

Schipper, Burkhard C.

**Working Paper**

**Preference-dependent unawareness**

Working Paper, No. 11-6

**Provided in Cooperation with:**

University of California Davis, Department of Economics

*Suggested Citation:* Schipper, Burkhard C. (2011) : Preference-dependent unawareness, Working Paper, No. 11-6, University of California, Department of Economics, Davis, CA

This Version is available at:

<https://hdl.handle.net/10419/58369>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

# UC DAVIS

---

## DEPARTMENT OF ECONOMICS

### Working Paper Series

Preference-Dependent Unawareness

Burkhard Schipper  
University of California, Davis

April 07, 2011

Paper # 11-6

Morris (1996, 1997) introduced preference-based definitions of knowledge of belief in standard state-space structures. This paper extends this preference-based approach to unawareness structures (Heifetz, Meier, and Schipper, 2006, 2008). By defining unawareness and knowledge in terms of preferences over acts in unawareness structures and showing their equivalence to the epistemic notions of unawareness and knowledge, we try to build a bridge between decision theory and epistemic logic. Unawareness of an event is behaviorally characterized as the event being null and its negation being null.

Department of Economics  
One Shields Avenue  
Davis, CA 95616  
(530)752-0741

[http://www.econ.ucdavis.edu/working\\_search.cfm](http://www.econ.ucdavis.edu/working_search.cfm)

# PREFERENCE-BASED UNAWARENESS

Burkhard C. Schipper\*

April 7, 2011

## Abstract

Morris (1996, 1997) introduced preference-based definitions of knowledge of belief in standard state-space structures. This paper extends this preference-based approach to unawareness structures (Heifetz, Meier, and Schipper, 2006, 2008). By defining unawareness and knowledge in terms of preferences over acts in unawareness structures and showing their equivalence to the epistemic notions of unawareness and knowledge, we try to build a bridge between decision theory and epistemic logic. Unawareness of an event is behaviorally characterized as the event being null and its negation being null.

**Keywords:** Unawareness, awareness, knowledge, preferences, subjective expected utility theory, decision theory, null event.

**JEL-Classifications:** C70, C72, D80, D82.

---

\*Department of Economics, University of California, Davis. Email: bcschipper@ucdavis.edu  
This paper is closely related to prior and ongoing work on unawareness with my friends, Aviad Heifetz and Martin Meier. All mistakes are mine. Financial support through NSF SES-0647811 is gratefully acknowledged.

# 1 Introduction

Unawareness refers to the lack of conception rather than the lack of information. There is a fundamental difference between not knowing about which events obtain and the inability to conceive of some events. Unawareness is an interdisciplinary topic that fascinates economists, computer scientists, logicians, and philosophers alike. Traditionally, computer scientists, logicians and philosophers are interested in epistemic models while most economists are mainly interested in the behavioral implications. In the literature, unawareness has been defined epistemically using syntactic and semantic approaches.<sup>1</sup> While epistemic characterizations are conceptually insightful, the behavioral content of unawareness is less clear. Can unawareness be characterized behaviorally?

A first attempt to answer this question is presented in Schipper (2010). There we apply the lattice state-space structure of Heifetz, Meier, and Schipper (2006, 2008, 2010) to the Anscombe-Aumann approach to subjective utility theory and characterize awareness-dependent subjective expected utility. This framework allows to distinguish unawareness of an event from the event being Savage null and thus provides already behavioral implications of unawareness. Yet, our approach in Schipper (2010) has many shortcomings. For instance, as in almost any decision theoretic framework, we consider states of nature rather than states of the world. That is, states describe just lists of physical events but are silent on the state of mind of the decision maker. While such an approach is sensible in a purely decision theoretic framework, where bets are placed on physical events rather than (higher order) beliefs of some players, it does not allow to characterize behaviorally - at least in principle - the intricate epistemic properties of knowledge and awareness across the lattice structure introduced in Heifetz, Meier, and Schipper (2006, 2008). For instance, introspection of knowledge requires bets on events in which the decision maker knows an event.

Morris (1996, 1997) showed for standard state-spaces with states of the world that properties of knowledge and belief can be - at least in principle - characterized behaviorally. In this paper, we extend his approach to unawareness structures and show how to characterize behaviorally properties of the possibility correspondence introduced in Heifetz, Meier and Schipper (2006). Moreover, we show how to characterize preference-based operators of knowledge, awareness and unawareness.

This work is directly related to a growing literature on unawareness. Heifetz, Meier, and Schipper (2006, 2008, 2010) introduced a syntax-free semantics of unawareness using state-spaces familiar to economists, decision theorists, and game theorists. Apart from having a syntax-free semantics, Heifetz, Meier, and Schipper (2006, 2008) generalize Modica and Rustichini (1999) and a version of Fagin and Halpern (1988) to the multi-agent case. The precise connection between Fagin and Halpern (1988), Modica and Rustichini (1999), Halpern (2001) and Heifetz, Meier, and Schipper (2006) is understood from Halpern and Rêgo (2008, 2010) and Heifetz, Meier, and Schipper (2008). Galanis

---

<sup>1</sup>For a bibliography see <http://www.econ.ucdavis.edu/faculty/schipper/unaw.htm>

(2007) relaxes some properties of the possibility correspondence in Heifetz, Meier, and Schipper (2006). The connection between Heifetz, Meier, and Schipper (2006, 2008) and Galanis (2007) is explored in Galanis (2010). Li (2009) and Feinberg (2009) also present approaches to unawareness. The connections between Li (2009) and Feinberg (2009) and other models in the literature are yet to be explored. The literature addressed also awareness of unawareness, see Board and Chung (2009), Halpern and Rego (2009 a, b), Sillari (2008), and Agotnes and Alechina (2007). The relationship between Board and Chung (2009) and Heifetz, Meier, and Schipper (2006) is studied in Board, Chung, and Schipper (2011). The relationship between Halpern and Rego (2009 a, b) and Heifetz, Meier and Schipper (2006) is explored in Halpern and Rego (2009b). See Section 4 for a further discussion of the related literature.

In the next section we provide a brief exposition of unawareness structures and review properties of knowledge and awareness. In Section 3 we introduce the decision theoretic framework and provide the results. We conclude with a discussion in Section 4. Proofs are collected in the appendix.

## 2 Unawareness Structures

### 2.1 State-Spaces

Let  $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$  be a finite lattice of disjoint *state-spaces*, with the partial order  $\preceq$  on  $\mathcal{S}$ . For simplicity we assume in this paper that each  $S$  is finite. If  $S_\alpha$  and  $S_\beta$  are such that  $S_\alpha \succeq S_\beta$  we say that “ $S_\alpha$  is more expressive than  $S_\beta$  – states of  $S_\alpha$  describe situations with a richer vocabulary than states of  $S_\beta$ ”.<sup>2</sup> Denote by  $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$  the union of these spaces.

Spaces in the lattice can be more or less “rich” in terms of facts that may or may not obtain in them. The partial order relates to the “richness” of spaces. The upmost space of the lattice may be interpreted as the “objective” state-space. Its states encompass full descriptions.

### 2.2 Projections

For every  $S$  and  $S'$  such that  $S' \succeq S$ , there is a surjective projection  $r_S^{S'} : S' \rightarrow S$ , where  $r_S^S$  is the identity. (“ $r_S^{S'}(\omega)$  is the restriction of the description  $\omega$  to the more limited vocabulary of  $S$ .”) Note that the cardinality of  $S$  is smaller than or equal to the cardinality of  $S'$ . We require the projections to commute: If  $S'' \succeq S' \succeq S$  then  $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$ . If  $\omega \in S'$ , denote  $\omega_S = r_S^{S'}(\omega)$ . If  $D \subseteq S'$ , denote  $D_S = \{\omega_S : \omega \in D\}$ .

---

<sup>2</sup>Here and in what follows, phrases within quotation marks hint at intended interpretations, but we emphasize that these interpretations are not part of the definition of the set-theoretic structure.

Projections “translate” states in “more expressive” spaces to states in “less expressive” spaces by “erasing” facts that can not be expressed in a lower space.

These surjective projections may embody Savage’s idea that “(i)t may be well, however, to emphasize that a state of the smaller world corresponds not to a state of the larger, but to a set of states” (Savage, 1954, p. 9).

## 2.3 Events

Denote  $g(S) = \{S' : S' \succeq S\}$ . For  $D \subseteq S$ , denote  $D^\dagger = \bigcup_{S' \in g(S)} (r_S^{S'})^{-1}(D)$ . (“All the extensions of descriptions in  $D$  to at least as expressive vocabularies.”)

An *event* is a pair  $(E, S)$ , where  $E = D^\dagger$  with  $D \subseteq S$ , where  $S \in \mathcal{S}$ .  $D$  is called the *base* and  $S$  the *base-space* of  $(E, S)$ , denoted by  $S(E)$ . If  $E \neq \emptyset$ , then  $S$  is uniquely determined by  $E$  and, abusing notation, we write  $E$  for  $(E, S)$ . Otherwise, we write  $\emptyset^S$  for  $(\emptyset, S)$ . Note that not every subset of  $\Omega$  is an event.

Some fact may obtain in a subset of a space. Then this fact should be also “expressible” in “more expressive” spaces. Therefore the event contains not only the particular subset but also its inverse images in “more expressive” spaces.

Let  $\Sigma$  be the set of events of  $\Omega$ . Note that unless  $\mathcal{S}$  is a singleton,  $\Sigma$  is not an algebra because it contains distinct vacuous events  $\emptyset^S$  for all  $S \in \mathcal{S}$ . These vacuous events correspond to contradictions with differing “expressive power”.

## 2.4 Negation

If  $(D^\dagger, S)$  is an event where  $D \subseteq S$ , the negation  $\neg(D^\dagger, S)$  of  $(D^\dagger, S)$  is defined by  $\neg(D^\dagger, S) := ((S \setminus D)^\dagger, S)$ . Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write  $\neg D^\dagger := (S \setminus D)^\dagger$ . Note that by our notational convention, we have  $\neg S^\dagger = \emptyset^S$  and  $\neg \emptyset^S = S^\dagger$ , for each space  $S \in \mathcal{S}$ . The event  $\emptyset^S$  should be interpreted as a “logical contradiction phrased with the expressive power available in  $S$ .”  $\neg D^\dagger$  is typically a proper subset of the complement  $\Omega \setminus D^\dagger$ . That is,  $(S \setminus D)^\dagger \subsetneq \Omega \setminus D^\dagger$ .

Intuitively, there may be states in which the description of an event  $D^\dagger$  is both expressible and true – these are the states in  $D^\dagger$ ; there may be states in which its description is expressible but false – these are the states in  $\neg D^\dagger$ ; and there may be states in which neither its description nor its negation are expressible – these are the states in

$$\Omega \setminus (D^\dagger \cup \neg D^\dagger) = \Omega \setminus S(D^\dagger)^\dagger.$$

Thus our structure is not a standard state-space model in the sense of Dekel, Lipman, and Rustichini (1998).

## 2.5 Conjunction and Disjunction

If  $\left\{ \left( D_\lambda^\uparrow, S_\lambda \right) \right\}_{\lambda \in L}$  is a collection of events (with  $D_\lambda \subseteq S_\lambda$ , for  $\lambda \in L$ ), their conjunction  $\bigwedge_{\lambda \in L} \left( D_\lambda^\uparrow, S_\lambda \right)$  is defined by  $\bigwedge_{\lambda \in L} \left( D_\lambda^\uparrow, S_\lambda \right) := \left( \left( \bigcap_{\lambda \in L} D_\lambda^\uparrow \right), \sup_{\lambda \in L} S_\lambda \right)$ . Note, that since  $\mathcal{S}$  is a *complete* lattice,  $\sup_{\lambda \in L} S_\lambda$  exists. If  $S = \sup_{\lambda \in L} S_\lambda$ , then we have  $\left( \bigcap_{\lambda \in L} D_\lambda^\uparrow \right) = \left( \bigcap_{\lambda \in L} \left( r_{S_\lambda}^S \right)^{-1} \left( D_\lambda \right) \right)^\uparrow$ . Again, abusing notation, we write  $\bigwedge_{\lambda \in L} D_\lambda^\uparrow := \bigcap_{\lambda \in L} D_\lambda^\uparrow$  (we will therefore use the conjunction symbol  $\wedge$  and the intersection symbol  $\cap$  interchangeably).

We define the relation  $\subseteq$  between events  $(E, S)$  and  $(F, S')$ , by  $(E, S) \subseteq (F, S')$  if and only if  $E \subseteq F$  as sets *and*  $S' \preceq S$ . If  $E \neq \emptyset$ , we have that  $(E, S) \subseteq (F, S')$  if and only if  $E \subseteq F$  as sets. Note however that for  $E = \emptyset^S$  we have  $(E, S) \subseteq (F, S')$  if and only if  $S' \preceq S$ . Hence we can write  $E \subseteq F$  instead of  $(E, S) \subseteq (F, S')$  as long as we keep in mind that in the case of  $E = \emptyset^S$  we have  $\emptyset^S \subseteq F$  if and only if  $S \succeq S(F)$ . It follows from these definitions that for events  $E$  and  $F$ ,  $E \subseteq F$  is equivalent to  $\neg F \subseteq \neg E$  only when  $E$  and  $F$  have the same base, i.e.,  $S(E) = S(F)$ .

The disjunction of  $\left\{ D_\lambda^\uparrow \right\}_{\lambda \in L}$  is defined by the de Morgan law  $\bigvee_{\lambda \in L} D_\lambda^\uparrow = \neg \left( \bigwedge_{\lambda \in L} \neg \left( D_\lambda^\uparrow \right) \right)$ . Typically  $\bigvee_{\lambda \in L} D_\lambda^\uparrow \subseteq \bigcup_{\lambda \in L} D_\lambda^\uparrow$ , and if all  $D_\lambda$  are nonempty we have that  $\bigvee_{\lambda \in L} D_\lambda^\uparrow = \bigcup_{\lambda \in L} D_\lambda^\uparrow$  holds if and only if all the  $D_\lambda^\uparrow$  have the same base-space. Note, that by these definitions, the conjunction and disjunction of events is a event.

## 2.6 Possibility Correspondence

For each individual there is a *possibility correspondence*  $\Pi : \Omega \longrightarrow 2^\Omega$  with the following properties:

0. *Confinement*: If  $\omega \in S$  then  $\Pi(\omega) \subseteq S'$  for some  $S' \preceq S$ .
1. *Generalized Reflexivity*:  $\omega \in \Pi^\uparrow(\omega)$  for every  $\omega \in \Omega$ .<sup>3</sup>
2. *Stationarity*:  $\omega' \in \Pi(\omega)$  implies  $\Pi(\omega') = \Pi(\omega)$ .
3. *Projections Preserve Awareness*: If  $\omega \in S'$ ,  $\omega \in \Pi(\omega)$  and  $S \preceq S'$  then  $\omega_S \in \Pi(\omega_S)$ .
4. *Projections Preserve Ignorance*: If  $\omega \in S'$  and  $S \preceq S'$  then  $\Pi^\uparrow(\omega) \subseteq \Pi^\uparrow(\omega_S)$ .
5. *Projections Preserve Knowledge*: If  $S \preceq S' \preceq S''$ ,  $\omega \in S''$  and  $\Pi(\omega) \subseteq S'$  then  $(\Pi(\omega))_S = \Pi(\omega_S)$ .

For 5., we could have assumed  $\supseteq$  and deduce  $=$  from  $\supseteq$ , 3., and the other properties.

---

<sup>3</sup>Here and in what follows, we abuse notation slightly and write  $\Pi^\uparrow(\omega)$  for  $(\Pi(\omega))^\uparrow$ .

**Remark 1** *Generalized Reflexivity implies that if  $S' \preceq S$ ,  $\omega \in S$  and  $\Pi(s) \subseteq S'$ , then  $r_{S'}^S(\omega) \in \Pi(\omega)$ . In particular, we have  $\Pi(\omega) \neq \emptyset$ , for all  $\omega \in \Omega$ .*

**Remark 2** *Property 4 and Confinement imply that if  $S' \preceq S$ ,  $\omega \in S$  and  $\Pi(\omega_{S'}) \subseteq S''$ , then  $\Pi(\omega) \subseteq S^*$  for some  $S^*$  with  $S'' \preceq S^*$ .*

**Remark 3** *Property 5 and Confinement imply Property 3.*

Generalized Reflexivity and Stationarity are the analogues of the partitional properties of the possibility correspondence in partitional information structures. In particular, Generalized Reflexivity will yield the truth property (that what an individual knows indeed obtains – property (iii) in Proposition 2); Stationarity will guarantee the introspection properties (that an individual knows what she knows – property (iv) in Proposition 2, and that an individual knows what she ignores provided she is aware of it – Property 5. in Proposition 3).

Properties 3. to 5. guarantee the coherence of the knowledge and the awareness of individuals down the lattice structure. They compare the possibility sets of an individual in a state  $\omega$  and its projection  $\omega_S$ . The properties guarantee that, first, at the projected state  $\omega_S$  the individual knows nothing she does not know at  $\omega$ , and second, at the projected state  $\omega_S$  the individual is not aware of anything she is unaware of at  $\omega$  (Projections Preserve Ignorance). Third, at the projected state  $\omega_S$  the individual knows every event she knows at  $\omega$ , provided that this event is based in a space lower than or equal to  $S$  (Projections Preserve Knowledge). Fourth, at the projected state  $\omega_S$  the individual is aware of every event she is aware of at  $\omega$ , provided that this event is based in a space lower than or equal to  $S$  (Projections Preserve Awareness).

## 2.7 Knowledge

**Definition 1** *The decision maker's knowledge operator on events  $E$  is defined, as usual, by*

$$K(E) := \{\omega \in \Omega : \Pi(\omega) \subseteq E\},$$

*if there is a state  $\omega$  such that  $\Pi(\omega) \subseteq E$ , and by*

$$K(E) := \emptyset^{S(E)}$$

*otherwise.*

The following two propositions are proved in Heifetz, Meier, and Schipper (2006):

**Proposition 1** *If  $E$  is an event, then  $K(E)$  is an  $S(E)$ -based event.*

**Proposition 2** *The Knowledge operator  $K$  has the following properties:*



- (i) *Necessitation*:  $K(\Omega) = \Omega$ ,
- (ii) *Conjunction*:  $K(\bigcap_{\lambda \in L} E_\lambda) = \bigcap_{\lambda \in L} K(E_\lambda)$ ,
- (iii) *Truth*:  $K(E) \subseteq E$ ,
- (iv) *Positive Introspection*:  $K(E) \subseteq KK(E)$ ,
- (v) *Monotonicity*:  $E \subseteq F$  implies  $K(E) \subseteq K(F)$ ,
- (vi)  $\neg K(E) \cap \neg K\neg K(E) \subseteq \neg K\neg K\neg K(E)$ .

Proposition 2 says that the knowledge operator has all the strong properties of knowledge in partitioned information structures, except for the weakening (vi) of the negative introspection property. Negative introspection – the property  $\neg K(E) \subseteq K\neg K(E)$  that when an individual does not know an event, she knows she does not know it – obtains only when the individual is also aware of the event (see property 5 of the next proposition).

## 2.8 Awareness and Unawareness

**Definition 2** *The decision maker's awareness operator from events to events is defined by*

$$A(E) = \{\omega \in \Omega : \Pi(\omega) \subseteq S(E)^\uparrow\}$$

*if there is a state  $\omega$  such that  $\Pi(\omega) \subseteq S(E)^\uparrow$ , and by*

$$A(E) = \emptyset^{S(E)}$$

*otherwise. The unawareness operator is then naturally defined by*

$$U(E) = \neg A(E).$$

Alternatively, we could follow Modica and Rustichini (1999) in defining the *unawareness operator* by

$$U(E) = \neg K(E) \cap \neg K\neg K(E).$$

Heifetz, Meier, and Schipper (2008, Remark 6) show that the two definitions are indeed equivalent. Note that by Proposition 1 and the definition of the negation, we have

$$A(E) = K(E) \cup K\neg K(E).$$

The following proposition is proved in Heifetz, Meier, and Schipper (2006):

**Proposition 3** *The following properties of knowledge and awareness obtain:*

1. *KU Introspection*:  $KU(E) = \emptyset^{S(E)}$ ,

2. *AU Introspection*:  $U(E) = UU(E)$ ,
3. *Weak Necessitation*:  $A(E) = K(S(E)^\uparrow)$ ,
4. *Strong Plausibility*:  $U(E) = \bigcap_{n=1}^{\infty} (\neg K)^n(E)$ ,
5. *Weak Negative Introspection*:  $\neg K(E) \cap A\neg K(E) = K\neg K(E)$ ,
6. *Symmetric*:  $A(E) = A(\neg E)$ ,
7. *A-Conjunction*:  $\bigcap_{\lambda \in L} A(E_\lambda) = A(\bigcap_{\lambda \in L} E_\lambda)$ ,
8. *AK-Self Reflection*:  $A(E) = AK(E)$ ,
9. *AA-Self Reflection*:  $A(E) = AA(E)$ ,
10. *A-Introspection*:  $A(E) = KA(E)$ .

Properties 1. to 4. have been proposed by Dekel, Lipman and Rustichini (1998), properties 6. to 9. by Modica and Rustichini (1999), and properties 5. to 9. by Fagin and Halpern (1988) and Halpern (2001). *A-Introspection* is the property that an individual is aware of an event if and only if she knows she is aware of it.

### 3 Preference-Based Knowledge and Awareness

So far, we just outlined unawareness structures introduced in Heifetz, Meier, and Schipper (2006, 2008). In this section, we add decision theoretic primitives and characterize the possibility correspondence, knowledge, awareness and unawareness by choices. As in Morris (1996, 1997), from now on we restrict ourselves to finite  $\Omega$  (see also Fn. 7). This assumption may be defended by the fact that most choice experiments take place in a finite context.

#### 3.1 Acts

An act is a function  $f : \Omega \rightarrow \mathbb{R}$ , where  $f(\omega) \in \mathbb{R}$  can be thought of the money prize at state  $\omega$ .

Note that different from Savage acts,  $f$  is not defined on just one state-space but on the union of spaces  $\Omega$ . This is interpreted as follows: Let's say an individual considers investing in a firm (e.g., the act  $f$ ). A firm can be viewed as a bundle of uncertain opportunities and liabilities. The decision maker may perceive only a subset of opportunities and liabilities depending on her awareness level that may be influenced by her prior experience or her reading of the "fine prints" of the share sales and purchase agreement. In our setting, the opportunities and liabilities are represented by money prizes contingent

on states, i.e. acts. Some of the opportunities and liabilities (i.e., events) may not be expressible in some of the spaces in  $\mathcal{S}$ . That is, they are not perceived when having certain awareness levels. So an act denotes simultaneously more or less rich descriptions of those opportunities and liabilities. It is essentially a label of the action “buy the firm”. Which opportunities and liabilities are perceived by the decision maker, i.e. the awareness level of the decision maker, will be captured by the preferences over acts introduced below.

For any event  $E$  and acts  $f$  and  $g$ , define a *composite act*  $f_Eg$  by

$$f_Eg(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in E \\ g(\omega) & \text{otherwise.} \end{cases}$$

Note that different from composite acts in the Savage approach,  $g$  is not only prescribed on the negation of  $E$  but also on all states that are neither in  $E$  nor in  $\neg E$ . Different from acts defined on a standard state-space, we have in general  $f_Eg \neq g_{\neg E}f$ .

For any collection of pairwise disjoint events  $E_1, E_2, \dots, E_n \in \Sigma$  and acts  $f^1, f^2, \dots, f^n, g \in \mathcal{A}$ , let  $f_{E_1}^1 f_{E_2}^2 \dots f_{E_n}^n g$  denote the composite act that yields  $f^i(\omega)$  if  $\omega \in E_i$  for  $i = 1, \dots, n$ , and  $g(\omega)$  otherwise.

Let  $\mathcal{A}$  denote the set of all acts.

Note that we do not impose a measurability condition on acts in the sense that for any  $f \in \mathcal{A}$  and  $x \in \mathbb{R}$ , the set of states  $\{\omega \in \Omega : f(\omega) = x\}$  is an event in  $\Sigma$  as defined previously. While such a measurability assumption may be justified in some applications, it may not be applicable in general. The speculative trade example in Heifetz, Meier, and Schipper (2006) is one instance.<sup>4</sup> A practical framework of decision making under unawareness should not rule out such examples.

In the following we will consider also composite acts of the form

$$f_{\{\omega\}}g(\omega') = \begin{cases} f(\omega') & \text{if } \omega = \omega' \\ g(\omega') & \text{otherwise.} \end{cases}$$

Although  $\{\omega\}$  may not be an event in the unawareness structure, we still have  $f_{\{\omega\}}g \in \mathcal{A}$ . To see this note that for every  $f, g \in \mathcal{A}$  we can define  $h \in \mathcal{A}$  such that  $h(\omega) = f(\omega)$  and  $h(\omega') = g(\omega')$  for  $\omega' \neq \omega$ . Then  $f_{\{\omega\}}g = h_{\{\omega\}\uparrow}g$  and clearly  $h_{\{\omega\}\uparrow}g \in \mathcal{A}$ .

## 3.2 Preferences

We follow Morris (1996, 1997) in defining a binary relation  $\succsim(\omega)$  on acts in  $\mathcal{A}$  for each  $\omega \in \Omega$ . We assume that for any  $\omega \in \Omega$ ,  $\succsim(\omega)$  is a preference relation, i.e.,  $\succsim(\omega)$  is reflexive, complete, and transitive. That is, at each state the decision maker is assumed

---

<sup>4</sup>There, the set of set of states where the value of the firm is 90 dollars (for instance) is not an event in the sense of our event structure. Thus, the act “buy the firm” would not satisfy such a measurability condition. See also Heifetz, Meier, and Schipper (2010) and Meier and Schipper (2010) for a detailed analysis of speculative trade under unawareness.

to have a preference relation.  $f \succsim(\omega) g$  means that at state  $\omega$ , the decision maker chooses  $f$  over  $g$ .<sup>5</sup> Reflexivity means that every act is as good as itself. Completeness is a strong assumption. It requires that the decision maker can rank any two acts. Transitivity is sufficient to rule out “money pumps”, i.e., choice cycles along which the decision maker could be exploited.

As usual, strict preference,  $\succ(\omega)$ , is defined on  $\mathcal{A}$  by  $\succsim(\omega)$  and not  $\succsim(\omega)$ . Indifference,  $\sim(\omega)$ , is defined on  $\mathcal{A}$  by  $\succsim(\omega)$  and  $\precsim(\omega)$ .

### 3.3 Preference-Based Possibility Correspondence

Define a preference-based correspondence  $\tilde{\Pi} : \Omega \rightarrow 2^\Omega$  by

$$\tilde{\Pi}(\omega) = \{\omega' \in \Omega : f_{\{\omega'\}}g \succ(\omega) h_{\{\omega'\}}g \text{ for some } f, g, h \in \mathcal{A}\}. \quad (1)$$

Our aim is to analyze when  $\tilde{\Pi}$  is a possibility correspondence satisfying properties 0. to 5. To this extent, we need to consider the following properties on  $\succsim$ .

We slightly abuse notation and denote by  $S_\omega$  the space  $S \in \mathcal{S}$  for which  $\omega \in S$ . Since any two spaces in  $\mathcal{S}$  are disjoint,  $S_\omega$  is unique, for all  $\omega \in \Omega$ .

**Property 1** *If  $f_{\{\omega'\}}g \succ(\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$  and  $S_{\omega''} \neq S_{\omega'}$ , then  $f'_{\{\omega''\}}g' \sim(\omega) h'_{\{\omega''\}}g'$  for all  $f', g', h' \in \mathcal{A}$ .*

**Property 2** *If  $f_{\{\omega'\}}g \succ(\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$ , then  $S_{\omega'} \preceq S_\omega$ .*

These two properties characterize Confinement of the preference-based possibility correspondence.

**Lemma 1** *For all  $\omega \in \Omega$ ,  $\succsim(\omega)$  satisfies Properties 1 and 2 if and only if  $\tilde{\Pi}$  satisfies Confinement.*

Close inspection of Confinement reveals that it consists of two properties of the possibility correspondence. First, for each state  $\omega$  it confines the value of the correspondence to a single space. Second, it requires that this space is weakly lower than the space  $S_\omega$ . The proof of Lemma 1 in the appendix reveals that these two features are mimicked by Properties 1 and 2 respectively.

Next we turn to Generalized Reflexivity.

**Property 3**  *$f_{\{\omega_S\}}g \succ(\omega) h_{\{\omega_S\}}g$  for some  $S \preceq S_\omega$  and some  $f, g, h \in \mathcal{A}$ .*

---

<sup>5</sup>Note that  $f \succsim(\omega) g$  does not necessarily imply that the decision maker would choose  $f$  over  $g$  if he believes that the state is  $\omega$ .

**Lemma 2** For all  $\omega \in \Omega$ ,  $\succsim(\omega)$  satisfies Property 3 if and only if  $\tilde{\Pi}$  satisfies Generalized Reflexivity.

The proof is contained in the appendix.

Morris (1996, Lemma 3) showed that  $f_{\{\omega\}}g \succ(\omega) h_{\{\omega\}}g$  for some  $f, g, h \in \mathcal{A}$  and for all  $\omega \in \Omega$  characterizes the truth axiom of knowledge in standard states-space structures. Property 3 and Lemma 2 should be understood as a generalization to unawareness structures.

**Property 4** If  $f_{\{\omega'\}}g \succ(\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$  and  $f'_{\{\omega''\}}g' \succ(\omega') h'_{\{\omega''\}}g'$  for some  $f', g', h' \in \mathcal{A}$ , then  $f''_{\{\omega''\}}g'' \succ(\omega) h''_{\{\omega''\}}g''$  for some  $f'', g'', h'' \in \mathcal{A}$ .

**Property 5** If  $f_{\{\omega'\}}g \succ(\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$  and  $f'_{\{\omega''\}}g' \succ(\omega') h'_{\{\omega''\}}g'$  for some  $f', g', h' \in \mathcal{A}$ , then  $f''_{\{\omega''\}}g'' \succ(\omega') h''_{\{\omega''\}}g''$  for some  $f'', g'', h'' \in \mathcal{A}$ .

**Lemma 3** For all  $\omega \in \Omega$ ,  $\succsim(\omega)$  satisfies Properties 4 and 5 if and only if  $\tilde{\Pi}$  satisfies Stationarity.

Note that  $\tilde{\Pi}$  satisfies stationarity if and only if it satisfies Transitivity (i.e., if  $\omega' \in \tilde{\Pi}(\omega)$  then  $\tilde{\Pi}(\omega') \subseteq \tilde{\Pi}(\omega)$ ) and Euclideaness (i.e., if  $\omega' \in \tilde{\Pi}(\omega)$  then  $\tilde{\Pi}(\omega') \supseteq \tilde{\Pi}(\omega)$ ). The proof contained in the appendix verifies that Property 4 is a “translation” of Transitivity and that Property 5 is a “translation” of Euclideaness.

The next two properties relate preferences across spaces. E.g., they relate a decision maker’s preference at  $\omega$  to his preference at the projection to a lower space  $\omega_S$ . The proofs of the following two lemmata are contained in the appendix.

**Property 6** If  $\omega \in S'$ ,  $S \preceq S'$ , and  $f_{\{\omega'\}}g \succ(\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$ , then there exists  $\omega'' \in \Omega$  with  $\omega' \in \{\omega''\}^\uparrow$  such that  $f'_{\{\omega''\}}g' \succ(\omega_S) h'_{\{\omega''\}}g'$  for some  $f', g', h' \in \mathcal{A}$ .

**Lemma 4** For all  $\omega \in \Omega$ ,  $\succsim(\omega)$  satisfies Property 6 if and only if  $\tilde{\Pi}$  satisfies Projections Preserve Ignorance.

**Property 7** For  $S \preceq S_{\omega'} \preceq S_\omega$ ,  $f_{\{\omega'\}}g \succ(\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$  if and only if  $f'_{\{\omega'_S\}}g' \succ(\omega_S) h'_{\{\omega'_S\}}g'$  for some  $f', g', h' \in \mathcal{A}$ .

**Lemma 5** For all  $\omega \in \Omega$ ,  $\succsim(\omega)$  satisfies Property 7 if and only if  $\tilde{\Pi}$  satisfies Projections Preserve Knowledge.

We summarize our observations in the following proposition:

**Proposition 4** The preference-based correspondence  $\tilde{\Pi}$  is a possibility correspondence satisfying 0. to 5. if and only if for any  $\omega \in \Omega$  the preference order  $\succsim(\omega)$  satisfies Properties 1 to 7.

### 3.4 Preference-Based Knowledge and Awareness

Assume now that for any  $\omega \in \Omega$ ,  $\succsim(\omega)$  is a preference relation satisfying Properties 1 to 7. We now define knowledge, awareness, and unawareness directly in terms of preferences.

**Definition 3** *The preference-based knowledge operator on  $\Sigma$  is defined by*

$$\tilde{K}(E) := \left\{ \omega \in \Omega : \begin{array}{l} (i) f_{-EG} \sim (\omega) h_{-EG} \text{ for all } f, g, h \in \mathcal{A}, \text{ and} \\ (ii) f_{EG} \succ (\omega) h_{EG} \text{ for some } f, g, h \in \mathcal{A} \end{array} \right\}$$

if there is a state  $\omega \in \Omega$  such that (i) and (ii) hold, and by  $\tilde{K}(E) := \emptyset^{S(E)}$  otherwise.

Property (i) is familiar from the definition of Savage null event (Savage, 1954). An event  $E$  is known if its negation is null. Morris (1996, 1997) defines “standard” knowledge by this property alone.<sup>6</sup> Yet, in unawareness structures it is possible that both the event and its negation are null, in which case - as we will see below - the decision maker is unaware of the event. That’s why we need to add the second requirement, property (ii), to the definition.

Given the possibility correspondence  $\tilde{\Pi}$  and Definition 1, the knowledge operator on  $\Sigma$  is

$$K(E) = \{\omega \in \Omega : \tilde{\Pi}(\omega) \subseteq E\}$$

if there is a state  $\omega \in \Omega$  such that  $\tilde{\Pi}(\omega) \subseteq E$ , and  $K(E) = \emptyset^{S(E)}$  otherwise.

**Proposition 5** *For any event  $E \in \Sigma$ ,  $\tilde{K}(E) = K(E)$ .*

The proof is contained in the appendix.<sup>7</sup>

**Definition 4** *The preference-based awareness operator on  $\Sigma$  is defined by*

$$\tilde{A}(E) := \left\{ \omega \in \Omega : \begin{array}{l} (ii) f_{EG} \succ (\omega) h_{EG} \text{ for some } f, g, h \in \mathcal{A}, \text{ or} \\ (iii) f_{-EG} \succ (\omega) h_{-EG} \text{ for some } f, g, h \in \mathcal{A} \end{array} \right\}$$

if there is a state  $\omega \in \Omega$  such that (ii) or (iii) hold, and by  $\tilde{A}(E) := \emptyset^{S(E)}$  otherwise.

Given the possibility correspondence  $\tilde{\Pi}$  and Definition 2, the decision maker’s *awareness operator* in  $\Sigma$  is

$$A(E) = \{\omega \in \Omega : \tilde{\Pi}(\omega) \subseteq S(E)^\uparrow\}$$

if there is a state  $\omega$  such that  $\tilde{\Pi}(\omega) \subseteq S(E)^\uparrow$ , and  $A(E) = \emptyset^{S(E)}$  otherwise.

---

<sup>6</sup>To be precise, Morris (1996, 1997) defines “Savage belief” by property (i) alone. Since in our context we impose Property 3 which implies the Truth, we think it is justified to call it knowledge.

<sup>7</sup>The proof makes use of the assumption that  $\Omega$  is finite. We were not be able to prove it for more general  $\Omega$  without imposing further assumptions. We suspect that this is one of the reasons why Morris (1996, 1997) assumed  $\Omega$  to be finite.

**Proposition 6** For any event  $E \in \Sigma$ ,  $\tilde{A}(E) = A(E)$ .

The proof follows from Proposition 5 and Weak Necessitation.

**Definition 5** The preference-based unawareness operator on  $\Sigma$  is defined by

$$\tilde{U}(E) := \left\{ \omega \in \Omega : \begin{array}{l} (0) f_{EG} \sim (\omega) h_{EG} \text{ for all } f, g, h \in \mathcal{A}, \text{ and} \\ (i) f_{-EG} \sim (\omega) h_{-EG} \text{ for all } f, g, h \in \mathcal{A} \end{array} \right\}$$

if there is a state  $\omega \in \Omega$  such that (0) and (i) hold, and by  $\tilde{U}(E) := \emptyset^{S(E)}$  otherwise.

This behavioral definition of unawareness means that a decision maker is unaware of an event if this event is null and the negation of the event is null.<sup>8</sup> Indeed, this behavioral definition of unawareness is equivalent to the epistemic notion of unawareness.

**Corollary 1** For any event  $E \in \Sigma$ ,  $\tilde{U}(E) = U(E)$ .

Note that we are able to characterize all epistemic operators by conjunction and disjunction of choices experiments corresponding to properties (0) to (iii) in Definitions 3 to 5.

## 4 Discussion

In what sense do our results really provide a *behavioral* characterization of properties of the possibility correspondence, knowledge, awareness, and unawareness? First notice that in order to “reveal” whether a decision maker’s possibility correspondence satisfies properties 0. to 5., many counterfactual choice experiments have to be performed. For instance, we first may want to conduct choice experiments at state  $\omega$  at which the decision maker may be unaware of both events  $E$  and  $F$ . Then we may want to consider choice experiments at state  $\omega'$ , at which the decision maker may be aware of  $E$  but unaware of  $F$ , and then at yet another state  $\omega''$  at which he is unaware of  $E$  but aware of  $F$  etc. But how should this be practically done? If these choice experiments are conducted sequentially, then it requires that after becoming aware of the  $E$  the decision must become unaware of  $E$  before we can conduct the last set of choice experiments.<sup>9</sup> More natural is the interpretation of the model as the *belief of another player (or the modeler)* about the decision maker’s choices at various states.

---

<sup>8</sup>This condition characterizes behaviorally unawareness in Schipper (2010) using a states-of-nature lattice structure and the Anscombe-Aumann approach.

<sup>9</sup>Another difficulty that this model shares with Morris (1996, 1997) is how to phrase practically bets on events corresponding to (even higher order) beliefs of some events.

While this interpretation may appear unusual at a first glance, we would argue that it corresponds rather closely to the actual treatment of decision theoretic models by decision theorists. Despite the emphasis on preferences revealed by choices, most decision theorists chose not to conduct choice experiments and many choice theoretic “axioms” are just of technical nature (e.g. continuity axioms) or impractical to implement in actual choice experiments (e.g. axioms on conditional preferences). Such “axioms” are testable just in “principle”. That is, they represent the belief of the modeler about how a decision maker would perform in certain choice experiments when he is ascribed a certain utility representation.<sup>10</sup>

Our analysis is subject to another caveat raised previously in Schipper (2010). Property 1 implies that the events of which the decision maker is unaware of do not affect her ranking of acts. This holds even for composite acts conditioned on events that the decision maker is unaware of. More generally, it rules out that a decision maker becomes aware of an event merely from facing an act. This is also the implicit assumption in standard decision theory (i.e., different acts do not change the subset of “small worlds” that a decision maker pays attention to). Yet, there it is less problematic since there is just one set of “small worlds”. But it may be unrealistic in some situations with unawareness. Sometimes, when facing an act, a decision maker may become in very subtle ways a bit more careful with the “fine prints” of acts, and this care may lead her to become aware of events. E.g., a buyer facing a decision about whether or not to enter into a certain purchase contract may become aware of events that she was previously unaware of when reading all the fine prints of the contract. If ex ante an outside observer does not know how acts affect the awareness of a decision maker, then it may be impossible to design choice experiments required to elicit properties of the static possibility correspondence.

Our analysis differs from the one presented in Schipper (2010) in several aspects. First, we mentioned already that Schipper (2010) considers states of nature while we consider state of the world. Latter approach allows us to talk about knowledge of knowledge etc. Second, while Schipper (2010) considers Anscombe-Aumann acts, i.e. mappings from states to mixtures over outcomes, we consider here real-valued Savage acts, i.e. mappings from states to real numbers. Third, Schipper (2010) aims at subjective expected utility representation theorems with probabilistic beliefs while we focus on characterizing the qualitative notion of knowledge, awareness and unawareness. Our analysis is related to Li (2008) who seeks to study the difference between unawareness and zero probability as well.

Morris (1996) also related dynamic preferences to properties of knowledge change. While various frameworks of dynamic unawareness have been presented in van Dittmarsch and French (2009, 2010), Feinberg (2009), Grant and Quiggin (2010), Halpern and Rego

---

<sup>10</sup>Our interpretation of the decision theoretic model is also reminiscent of the interpretation of the notion of strategy in extensive-form games. As Rubinstein (1991) pointed out, an action prescribed by a strategy at an information set which is excluded by an earlier move of that very strategy is implicitly interpreted in game theory as the beliefs the other players entertain regarding the player’s move if that information set were reached.



(2006), Rego and Halpern (2007), Heifetz, Meier and Schipper (2011a, b, c), Hill (2010), and Li (2006), we leave a decision theoretic analysis of dynamic unawareness to future research.

## A Proofs

### A.1 Proof of Lemma 1

“ $\Rightarrow$ ”: Suppose there exists  $S \in \mathcal{S}$  and  $\omega \in S$  such that  $\tilde{\Pi}(\omega) \subseteq S'$  with  $S' \not\subseteq S$ . Then for  $\omega' \in \tilde{\Pi}(\omega)$  we have  $f_{\{\omega'\}}g \succ (\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$ . But Property 2 implies  $S_{\omega'} \preceq S_\omega$ . Note that  $S' = S_{\omega'}$ , a contradiction.

Suppose now that there exists  $S \in \mathcal{S}$  and  $\omega \in S$  such that  $\tilde{\Pi}(\omega) \not\subseteq S'$ , for all  $S' \in \mathcal{S}$ . Then there exist  $\omega', \omega'' \in \tilde{\Pi}(\omega)$  such that  $S_{\omega'} \neq S_{\omega''}$ . If  $\omega' \in \tilde{\Pi}(\omega)$  then by Property 1 we have  $f_{\{\omega'\}}g \succ (\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$  and for any  $\omega'' \neq \omega'$  with  $S_{\omega''} \neq S_{\omega'}$ ,  $f'_{\{\omega''\}}g' \sim (\omega) h'_{\{\omega''\}}g'$  for all  $f', g', h' \in \mathcal{A}$ . Hence  $\omega'' \notin \tilde{\Pi}(\omega)$ , a contradiction.

“ $\Leftarrow$ ”: Suppose there exist  $\omega', \omega'' \in \Omega$  such that Property 1 is violated, i.e.,  $f_{\{\omega'\}}g \succ (\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$ ,  $S_{\omega''} \neq S_{\omega'}$ , and  $f'_{\{\omega''\}}g' \approx (\omega) h'_{\{\omega''\}}g'$  for some  $f', g', h' \in \mathcal{A}$ . Then  $\omega', \omega'' \in \tilde{\Pi}(\omega)$ , a contradiction to  $\tilde{\Pi}(\omega) \subseteq S'$  for some  $S' \in \mathcal{S}$ .

Suppose there exist  $\omega, \omega', \omega'' \in \Omega$  such that Property 2 is violated, i.e.,  $f_{\{\omega'\}}g \succ (\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$  and  $S_{\omega'} \not\subseteq S_\omega$ . Then by previous arguments  $\tilde{\Pi}(\omega) \subseteq S'$  with  $S' = S_{\omega'} \not\subseteq S_\omega = S$ , a contradiction of Confinement.  $\square$

### A.2 Proof of Lemma 2

“ $\Leftarrow$ ”: Suppose by contradiction that  $f_{\{\omega_S\}}g \sim (\omega) h_{\{\omega_S\}}g$  for all  $S \preceq S_\omega$  and all  $f, g, h \in \mathcal{A}$ . Thus,  $r_S^{\omega_S}(\omega) \notin \tilde{\Pi}(\omega)$  for all  $S \preceq S_\omega$ , a contradiction.

“ $\Rightarrow$ ”: Suppose  $\omega \notin \tilde{\Pi}^\uparrow(\omega)$  and  $S_\omega \succeq S$ . If  $\tilde{\Pi}(\omega) \subseteq S$ , then  $\omega_S \notin \tilde{\Pi}(\omega)$ . But  $f_{\{\omega_S\}}g \succ (\omega) h_{\{\omega_S\}}g$  for some  $f, g, h \in \mathcal{A}$  and  $S' \preceq S_\omega$ , a contradiction.  $\square$

### A.3 Proof of Lemma 3

We first prove that Transitivity is equivalent to Property 4:  $f_{\{\omega''\}}g \succ (\omega) h_{\{\omega''\}}g$  for some  $f, g, h \in \mathcal{A}$  if and only if  $\omega'' \in \tilde{\Pi}(\omega)$ .  $f'_{\{\omega'\}}g' \succ (\omega) h'_{\{\omega'\}}g'$  for some  $f', g', h' \in \mathcal{A}$  if and only if  $\omega' \in \tilde{\Pi}(\omega)$ .  $f''_{\{\omega''\}}g'' \succ (\omega') h''_{\{\omega''\}}g''$  for some  $f'', g'', h'' \in \mathcal{A}$  if and only if  $\omega'' \in \tilde{\Pi}(\omega')$ . Thus, Property 4 is equivalent to if  $\omega' \in \tilde{\Pi}(\omega)$  and  $\omega'' \in \tilde{\Pi}(\omega')$ , then  $\omega'' \in \tilde{\Pi}(\omega)$ . This is equivalent to Transitivity.

Second, we prove that Euclideaness is equivalent to Property 5:  $f_{\{\omega''\}}g \succ (\omega') h_{\{\omega''\}}g$  for some  $f, g, h \in \mathcal{A}$  if and only if  $\omega'' \in \tilde{\Pi}(\omega')$ .  $f'_{\{\omega'\}}g' \succ (\omega) h'_{\{\omega'\}}g'$  for some  $f', g', h' \in \mathcal{A}$

if and only if  $\omega' \in \tilde{\Pi}(\omega)$ .  $f''_{\{\omega''\}}g'' \succ (\omega) h''_{\{\omega''\}}g''$  for some  $f'', g'', h'' \in \mathcal{A}$  if and only if  $\omega'' \in \tilde{\Pi}(\omega)$ . Thus, Property 5 is equivalent to if  $\omega' \in \tilde{\Pi}(\omega)$  and  $\omega'' \in \tilde{\Pi}(\omega)$ , then  $\omega'' \in \tilde{\Pi}(\omega')$ . This is equivalent to Euclideanness.  $\square$

#### A.4 Proof of Lemma 4

“ $\Rightarrow$ ”: If  $f_{\{\omega'\}}g \succ (\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$ , then  $\omega' \in \tilde{\Pi}(\omega)$ . If  $f'_{\{\omega''\}}g' \succ (\omega_S) h'_{\{\omega''\}}g'$  for some  $f', g', h' \in \mathcal{A}$ , then  $\omega'' \in \tilde{\Pi}(\omega_S)$ . Moreover, according to Property 6,  $\omega' \in \{\omega''\}^\uparrow$ . Thus  $\omega' \in \tilde{\Pi}^\uparrow(\omega_S)$ . Hence,  $\tilde{\Pi}^\uparrow(\omega) \subseteq \tilde{\Pi}^\uparrow(\omega_S)$ .

“ $\Leftarrow$ ”:  $\omega' \in \tilde{\Pi}(\omega)$  if and only if  $f_{\{\omega'\}}g \succ (\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$ . Since  $\omega' \in \tilde{\Pi}^\uparrow(\omega_S)$  by Projections Preserve Ignorance, there exists  $\omega'' \in \tilde{\Pi}(\omega_S)$  such that  $\omega' \in \{\omega''\}^\uparrow$ . Moreover,  $f'_{\{\omega''\}}g' \succ (\omega_S) h'_{\{\omega''\}}g'$  for some  $f', g', h' \in \mathcal{A}$   $\square$

#### A.5 Proof of Lemma 5

Note that by definition of  $\tilde{\Pi}$ , Property 7 is equivalent to: For  $S \preceq S_{\omega'} \preceq S_\omega$ ,  $\omega' \in \tilde{\Pi}(\omega)$  if and only if  $\omega'_S \in \tilde{\Pi}(\omega_S)$ . We show that this is equivalent to Projections Preserve Knowledge. Let for some  $S \preceq S_{\omega'} \preceq S_\omega$ ,  $\omega' \in \tilde{\Pi}(\omega)$ . Then  $\omega'_S \in \left(\tilde{\Pi}(\omega)\right)_S$ . By above equivalent statement of Property 7,  $\omega'_S \in \tilde{\Pi}(\omega_S)$ .

Conversely, let for some  $S \preceq S_{\omega'} \preceq S_\omega$ ,  $\omega'_S \in \tilde{\Pi}(\omega_S)$ . Then by the equivalent statement of Property 7,  $\omega' \in \tilde{\Pi}(\omega)$ . Then  $\omega'_S \in \left(\tilde{\Pi}(\omega)\right)_S$ .  $\square$

#### A.6 Proof of Proposition 5

“ $\supseteq$ ”:  $\omega \in K(E)$  if and only if  $\tilde{\Pi}(\omega) \subseteq E$ . It follows that  $f_{\{\omega'\}}g \succ (\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$  implies  $\omega' \in E$ . Define

$$\begin{aligned} f'(\omega) &= \begin{cases} f(\omega) & \text{if } \omega = \omega' \\ g(\omega) & \text{if } \omega \neq \omega' \end{cases} \\ h'(\omega) &= \begin{cases} h(\omega) & \text{if } \omega = \omega' \\ g(\omega) & \text{if } \omega \neq \omega' \end{cases} \end{aligned}$$

and  $g' = g$ . Note that  $f', h', g' \in \mathcal{A}$ . Then from  $f_{\{\omega'\}}g \succ (\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$  implies  $\omega' \in E$  follows that  $f'_E g' \succ (\omega) h'_E g'$  for some  $f', g', h' \in \mathcal{A}$ , yielding property (ii).

Note that property (i) in the definition of  $\tilde{K}$  holds trivially if  $E = S(E)^\uparrow$ . Assume  $E \subsetneq S(E)^\uparrow$ . From  $\tilde{\Pi}(\omega) \subseteq E$  follows that  $\omega' \in \neg E$  implies  $\omega' \notin \tilde{\Pi}(\omega)$ . Thus  $f_{\{\omega'\}}g \sim (\omega) h_{\{\omega'\}}g$  for all  $f, g, h \in \mathcal{A}$ . Since  $\Omega$  is finite, we can enumerate all states in  $\neg E$  from lets say 1 to  $n$ . Then for any  $f, g, h$  we have  $f_{\{\omega_1\}}g \sim (\omega) f_{\{\omega_1, \omega_2\}}g \sim (\omega) \dots \sim (\omega) f_{\{\omega_1, \omega_2, \dots, \omega_n\}}g = f_{\neg E}g$  as well as  $h_{\{\omega_1\}}g \sim (\omega) h_{\{\omega_1, \omega_2\}}g \sim (\omega) \dots \sim (\omega) h_{\{\omega_1, \omega_2, \dots, \omega_n\}}g = h_{\neg E}g$  and thus by

transitivity of  $\succeq(\omega)$  also  $f_{\neg E}g \sim(\omega) h_{\neg E}g$ , yielding property (i) of the definition of  $\tilde{K}$ . Hence  $\omega \in \tilde{K}(E)$ .

“ $\subseteq$ ”:  $\omega \in \tilde{K}(E)$  if and only if (i)  $f_{\neg E}g \sim(\omega) h_{\neg E}g$  for all  $f, g, h \in \mathcal{A}$ , and (ii)  $f_Eg \succ(\omega) h_Eg$  for some  $f, g, h \in \mathcal{A}$ . Consider first (i). For any  $\omega' \in \neg E$ , arguments made above imply  $f_{\{\omega'\}}g \sim(\omega) h_{\{\omega'\}}g$  for all  $f, g, h \in \mathcal{A}$  if we let  $\omega' = \omega_1$ . It follows that  $\tilde{\Pi}(\omega) \cap \neg E = \emptyset$ .

Consider now property (ii) of the definition of  $\tilde{K}$ . We claim that for  $\omega' \in E$ ,  $f_{\{\omega'\}}g \succ(\omega) h_{\{\omega'\}}g$  for some  $f, g, h \in \mathcal{A}$ . Suppose to the contrary that  $f_{\{\omega'\}}g \sim(\omega) h_{\{\omega'\}}g$  for all  $f, g, h \in \mathcal{A}$  and all  $\omega' \in E$ . Since  $\Omega$  is finite, we can enumerate states in  $E$  from lets say 1 to  $m$ . Previous arguments imply  $f_Eg \sim(\omega) h_Eg$  for all  $f, g, h \in \mathcal{A}$ , a contradiction to property (ii). Thus  $\tilde{\Pi}(\omega) \subseteq E$  which is equivalent to  $\omega \in K(E)$ .  $\square$

## References

- [1] Agotnes, T. and N. Alechina (2007). Full and relative awareness: A decidable logic for reasoning about knowledge of unawareness, in: *Theoretical Aspects of Rationality and Knowledge, Proceedings of the 11th Conference (TARK 2007)*, Dov Samet (Ed.), Presses Universitaires De Louvain, 6-14.
- [2] Board, O. and K.S. Chung (2009). Object-based unawareness: Theory and applications, mimeo., University of Minnesota.
- [3] Board, O., Chung, K.S., and B.C. Schipper (2011). Two Models of Unawareness: Comparing the Object-Based and Subjective-State-Space Approaches, *Synthese (Knowledge, Rationality and Action)*, forthcoming.
- [4] Dekel, E., Lipman, B. and A. Rustichini (1998). Standard state-space models preclude unawareness, *Econometrica* **66**, 159-173.
- [5] van Ditmarsch, H. and T. French (2009). Awareness and forgetting of facts and agents, in: P. Boldi, G. Vizzari, G. Pasi and R. Baeza-Yates, *Proceedings of WLIAT Workshops (WLIAMAS) 2009*, IEEE Press, 478-483.
- [6] van Ditmarsch, H. and T. French (2010). Becoming Aware, *Proceedings of 3rd Workshop on Logical Aspects of Multi-Agent Systems (LAMAS)*, Toronto.
- [7] Fagin, R. and J.Y. Halpern (1988). Belief, awareness and limited reasoning, *Artificial Intelligence* **34**, 39-76.
- [8] Feinberg, Y. (2009). Games with unawareness, mimeo., Stanford University.
- [9] Galanis, S. (2010). Syntactic foundations of unawareness of theorems, *Theory and Decision*, forthcoming.

- [10] Galanis, S. (2007). Unawareness of theorems, mimeo., The University of Southampton.
- [11] Grant, S. and J. Quiggin (2010). Inductive reasoning about unawareness, mimeo., Rice University.
- [12] Halpern, J.Y. (2001). Alternative semantics for unawareness, *Games and Economic Behavior* **37**, 321-339.
- [13] Halpern, J. and L.C. Rego (2006). Extensive games with possibly unaware players, in: *Proc. Fifth International Joint Conference on Autonomous Agents and Multiagent Systems*, 744-751.
- [14] Halpern, J. and L. Rêgo (2008). Interactive unawareness revisited, *Games and Economic Behavior* **62**, 232-262.
- [15] Halpern, J. and L. Rêgo (2009a). Reasoning about knowledge of unawareness, *Games and Economic Behavior*, forthcoming.
- [16] Halpern, J. and L. Rêgo (2009a). Reasoning about knowledge of unawareness revisited, *Proceedings of Twelfth Conference on Theoretical Aspects of Rationality and Knowledge*, 166-173.
- [17] Heifetz, A., Meier, M. and B. C. Schipper (2006). Interactive unawareness, *Journal of Economic Theory* **130**, 78-94.
- [18] Heifetz, A., Meier, M. and B. C. Schipper (2008). A canonical model for interactive unawareness, *Games and Economic Behavior* **62**, 304-324.
- [19] Heifetz, A., Meier, M. and B. C. Schipper (2010). Unawareness, beliefs and speculative trade, mimeo., The University of California, Davis.
- [20] Heifetz, A., Meier, M. and B.C. Schipper (2011a). Dynamic unawareness and rationalizable behavior, mimeo., The University of California, Davis.
- [21] Heifetz, A., Meier, M. and B.C. Schipper (2011b). Prudent rationalizability in generalized extensive-form games, mimeo., The University of California, Davis.
- [22] Heifetz, A., Meier, M. and B.C. Schipper (2011c). Conditional dominance in dynamic games with unawareness, mimeo., The University of California, Davis.
- [23] Hill, B. (2010). Awareness dynamics, *Journal of Philosophical Logic* **39**, 113-137.
- [24] Li J. (2006). Dynamic games of complete information with unawareness, mimeo.
- [25] Li J. (2008). A note on unawareness and zero probability, mimeo.
- [26] Li, J. (2009). Information structures with unawareness, *Journal of Economic Theory* **144**, 977-993.

- [27] Modica, S. and A. Rustichini (1999). Unawareness and partitional information structures, *Games and Economic Behavior* **27**, 265-298.
- [28] Morris, S. (1997). Alternative definitions of knowledge, in: Bacharach, M.O.L. et al. (eds.), *Epistemic logic and the theory of games and decisions*, Kluwer Academic Publishers, 217-233.
- [29] Morris, S. (1996). The logic of belief and belief change: A decision theoretic approach, *Journal of Economic Theory* **69**, 1-23.
- [30] Meier, M. and B. C. Schipper (2010). Speculative trade under unawareness – The infinite case, mimeo., The University of California, Davis.
- [31] Rego, L. C. and J. Halpern (2007). Generalized solution concepts in games with possibly unaware players, in: *Theoretical Aspects of Rationality and Knowledge, Proceedings of the 11th Conference (TARK 2007)*, Dov Samet (Ed.), Presses Universitaires De Louvain, 253-262.
- [32] Rubinstein, A. (1991), Comments on the interpretation of game theory, *Econometrica* **59**, 909-924.
- [33] Savage, L.J. (1954). *The foundations of statistics*, New York: Dover.
- [34] Schipper, B.C. (2010). Awareness-dependent subjective expected utility theory, mimeo., University of California, Davis.
- [35] Sillari, G. (2008). Quantified logic of awareness and impossible possible worlds, *Review of Symbolic Logic* **1**, 514-529.