) \equiv f(\theta_1) \left[ \begin{array}{c} \sum_{s=1}^T g(u_s) - \frac{\alpha}{2} \left( \sum_{s=1}^T \psi(g(u_s)) + \sum_{s=1}^T J_1^s x_s \right)^2 \\ - \frac{\alpha}{2} \sum_{s=2}^T \sigma_s^2 \left( \sum_{\tau=s}^T J_\tau^s u_\tau \right)^2 - \beta \left( \sum_{s=1}^T \psi(g(u_s)) + \sum_{s=1}^T J_1^s x_s \right) \end{array} \right]$$

under the constraint that

$$x_t(\theta_1) = \int_{\theta_1}^{\theta_1} u_t(z) dz, \quad \forall t = 1, \dots, T, \quad \forall \theta_1 \in \Theta_1. \quad (32)$$



We solve this problem with optimal control treating  $u$  as the vector of control variables and  $x$  as the vector of state variables. First we verify that a solution to this optimal control problem exists by applying the Tonelli existence theorem.<sup>32</sup> To this aim, we first show that, for any  $(\theta_1, x) \in \Theta_1 \times \mathbb{R}_+^T$ , the function  $L(\theta_1, \cdot, x)$  is strictly concave. Note that, for any  $y \in (0, K)$ ,

$$g''(y) = \frac{d}{dy} \left[ \frac{1}{\psi''(g(y))} \right] = \frac{-\psi'''(g(y))}{[\psi''(g(y))]^3} < 0.$$

This implies that  $\sum_{s=1}^T g(u_s)$  is strictly concave in  $u$ . Next, note that, for any  $y \in (0, K)$ ,

$$\frac{d^2}{dy^2} \psi(g(y)) = \frac{d}{du} \left[ \frac{\psi'(g(y))}{\psi''(g(y))} \right] = \frac{h'(g(y))}{\psi''(g(y))}$$

where  $h(z) \equiv \frac{\psi'(z)}{\psi''(z)}$ , and so  $h'(z) \equiv \frac{d}{dz} \left( \frac{\psi'(z)}{\psi''(z)} \right) = \frac{d}{dz} \left( \left\{ \frac{d \log \psi'(z)}{dz} \right\}^{-1} \right) > 0$ , since  $\log \psi'$  is concave.

Hence  $\psi(g(y))$  is convex. Therefore,  $\sum_{s=1}^T \psi(g(u_s))$  is convex in  $u$ . Moreover, since  $(\cdot)^2$  is convex and is increasing whenever its argument is non-negative,  $-\left( \sum_{s=1}^T \psi(g(u_s)) + \sum_{s=1}^T J_1^s x_s \right)^2$  is concave in  $u$  as well. The same argument implies that, for any  $s$ , the function  $-\left( \sum_{\tau=s}^T J_s^\tau u_\tau \right)^2$  is concave in  $u$  so that  $-\frac{\alpha}{2} \sum_{s=2}^T \sigma_s^2 \left( \sum_{\tau=s}^T J_s^\tau u_\tau \right)^2$  is weakly concave. Together these observations imply that  $L$  is strictly concave in  $u$ , as required. Moreover,  $L$  is continuous.<sup>33</sup> Finally, that  $u$  is bounded renders the ‘‘coercivity’’ condition of Tonelli’s theorem unnecessary.<sup>34</sup> Thus a solution exists. Finally, that  $L$  is weakly jointly concave in  $(u, x)$  and strictly concave in  $u$  implies that the solution is essentially unique.

By Proposition 2.1 of Clarke (1989), that  $u$  is bounded and that  $L$  is strictly concave in  $u$  for each  $x$ , then guarantees that each  $x_t(\cdot)$  is continuously differentiable and that the Pontryagin principle applies.

The Hamiltonian function is given by

$$\mathcal{H} = L(\theta_1, u(\theta_1), x(\theta_1)) + \mu(\theta_1)^\top u(\theta_1).$$

<sup>32</sup>See, for example, Theorem 3.7 of Buttazzo, Giaquinta and Hildebrandt (1998).

<sup>33</sup>All that is required for a Tonelli-type existence theorem is that  $L$  be measurable in  $\theta_1$  for all admissible  $x$  and  $u \in [0, K]^T$ , and continuous in  $(x, u)$  for almost every  $\theta_1$ . See Theorem 3.6 of Buttazzo, Giaquinta and Hildebrandt (1998).

<sup>34</sup>See, for instance, Theorem 3.7 of Buttazzo, Giaquinta and Hildebrandt (1998). The role of the coercivity condition in Tonelli’s result is exactly to guarantee that the controls  $u$  are essentially bounded.

where  $\mu$  is the vector of co-state variables associated with the law of motions given by

$$\dot{x}(\theta_1) = u(\theta_1) \quad \text{a.e. } \theta_1. \quad (33)$$

Pointwise maximization of the Hamiltonian then requires that for almost every  $\theta_1 \in \Theta_1$ , all  $t = 1, \dots, T$ ,<sup>35</sup>

$$f(\theta_1) \left[ \begin{array}{l} g'(u_t(\theta_1)) - \psi'(g(u_t(\theta_1))) g'(u_t(\theta_1)) \left[ \alpha \left( \sum_{s=1}^T \psi(g(u_s(\theta_1))) + \sum_{s=1}^T J_1^s x_s(\theta_1) \right) + \beta \right] \\ -\alpha \sum_{s=2}^t \left[ \left( \sum_{\tau=s}^T J_s^\tau u_\tau(\theta_1) \right) J_s^\tau \sigma_s^2 \right] \end{array} \right] + \mu_t(\theta_1) \leq 0, \quad (34)$$

with inequality satisfied as equality if  $u_t(\theta_1) > 0$ . Furthermore, for almost every  $\theta_1 \in \Theta_1$ , any  $t = 1, \dots, T$ , the adjoint equations

$$\dot{\mu}_t(\theta_1) = f(\theta_1) J_1^t \left[ \alpha \left( \sum_{s=1}^T \psi(g(u_s(\theta_1))) + \sum_{s=1}^T J_1^s x_s(\theta_1) \right) + \beta \right] \quad (35)$$

must hold. Finally, the following boundary and transversality conditions must be satisfied:

$$\mu_t(\bar{\theta}_1) = x_t(\underline{\theta}_1) = 0, \quad t = 1, \dots, T. \quad (36)$$

Combining together (34)-(36), and using absolute continuity of the co-state variables, gives (14). ■

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<sup>35</sup>We abstract from the constraints that  $u_t \leq K$ . It is in fact immediate from the fact that  $K = \psi'(\bar{e}) > 1/\beta$  that these constraints never bind.

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