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Working Paper

Refinements of Nash equilibrium in potential games

Working Paper, No. 2011-25

Provided in Cooperation with:
Department of Economics, Rutgers University

Refinements of Nash equilibrium in potential games

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May 2011

Abstract

We prove the existence of strategically stable sets of pure-strategy Nash equilibria (and hence the existence of pure-strategy trembling-hand perfect equilibria) in potential games that admit an upper semicontinuous potential, and we show that generic potential games possess pure-strategy strictly perfect and essential equilibria. In addition, we provide a link between upper semicontinuity of a potential and conditions defined directly on the payoff functions of a potential game. Finally, we show that stable sets and (strictly) perfect equilibria are related to the set of maximizers of a potential, which refines the set of Nash equilibria. Specifically, the set of maximizers of a potential contains a strategically stable set of pure-strategy Nash equilibria (and hence a pure-strategy trembling-hand perfect equilibrium) and, for generic games, any maximizer of a potential is a pure-strategy strictly perfect and essential equilibrium.

Keywords: discontinuous game, potential game, trembling-hand perfect equilibrium, stable set, essential equilibrium.

JEL classification: C72.

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1 Introduction

This paper studies refinements of Nash equilibrium in potential games. A strategic-form game is a potential game if the incentive of all players to change their strategy can be expressed in one global function, called the game's potential. Potential games have many applications in Economics and other disciplines (cf. Rosenthal [26], Monderer and Shapley [21], Ostrovsky and Schwarz [24], Armstrong and Vickers [3], Myatt and Wallace [22], *inter alia*). The contribution is twofold. First, we prove the existence of strategically stable sets of pure-strategy Nash equilibria (and hence the existence of pure-strategy trembling-hand perfect equilibria) in possibly discontinuous potential games, and show that generic potential games possess pure-strategy strictly perfect and essential equilibria. Second, stable sets and (strictly) perfect equilibria are related to the set of maximizers of a potential, which refines the set of Nash equilibria (cf. Monderer and Shapley [21]). Specifically, the set of maximizers of a potential contains a strategically stable set of pure-strategy Nash equilibria (and hence a pure-strategy trembling-hand perfect equilibrium) and, for generic games, any maximizer of a potential is a pure-strategy strictly perfect and essential equilibrium. This justification of the set of maximizers of a potential as a refinement specification is analogous to that furnished in Ui [29], where it is shown that maximizers of a potential (in finite potential games) are generically robust to the presence of incomplete information in the sense of Kajii and Morris [17, 18]. We slightly perturb games in terms of the actions that players can take, while Ui [29] considers perturbations in terms of the information the players might have. Our analysis complements that of Ui [29] as well as alternative formal justifications for global maximizers of a potential as an equilibrium selection device, such as those in Blume [7, 8] and Hofbauer and Sorger [16].

By confining attention to the collection of potential games, elementary machinery from variational analysis can be used to prove results under fairly simple assumptions.[1]

Theorems 1 and 2 state that the set of maximizers of an upper semicontinuous potential contains a pure-strategy trembling-hand perfect equilibrium and a stable set of pure strategies, according to notions of trembling-hand perfection and stability that extend the standard equilibrium concepts for fi-

[1]The existence of pure-strategy trembling-hand perfect equilibria in general (possibly discontinuous) strategic-form games requires more structure (cf. Carbonell-Nicolau [11]).

nite strategic-form games (cf. Selten [27] and Kohlberg and Mertens [19]) to infinite strategic-form games (cf. Simon and Stinchcombe [28], Al-Najjar [2], and Carbonell-Nicolau [9, 10, 11, 13]). Theorems 1 and 2 generalize the Nash existence results of Monderer and Shapley [21]. Proposition 2 and Corollary 1 provide a link between upper semicontinuity of a potential and conditions defined directly on the payoff functions of a potential game: it is shown that for potential games, a subset of the standard conditions for existence of Nash equilibrium based on Ky Fan inequalities gives a pure-strategy trembling-hand perfect equilibrium or a stable set of pure strategies. Proposition 5 states that for generic games in the class of games that admit an upper semi-continuous potential, maximizers of the potential are pure-strategy essential and strictly perfect equilibria.

Finally, Example 1 shows that the main results are tight: assuming the existence of a maximizer for the potential (rather than imposing upper semi-continuity) need not imply even the existence of a trembling-hand perfect equilibrium.

2 Preliminaries

A ***strategic-form game*** is a tuple $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty set of actions for player i, and u_i is a real-valued payoff function defined on $X := \times_{i=1}^N X_i$. A game $G = (X_i, u_i)_{i=1}^N$ is a ***compact metric game*** if it satisfies the following assumptions:

(i) Each X_i a compact metric space.

(ii) Each u_i bounded and Borel measurable.

In this paper, we assume that *all* games are compact, metric games. These games will be referred to simply as games. We will however make further assumptions later regarding, e.g., continuity of the payoffs.

Throughout the paper, we will view a payoff profile $u = (u_1, .., u_2)$ as an element of the complete metric space $(B(X)^N, d)$, where $B(X)$ denotes the space of bounded real-valued functions on X and the metric $d : B(X)^N \times B(X)^N \to \mathbb{R}$ is defined by

$$d\left((f_1, ..., f_N), (g_1, ..., g_N)\right) := \sum_{i=1}^N \sup_{x \in X} |f_i(x) - g_i(x)|.$$

Let $X_{-i} := \times_{j \neq i} X_j$ for each i. Given i and $(x_i, x_{-i}) \in X_i \times X_{-i}$, we employ the standard convention and write $(x_1, ..., x_N)$ in X as (x_i, x_{-i}). As usual, X is endowed with the product metric toplogy. Subsection 2.1 defines potential games.

Subsection 2.2 defines a trembling-hand perfect equilibrium and a strategically stable set.

2.1 Potential games

Given $G = (X_i, u_i)_{i=1}^N$, a map $P : X \to \mathbb{R}$ is a ***potential*** for G if for each i and every $x_{-i} \in X_{-i}$,

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}), \quad \text{for all } \{x_i, y_i\} \subseteq X_i.$$

Definition 1. A game is a ***potential game*** if it admits a potential. A game is an ***upper semicontinuous potential game*** if it admits an upper semicontinuous potential.

Potential games possess an important and convenient feature: a maximizer of a potential function is a pure-strategy Nash equilibrium. Stability of certain equilibria can now be defined in terms of stability of optimizers and we will exploit this in our treatment of equilibrium refinements. We conclude this section with the following lemma.

Lemma 1. *If $G = (X_i, u_i)_{i=1}^N$ is a potential game with potential $P : X \to \mathbb{R}$, then P is bounded and Borel measurable.*

Proof. Suppose $G = (X_i, u_i)_{i=1}^N$ is potential game with potential $P : X \to \mathbb{R}$. Fix $\overline{x} = (\overline{x}_1, .., \overline{x}_n) \in X$. It is straightforward to verify that $P^* : X \to \mathbb{R}$ defined as

$$P^*(x) := P(x) - P(\overline{x})$$

is also a potential for G. Writing

$$\begin{aligned} P^*(x_1, .., x_N) &= \sum_{i=1}^N \left[P(x_1, .., x_i, \overline{x}_{i+1}, .., \overline{x}_N) - P(x_1, ..x_{i-1}, \overline{x}_i, .., \overline{x}_N)\right] \\ &= \sum_{i=1}^N \left[u_i(x_1, .., x_i, \overline{x}_{i+1}, .., \overline{x}_N) - u_i(x_1, ..x_{i-1}, \overline{x}_i, .., \overline{x}_N)\right], \end{aligned}$$

it follows that P^* is bounded and measurable since each u_i is bounded and measurable. Consequently, P is bounded and measurable. ■

2.2 Perfect and strictly perfect equilibrium

If X_i is a compact, metric space, let $\Delta(X_i)$ represent the set of regular Borel probability measures on X_i, endowed with the topology of weak convergence. Since each X_i is a compact metric space, it follows that the topology of weak convergence is metrizable and that $\Delta(X_i)$ is a compact metric space. In particular, a sequence in $\Delta(X_i)$ is weakly convergent if and only if the sequence is convergent with respect to the Prokhorov metric. Next, extend u_i to $\Delta(X) := \times_{i=1}^N \Delta(X_i)$ in the usual manner by using Fubini's Theorem (recall that u_i is bounded and Borel measurable) and defining

$$u_i(\mu) := \int_X u_i d(\mu_1 \otimes \cdots \otimes \mu_N) = \int_{X_1} \cdots \int_{X_N} u_i d\mu_1 \cdots d\mu_N.$$

The usual ***mixed extension*** of G is the strategic form game

$$\overline{G} = (\Delta(X_i), u_i)_{i=1}^N .$$

For notational simplicity, we will adopt a standard convention and will not distinguish notationally between the pure strategy $x_i \in X_i$ and corresponding Dirac measures in $\Delta(X_i)$.

For the remainder of this paper, we will treat each player as having a fixed compact, metric strategy set X_i. However, we will treat payoff functions as parameters in various places. Consequently, $\pi(u)$ will denote the set of pure-strategy Nash equilibria of a game $G = (X_i, u_i)_{i=1}^N$ and $\xi(u)$ will denote the set of mixed-strategy Nash equilibria of G, i.e., the Nash equilibria of the mixed extension $\overline{G} = (\Delta(X_i), u_i)_{i=1}^N$.

Let $B_\varepsilon(x)$ denote the open ball centered at $x \in X$ with radius $\varepsilon > 0$ (defined with respect to the product metric on X) and let $B_\varepsilon(\sigma)$ denote the open ball of radius ε centered at $\sigma \in \Delta(X)$ (defined with respect to the product Prokhorov metric on $\Delta(X)$). We will also write $B_\varepsilon(x)$ for the open ball of radius ε in $\Delta(X)$ when $x \in X$ is identified with the associated profile of Dirac measures in $\Delta(X)$. This will not cause confusion since the context will be clear.

Let $M_+(X_i)$ denote the set of all regular measures defined on the Borel sets in X_i. A measure $\mu_i \in M_+(X_i)$ is ***strictly positive*** if $\mu_i(U) > 0$ for every nonempty open set U in X_i. Let $M_{++}(X_i)$ denote the set of all strictly positive measures in $M_+(X_i)$, let $\widehat{\Delta}(X_i)$ denote the set of all strictly positive probability measures in $M_+(X_i)$, and let $\widehat{\Delta}(X) := \times_{i=1}^N \widehat{\Delta}(X_i)$. Given $\delta =$

$(\delta_1, ..., \delta_N) \in (0,1)^N$ and $\mu = (\mu_1, ..., \mu_N) \in \widehat{\Delta}(X)$, define $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ as

$$u_i^{(\delta,\mu)}(x) := u_i\left((1-\delta_1)x_1 + \delta_1\mu_1, ..., (1-\delta_N)x_N + \delta_N\mu_N\right).$$

Here, $(1-\delta_i)x_i + \delta_i\mu_i$ denotes the probability measure $\sigma_i \in \Delta(X_i)$ for which $\sigma_i(B) := 1 - \delta_i + \delta_i\mu_i(B)$ if $x_i \in B$ and $\sigma_i(B) := \delta_i\mu_i(B)$ otherwise. Note that $u_i^{(\delta,\mu)}$ is bounded and Borel measurable as a consequence of Fubini's Theorem. Let $G_{(\delta,\mu)}$ denote the game defined as

$$G_{(\delta,\mu)} := (X_i, u_i^{(\delta,\mu)})_{i=1}^N.$$

Using the notational convention established above, $\pi(u^{(\delta,\mu)})$ denotes the set of pure-strategy Nash equilibria of the game $G_{(\delta,\mu)} = (X_i, u_i^{(\delta,\mu)})_{i=1}^N$ and $\xi(u^{(\delta,\mu)})$ denotes the set of mixed-strategy Nash equilibria of $G_{(\delta,\mu)}$, i.e., the Nash equilibria of the mixed extension $\overline{G}_{(\delta,\mu)} = (\Delta(X_i), u_i^{(\delta,\mu)})_{i=1}^N$.

Definition 2. A strategy profile $\sigma \in \xi(u)$ is a ***trembling-hand perfect (thp) equilibrium*** in $G = (X_i, u)_{i=1}^N$ if there exist sequences (δ^n), (μ^n), and (σ^n) such that $(0,1)^N \ni \delta^n \to 0$, $\mu^n \in \widehat{\Delta}(X)$, $\sigma^n \to \sigma$, and $\sigma^n \in \xi(u^{(\delta,\mu)})$ for each n.

Definition 3. A strategy profile $\sigma \in \xi(u)$ is a ***strictly perfect equilibrium*** in $G = (X_i, u)_{i=1}^N$ if for all sequences (δ^n) and (μ^n) such that $(0,1)^N \ni \delta^n \to 0$ and $\mu^n \in \widehat{\Delta}(X)$, there exists a sequence (σ^n) satisfying $\sigma^n \in \xi(u^{(\delta^n,\mu^n)})$ for each n and $\sigma^n \to \sigma$.

Every strictly perfect equilibrium is a thp perfect equilibrium. Furthermore, this definition of trembling-hand perfection is equivalent to an alternative definition in terms of perturbed sets of mixed strategies. If $\eta_i \in M_{++}(X_i)$ and $\eta_i(X_i) < 1$, we define the perturbed mixed-strategy set of player i as

$$\Delta(X_i, \eta_i) := \{\nu_i \in \Delta(X_i) : \nu_i \geq \eta_i\}.$$

Given a profile $\eta = (\eta_1, ..., \eta_N)$ of perturbations, we define the associated ***Selten perturbation*** of G to be the game

$$\overline{G}_\eta = (\Delta(X_i, \eta_i), u_i)_{i=1}^N.$$

Lemma 2. *Let $G = (X_i, u_i)_{i=1}^N$ be a game and let $\sigma \in \Delta(X)$ be a strategy profile. The following are equivalent:*

(i) The profile σ is a trembling-hand perfect equilibrium in G.

(ii) There exist sequences (η^n) and (σ^n) such that $\eta^n \to 0$, $\sigma^n \to \sigma$, and σ^n is an equilibrium in the Selten perturbed game $\overline{G}_{\eta^n}$ for each n.

Next we record a useful characterization of strict perfection.

Lemma 3. *Let $G = (X_i, u_i)_{i=1}^N$ be a game and let $\sigma \in \Delta(X)$ be a strategy profile. The following are equivalent:*

(i) The profile σ is a strictly perfect equilibrium in G.

(ii) For every $\varepsilon > 0$, there exists a $\kappa > 0$ such that the following holds: if $0 < \delta_i < \kappa$ for each i and if $\mu \in \widehat{\Delta}(X)$, then $\xi(u^{(\delta,\mu)}) \cap B_\varepsilon(\sigma) \neq \varnothing$.

3 Existence of pure-strategy perfect equilibrium

We begin by establishing that certain perturbations of upper semicontinuous potential games are also upper semicontinuous potential games.

Lemma 4. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P and suppose that $(\delta, \mu) \in (0,1)^N \times \widehat{\Delta}(X)$. For each $x = (x_1, .., x_N) \in X$, define $\nu_i^{x_i} \in \Delta(X_i)$ as*

$$\nu_i^{x_i} := (1 - \delta_i)x_i + \delta_i \mu_i.$$

Then $P^{(\delta,\mu)} : X \to \mathbb{R}$ defined as

$$P^{(\delta,\mu)}(x_1, ..., x_N) := \int_X P d\nu_1^{x_1} \cdots d\nu_N^{x_N}$$

is an upper semicontinuous potential for $G_{(\delta,\mu)}$. Therefore, $G_{(\delta,\mu)}$ has a pure-strategy Nash equilibrium, i.e., $\pi(u^{(\delta,\mu)}) \neq \varnothing$.

Proof. Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P and suppose that $(\delta, \mu) \in$

$(0,1)^N \times \widehat{\Delta}(X)$. Define $\nu_i^{x_i}$ and $P : X \to \mathbb{R}$ as in the statement of the lemma and let $\nu_{-i}^{x_{-i}} = (\nu_j^{x_j})_{j \neq i}$. Given i, $z_{-i} \in X_{-i}$, and $\{x_i, y_i\} \subseteq X_i$, we have

$$\begin{aligned}
&u_i((1-\delta_i)x_i + \delta_i\mu_i, z_{-i}) - u_i((1-\delta_i)y_i + \delta_i\mu_i, z_{-i}) \\
&\quad = (1-\delta_i)u_i(x_i, z_{-i}) + \delta_i u_i(\mu_i, z_{-i}) - (1-\delta_i)u_i(y_i, z_{-i}) - \delta_i u_i(\mu_i, z_{-i}) \\
&\quad = (1-\delta_i)(u_i(x_i, z_{-i}) - u_i(y_i, z_{-i})) \\
&\quad = (1-\delta_i)(P(x_i, z_{-i}) - P(y_i, z_{-i})) \\
&\quad = (1-\delta_i)(P(x_i, z_{-i}) - P(y_i, z_{-i})) + \delta_i \left(\int_{X_i} P(\cdot, z_{-i}) d\mu_i - \int_{X_i} P(\cdot, z_{-i}) d\mu_i \right) \\
&\quad = \left[(1-\delta_i)P(x_i, z_{-i}) + \delta_i \int_{X_i} P(\cdot, z_{-i}) d\mu_i \right] \\
&\qquad\qquad\qquad - \left[(1-\delta_i)P(y_i, z_{-i}) + \delta_i \int_{X_i} P(\cdot, z_{-i}) d\mu_i \right] \\
&\quad = P((1-\delta_i)x_i + \delta_i\mu_i, z_{-i}) - P((1-\delta_i)y_i + \delta_i\mu_i, z_{-i}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
u_i^{(\delta,\mu)}(x_i, x_{-i}) - u_i^{(\delta,\mu)}(y_i, x_{-i}) &= \int_{X_{-i}} [u_i((1-\delta_i)x_i + \delta_i\mu_i, \cdot) - u_i((1-\delta_i)y_i + \delta_i\mu_i, \cdot)] d\nu_{-i}^{x_{-i}} \\
&= \int_{X_{-i}} [P((1-\delta_i)x_i + \delta_i\mu_i, \cdot) - P((1-\delta_i)y_i + \delta_i\mu_i, \cdot)]\, d\nu_{-i}^{x_{-i}} \\
&= P^{(\delta,\mu)}(x_i, x_{-i}) - P^{(\delta,\mu)}(y_i, x_{-i}),
\end{aligned}$$

so $P^{(\delta,\mu)}$ is a potential for $G_{(\delta,\mu)}$.

To see that $P^{(\delta,\mu)}$ is upper semicontinuous, suppose that

$$x^n = (x_1^n, .., x_N^n) \to (x_1, .., x_N) = x.$$

Then $\nu_i^{x_i^n} \to \nu_i^{x_i}$ in the topology of weak convergence on $\Delta(X_i)$. Consequently, $(\nu_1^{x_1^n}, .., \nu_N^{x_N^n}) \to (\nu_1^{x_1}, .., \nu_N^{x_N})$ in the product topology on $\Delta(X)$. Applying Theorem 1 in Glycopantis and Muir [15] or Theorem 3.2 in Billingsley [6] for example, we conclude that

$$\nu_1^{x_1^n} \otimes \cdots \otimes \nu_N^{x_N^n} \to \nu_1^{x_1} \otimes \cdots \otimes \nu_N^{x_N},$$

so applying Fubini's Theorem and Theorem 14.5 in Aliprantis and Border [1], we obtain

$$\begin{aligned}
\limsup_{n\to\infty} P^{(\delta,\mu)}(x_1^n, ..., x_N^n) &= \limsup_{n\to\infty} \int_X P d\nu_1^{x_1^n} \cdots d\nu_N^{x_N^n} \\
&= \limsup_{n\to\infty} \int_X P d(\nu_1^{x_1^n} \otimes \cdots \otimes \nu_N^{x_N^n}) \\
&\leq \int_X P d(\nu_1^{x_1} \otimes \cdots \otimes \nu_N^{x_N}) \\
&= \int_X P d\nu_1^{x_1} \cdots d\nu_N^{x_N} \\
&= P^{(\delta,\mu)}(x_1, ..., x_N).
\end{aligned}$$

Since $P^{(\delta,\mu)}$ is an upper semicontinuous potential for $G_{(\delta,\mu)}$, $P^{(\delta,\mu)}$ attains a maximum at a pure-strategy Nash equilibrium of $G_{(\delta,\mu)}$. ■

To prove the existence of pure-strategy thp equilibria in upper semicontinuous potential games, we require a few basic results from variational analysis that we record here.

Definition 4. Suppose that S is a metric space. A sequence (f^n) of real-valued functions on S is *hypoconvergent with hypo-limit* f if for each $x \in S$, the following conditions hold:

(i) There exists a sequence (z^n) such that $z^n \to x$ and

$$f(x) = \lim_{n\to\infty} f^n(z^n).$$

(ii) For every sequence (x^n) such that $x^n \to x$, we have

$$\limsup_{n\to\infty} f^n(x^n) \leq f(x).$$

The next lemma is proved for $S \subseteq \mathbb{R}^k$ in Rockafellar and Wets [25] (Proposition 7.15) and we include a simple direct proof when S is a metric space for the sake of completeness.

Lemma 5. *Suppose that S is a metric space and suppose that (f^n) is a uniformly convergent sequence of upper semicontinuous real-valued functions on S with uniform limit f. Then f is upper semicontinuous and (f^n) is hypoconvergent with hypo-limit f.*

Proof. Choose $x \in S$ and suppose that (x^n) is convergent in S with limit x and choose $\varepsilon > 0$. Uniform convergence implies that there exists an m such that

$$|f^m(x) - f(x)| < \tfrac{\varepsilon}{2}$$

for all $x \in X$. Since f^m is upper semicontinuous, there exists an $\widehat{n}$ such that

$$f^m(x^n) < f(x) + \tfrac{\varepsilon}{2}$$

whenever $n > \widehat{n}$. Therefore, $n > \widehat{n}$ implies that

$$f(x^n) - f(x) = [f(x^n) - f^m(x^n)] + [f^m(x^n) - f(x)] < \varepsilon$$

and we conclude that f is upper semicontinuous. Next, note that uniform convergence implies that there exists an $\widehat{n}$ such that

$$|f^n(x^n) - f(x^n)| < \varepsilon$$

for all $n > \widehat{n}$. Therefore,

$$f^n(x^n) < f(x^n) + \varepsilon$$

whenever $n > \widehat{n}$. The upper semicontinuity of f implies that

$$\limsup_{n\to\infty} f^n(x^n) \leq \limsup_{n\to\infty} f(x^n) + \varepsilon \leq f(x) + \varepsilon$$

and it follows that

$$\limsup_{n\to\infty} f^n(x^n) \leq f(x).$$

To show that condition (ii) is satisfied, define $z^n = x$ for all n. Noting that uniform convergence implies pointwise convergence, it follows that

$$f(x) = \lim f^n(x) = \lim f^n(z^n),$$

proving that (f^n) is hypoconvergent with hypo-limit f. ■

The next result is also well-known and follows immediately from Lemma 5, together with Theorems 5.3.5 and 5.3.6 in Beer [5].

Proposition 1. *Suppose that S is a metric space and suppose that (f^n) is a uniformly convergent sequence of upper semicontinuous real-valued functions on S with uniform limit f. If $x^n \in \arg\max_{x\in X} f^n(x)$ for each n and $x^n \to x$, then $x \in \arg\max_{x\in X} f(x)$.*

Lemma 6. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P. For every $\varepsilon > 0$ there exists a $\kappa \in (0, 1)$ such that the following condition holds: for every $(\delta_1, .., \delta_N)$ with $0 < \delta_i < \kappa$ for each i, and for every $(\mu_1, .., \mu_N)$ with $\mu_i \in \widehat{\Delta}(X_i)$ for each i,*

$$\sup_{x \in X} |P(x) - P^{(\delta,\mu)}(x)| < \varepsilon.$$

Proof. Let $\widehat{N} = \{1, ..., N\}$. Applying an induction argument, it follows that

$$P^{(\delta,\mu)}(z) = \sum_{I \subseteq \widehat{N}} \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i \right] P(z_I, \mu_{\widehat{N} \setminus I})$$

so that

$$P^{(\delta,\mu)}(z) = \left[\prod_{i \in \widehat{N}} (1 - \delta_i) \right] P(z) + \sum_{\substack{I \subseteq \widehat{N} \\ : I \neq \widehat{N}}} \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i \right] P(z_I, \mu_{\widehat{N} \setminus I}).$$

Let $M = \sup_{x \in X} |P(x)|$ (recall that P is bounded (Lemma 1)), choose $\varepsilon > 0$ and choose $\kappa \in (0, 1)$ so that

$$\left[\left[1 - (1 - \kappa)^N \right] + \kappa (2^N - 1) \right] M < \varepsilon.$$

If $I \neq \widehat{N}$, then there exists a $j \in \widehat{N} \setminus I$ such that

$$\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i = \delta_j \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus (I \cup \{j\})} \delta_i \right]$$

implying (since $0 < \delta_i < \kappa < 1$ for each i) that

$$\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i = \delta_j \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus (I \cup \{j\})} \delta_i \right] < \delta_j < \kappa.$$

Then for each $z \in X$, it follows that

$$
\begin{aligned}
&|P(z) - P^{(\delta,\mu)}(z)| \\
&\quad \leq \left[1 - \prod_{i \in \widehat{N}} (1-\delta_i)\right] |P(z)| + \sum_{\substack{I \subseteq \widehat{N} \\ : I \neq \widehat{N}}} \left[\prod_{i \in I}(1-\delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i\right] |P(z_I, \mu_{\widehat{N}\setminus I})| \\
&\quad \leq \left[\left[1 - \prod_{i \in \widehat{N}} (1-\delta_i)\right] + \sum_{\substack{I \subseteq \widehat{N} \\ : I \neq \widehat{N}}} \left[\prod_{i \in I}(1-\delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i\right]\right] M \\
&\quad \leq \left[\left[1 - (1-\kappa)^N\right] + \kappa(2^N - 1)\right] M < \varepsilon,
\end{aligned}
$$

as desired. ■

Lemma 6 asserts that an upper semicontinuous potential P can be uniformly approximated by an upper semicontinuous potential $P^{(\delta,\mu)}$ when δ is close to zero. This property is crucial for the next result.

Theorem 1. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P. Then G possesses a pure-strategy trembling-hand perfect equilibrium in* $\arg\max_{x \in X} P(x)$.

Proof. Suppose that P is an upper semicontinuous potential for the game $G = (X_i, u_i)_{i=1}^N$. Choose $\mu \in \widehat{\Delta}(X)$ and a sequence $\delta^n = (\delta_1^n, .., \delta_N^n) \in (0,1)^N$ with $\delta^n \to 0$. Applying Lemma 4, it follows that $P^{(\delta^n,\mu)}$ is an upper semicontinuous potential for $G_{(\delta,\mu)}$. Applying Lemma 6, we conclude that P is the uniform limit of the sequence $(P^{(\delta^n,\mu)})$ so, applying Lemma 5, it follows that $(P^{(\delta^n,\mu)})$ is hypoconvergent with hypo-limit P. For each n, choose

$$x^n \in \arg\max_{x \in X} P^{(\delta^n,\mu)}(x).$$

Then x^n is a pure-strategy equilibrium in $G_{(\delta^n,\mu)}$, i.e., $x^n \in \pi(u^{(\delta^n,\mu)})$. Since X is compact, there exists a subsequence (x^{n_k}) of (x^n) and a pure-strategy profile $x \in X$ such that $x^{n_k} \to x$. From Proposition 1, we conclude that

$$x \in \arg\max_{x \in X} P(x),$$

from which it follows that x is a pure-strategy equilibrium in G, i.e., $x \in \pi(u)$. Finally, note that since $x^{n_k} \in \pi(u^{(\delta^{n_k},\mu)})$ and $\delta^{n_k} \to 0$, we conclude that x is a pure-strategy thp equilibrium in G. ■

It is of course helpful to identify conditions on the underlying payoffs of a potential game that will guarantee that the potential is upper semicontinuous.

Proposition 2. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact, metric game and suppose that for each i, the following conditions are satisfied:*

- (i) *$(x_1, .., x_N) \mapsto \sum_{i=1}^N u_i(x_1, .., x_N)$ is upper semicontinuous on X.*
- (ii) *There exists a strategy profile $(\overline{x}_1, .., \overline{x}_N) \in X$ such that the map $(x_1, .., x_N) \mapsto \sum_{i=1}^N u_i(\overline{x}_i, x_{-i})$ is lower semicontinuous on X.*

If P is a potential for G, then P is upper semicontinuous.

Proof. Fix a strategy profile $(\overline{x}_1, .., \overline{x}_N) \in X$ satisfying condition (ii). If $(x_1, .., x_N) \in X$, then

$$\begin{aligned}
&P(x_1, .., x_N)\\
&\quad = P(\overline{x}_1, .., \overline{x}_N) + \sum_{i=1}^N [P(x_{1,..,}x_i, \overline{x}_{i+1}, .., \overline{x}_N) - P(x_{1,..,}x_{i-1}, \overline{x}_i, .., \overline{x}_N)]\\
&\quad = P(\overline{x}_1, .., \overline{x}_N) + \sum_{i=1}^N [u_i(x_1, .., x_i, \overline{x}_{i+1}, .., \overline{x}_N) - u_i(x_1, ..x_{i-1}, \overline{x}_i, .., \overline{x}_N)]\\
&\quad = P(\overline{x}_1, .., \overline{x}_N)\\
&\qquad\qquad + \left[\sum_{i=1}^N u_i(x_1, .., x_i, \overline{x}_{i+1}, .., \overline{x}_N)\right] - \left[\sum_{i=1}^N u_i(x_1, ..x_{i-1}, \overline{x}_i, .., \overline{x}_N)\right].
\end{aligned}$$

Condition (i) implies that the first bracketed term defines an upper semicontinuous function while condition (ii) implies that the second bracketed term defines a lower semicontinuous function. Therefore, $P : X \to \mathbb{R}$ is upper semicontinuous. ■

Corollary 1. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a game satisfying the following conditions:*

(*i*) $(x_1, .., x_N) \mapsto \sum_{i=1}^N u_i(x_1, .., x_N)$ *is upper semicontinuous on* X.

(*ii*) *For each* i *and for every* $x_i \in X_i$, *the function* $u_i(x_i, \cdot) : X_{-i} \to \mathbb{R}$ *is lower semicontinuous.*

If P *is a potential for* G, *then* P *is upper semicontinuous.*

The conditions of the corollary are precisely those that appear in the approach to equilibrium existence via Ky Fan inequalities. For a game $G = (X_i, u_i)_{i=1}^N$, define a function $F : X \times X \to R$ as follows:

$$F(x, y) = \sum_{i=1}^{N} [u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})].$$

The *Ky Fan inequality problem* for the pair (F, X) may be stated as follows: find $x \in X$ such that $F(x, y) \leq 0$ for all $y \in X$. From this definition, it is immediate that x solves the Ky Fan inequality problem for (F, X) if and only if $x \in \pi(u)$. The Ky Fan inequality associated with a strategic form game was formulated by Nikaido and Isoda [23] and the following generalization of their result follows from, e.g., Theorem 3.1.1 in Aubin and Frankowska [4].

Proposition 3. *Suppose that* $G = (X_i, u_i)_{i=1}^N$ *is game satisfying assumptions* (*i*) *and* (*ii*) *of Corollary 1. In addition, suppose that each* X_i *is a compact convex nonempty subset of a Hausdorff locally convex topological vector space, and suppose that* $u_i(\cdot, x_{-i}) : X_i \to R$ *is concave for each* $x_{-i} \in X_{-i}$. *Then the Ky Fan inequality problem for* (F, X) *has a solution. That is, the game* G *has an equilibrium.*

If a game G satisfies conditions (*i*) and (*ii*) of Corollary 1, then G has an equilibrium if G is a potential game or if the strategy sets and payoffs satisfy the appropriate convexity and concavity assumptions of Proposition 3. Finally, note that, if P is a potential for $G = (X_i, u_i)_{i=1}^N$, then

$$F(x, y) = \sum_{i-1}^{N} [u_i(x_{-i}, y_i) - u_i(x_{-i}, x_i)] = \sum_{i-1}^{N} [P(x_{-i}, y_i) - P(x_{-i}, x_i)]$$

and it follows that any maximizer of P on X will be a solution to the Ky Fan inequality problem for (F, X).

A game $G = (X_i, u_i)_{i=1}^N$ is ***continuous*** if each u_i is continuous. Since a continuous potential game is an upper semicontinuous potential game, the following is an immediate corollary of Theorem 1.

Corollary 2. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a potential game with continuous potential P. Then G possesses a pure-strategy trembling-hand perfect equilibrium.*

We conclude this section by noting that one cannot drop upper semicontinuity of the potential in the hypothesis of Theorem 1. In fact, Example 1 below presents a potential game that lacks a pure-strategy thp equilibrium (and hence, by Theorem 1, the game does not admit an upper semicontinuous potential).

Assuming that the potential of G is upper semicontinuous ensures that the corresponding potentials for perturbed games of the form $G_{(\delta,\mu)}$ are upper semicontinuous, and this, in turn, guarantees the existence of a global maximizer for the potentials of the perturbations. Simply assuming that G admits a potential that can be maximized will generally *not* be sufficient for perturbations of the form $G_{(\delta,\mu)}$ to have potentials that attain a maximum. In fact, while the game G in Example 1 (below) does not admit an upper semicontinuous potential, the game does admit a potential that attains a maximum in X. Nevertheless, no sequence $(G_{(\delta^n,\mu^n)})$ of perturbations (with $(0,1)^N \ni \delta^n \to 0$ and $\mu^n \in \widehat{\Delta}(X))$ can be obtained such that each $G_{(\delta^n,\mu^n)}$ admits a potential that can be maximized.

Example 1. For each $k \geq 1$, let $\alpha(k) = \frac{k+1}{k+2}$. Consider the game $G = (X_i, u_i)_{i=1}^2$ where $X_1 = \{1\} \cup \{\alpha^k : k \geq 1\}, X_2 = [0,1]$, and $u_1 = u_2 = u$ where

$$u(x_1, x_2) := \begin{cases} 1 & \text{if } (x_1, x_2) = (1, 0), \\ 0 & \text{if } x_1 = 1 \text{ and } x_2 \neq 0, \\ \alpha^k & \text{if } (x_1, x_2) = (\alpha^k, 0), \\ \frac{1}{2} & \text{if } x_1 = \alpha^k \text{ and } x_2 \neq 0, \end{cases}$$

Note that each X_i is compact in the Euclidean metric topology. Since $u_1 = u_2$, it follows that G is a potential game with potential $P = u$. From Lemma 2.7 in Monderer and Shapley [21], it follows that if $\widehat{P}$ is any other potential for G, then $\widehat{P} = P + c = u + c$ for some constant c. The payoff function u is not upper semicontinuous since $(\alpha^n, 1) \to (1, 1)$ but

$$\limsup_{n\to\infty} u(\alpha^n, 1) = \tfrac{1}{2} > 0 = u(1, 1).$$

Therefore, G has a potential but no potential for G is upper semicontinuous. To see that no equilibrium in G is trembling-hand perfect, first observe that

$x_2 = 0$ is a (strictly) dominant strategy for player 2 in G implying that the pure-strategy profile $(x_1, x_2) = (1, 0)$ is the unique equilibrium in G. Furthermore, $(x_1, x_2) = (1, 0)$ is the unique maximizer of any potential function for G. Next, choose sequences (δ^n) and (μ^n) with $\delta^n \in (0,1)^2$ and $\delta^n \to (0,0)$ and $\mu^n \in \widehat{\Delta}(X)$. First, we claim that $x_2 = 0$ is also the unique best response (pure or mixed) of player 2 in the game $G_{(\delta^n,\mu^n)}$. To see this, note that:

If $x_1 = 1$ and $0 < y_2 \leq 1$, then $u_2(x_1, y_2) = 0$ and $\frac{1}{2} < \alpha^k$ for every k so that

$$\begin{aligned}
&u_2((1-\delta_1^n)x_1 + \delta_1^n \mu_1^n, 0)\\
&\qquad = (1-\delta_1^n)u_2(1,0) + \delta_1^n u_2(\mu_1^n, 0)\\
&\qquad = (1-\delta_1^n) + \delta_1^n \left[\sum_k u_2(\alpha^k, 0)\mu_1^n(\alpha^k) + u_2(1,0)\mu_1^n(1)\right]\\
&\qquad = (1-\delta_1^n) + \delta_1^n \left[\sum_k \alpha^k \mu_1^n(\alpha^k) + \mu_1^n(1)\right]\\
&\qquad > 0 + \delta_1^n \left[\sum_k \tfrac{1}{2}\mu_1^n(\alpha^k) + 0\right]\\
&\qquad = (1-\delta_1^n)u_2(1, y_2) + \delta_1^n \left[\sum_k u_2(\alpha^k, y_2)\mu_1^n(\alpha^k) + u_2(1, y_2)\mu_1^n(1)\right]\\
&\qquad = u_2((1-\delta_1^n)x_1 + \delta_1^n \mu_1^n, y_2).
\end{aligned}$$

If $x_1 = \alpha^m$ for some m and $0 < y_2 \leq 1$, then $u_2(x_1, y_2) = \frac{1}{2} < \alpha^k$ for every k so that

$$\begin{aligned}
&u_2((1-\delta_1^n)x_1 + \delta_1^n \mu_1^n, 0)\\
&\qquad = (1-\delta_1^n)u_2(\alpha^m,0) + \delta_1^n u_2(\mu_1^n, 0)\\
&\qquad = (1-\delta_1^n)\alpha^m + \delta_1^n \left[\sum_k u_2(\alpha^k, 0)\mu_1^n(\alpha^k) + u_2(1,0)\mu_1^n(1)\right]\\
&\qquad = (1-\delta_1^n)\alpha^m + \delta_1^n \left[\sum_k \alpha^k \mu_1^n(\alpha^k) + \mu_1^n(1)\right]\\
&\qquad > (1-\delta_1^n)\tfrac{1}{2} + \delta_1^n \left[\textstyle\sum_k \tfrac{1}{2}\mu_1^n(\alpha^k) + 0\right]\\
&\qquad = (1-\delta_1^n)u_2(\alpha_k, y_2) + \delta_1^n \left[\sum_k u_2(\alpha^k, y_2)\mu_1^n(\alpha^k) + u_2(1, y_2)\mu_1^n(1)\right]
\end{aligned}$$

$$= u_2((1-\delta_1^n)x_1 + \delta_1^n\mu_1^n, y_2).$$

From these observations, it follows that $x_2 = 0$ is the unique best response of player 2 in the game $G_{(\delta^n,\mu^n)}$. To complete the argument, we show that player 1 has no best response (pure or mixed) to $x_2 = 0$ in the game $G_{(\delta^n,\mu^n)}$ implying that the game $G_{(\delta^n,\mu^n)}$ does not have an equilibrium. First, observe that $(0,1]$ is open in X_2 and $\mu_2^n \in \widehat{\Delta}(X_2)$ implying that $\mu_2^n((0,1]) > 0$. Therefore,

$$u_1(1,\mu_2^n) = \int_{\{0\}} u_1(1,y_2)d\mu_2^n + \int_{(0,1]} u_1(1,y_2)d\mu_2^n = u_1(1,0)\mu_2^n(\{0\}) = \mu_2^n(\{0\})$$

and

$$\begin{aligned} u_1(\alpha^m,\mu_2^n) &= \int_{\{0\}} u_1(\alpha^m,y_2)d\mu_2^n + \int_{(0,1]} u_1(\alpha^m,y_2)d\mu_2^n \\ &= u_1(\alpha^m,0)\mu_2^n(\{0\}) + \int_{(0,1]} \tfrac{1}{2}d\mu_2^n \\ &= x\alpha^m\mu_2^n(\{0\}) + \tfrac{1}{2}\mu_2^n((0,1]). \end{aligned}$$

As a result,

$$u_1(\alpha^m,\mu_2^n) < u_1(\alpha^{m+1},\mu_2^n)$$

for each m, implying that

$$u_1((1-\delta_1^n)\alpha^m+\delta_1^n\mu_1^n,(1-\delta_2^n)0+\delta_1^n\mu_2^n) < u_1((1-\delta_1^n)\alpha^{m+1}+\delta_1^n\mu_1^n,(1-\delta_2^n)0+\delta_1^n\mu_2^n).$$

Therefore, there does not exist an m such that α^m is best response to $x_2 = 0$ in $G_{(\delta^n,\mu^n)}$. In addition, there exists an $\widehat{m}$ such that

$$(1-\alpha^{\widehat{m}})\mu_2^n(\{0\}) < \tfrac{1}{2}\mu_2^n((0,1])$$

implying that

$$u_1(1,\mu_2^n) = \mu_2^n(\{0\}) < \alpha^{\widehat{m}}\mu_2^n(\{0\}) + \tfrac{1}{2}\mu_2^n((0,1]) = u_1(\alpha^{\widehat{m}},\mu_2^n)$$

and, consequently,

$$u_1((1-\delta_1^n)1+\delta_1^n\mu_1^n,(1-\delta_2^n)0+\delta_1^n\mu_2^n) < u_1((1-\delta_1^n)\alpha^{\widehat{m}}+\delta_1^n\mu_1^n,(1-\delta_2^n)0+\delta_1^n\mu_2^n).$$

Therefore, $x_1 = 1$ is not best response to $x_2 = 0$ in $G_{(\delta^n,\mu^n)}$. This proves that the game $G_{(\delta^n,\mu^n)}$ has no Nash equilibrium and applying Lemma 2, we conclude that G has no thp equilibrium.

4 Stable sets of equilibria

If G is a potential game with potential P, let

$$\arg\max_X P := \arg\max_{x \in X} P(x)$$

Then $\arg\max_X P \subseteq \pi(u)$, i.e., every maximizer of P is a pure-strategy Nash equilibrium in G. Therefore, $\arg\max_X P$ defines a refinement of the set of equilibria. We have shown that $\arg\max_X P$ contains a pure-strategy perfect equlibrium and it is our goal to provide a relationship between $\arg\max_X P$ and strategically stable sets.

Definition 5. Suppose that $G = (X_i, u_i)_{i=1}^N$ is a game. A subset $E \subseteq \xi(u)$ is *KM prestable* if E is closed and the following condition is satisfied: for every open set U containing E, there exists a $\kappa > 0$ such that, for every $\delta = (\delta_1, .., \delta_N)$ with $0 < \delta_i < \kappa$ and for every $\mu = (\mu_1, .., \mu_N)$ with $\mu_i \in \widehat{\Delta}(X_i)$ for each i,

$$\xi(u^{(\delta,\mu)}) \cap U \neq \varnothing.$$

A subset $E \subseteq \xi(u)$ is a ***KM stable set*** if E is a minimal (with respect to set inclusion) KM prestable set.

Remark 1. As a consequence of Definition 3, an equilibrium $\sigma \in \xi(u)$ is strictly perfect if and only if the set $E = \{\sigma\}$ is a KM stable set.

Theorem 2. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a game with upper semicontinuous potential P. Then $\arg\max_X P$ contains a KM stable set for G.*

Proof. First we show that $\arg\max_X P$ is KM prestable. For each $(\delta, \mu) \in (0,1)^N \times \widehat{\Delta}(X)$, let $G_{(\delta,\mu)}$ be the game defined in Section 2.2 as

$$G_{(\delta,\mu)} = (X_i, u_i^{(\delta,\mu)})_{i=1}^N,$$

where $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ is given by

$$u_i^{(\delta,\mu)}(x) := u_i\left((1-\delta_1)x_1 + \delta_1\mu_1, ..., (1-\delta_N)x_N + \delta_N\mu_N\right).$$

Since $\arg\max_X P^{(\delta,\mu)} \subseteq \pi(u^{(\delta,\mu)}) \subseteq \xi(u^{(\delta,\mu)})$, it suffices to prove that, for every open set U containing $\arg\max_{x\in X} P$, there exists a $\kappa > 0$ such that the following condition holds: for every $(\delta_1, .., \delta_N)$ with $0 < \delta_i < \kappa$ for each i and for every $(\mu_1, .., \mu_N)$ with $\mu_i \in \widehat{\Delta}(X_i)$ for each i,

$$\left(\arg\max_{x\in X} P^{(\delta,\mu)}\right) \cap U \neq \varnothing.$$

To see this, suppose not. Then there exists an open set U containing $\arg\max_X P$ and for each n numbers $0 < \delta_i^n < \frac{1}{n}$ and probability measures $\mu_i^n \in \widehat{\Delta}(X_i)$ such that $\arg\max_X P^{(\delta^n,\mu^n)} \cap U = \varnothing$. Since P is the uniform limit of the sequence $(P^{(\delta^n,\mu^n)})$ (apply Lemma 6) and X is compact, we can apply the same argument as that used in the proof of Theorem 1 and conclude that there exists a subsequence $(P^{(\delta^{n_k},\mu^{n_k})})$ and a sequence $x^k \in \arg\max_X P^{(\delta^{n_k},\mu^{n_k})}$ such that $x^k \to x$ and $x \in \arg\max_X P$. This contradiction establishes the claim. Since $\arg\max_X P$ is closed (since P is upper semicontinuous), it follows that $\arg\max_X P$ is KM prestable.

To complete the proof, we show that $\arg\max_X P$ contains a minimal KM prestable set by applying Zorn's Lemma in a standard way. Let $\mathcal{F}$ be defined as the collection of sets of Nash equilibria of G satisfying (i) $E \subseteq \arg\max_X P$ and (ii) E is KM prestable in G. Next, suppose that $\mathcal{F}$ is ordered by set inclusion and suppose that $\mathcal{C}$ is a totally ordered subcollection of $\mathcal{F}$. The collection $\mathcal{C}$ has the finite intersection property. Therefore, $S = \cap\{E : E \in \mathcal{C}\}$ is compact and nonempty since each member of $\mathcal{C}$ is closed and $\arg\max_X P$ is compact. To show that S is KM prestable, suppose that U is open and $S \subseteq U$. Then there exist $E' \in \mathcal{C}$ such that $E' \subseteq U$. Otherwise, $\{E\backslash U : E \in \mathcal{C}\}$ is a collection of closed subsets of $\arg\max_X P$ satisfying the finite intersection property. This implies that $S\backslash U = \cap\{E\backslash U : E \in \mathcal{C}\} \neq \varnothing$, an impossibility. Since E' is KM prestable, it follows that S is KM prestable. The existence of a minimal KM prestable set in G contained in $\arg\max_X P$ now follows from Zorn's Lemma. ■

While $\arg\max_X P$ contains a KM stable set, the next example shows that $\arg\max_X P$ itself need not be KM stable. This game is (trivially) continuous and also demonstrates that a continuous potential game need not have a strictly perfect equilibrium.

Example 2. Consider the finite two-player game G defined as

	L	C	R
T	$1,1$	$1,1$	$0,0$
B	$1,1$	$0,0$	$1,1$

The game G is a potential game and the value of the potential P at each strategy pair is indicated in the table below

	L	C	R
T	1	1	0
B	1	0	1

In this example

$$\arg\max_X P = \{(T, L), (T, C), (B, L), (B, R)\}.$$

However, the unique KM stable set for G is $\{(T, L), (B, L)\}$. Finally, we note that G has no strictly perfect equilibria.

To complete the discussion of stategic stability, we show that a strategically stable set contained in $\arg\max_X P$ consists of trembling-hand perfect pure-strategy equilibria.

Theorem 3. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a game with upper semicontinuous potential P. If $E \subseteq \arg\max_X P$ is a KM stable set, then each element of E is a pure-strategy trembling-hand perfect equilibrium*

Proof. If $|E| = 1$, then the one member of E is a strictly perfect equilibrium, hence a trembling-hand perfect equilibrium. So suppose that $|E| > 1$. Choose $x \in E$ and choose $\varepsilon > 0$ so that $E\backslash B_\varepsilon(x) \neq \varnothing$. Since E is KM stable and $E\backslash B_\varepsilon(x)$ is closed and nonempty, it follows from minimality that $E\backslash B_\varepsilon(x)$ is not KM stable. Therefore, there exists an open set U containing $E\backslash B_\varepsilon(x)$ such that, for every k, there exist $0 < \delta_i^k < \frac{1}{k}$ and $\mu_k \in \widehat{\Delta}(X)$ such that $\xi(u^{(\delta^k,\mu^k)}) \cap U = \varnothing$. Next, note that $E \subseteq U \cup B_\varepsilon(x)$ and $U \cup B_\varepsilon(x)$ is open. Since E is prestable, it follows that $\xi(u^{(\delta^k,\mu^k)}) \cap [U \cup B_\varepsilon(x)] \neq \varnothing$ for sufficiently large k. In particular, $\xi(u^{(\delta^k,\mu^k)}) \cap B_\varepsilon(x) \neq \varnothing$ for sufficiently large k and we conclude that x is trembling-hand perfect. ■

5 Essential equilibria

We conclude the paper with a discussion of essential equilibria in potential games. For the fixed compact, metric strategy spaces X_i, let $USC(X)$ denote the space of upper semicontinuous real-valued functions on $X = X_1 \times \cdots \times X_N$ and define $\mathcal{P}(X)$ to be the set of payoff profiles $u = (u_1, .., u_N)$ such that

$(X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game. If $u = (u_1, .., u_N) \in \mathcal{P}(X)$, then we will refer to u as an usc potential game and we will refer to an usc potential for $(X_i, u_i)_{i=1}^N$ as an usc potential for u. Furthermore, we view $\mathcal{P}(X)$ as a subset of the metric space $(B(X)^N, d)$, as defined in Section 2. Suppose that $u = (u_1, .., u_N) \in \mathcal{P}(X)$, let P be a potential for u and define

$$\eta(u) := \arg\max_{x \in X} P(x).$$

This definition is unambiguous since two potentials for u give rise to the same set of maximizers. Recalling that $\pi(u)$ (resp. $\xi(u)$) denotes the set of pure-strategy Nash equilibria (resp. mixed-strategy Nash equilibria) for u, it is clear that $\eta(u) \subseteq \pi(u) \subseteq \xi(u)$.

Definition 6. Suppose that $u \in \mathcal{P}(X)$. An equilibrium $\sigma \in \xi(u)$ is ***essential*** if the following condition is satisfied: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\xi(v) \cap B_\varepsilon(\sigma) \neq \varnothing$ whenever $v \in \mathcal{P}(X)$ and $d(v, u) < \delta$.

In the case of finite games, it is well-known that essential equilibria are strictly perfect and this result extends to upper semicontinuous potential games.

Proposition 4. *Suppose that $u \in \mathcal{P}(X)$. If $\sigma \in \xi(u)$ is an essential equilibrium in $G = (X_i, u_i)_{i=1}^N$, then σ is a strictly perfect equilibrium.*

Proof. Suppose that $\sigma \in \xi(u)$ is an essential equilibrium in $G = (X_i, u_i)_{i=1}^N$. Fix $\varepsilon > 0$. Then there exists a $\beta > 0$ such that $\xi(v) \cap B_\varepsilon(\sigma) \neq \varnothing$ whenever $v \in \mathcal{P}(X)$ and $d(v, u) < \beta$. Next, we can duplicate the proof of Lemma 5 and conclude that there exists a $\kappa > 0$ such that the following condition holds for each player j: for every $(\delta_1, .., \delta_N)$ with $0 < \delta_i < \kappa$ for each i, and for every $(\mu_1, .., \mu_N)$ with $\mu_i \in \widehat{\Delta}(X_i)$ for each i,

$$\sup_{x \in X} |u_j(x) - u_j^{(\delta,\mu)}(x)| < \tfrac{\beta}{n}.$$

Since $u \in \mathcal{P}(X)$ admits at least one upper semicontinuous potential P, it follows that $u^{(\delta,\mu)} \in \mathcal{P}(X)$ since $P^{(\delta,\mu)}$ is an upper semicontinuous potential for $u^{(\delta,\mu)}$ as a consequence of Lemma 3. Consequently, $u^{(\delta,\mu)} \in \mathcal{P}(X)$ and $d(u^{(\delta,\mu)}, u) < \beta$ whenever $0 < \delta_i < \kappa$ for each i and $\mu \in \widehat{\Delta}(X)$. Therefore, $\xi(u^{(\delta,\mu)}) \cap B_\varepsilon(\sigma) \neq \varnothing$, and we deduce from Lemma 2 that σ is strictly perfect. $\blacksquare$

Finally, it is our goal to show that, for "most" potential games $u \in \mathcal{P}(X)$ the set $\eta(u)$ consists of pure-strategy Nash equilibria for u, all of which are essential, strictly perfect and KM stable as singleton sets. To accomplish this, we need more notation and a few lemmas. Since a given potential game can be identified with an equivalence class of potentials that only differ by a constant, it will be convenient to specify a particular normalized potential with each $u \in \mathcal{P}(X)$. Fix $\overline{x} \in X$. For each $u \in \mathcal{P}(X)$, let $F(u) \in USC(X)$ denote the potential for u defined as

$$F(u)(x_1, .., x_N) = \sum_{i=1}^{N} \left[u_i(x_1, .., x_i, \overline{x}_{i+1}, .., \overline{x}_N) - u_i(x_1, ..x_{i-1}, \overline{x}_i, .., \overline{x}_N)\right].$$

Consequently,

$$\eta(u) = \arg\max_X F(u).$$

We will suppress the dependence of F on $\overline{x}$ to lighten the notation.

Lemma 7. *The mapping $F : \mathcal{P}(X) \to USC(X)$ is uniformly continuous.*

Proof. Choose $\varepsilon > 0$, choose $0 < \delta < \frac{\varepsilon}{2}$ and suppose that $\{u, v\} \subseteq \mathcal{P}(X)$ and $d(v, u) < \delta$. Then for each $(x_1, .., x_N) \in X$, we have

$$\begin{aligned}
&|F(u)(x_1, .., x_N) - F(v)(x_1, .., x_N)| \\
&\qquad \le \sum_{i=1}^{N} |u_i(x_1, .., x_i, \overline{x}_{i+1}, .., \overline{x}_N) - v_i(x_{1,..,}x_i, \overline{x}_{i+1}, .., \overline{x}_N)| \\
&\qquad\qquad + \sum_{i=1}^{N} |v_i(x_1, .., x_{i-1}, \overline{x}_i, .., \overline{x}_N) - u_i(x_{1,..,}x_{i-1}, \overline{x}_i, .., \overline{x}_N)| \\
&\qquad \le 2\delta \\
&\qquad < \varepsilon,
\end{aligned}$$

so F is uniformly continuous. ■

To show that all members of $\eta(u)$ are essential for all u in a "topologically large" subset of $\mathcal{P}(X)$, we first observe that all members of $\eta(u)$ are essential equilibria for any $u \in \mathcal{P}(X)$ at which the correspondence $\eta : \mathcal{P}(X) \rightrightarrows X$ is lower hemicontinuous. The key result for establishing our genericity theorem is a classic result of Fort [14] which we now state.

Theorem 4 (Fort's Theorem)**.** *Suppose that S is a topological space and Y is a metric space. If the correspondence $\varphi : S \rightrightarrows Y$ is nonempty-valued, compact-valued and upper hemicontinuous, then φ is lower hemicontinuous at all all points in a residual subset of S.*

Fort's theorem has been used is a number of papers to establish genericity of essential equilibria and essential components of equilibria in strategic-form games (e.g., Zhou *et al.* [30] and the references cited there, and Carbonell-Nicolau [12]).

Proposition 5. *There exists a dense, residual subset $Z \subseteq \mathcal{P}(X)$ such that $\eta : \mathcal{P}(X) \rightrightarrows X$ is lower hemicontinuous at each $u \in Z$. If $u \in Z$, then each $x \in \eta(u)$ is an essential equilibrium, hence a strictly perfect equilibrium and $\{x\}$ is a singleton stable set.*

Proof. Obviously, $\eta(x) = \arg\max_X F(u)$ is nonempty and compact for each $u \in \mathcal{P}(X)$. Next, we claim that the correspondence $\eta : \mathcal{P}(X) \rightrightarrows X$ is upper hemicontinuous. Since X is compact, it suffices to show that η has closed graph. To see this, suppose that $u^n \to u, x^n \to x$ and $x^n \in \eta(u^n) = \arg\max_X F(u^n)$ for each n. Applying Lemma 7, it follows that $F(u^n) \to F(u)$ uniformly on X, so from Proposition 1 we conclude that $x \in \eta(u)$. Applying Theorem 4 to the upper hemicontinuous correspondence η, there exists a residual subset $Z \subseteq \mathcal{P}(X)$ such that $\eta : \mathcal{P}(X) \rightrightarrows X$ is lower hemicontinuous at each $u \in Z$. It follows that every member of $\eta(Z)$ is an essential equilibrium, and hence a strictly perfect equilibrium (Proposition 4). To complete the proof, we show that Z is dense. From Lemma 5, we conclude that $\mathcal{P}(X)$ is a closed subset of the Banach space $[B(X)]^N$ implying that $\mathcal{P}(X)$ is a complete metric space. Therefore, Z is dense as a consequence of the Baire Category Theorem. ■

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