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# Equilibria Existence in Regular Discontinuous Games\*

Luciano I. de Castro<sup>†</sup>

## Abstract

Many conditions have been introduced to weaken the continuity requirements for equilibrium existence in games. We introduce a new condition, called regularity, that is simple and easy to verify. It is implied both by Reny's better-reply security and Simon and Zame's endogenous sharing rule method. Regularity implies that the limits of  $\epsilon$ -equilibria are equilibria. Since this condition is weak, it is yet not enough to ensure pure strategy equilibrium existence, but we are able to identify extra conditions that, together with regularity, are sufficient for equilibrium existence. One is the marginal continuity property introduced by Prokopovych (2008), while the second is the well behavior of a sequence of approximating continuous functions. In this way, we provide new equilibrium existence results for discontinuous games under conditions that are simpler and easier to check than most of the available alternatives.

**Keywords:** Discontinuous games; weak payoff security; better-reply security; regularity; regular games; epsilon-equilibria; approximating sequence.

**JEL classification numbers:** C72; C73.

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# 1 Introduction

Many classical problems in economics are modeled as games with a continuum of actions, but discontinuous payoffs. In this category are Bertrand's duopolistic competition, Hotelling's spatial competition, auctions and many other games. The discontinuities in these games pose a difficulty for establishing equilibrium existence. Given the importance of discontinuous games, such difficulties have been stimulating efforts towards weakening the sufficient conditions for (pure strategy) equilibrium existence.

In an innovative paper, Simon and Zame (1990) observe that many cases of discontinuities arise from the specification of a tie-breaking or sharing rule. Although there is a usual way of breaking ties (splitting the prize in equal proportions), this is not always the only natural sharing rule. To illustrate this point, they offer the example of two psychologists choosing locations on a portion of Interstate 5 running through California and Oregon. The relevant position is represented by a point in the interval  $[0,4]$ ; the California portion is represented by  $[0,3]$  and the Oregon part by  $[3,4]$ . There is a continuum of potential clients uniformly distributed along the Interstate and, as in the classical Hotelling's model, each client chooses the psychologist located closest to him. In Simon and Zame's example, the psychologists are constrained to be in their own state. In this game, the natural equilibrium seems to be for both to be in the border (point 3). However, the standard sharing rule (that splits in equal proportion the clients) does not support this choice as equilibrium. In fact, with this sharing rule the game does not have equilibrium. Simon and Zame (1990) then propose that the sharing rule is modified to reflect the limit of the proportion of clients (the psychologists' payoffs) from strategies that approximate the point 3, but that are not in a tie. In this way, one obtains a sharing rule that supports equilibrium.

The use of special tie-breaking rules in auctions goes back at least to Maskin and Riley (2000), who used a “second price auction tie-breaking rule”, which consisted in running a second price auction in case of a tie. Jackson, Simon, Swinkels, and Zame (2002) provided an example of an auction where no equilibrium exists under the standard tie-breaking rule. From this, they extended Simon and Zame (1990)’s idea to games with incomplete information. Jackson and Swinkels (2005) applied this method of proof for establishing equilibrium existence in multi-unit private value auctions.<sup>1</sup> Araujo, de Castro, and Moreira (2008) showed that special tie-breaking rules may also be necessary when types are multidimensional and utilities are non-monotonic, even in the symmetric case. They showed equilibrium existence under the all-pay auction tie-breaking rule, which consisted in running an all-pay auction as the tie-breaking mechanism. Araujo and de Castro (2009) considered asymmetric single and double auctions, and showed that special tie-breaking rules are necessary in general. They were able to show that monotonic tie-breaking rules are sufficient for equilibrium existence.

A usual criticism of this approach relies on Simon and Zame (1990)’s insistence on the *endogenous* definition of the sharing rule. This problem was explicitly indicated by Reny (1999, p. 1050): “in a mechanism design environment where discontinuities are sometimes deliberately introduced (auction design, for example), the participants must be presented with a game that fully describes the strategies and payoffs. One cannot leave some of the payoffs unspecified, to somehow be endogenously determined. In addition, this method is only useful in establishing the existence of a mixed, as opposed to pure, strategy equilibrium.” However, these two shortcomings are not essential to the “special tie-breaking rule” approach, broadly defined.<sup>2</sup> Indeed, Araujo, de Castro, and

<sup>1</sup>Since they worked with private values, they were able to prove that the definition of the tie-breaking rule was not important.

<sup>2</sup>Reny (1999)’s comments are fair, nevertheless, since these two shortcomings were essential

Moreira (2008) showed that an exogenously specified tie-breaking rule (the all-pay auction tie-breaking rule) is enough to guarantee equilibrium existence in a class of discontinuous games. Also, both Araujo, de Castro, and Moreira (2008) and Araujo and de Castro (2009) present results in pure strategy equilibrium.

Another approach to equilibrium existence in discontinuous games was developed by Reny (1999). See also Simon (1987), Dasgupta and Maskin (1986) and Baye, Tian, and Zhou (1993).<sup>3</sup> This approach is based in the better-reply security condition, which roughly requires that whenever a point is not equilibrium, one player can secure a payoff above her limit of payoffs, even if other players are allowed to slightly change their actions. This method seems to have absolutely no connection with the “special tie-breaking rule” approach described above. However, Jackson and Swinkels (2005, p.121) noticed that there are, indeed, a deep connection between these two methods:

It is interesting that the tricky part of the proof using better-reply-security is to get a handle on the  $u^*$ 's in the closure of the game. The fact that they are those generated by omniscient tie-breaking suggests a deeper connection between the machinery of Reny and that of JSSZ. That is, a proof of existence via “apply JSSZ and check that some equilibria correspond to nice tie-breaking” and “check better-reply-security” are closely related. Because of the requirement that better-reply-security apply relative to all points in the closure of the graph, rather than just the graph, one has to understand exactly what might be in that closure; and the points in the closure are precisely the points that come from omniscient choices at points of

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in Simon and Zame (1990) and also in Jackson, Simon, Swinkels, and Zame (2002).

<sup>3</sup>There is yet a third approach to equilibrium existence in discontinuous games, which use order and lattice-theoretical concepts. See for instance Vives (1990), Milgrom and Roberts (1990), Athey (2001), McAdams (2003) and Fudenberg, Mobius, and Szeidl (2007). Although we will not discuss much this third approach, our regularity condition seems very important for this method to apply.

discontinuity. On the other hand, in applying JSSZ, one has to understand the equilibria that might be generated under omniscient choices at points of discontinuity. In the auction setting, these two tasks are closely related. How these approaches turn out to be related and which might be more efficient in other settings is an *open question*.<sup>4</sup>

The purpose of this paper is to offer a solution to this open question. Our point is that the condition that connects both methods is what we call “regularity”. This condition is very simple, easy to verify and, at the same time, very weak, since it is satisfied for most games with equilibria. In particular, it is trivially implied by Reny (1999)’s better reply security and it is at the heart of the idea of special sharing rules introduced by Simon and Zame (1990). Regularity requires that if a strategy  $s^*$  is such that there is a sequence of strategies converging to  $s^*$  whose payoffs approximate the best-replies payoffs at  $s^*$ , then  $s^*$  is an equilibrium. This captures the idea that the payoffs at discontinuous points should be equilibrium payoffs, if there is a sequence of “almost” optimal points approximating to it. In particular, regularity implies that a sequence of  $\epsilon$ -equilibria is equilibrium. In fact, this is the spirit of the condition, although the condition is easily verified.

By analyzing some examples that do not have equilibrium but satisfy other standard assumptions, we show that the failure of regularity may explain the failure of equilibrium existence in many cases. From this, one can learn what should be the adaptation in the standard rule of some games (such as the tie-breaking rules in auctions) necessary to ensure equilibrium existence.

On the other hand, since regularity is a so weak condition, it is not sufficient to guarantee equilibrium existence. However, whenever one has  $\epsilon$ -equilibria for

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<sup>4</sup>Jackson and Swinkels (2005, p.121). The emphasis is ours.

all  $\epsilon > 0$ , regularity implies equilibrium existence. Thus, by assuming the (also easily checked) marginal continuity property introduced by Prokopovych (2008), we are able to deliver a very simple pure strategy equilibrium result.

Another variation of this result requires less than the marginal continuity property. It is sufficient to require the existence of an approximating sequence of continuous functions, which is also implied by a weak form of payoff security, called weak payoff security. Payoff security was introduced by Reny (1999) to characterize better-reply security and requires that any player is able to choose a single strategy and yet ensure that his payoff is at least  $v_i(s_i, s_{-i}) - \epsilon$ , even if the opponents choose strategies  $s'_i$  in a neighborhood of  $s_{-i}$ . Weak payoff security allows the player to choose different strategies for each  $s'_{-i}$  in the neighborhood of  $s_{-i}$ . We show (Lemma 5) that the weak payoff security implies the existence of an approximating sequence of continuous functions. Then, if this approximating sequence is sufficiently well-behaved (a property that we call well approximation), then compact, regular, quasiconcave games have an equilibrium.

The technique of approximating discontinuous games by sequence of continuous ones is clearly not new. It probably goes back to the first attempts to prove equilibrium existence in discontinuous games. However, the regularity condition introduced in this paper seems to be new in the literature. For a discussion of the literature, see section 6.

The rest of this paper is organized as follows. In section 2 we describe the basic setup and introduce the notation. Regularity is introduced in section 3, that also discuss its basic properties. Section 4 collects our equilibrium existence results, while section 5 illustrates the assumptions with some examples. A review of related literature is to be found in section 6 and a conclusion, in section 7.

## 2 Preliminaries

Let  $I = \{1, \dots, N\}$  be the set of players. Each player chooses a strategy from a compact convex subset  $S_i$  of a locally convex Hausdorff topological vector space.<sup>5</sup> We summarize the profile of strategies by  $s = (s_i, s_{-i}) \in S = S_i \times S_{-i}$ , where  $S_{-i} = \prod_{j \neq i} S_j$ . Naturally, we endow  $S$  with the product topology. Since each  $S_i$  is compact,  $S$  is also compact, by Tychonoff Theorem.

The payoff of player  $i$  is given by the function  $v_i : S \rightarrow \mathbb{R}$ , bounded above.<sup>6</sup> Occasionally, we will refer to  $v : S \rightarrow \mathbb{R}^N$ , understanding that the  $i$ -th coordinate of  $v(s)$  is denoted as  $v_i(s)$ . We denote the game by  $(S_i, v_i)_{i \in I}$  but we may refer to it briefly only as  $v$ .

We say that  $v$  is quasiconcave if the sets  $\{s_i \in S_i : v_i(s_i, s_{-i}) \geq \alpha\}$  are convex for all  $i \in I$ ,  $s_{-i} \in S_{-i}$  and  $\alpha \in \mathbb{R}$ . We say that  $v$  is compact if  $S$  is as described above. We denote the set of equilibrium points of  $(S_i, v_i)_{i \in I}$  by  $E(v)$ , that is,

$$E(v) \equiv \{s \in S : v_i(s) \geq v_i(s'_i, s_{-i}), \forall i \in I, s'_i \in S_i\}.$$

It will be convenient to define the correspondence of best-reply as follows:

$$\Gamma_v(s) \equiv \left\{ \tilde{s} \in S : \forall i \in I, v_i(\tilde{s}_i, s_{-i}) = \sup_{s'_i \in S_i} v_i(s'_i, s_{-i}) \right\}.$$

Of course,  $s \in E(v)$  if and only if  $s$  is a fixed point of  $\Gamma_v$ .

<sup>5</sup>A vector space is topological if it is endowed with a topology where the addition and multiplication by scalars are continuous transformations. A topological vector space is said to be locally convex if it possesses a base for its topology consisting of convex sets.

<sup>6</sup>We use this assumption mainly for convenience. Since we are concerned with the points that maximizes the function, it is convenient that the value at this point is not infinite. As noted by Reny (1999), we can transform unbounded payoffs  $u_i$  in bounded ones, by adopting  $v_i = \exp u_i / (1 + \exp u_i)$ .

### 3 Regular games

Given  $v : S \rightarrow \mathbb{R}^N$ , let us denote by  $\hat{v} : S \rightarrow \mathbb{R}^N$  the function whose coordinates  $v_i : S_{-i} \rightarrow \mathbb{R}$  are given by  $\hat{v}_i(s_{-i}) \equiv \sup_{s'_i \in S_i} v_i(s'_i, s_{-i})$ , for each  $i \in I$ . Although  $\hat{v}_i$  is a function only of  $s_{-i}$ , it will be convenient to abuse notation by considering  $\hat{v}_i$  as a function of  $s$ . This shall not cause confusion. Of course,  $v(s) \leq \hat{v}(s)$ , where the inequality is in the coordinate-wise sense, that is,  $v_i(s) \leq \hat{v}_i(s)$ , for each  $i \in I$ . This function is used in the following:

**Definition 1** *Given  $v : S \rightarrow \mathbb{R}^N$ , the regularization of  $v$  is the function  $\bar{v} : S \rightarrow \mathbb{R}^N$  defined by:*

$$\bar{v}(s) = \begin{cases} \hat{v}(s), & \text{if } \exists s^n \rightarrow s \text{ such that } \lim_n v(s^n) \geq \hat{v}(s) \\ v(s), & \text{otherwise} \end{cases}$$

*If  $v = \bar{v}$ , we say that  $v$  is regular.*

The functions  $v(s)$  and  $\bar{v}(s)$  have different values only in the points that are not equilibrium, but would be equilibrium if the game was continuous. In fact, if  $v$  is continuous at  $s$  then  $v(s) = \bar{v}(s)$ . Now, regularity does not require  $v$  to be continuous, but if a point is a candidate to be equilibrium because there is a sequence of points approximating a candidate for equilibrium, then it must be an equilibrium. This condition captures Simon and Zame (1990)'s idea of the allocation at a point of discontinuity: if one takes a sequence of strategies that converge to a "good" point, then the outcome at this point is selected to be the limit of the approximating points' outcomes.

This condition is also related to the better reply security introduced by Reny (1999). To see this, let us recall some definitions. A player  $i$  can secure a payoff of  $\alpha \in \mathbb{R}$  at  $s \in S$  if there exists  $s_i^* \in S_i$  such that  $v_i(s_i^*, s'_{-i}) \geq \alpha$  for all  $s'_{-i}$  in some open neighborhood of  $s_{-i}$ . A game  $(v_i, S_i)_{i \in I}$  (or, abbreviated,  $v$ )

is better reply security if whenever  $(s^*, v^*) \in cl(graph(v))$ , and  $s^*$  is not an equilibrium, some player  $i$  can secure a payoff strictly above  $v_i^*$  at  $s^*$ .

**Proposition 1** *If  $v$  satisfies better reply security, then  $v$  is regular, that is,  $v = \bar{v}$ .*

**Proof.** *It is clear that  $\bar{v} \geq v$ . Suppose that  $\bar{v}(s^*) = \hat{v}(s^*) > v(s^*)$ . Thus,  $s^*$  is not an equilibrium point. By definition, there is  $s^n \rightarrow s^*$  such that  $u^* \equiv \lim_n v(s^n) \geq \hat{v}(s^*)$ . Thus,  $(s^*, u^*) \in cl(gr(v))$ . By better reply security, there is a player  $i \in I$ ,  $s'_i \in S_i$ , a neighborhood  $U$  of  $s_{-i}^*$  and  $\delta > 0$  such that  $v_i(s'_i, s_{-i}^*) > u_i^* + \delta$  for all  $\tilde{s}_{-i} \in U$ . This is impossible, because  $u_i^* \geq \hat{v}_i(s^*) = \sup_{s'_i \in S_i} v_i(s'_i, s_{-i})$  and  $s_{-i} \in U$ . The contradiction establishes that  $\bar{v}(s^*) = \hat{v}(s^*) = v(s^*)$ , that is,  $v$  is regular. ■*

The regularity of a function is a very weak assumption, since it is almost equivalent to the existence of equilibrium points. The following summarizes the relation:

**Proposition 2**  $E(v) = E(\bar{v}) \cap \{s : \bar{v}(s) = v(s)\}$ .

**Proof.** *It is easy to see that whenever  $s$  is an equilibrium point,  $v(s) = \bar{v}(s)$  and  $s$  is also an equilibrium of  $\bar{v}$ , that is,  $E(v) \subset E(\bar{v}) \cap \{s : \bar{v}(s) = v(s)\}$ . Now if  $s \in E(\bar{v}) \cap \{s : \bar{v}(s) = v(s)\}$ , then  $v_i(s) = \bar{v}_i(s) \geq \bar{v}_i(s'_i, s_{-i}) \geq v_i(s'_i, s_{-i})$ , for all  $s'_i \in S_i$  and  $i \in I$ , which shows that  $s \in E(v)$ . ■*

This shows that regularity is a basic property that games with equilibrium “almost” need to satisfy. By this we mean that the condition is a little stronger than the existence of equilibrium, since it is possible that for some  $s \in S$ ,  $v(s) < \bar{v}(s)$ , but the game has an equilibrium in another point  $s'$ . However, if the game is not regular, standard topological methods will need adaptations for ensuring equilibrium existence in these games. A way to do this is to require

the kind of “transfer” assumptions, introduced by Baye, Tian, and Zhou (1993), which require that a property holds not in a point but in some “transfer” point.

## 4 Pure Strategy Equilibrium Existence

Following Prokopovych (2008), we introduce the following:

**Definition 2 (Marginal continuity property)** *A game  $(S_i, v_i)_{i \in I}$  has the marginal continuity property if  $\hat{v}$  is continuous.*

For the next result, we need the following definition, due to Reny (1999): the game  $(S_i, v_i)_{i \in I}$  is payoff secure if for all  $\epsilon > 0$ , each player  $i$  can secure a payoff of  $v_i(s) - \epsilon$  at  $s$ , that is, there exists  $s'_i \in S_i$  and a neighborhood  $U$  of  $s_{-i}$  such that  $v_i(s'_i, s_{-i}) > v_i(s) - \epsilon$  for all  $s'_{-i} \in U$ . Our first main result is the following:

**Theorem 1** *If a game  $(S_i, v_i)_{i \in I}$  is compact, quasiconcave, regular, payoff secure and has the marginal continuity property then it has a pure strategy equilibrium.*

The proof is very simple and based in two facts, which are of interest by their own:

**Proposition 3** *If  $v$  is regular and has the marginal continuity property then the limit of  $\epsilon$ -equilibria when  $\epsilon$  goes to 0 is an equilibrium.*

**Proof.** *Suppose that  $s^n$  is a sequence of  $\frac{1}{n}$ -equilibria, that is,  $\hat{v}(s^n) - \frac{1}{n} \leq v(s^n) \leq \hat{v}(s^n)$ . Since  $\hat{v}$  is continuous,  $s^n \rightarrow s^* \in S$  implies  $\hat{v}(s^n) \rightarrow \hat{v}(s^*)$  and the inequality gives  $v(s^n) \rightarrow \hat{v}(s^*)$ . Since  $v$  is regular, this means that  $v(s^*) = \bar{v}(s^*) = \hat{v}(s^*)$ , that is,  $s^* \in E(v)$ . ■*

If  $s^n$  is a sequence of  $\epsilon$ -equilibria, then compactness allows us to assume that  $s^n \rightarrow s^*$  for some  $s^*$ , passing to a subsequence if needed. Therefore, the proof is completed with the following:

**Proposition 4 (Prokopovych (2008))** *If  $v$  is compact, quasiconcave and has the marginal continuity property, then it possesses a pure strategy  $\epsilon$ -equilibrium for every  $\epsilon > 0$ .*

Although the above setup is simple and suitable for applications, we have another result that uses a condition that is weaker than marginal continuity, but yet sufficiently simple. This is introduced by the following:

**Definition 3 (Approximating sequence)** *Given a game  $(S_i, v_i)_{i \in I}$ , an approximating sequence is a sequence of continuous and quasiconcave functions  $v^n : S \rightarrow \mathbb{R}^N$  that satisfy the following:*

1.  $v^n(s) \leq \hat{v}(s)$  for all  $s \in S$ ;
2. If  $s^n \rightarrow s^*$ , then  $\liminf_n v_i^n(s^n) \geq \hat{v}_i(s^*)$ .

It is clear that if  $v$  has the marginal property, then  $v^n = \hat{v}$  is an approximating sequence. The existence of approximating sequences require less than this, however. Consider the following definition, which is weaker than payoff security:

**Definition 4 (Weakly payoff secure)** *We say that  $v$  is weakly payoff secure if for all  $i \in I$ ,  $\epsilon > 0$  and  $s \in S$ , there exists an open neighborhood  $U$  of  $s_{-i}$  such that for each  $s'_{-i} \in U$ , there exists  $s'_i \in S_i$  such that  $v_i(s'_i, s'_{-i}) \geq v_i(s) - \epsilon$ .<sup>7</sup>*

Weak payoff security implies the existence of an approximating sequence:

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<sup>7</sup>In other words, there is a nonempty correspondence  $\varphi_i : U \rightarrow S_i$  that gives  $s'_i \in \varphi_i(s'_{-i})$  for each  $s'_{-i} \in U$  satisfying the required inequality. Note, however, that we do not require that this correspondence has convex values or it is upper or lower semicontinuous.

**Proposition 5** *Assume that  $v$  is weakly payoff secure and  $S$  is compact metric. Then,  $v$  has an approximating sequence.*

**Proof.** First, let us show that  $\hat{v}_i$  is lower semicontinuous, that is, the set  $\{s_{-i} \in S_{-i} : \hat{v}_i(s_{-i}) > \alpha\}$  is open for all  $\alpha \in \mathbb{R}$  and  $i \in I$ . Indeed, fix  $s_{-i}$  in this set and choose  $\epsilon > 0$  such that  $\hat{v}_i(s_{-i}) - \epsilon > \alpha$ . Recall that  $\hat{v}_i(s_{-i}) = \sup_{\tilde{s}_i \in S_i} v_i(\tilde{s}_i, s_{-i})$ . Then, there exists  $\tilde{s}_i$  such that  $v_i(\tilde{s}_i, s_{-i}) > \hat{v}_i(s_{-i}) - \epsilon$ . Since the game is weakly payoff secure, there exists open neighborhood  $U_{s_{-i}}$  of  $s_{-i}$  such that:

$$\hat{v}_i(s'_{-i}) \geq v_i(s'_i, s'_{-i}) \geq v_i(\tilde{s}_i, s_{-i}) > \hat{v}_i(s_{-i}) - \epsilon, \forall s'_{-i} \in U_{s_{-i}}.$$

This shows that all  $s'_{-i}$  in  $U$  are also in  $\{s_{-i} \in S_{-i} : \hat{v}_i(s_{-i}) > \alpha\}$ , that is,  $\hat{v}_i$  is lower semicontinuous.

By Reny (1999), Lemma 3.5, there exist a sequence of continuous functions  $v_i^n : S_{-i} \rightarrow \mathbb{R}$  satisfying conditions 1 and 2 of the definition of approximating sequence. Since each  $v_i^n$  does not depend on  $s_i$ , they are quasiconcave. This concludes the proof. ■

Now, consider the following:

**Definition 5 (Well approximated game)** *We say that  $v$  is well approximated if there is an approximating sequence  $v^n$  such that, if  $s^n$  is a sequence of equilibria for  $(S_i, v_i^n)_{i \in I}$  then for each  $\epsilon > 0$ , there exists  $n$  such that  $v^n(s^n) \leq v(s^n) + \epsilon$ .*

This condition guarantees that the approximating sequence can be taken not too above  $v$ , at least for equilibrium points of the approximating game. With these definitions in place, we have the following:

**Theorem 2** *If a game  $(S_i, v_i)_{i \in I}$  is compact, regular and well approximated, then it has a pure strategy equilibrium.*

**Proof.** Take an approximating sequence  $v^n : S \rightarrow \mathbb{R}^N$  that well approximates  $v$ . Since  $(S_i, v_i^n)_{i \in I}$  is compact and quasiconcave and the function  $v^n$  is continuous, there exists a pure strategy equilibrium  $s^n$ . By compactness, we may assume (passing to subsequences if necessary) that  $s^n \rightarrow s^*$ . Since  $v^n$  is an approximating sequence,  $\hat{v}(s^*) \leq \limsup_n \hat{v}^n(s^n)$ . Using the fact that  $s^n$  is equilibrium for  $v^n$ , we have  $v^n(s^n) = \hat{v}^n(s^n)$ . Therefore, choosing  $\epsilon_n \downarrow 0$ ,

$$\hat{v}(s^*) \leq \limsup_n v^n(s^n) \leq \limsup_n (v(s^n) + \epsilon_n) = \limsup_n v(s^n),$$

where the last inequality and equality hold because  $v$  is well approximated by  $v^n$ . Since  $v$  is regular, this implies that  $v(s^*) = \hat{v}(s^*)$ , which establishes that  $s^*$  is equilibrium. ■

From the proof, we can see that well approximation and regularity guarantee that the limit of equilibrium points in the approximating games is an equilibrium of the original game.

## 5 Examples

In this section, we illustrate how the failure of equilibrium existence is related to the failure of regularity. The first example is example 1 of Carmona (2005):

**Example 1** Let  $I = \{1, 2\}$ ,  $S_1 = S_2 = [0, 1]$ ,  $v_1 : S \rightarrow \mathbb{R}$  given by:

$$v_1(s_1, s_2) = \begin{cases} 0, & \text{if } s_2 \leq \frac{1}{2} - s_1; \\ 2, & \text{if } s_1 = 0 \text{ and } s_2 > \frac{1}{2}; \\ 1, & \text{otherwise} \end{cases}$$

and  $v_2 : S \rightarrow \mathbb{R}$  given by:

$$v_2(s_1, s_2) = \begin{cases} 0, & \text{if } s_1 \leq \frac{1}{2} \text{ and } s_2 > 0; \\ 1, & \text{if } s_1 \leq \frac{1}{2} \text{ and } s_2 = 0; \\ 1, & \text{if } s_1 > \frac{1}{2} \text{ and } s_2 \leq \frac{1}{2}; \\ 2, & \text{if } s_1 > \frac{1}{2} \text{ and } s_2 > \frac{1}{2}; \end{cases}$$

Carmona (2005, Proposition 1) shows that the game in example 1 is quasi-concave and payoff secure, but has no pure strategy equilibrium or  $\epsilon$ -equilibrium for  $\epsilon > 0$  sufficiently small. It is not difficult to see that:

$$\hat{v}_1(s_2) = \begin{cases} 2, & \text{if } s_2 > \frac{1}{2}; \\ 1, & \text{otherwise} \end{cases}$$

and

$$\hat{v}_2(s_1) = \begin{cases} 1, & \text{if } s_1 \leq \frac{1}{2}; \\ 2, & \text{if } s_1 > \frac{1}{2}; \end{cases}$$

and the marginal continuity property is not satisfied. On the other hand, since the game is payoff secure, it is also weakly payoff secure and it has an approximating sequence. However, it is not regular. To see this, observe that  $\hat{v}(\frac{1}{2}, \frac{1}{2}) = (1, 1)$ , but  $\limsup_n v(\frac{1}{2} + \frac{1}{n}, \frac{1}{2} + \frac{1}{n}) = (1, 2) > (1, 1)$ ,

Now we consider example 3 of Prokopovych (2008).

**Example 2** Let  $I = \{1, 2\}$ ,  $S_1 = S_2 = [0, 1]$ ,  $v_1 : S \rightarrow \mathbb{R}$  given by:

$$v_1(s_1, s_2) = \begin{cases} 1 - s_1, & \text{if } s \in [0, 1] \times \{0\}; \\ 1 + s_1 s_2, & \text{if } s \in [0, 1] \times (0, 1]; \end{cases}$$

and  $v_2 : S \rightarrow \mathbb{R}$  given by:

$$v_2(s_1, s_2) = \begin{cases} s_2, & \text{if } s \in \{0\} \times [0, 1]; \\ 1 + s_1(1 - s_2), & \text{if } s \in (0, 1] \times [0, 1]; \end{cases}$$

As Prokopovych (2008) observes, this game is compact, quasiconcave, payoff secure and satisfies the marginal continuity property, but it does not have a pure strategy equilibrium. Let us verify that it is not regular. It is easy to see that:

$$\hat{v}_1(s_2) = \begin{cases} 1, & \text{if } s \in [0, 1] \times \{0\}; \\ 1 + s_2, & \text{if } s \in [0, 1] \times (0, 1]; \end{cases}$$

and

$$\hat{v}_2(s_1) = \begin{cases} 1, & \text{if } s \in \{0\} \times [0, 1]; \\ 1 + s_1, & \text{if } s \in (0, 1] \times [0, 1]; \end{cases}$$

and  $\hat{v}(0, 0) = (1, 1) > (1, 0) = \lim_n v(\frac{1}{n}, \frac{1}{n})$ , that is, regularity is not satisfied.

## 6 Relation with other methods

As Proposition 1 shows, better-reply security implies regularity. In the case that  $S$  is a metric space, Proposition 5 shows that (weak) payoff security implies the existence of an approximating sequence. Although payoff security and weak reciprocal upper semicontinuity implies better-reply security, they are not necessary for a game to be better-reply security (see Bagh and Jofre (2006)). It is also not clear that the property of well approximation is satisfied by all games that satisfy better-reply security. Barelli and Soza (2009) generalizes Reny (1999) results for Hausdorff locally convex spaces. The previous comments also apply to their work.

In a very recent paper, Carmona (2009) generalizes the pure strategy equi-

librium existence results of Reny (1999) and Barelli and Soza (2009) for metric spaces. First, he defines a game  $(S_i, v_i)_{i \in I}$  to be better-reply closed relative to a function  $\underline{u} : S \rightarrow \mathbb{R}^N$  if  $s^*$  is an equilibrium whenever  $(s^*, u^*) \in cl(graph(v))$  and  $u_i^* \geq \hat{u}_i(s_{-i}^*)$  for all  $i \in I$ . Maybe the weakest form of better-reply closeness occurs when the game  $(S_i, v_i)_{i \in I}$  is better-reply close relative to  $v$  itself. However, this condition is sufficient for regularity. Indeed, let  $u^* \equiv \lim_n v(s^n)$ , for some sequence  $s^n \rightarrow s^*$ , which means that  $(s^*, u^*) \in cl(graph(v))$ . If  $u_i^* \geq \hat{v}_i(s^*)$  for all  $i$  implies that  $s^*$  is equilibrium, then the regularity condition is satisfied.

Carmona (2009) says that  $(S_i, v_i)_{i \in I}$  is *upper (resp. lower) payoff secure* if for all  $i \in I$ ,  $\epsilon > 0$  and  $s \in S$ , there exists an open neighborhood  $V_{s_{-i}}$  and a nonempty, closed, convex valued, upper (resp. lower) hemicontinuous correspondence  $\varphi_i : V_{s_{-i}} \rightrightarrows S_i$  such that  $v_i(s') \geq v_i(s) - \epsilon$  for all  $s' \in graph(\varphi_i)$ . A game  $(S_i, v_i)_{i \in I}$  is approximately payoff secure relative to  $\underline{u}$  if, for all  $i \in I$ ,  $\underline{u}_i \leq v_i$ ,  $\underline{u}_i$  is quasiconcave and  $(S_i, \underline{u}_i)_{i \in I}$  is upper or lower payoff secure. As Carmona (2009) shows in his Lemma 2, if  $(S_i, v_i)_{i \in I}$  is approximately payoff relative to  $v$  itself, then  $\hat{v}_i$  is lower semi-continuous. Therefore, by the proof of our Lemma 5 this property is sufficient for the existence of an approximating sequence. (Incidentally, weakly payoff secure is strictly weaker than upper or lower payoff secure.) A close look at Carmona (2009)'s proof suggests that the well approximation property is also implied by his assumptions, although such assumptions refer to a function  $\underline{u}$ , which makes the comparison not straightforward.

## 7 Conclusion

This paper offered a new assumption, regularity, which is both simple, easy to verify and central to equilibrium existence. Many examples that fail to have equilibrium, fail precisely because of the failure of satisfying regularity. However,

regularity is too weak for being sufficient for equilibrium existence. Using the easily checked marginal continuity property introduced by Prokopovych (2008), we are able to provide an extremely simple equilibrium existence result. Using other assumptions, yet simple, we provide an alternative equilibrium existence theorem, also using regularity.

As we have argued, regularity is a property implied both by better-reply security and by the Simon and Zame (1990)'s approach. Jackson and Swinkels (2005) have previously noted the connection between the endogenous tie-breaking method and Reny's better-reply, but they left open the understanding what both methods have in common: "How these approaches turn out to be related and which might be more efficient in other settings is an open question." (p.121) We believe that regularity points out to a clarification and a better understanding of this matter.

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