

Kempf, Alexander; Memmel, Christoph

**Working Paper**

## On the estimation of the global minimum variance portfolio

CFR Working Paper, No. 05-02

**Provided in Cooperation with:**

Centre for Financial Research (CFR), University of Cologne

*Suggested Citation:* Kempf, Alexander; Memmel, Christoph (2005) : On the estimation of the global minimum variance portfolio, CFR Working Paper, No. 05-02, University of Cologne, Centre for Financial Research (CFR), Cologne

This Version is available at:

<https://hdl.handle.net/10419/57730>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

**CFR-working paper no. 05-02**

**on the estimation of the global  
minimum variance portfolio**

**alexander kempf • christoph memmel**

**centre for financial research**  
Look deeper

# On the Estimation of the Global Minimum Variance Portfolio

Alexander Kempf and Christoph Memmel\*

## Abstract

Expected returns can hardly be estimated from time series data. Therefore, many recent papers suggest investing in the global minimum variance portfolio. The weights of this portfolio depend only on the return variances and covariances, but not on the expected returns. The weights of the global minimum variance portfolio are usually estimated by replacing the true return covariance matrix by its time series estimator. However, little is known about the distributions of the estimated weights and return parameters of this portfolio. Our contribution is to determine these distributions. The knowledge of these distributions allows us to calculate the extent of the estimation risk an investor faces and to answer important questions in asset management.

Keywords: Global Minimum Variance Portfolio, Weight Estimation, Estimation  
Risk

JEL classification: C22, G11

---

\*University of Cologne, Department of Finance, Albertus-Magnus-Platz, 50923 Cologne, Germany, Tel.: +49-221-4702741, Fax: +49-221-4703992, Email: kempf@wiso.uni-koeln.de.

# On the Estimation of the Global Minimum Variance Portfolio

Expected stock returns are hard to estimate [see, e.g., Merton (1980)]. Typically, the estimated values differ largely from the true ones. These estimation errors lead to a suboptimal portfolio composition and thus to a poor portfolio performance [see, e.g., Jorion (1991) and Chopra and Ziemba (1993)]. Therefore, several recent papers suggest avoiding the estimation of expected returns [see, e.g., Ledoit and Wolf (2003) and Jagannathan and Ma (2003)]. Instead, they assume that all stocks have equal expected returns. Under this assumption, all stock portfolios differ only with respect to their risk, but not with respect to expected returns. Therefore, the only efficient stock portfolios is the one with the smallest risk, i.e. the global minimum variance portfolio. All investors which optimize the tradeoff between expected return and risk of their portfolio should then combine the global minimum variance portfolio with the risk free asset.

The composition of the global minimum variance portfolio depends only on the covariance matrix of stock returns. Since the covariance matrix can be estimated much more precisely than the expected returns (see Section 1), the estimation risk of the investor is expected to be reduced by focusing on the global minimum variance portfolio.<sup>1</sup> However, little is known about the distribution of the estimated portfolio weights and the extent of the estimation risk. Dickinson (1974) calculates the unconditional distribution of the portfolio weights in the special case of two uncorrelated assets. Ohkrin and Schmid (2004) generalize this result by allowing  $N$  assets with arbitrary correlations. However, the conditional distribution is yet unknown, but it is necessary for calculating test statistics and confidence intervals in small samples. The main contribution of our paper is to derive the conditional distributions of the estimated weights of the global minimum variance portfolio, its estimated expected return and its estimated return variance. Knowing the conditional distributions allows us to answer important questions in asset management, for example: (i) What determines the extent of estimation risk? (ii) Can an investor reduce the portfolio risk significantly by including additional assets in his portfolio? The paper is organized as follows. In Section 1 we show that the covariance matrix can be estimated much more precisely than expected returns. This provides

the motivation for focusing on the global minimum variance portfolio. In Section 2 we briefly review the traditional approach of estimating the weights of the global minimum variance portfolio. In Section 3 we present an alternative OLS estimation approach, which leads to identical weight estimates. Using this alternative estimation approach we derive in Section 4 the conditional distributions of the estimated portfolio weights and the estimated return parameters of the global minimum variance portfolio. In Section 5 we show that our OLS approach can also be applied when we give up the assumption of normally distributed asset returns. In Section 6 we apply our results to calculate the estimation risk associated with the estimation of the global minimum variance portfolio. In Section 7 we give some examples of how to apply our results in international asset management. Section 8 concludes.

## 1 Precision of Parameter Estimates

Assume that there are  $N$  stocks in the capital market. We denote the return of stock  $i$  from time  $t - 1$  to  $t$  by  $r_{t,i}$ . The vector  $\mu$  contains the expected returns of the  $N$  stocks. The  $N \times N$  matrix  $\Sigma$  contains the return variances and covariances  $\sigma_{ij}$ . We assume that the returns are multivariate normally distributed. In addition, the returns are identically and independently distributed. Thus, we assume the best possible situation for an investor who wants to estimate the returns distribution parameters. The investor can increase the precision of the estimate by using longer time series. If the length of the time series goes to infinity, both, the expected returns and the covariance matrix, can be estimated exactly. There is no estimation risk. However, real time series are not that long and the distribution parameters cannot be estimated exactly. Estimation risk occurs - even in the best possible situation an investor can face.

Assume that there are  $\tau \geq 1$  years of data available to estimate the expected returns  $\mu$  and the covariance matrix  $\Sigma$ . There are  $n \geq 1$  subperiods of equal length per year. Thus, the number of observations is  $T = \tau n$ .<sup>2</sup> The precision of the estimates is given by the variance of the estimators. The variances of the estimators for the expected

return  $\mu_i$  and the standard deviation  $\sigma_i$  are:<sup>3</sup>

$$\text{var}(\hat{\mu}_i) = \frac{\sigma_i^2}{\tau}. \quad (1)$$

$$\text{var}(\hat{\sigma}_i) \approx \frac{1}{2} \frac{\sigma_i^2}{\tau n} \quad (2)$$

The precision of both estimates is the larger, the more years of data ( $\tau$ ) are available. For  $\tau \rightarrow \infty$  both variances go to zero, the estimation risk disappears. For a finite number of years ( $\tau < \infty$ ), the covariance matrix can be estimated more precisely than the estimated returns. The precision ratio is given as:

$$\frac{\text{var}(\hat{\mu}_i)}{\text{var}(\hat{\sigma}_i)} \approx 2 n \quad (3)$$

For stock markets, one typically uses daily ( $n = 250$ ), weekly ( $n = 52$ ) or monthly ( $n = 12$ ) observations. Therefore, typical precision ratios are within a range of 24 to 500. Thus, the covariance matrix can be estimated with a much higher precision than the vector of expected returns.

To illustrate the size of the estimation risk we provide a numerical example. Assume that the stock has a volatility  $\sigma_i$  of 25%. There are weekly return observations available. Table 1 shows the level of the stock risk  $\sigma_i$ , the estimation risk of the expected returns  $\sqrt{\text{var}(\hat{\mu}_i)}$  and the estimation risk of the volatility  $\sqrt{\text{var}(\hat{\sigma}_i)}$ .

Table 1: *Precision of Parameter Estimates*

	$T = 1$	$T = 5$	$T = 10$	$T = 20$	$T = 50$	$T = 100$
$\sigma_i$	0.250	0.250	0.250	0.250	0.250	0.250
$\sqrt{\text{var}(\hat{\mu}_i)}$	0.250	0.112	0.079	0.056	0.035	0.025
$\sqrt{\text{var}(\hat{\sigma}_i)}$	0.025	0.011	0.008	0.005	0.003	0.002

Table 1 shows that the estimation risk with respect to the volatility is comparably small. The opposite is true for the estimation risk with respect to expected returns. For short time periods, the estimation risk with respect to expected returns is huge, and even for long time series, it does not become negligible. Therefore, an investor might be well advised to abstain from estimating expected returns and to concentrate on the global minimum variance portfolio - even when stock returns have all the desirable features like normality and IID.

## 2 Traditional Approach

The global minimum variance portfolio ( $MV$ ) is the stock portfolio with the lowest return variance for a given covariance matrix  $\Sigma$ . It is the solution to the following minimization problem:

$$\min_{w=(w_1, \dots, w_N)'} w' \Sigma w \quad s.t. \quad w' \underline{1} = 1 \quad (4)$$

$\underline{1}$  is a column vector of appropriate dimension whose entries are ones and  $w = (w_1, \dots, w_N)'$  is a vector of portfolio weights. The weights  $w_{MV} = (w_{MV,1}, \dots, w_{MV,N})'$  of the global minimum variance portfolio are given as

$$w_{MV} = \frac{\Sigma^{-1} \underline{1}}{\underline{1}' \Sigma^{-1} \underline{1}}. \quad (5)$$

The expected return  $\mu_{MV}$  and the return variance  $\sigma_{MV}^2$  of the global minimum variance portfolio are given as

$$\mu_{MV} = \mu' w_{MV} = \frac{\mu' \Sigma^{-1} \underline{1}}{\underline{1}' \Sigma^{-1} \underline{1}} \quad (6)$$

and

$$\sigma_{MV}^2 = w_{MV}' \Sigma w_{MV} = \frac{1}{\underline{1}' \Sigma^{-1} \underline{1}}. \quad (7)$$

The lower variance bound (7) can only be attained if the covariance matrix  $\Sigma$  of the stock returns is known. As pointed out before, the covariance matrix  $\Sigma$  is not known but has to be estimated in real markets. Typically, historical return observations are used for this estimation.

The traditional estimation approach is to replace the expected returns  $\mu$  and the covariance matrix  $\Sigma$  by their maximum likelihood estimators  $\hat{\mu}$  and  $\hat{\Sigma}$  in the Equations (5) - (7). The estimated portfolio weights  $\hat{w}_{MV}$  and return parameters  $\hat{\mu}_{MV}$  and  $\hat{\sigma}_{MV}^2$  of the global minimum variance portfolio are non-linear functions of the stock return parameter estimates  $\hat{\mu}$  and  $\hat{\Sigma}$ . Therefore, the distributions of  $\hat{w}_{MV}$ ,  $\hat{\mu}_{MV}$  and  $\hat{\sigma}_{MV}^2$  are hard to determine, even if the distributions of the parameter estimates  $\hat{\mu}$  and  $\hat{\Sigma}$  are known. The calculation of these distributions is the main contribution of our paper.

### 3 OLS Approach

We use a regression based approach to determine the weights  $w_{MV}$ , the expected return  $\mu_{MV}$  and the return variance  $\sigma_{MV}^2$  of the global minimum variance portfolio. We rewrite the weights of the global minimum variance portfolio as regression coefficients. Without loss of generality we choose the return of stock  $N$  to be the dependent variable:

$$r_{t,N} = \alpha + \beta_1 (r_{t,N} - r_{t,1}) + \dots + \beta_{N-1} (r_{t,N} - r_{t,N-1}) + \varepsilon_t \quad t = 1, \dots, T > N \quad (8)$$

$\varepsilon_t$  is a noise term that satisfies the standard assumptions of the classical linear regression model regarding errors.<sup>4</sup> The returns are again normal and IID.<sup>5</sup> The



three statements in Proposition 1 describe the relation between the linear regression and the global minimum variance portfolio.

**Proposition 1**

1. *The regression coefficients  $\beta_1, \dots, \beta_{N-1}$  in Equation (8) correspond to the portfolio weights  $w_{MV,1}, \dots, w_{MV,N-1}$  of the global minimum variance portfolio:*

$$\beta_i = w_{MV,i} \quad (9)$$

2. *The coefficient  $\alpha$  in Equation (8) corresponds to the expected return  $\mu_{MV}$  of the global minimum variance portfolio:*

$$\alpha = \mu_{MV} \quad (10)$$

3. *The variance  $\sigma_\varepsilon^2$  of the noise term  $\varepsilon_t$  in Equation (8) corresponds to the variance  $\sigma_{MV}^2$  of the global minimum variance portfolio:*

$$\sigma_\varepsilon^2 = \sigma_{MV}^2 \quad (11)$$

To prove this proposition we define  $\beta^{ex}$ ,  $w_{MV}^{ex}$  and  $r_t^{ex}$  as column vectors of dimension  $N - 1$ .<sup>6</sup> These vectors contain the entries  $\beta_i$ ,  $w_{MV,i}$  and  $r_{t,i}$  with  $i = 1, \dots, N - 1$ . The  $(N - 1) \times (N - 1)$  matrix  $\Omega$  is the covariance matrix of the regressors of Equation (8):

$$\Omega := \text{var} (r_{t,N} \mathbf{1} - r_t^{ex}) \quad (12)$$

The regression coefficients  $\beta^{ex}$  are the standardized covariances of the regressors and the dependent variable:

$$\beta^{ex} = \Omega^{-1} \text{cov} (r_{t,N} \mathbf{1} - r_t^{ex}, r_{t,N}) \quad (13)$$

We have to show that the weights  $w_{MV}^{ex}$  of the global minimum variance portfolio correspond to the regression coefficients  $\beta^{ex}$ . The weight  $w_{MV,N}$  can then be computed as  $1 - (w_{MV}^{ex})' \mathbf{1}$ . To prove  $\beta^{ex} = w_{MV}^{ex}$  we consider an arbitrary portfolio  $P$ . Its return is determined by the weight vector  $w_P^{ex} = (w_{P,1}, \dots, w_{P,N-1})'$  and the stock

returns  $r_t^{ex}$  and  $r_{t,N}$ :

$$r_{t,P} = (w_P^{ex})' r_t^{ex} + (1 - (w_P^{ex})' \mathbf{1}) r_{t,N} = r_{t,N} - (w_P^{ex})' (r_{t,N} \mathbf{1} - r_t^{ex}) \quad (14)$$

The return variance of this arbitrary portfolio  $P$

$$\sigma_P^2 = \sigma_N^2 + (w_P^{ex})' \Omega w_P^{ex} - 2(w_P^{ex})' \text{cov}(r_{t,N} \mathbf{1} - r_t^{ex}, r_{t,N}) \quad (15)$$

is a function of the weights  $w_P^{ex}$ . To find the weights of the global minimum variance portfolio we minimize (15) with respect to the portfolio weights  $w_P^{ex}$ . This minimization leads to

$$w_{MV}^{ex} = \Omega^{-1} \text{cov}(r_{t,N} \mathbf{1} - r_t^{ex}, r_{t,N}). \quad (16)$$

The weights (16) correspond to the regression coefficients (13). This proves the first statement of Proposition 1. To prove our Statements 2 and 3 we rearrange (8) and use  $\beta_i = w_{MV,i}$ :

$$\alpha + \varepsilon_t = w_{MV,1} r_{t,1} + \dots + w_{MV,N-1} r_{t,N-1} + \left(1 - \sum_{i=1}^{N-1} w_{MV,i}\right) r_{t,N} \quad (17)$$

The right hand side of Equation (17) is the return of the global minimum variance portfolio. Applying the expectation and the variance operator to (17) proves our Statements 2 and 3.

Proposition 1 shows that the traditional approach and the OLS approach lead to identical portfolio weights. However, the result was based on the assumption of known parameters. Next we show that the identity result holds even if we have to estimate the parameters. We define the OLS estimates of the coefficients in Equation (8) as  $\hat{\alpha}, \hat{\beta}_1, \dots, \hat{\beta}_{N-1}$ .  $\hat{\sigma}_\varepsilon^2 = \frac{1}{T-N} \sum_{t=1}^T \hat{\varepsilon}_t^2$  is the OLS estimate of the variance of  $\varepsilon_t$ .

## Proposition 2

1. The traditional weight estimate  $\hat{w}_{MV,i}$  equals the OLS estimate:

$$\hat{w}_{MV,i} = \hat{\beta}_i \quad \forall i = 1, \dots, N-1 \quad (18)$$

$$\hat{w}_{MV,N} = 1 - \sum_{i=1}^{N-1} \hat{\beta}_i \quad (19)$$

2. The traditional estimate of the expected return of the global minimum variance portfolio  $\hat{\mu}_{MV}$  equals the OLS estimate:

$$\hat{\mu}_{MV} = \hat{\alpha} \quad (20)$$

3. The traditional estimate of the return variance of the global minimum variance portfolio  $\hat{\sigma}_{MV}^2$  is a multiple of the OLS estimate of the variance  $\hat{\sigma}_\varepsilon^2$ :

$$\hat{\sigma}_{MV}^2 = \frac{T-N}{T} \hat{\sigma}_\varepsilon^2 \quad (21)$$

First we prove Statement 1. The traditional approach is the solution to the minimization problem

$$\min_{w_1, \dots, w_N} \sum_{i=1}^N \sum_{j=1}^N w_i w_j \hat{\sigma}_{ij}. \quad (22)$$

In the OLS approach the regression coefficients are estimated by solving the following minimization problem

$$\min_{\alpha, \beta_1, \dots, \beta_{N-1}} \sum_{t=1}^T \varepsilon_t^2. \quad (23)$$

(23) can be rewritten as

$$\min_{\alpha, \beta_1, \dots, \beta_{N-1}} \sum_{t=1}^T \left[ -\alpha + \beta_1 r_{t,1} + \dots + \beta_{N-1} r_{t,N-1} + \left( 1 - \sum_{i=1}^{N-1} \beta_i \right) r_{t,N} \right]^2. \quad (24)$$

Since the coefficients  $\beta_i$  correspond to the portfolio weights  $w_i$  (Proposition 1) and since the  $N$  portfolio weights add up to one, we can rearrange Equation (24) as follows:

$$\min_{\alpha, w_1, \dots, w_N} \sum_{t=1}^T [-\alpha + w_1 r_{t,1} + \dots + w_N r_{t,N}]^2 \quad s.t. \quad \sum_{i=1}^N w_i = 1 \quad (25)$$

Differentiating (25) with respect to  $\alpha$  leads to the necessary condition for a minimum:

$$\alpha = w_1 \hat{\mu}_1 + \dots + w_N \hat{\mu}_N \quad (26)$$

Here  $\hat{\mu}_i = \frac{1}{T} \sum_{t=1}^T r_{t,i}$  is the estimated mean return of asset  $i$ . Using (26) we rewrite (25) as

$$\min_{w_1, \dots, w_N} \sum_{t=1}^T [w_1(r_{t,1} - \hat{\mu}_1) + \dots + w_N(r_{t,N} - \hat{\mu}_N)]^2 \quad (27)$$

subject to the condition that the  $N$  portfolio weights add up to one. Rearranging the sum in (27) yields another representation of the OLS approach (23):

$$\min_{w_1, \dots, w_N} T \sum_{i=1}^N \sum_{j=1}^N \left[ w_i w_j \frac{1}{T} \sum_{t=1}^T (r_{t,i} - \hat{\mu}_i)(r_{t,j} - \hat{\mu}_j) \right] = \min_{w_1, \dots, w_N} T \sum_{i=1}^N \sum_{j=1}^N w_i w_j \hat{\sigma}_{ij} \quad (28)$$

Thus, the sum of the squared residuals in (23) is equivalent to (28). Since (28) and (22) differ only by the positive factor  $T$ , both optimization problems produce the same portfolio weights. This proves the first statement of Proposition 2.

Statement 2 can be derived from the necessary condition (26). Replacing  $w_i$  by  $\hat{w}_{MV,i}$  makes  $\hat{\alpha}$  the estimated expected return of the global minimum variance portfolio, which leads to  $\hat{\alpha} = \hat{\mu}' \hat{w}_{MV}$ . The expression  $\hat{\mu}' \hat{w}_{MV}$  equals the traditional estimator  $\hat{\mu}_{MV}$ .

Statement 3 can be derived accordingly. The sum of the squared residuals (23) equals  $T \hat{\sigma}_{MV}^2$ . This can be easily seen by rewriting (28) as  $T \min_w w' \hat{\Sigma} w$ . Its solution  $T \hat{w}'_{MV} \hat{\Sigma} \hat{w}_{MV}$  equals  $T$  times the estimated variance of the global minimum variance portfolio.

Proposition 2 states that the OLS estimation technique and the traditional approach yield identical estimates of the portfolio weights of the global minimum variance portfolio. Therefore, the estimates of  $\hat{\mu}_{MV}$  are identical. The variance estimates differ only by the scalar  $(T - N)/T$ .

The equivalence of the two estimation approaches allows us to transfer all the distributional results of the OLS approach to the traditional approach. Therefore, we have a powerful yet simple way of deriving the conditional distributions of the estimated weights and return parameters. This is done in Section 4.

## 4 Conditional Distribution

We estimate the weights of the global minimum variance portfolio using the linear regression (8). We define the  $T \times N$  matrix  $Z$  which contains the regressors  $z_t = (r_{t,N} - r_{t,1}, \dots, r_{t,N} - r_{t,N-1})'$  of the linear regression (8):

$$Z := \begin{pmatrix} 1 & z'_1 \\ \vdots & \vdots \\ 1 & z'_T \end{pmatrix} = (\underline{1} \ z) \quad (29)$$

The vector  $\bar{z} = \frac{1}{T} \sum_{t=1}^T z_t$  consists of the arithmetic averages of the regressors. Proposition 3 gives the conditional distributions of the estimated portfolio weights and return parameters. The information set we condition on consists of the  $T \times (N - 1)$  matrix  $z$  of return differences .

### Proposition 3

1. The OLS estimates of the portfolio weights,  $\hat{\beta}^{ex}$ , are jointly normally distributed:

$$\hat{\beta}^{ex}|z \sim N(w_{MV}^{ex}; \sigma_{MV}^2(z'z - T\bar{z}\bar{z}')^{-1}) \quad (30)$$

2. The OLS estimate of the expected return,  $\hat{\alpha}$ , is normally distributed:

$$\hat{\alpha}|z \sim N(\mu_{MV}; \sigma_{MV}^2(1/T + \bar{z}'(z'z - T\bar{z}\bar{z}')^{-1}\bar{z})) \quad (31)$$

3. Let  $\hat{\sigma}_\varepsilon^2$  be the OLS estimate of the variance of the error term  $\varepsilon_t$ . The following expression is  $\chi^2$ -distributed:

$$(T - N) \frac{\hat{\sigma}_\varepsilon^2}{\sigma_{MV}^2} \sim \chi^2(T - N) \quad (32)$$

Proposition 3 is based on Proposition 1. The OLS estimator  $\hat{B} = (\hat{\alpha}, \hat{\beta}_1, \dots, \hat{\beta}_{N-1})' = (Z'Z)^{-1}Z'r_N$  with  $r_N = (r_{1,N}, \dots, r_{T,N})'$  is normally distributed:

$$\hat{B}|z \sim N(B; \sigma_\varepsilon^2(Z'Z)^{-1}). \quad (33)$$

$B = (\alpha, \beta_1, \dots, \beta_{N-1})'$  is the parameter vector. From (33) we see directly that the expectations of the conditional estimators  $\hat{\beta}^{ex}$  and  $\hat{\alpha}$  are  $\beta^{ex}$  and  $\alpha$ . According to Proposition 1, the variance  $\sigma_\varepsilon^2$  is equal to the variance of the global minimum variance portfolio  $\sigma_{MV}^2$ . Using (29) we partition the matrix  $Z'Z$ :

$$Z'Z = \begin{pmatrix} T & T\bar{z}' \\ T\bar{z} & z'z \end{pmatrix} \quad (34)$$

The inversion of the matrix  $Z'Z$  yields:<sup>7</sup>

$$(Z'Z)^{-1} = \begin{pmatrix} 1/T + \bar{z}'(z'z - T\bar{z}\bar{z}')^{-1}\bar{z} & \bar{z}'(z'z - T\bar{z}\bar{z}')^{-1} \\ (z'z - T\bar{z}\bar{z}')^{-1}\bar{z} & (z'z - T\bar{z}\bar{z}')^{-1} \end{pmatrix} \quad (35)$$

$\sigma_{MV}^2$  times the upper left element of the right hand side of (35) is the conditional variance of  $\hat{\alpha}$ .  $\sigma_{MV}^2$  times the lower right element is the conditional covariance matrix of  $\hat{\beta}^{ex}$ .

Proposition 3 states the core results of this paper. It allows us to calculate the estimation risk involved in estimating the global minimum variance portfolio (Section 6) and to carry out statistical tests concerning the estimated weights and return parameters (Section 7).

## 5 Non-Normal Returns

Throughout the paper we assumed that stock returns are normally distributed and IID. However, our OLS approach can also be used for non-normal returns. Instead of restricting ourselves to the multivariate normal distribution, we now consider the broader class of elliptical distributions. Among others, the class of elliptical distributions comprises the normal distribution and the Student-t-distribution. We choose this class of distributions for two reasons. Firstly, elliptical distributions support mean variance analysis since they fulfill the two requirements: (i) Elliptical distributions can be entirely characterized by their mean and variance and (ii) linear combinations of elliptically distributed random variables are again elliptically distributed. Secondly, elliptical distributions can describe empirical features of stock returns, especially the heavy tails of stock return distributions.

If we no longer assume normally distributed returns, but elliptically distributed returns, the noise term  $\varepsilon_t$  in Equation (8) will remain uncorrelated of the regressors  $z_t$ . However, the error term will not necessarily be independent of the regressors. For instance, the correlation of the squared noise term  $\varepsilon_t^2$  and the squared regressors  $z_{t,i}^2$  may be different from zero. This means that the standard assumptions of the linear regression model are no longer fulfilled because the noise terms  $\varepsilon_t$  are heteroskedastic. The variance of  $\varepsilon_t$  varies in a systematic way. Nevertheless, we can apply the OLS methodology.<sup>8</sup> Propositions 1 and 2 remain unaltered because their proofs do not depend on the normality assumption, but Proposition 3 has to be modified: The OLS estimates remain unbiased and consistent but the estimates  $\hat{\alpha}$  and  $\hat{\beta}$  are only asymptotically normally distributed. In addition, the estimated covariance matrix needs to be modified. To get correct standard errors in the regression, one has to adjust the covariance matrix e.g. by using the White (1980)-correction.<sup>9</sup>

## 6 Estimation Risk

We now apply our results to calculate the extent of the estimation risk. We again assume the best situation an investor can face: the returns are normal and IID. The estimation risk is the additional out-of-sample return variance due to errors in the estimated portfolio weights. In our Propositions 4 and 5 we calculate the conditional and unconditional estimation risk, respectively. In Proposition 6 we prove that the traditional weight estimator  $\hat{w}_{MV}$  leads to the lowest estimation risk of all unbiased estimators.<sup>10</sup>

We consider an investor who uses  $T$  return observations  $r_1, \dots, r_T$  to estimate  $\hat{w}_{MV}$ . Using the estimates  $\hat{w}_{MV}$ , the investor invests his funds for the period to follow. This strategy yields the out-of-sample return  $\hat{r}_{T+1,MV} = \hat{w}'_{MV} r_{T+1}$ . Its risk is  $\text{var}(\hat{r}_{T+1,MV} | r_1, \dots, r_T)$  which depends on the realizations of the stock returns from  $t = 1$  to  $t = T$ .

### Proposition 4

*If the portfolio weights are estimated traditionally, then the conditional out-of-*

sample return variance is given by

$$\text{var}(\hat{r}_{T+1,MV}|r_1, \dots, r_T) = \sigma_{MV}^2 + \tilde{R}(\hat{w}_{MV}) \quad (36)$$

with

$$\tilde{R}(\hat{w}_{MV}) = (\hat{w}_{MV} - w_{MV})' \Sigma (\hat{w}_{MV} - w_{MV}). \quad (37)$$

Proposition 4 (proved in Appendix 1) shows that the risk depends on two components. The first component,  $\sigma_{MV}^2$ , is the innovation risk, i.e. the risk due to the randomness of stock returns. The second component,  $\tilde{R}(\hat{w}_{MV})$ , is the estimation risk. If the investors knew all return distribution parameters, they would choose (5) as their weights when selecting the global minimum variance portfolio. In such a case there is no estimation risk and (36) reduces to (7). However, since the investor does not know the distribution parameters and has to estimate them instead, his estimated portfolio weights,  $\hat{w}_{MV}$ , differ from the true ones,  $w_{MV}$ . This difference leads to the conditional estimation risk  $\tilde{R}(\hat{w}_{MV})$ . Note that the  $\tilde{R}(\hat{w}_{MV})$  is a random variable which takes on only positive values. The more the estimated weights differ from the true ones, the larger  $\tilde{R}(\hat{w}_{MV})$  is. The unconditional estimation risk is obtained by applying the expectation operator to  $\text{var}(\hat{r}_{T+1,MV}|r_1, \dots, r_T)$ .

### Proposition 5

*If the portfolio weights are estimated traditionally, then the unconditional out-of-sample return variance is given by*

$$\text{E}(\text{var}(\hat{r}_{T+1,MV}|r_1, \dots, r_T)) = \sigma_{MV}^2 + \bar{R}(\hat{w}_{MV}) \quad (38)$$

with

$$\bar{R}(\hat{w}_{MV}) = \sigma_{MV}^2 \frac{N-1}{T-N-1}. \quad (39)$$

According to this proposition (proved in Appendix 2) the larger the innovation risk  $\sigma_{MV}^2$ , the larger the investment universe  $N$  and the shorter the estimation period  $T$  are, the higher is the unconditional estimation risk  $\bar{R}(\hat{w}_{MV})$ .<sup>11</sup>

Proposition 6 proves that the estimation risk cannot be reduced by choosing another unbiased weight estimator. The traditional weight estimator is the best unbiased estimator.



**Proposition 6**

*The traditional weight estimator  $\hat{w}_{MV}$  as given in Equation (18) has the lowest unconditional estimation risk  $\bar{R}(\cdot)$  of all unbiased weight estimators  $\check{w}_{MV}$ :*

$$\bar{R}(\hat{w}_{MV}) \leq \bar{R}(\check{w}_{MV}). \quad (40)$$

This proposition follows from the properties of OLS estimators. In the case of normally distributed error terms, the OLS estimator is the best unbiased weight estimator. According to Proposition 2 this statement is true for the traditional estimator, too. In Appendix 3 we show that this property implies the lowest estimation risk possible.

## 7 Statistical Inference

In this section we use our results to address problems in international asset allocation. We conduct an empirical study based on international stock data. Our data set consists of monthly MSCI total return indices of the G7 countries Canada, France, Germany, Italy, Japan, the United Kingdom, and the United States. These countries cover the major currency regions (Dollar, Euro, Pound, Yen). All indices are calculated in Euro, i.e. we take the view of an German investor. The data set covers the period from January 1984 to December 2003. We choose the return of the German index as the dependent variable  $r_{t,N}$  in the regression (8). We run the regression and obtain estimates of the portfolio weights of the global minimum variance portfolio. In Table 2 we report the weight estimates  $\hat{w}_{MV,i}$ , their standard errors and the  $t$ -statistics.<sup>12</sup> As the stock returns show excess kurtosis we applied the White (1980)-correction to calculate standard errors.

Table 2 shows that the UK market has the highest weight in the international global minimum variance portfolio, followed by Japan and USA. Only the weights for the indices of Japan, the UK and USA are significantly different from zero at the 10% level. This suggests that a German investor who only holds German stocks should add American, Japanese and British stocks to his domestic holdings.

To test whether a German investor can exclude several countries from his port-

Table 2: *Weight Estimates of the Global Minimum Variance Portfolio*

Country ( $i$ )	Weight $\hat{w}_{MV,i}$	Standard Error	$t$ -Statistics
Canada ( <i>Can</i> )	0.0146	0.0998	0.1467
France ( <i>Fra</i> )	0.0756	0.0772	0.9799
Germany ( <i>Ger</i> )	0.1418	0.0881	1.6088
Italy ( <i>Ita</i> )	0.0427	0.0512	0.8336
Japan ( <i>Jap</i> )	0.1909	0.0570	3.3512
United Kingdom ( <i>UK</i> )	0.3536	0.0946	3.7362
United States ( <i>USA</i> )	0.1807	0.1067	1.6947

folio without increasing the risk of his portfolio, we apply the  $F$ -test as shown in Appendix 4.<sup>13</sup> The  $F$ -test allows to test several linear restrictions concerning the portfolio weights simultaneously.<sup>14</sup> First we want to know whether a German investor can reduce his portfolio risk by diversifying internationally. We test the hypothesis:

$H_{0,1}$ : International diversification does not pay for German investors, i.e.  $w_{MV,Can} = w_{MV,Fra} = w_{MV,USA} = w_{MV,Ita} = w_{MV,UK} = w_{MV,Jap} = 0$ .

The null hypothesis is rejected at the 1%-level ( $F(6, 233)$ -statistic = 22.45). Thus, it pays for a German investor to diversify internationally. Whether adopting a naive diversification strategy or diversifying optimally makes a difference is analyzed next.

$H_{0,2}$ : Naive diversification ( $w_{MV,i} = 1/7 \forall i$ ) offers the same risk diversification effect as optimal diversification.

$H_{0,2}$  is rejected at the 10% level ( $F(6, 233)$ -statistic = 2.06). We conclude that a German investor is better off choosing the weights according to (5) than by investing equally in all countries. Finally, we want to know whether investing in only *one* country per currency region reduces the diversification effect significantly. The

countries invested in are Germany (Euro), Japan (Yen), the UK (Pound) and the United States (Dollar).

$H_{0,3}$ : Investing in one country per currency region ( $w_{MV,Can} = w_{MV,Fra} = w_{MV,Ita} = 0$ ) offers the same risk diversification as investing in all countries.

We cannot reject  $H_{0,3}$  ( $F(3, 233)$ –statistic = 0.58). The results suggest that covering the major currency regions by choosing only one country for each currency region provides sufficient diversification.

The three hypotheses tested above serve as examples of how to use the results of this paper. Obviously, one can easily find other hypotheses to test with our approach.

## 8 Conclusion

In this paper we show that the weights of the global minimum variance portfolio are equal to regression coefficients. This allows us to transfer the entire OLS methodology to the estimation of the weights and return parameters of the global minimum variance portfolio. From the OLS methodology we derive the conditional distributions of the estimated portfolio weights and estimated return parameters.

We discuss two applications of our distributional results. The first application is to assess the extent of the estimation risk involved in estimating the global minimum variance portfolio. Our second application is to test important hypotheses in international asset management. These two applications serve as an illustration of the usefulness of our approach.

## Appendix 1

Using  $\hat{w}_{MV} = w_{MV} + (\hat{w}_{MV} - w_{MV})$  we rewrite the conditional out-of-sample return variance as

$$\begin{aligned}\text{var}(\hat{r}_{T+1,MV} | r_1, \dots, r_T) &= \hat{w}_{MV}' \Sigma \hat{w}_{MV} \\ &= \sigma_{MV}^2 + (\hat{w}_{MV} - w_{MV})' \Sigma (\hat{w}_{MV} - w_{MV}) \\ &\quad + 2w_{MV}' \Sigma (\hat{w}_{MV} - w_{MV}).\end{aligned}\tag{41}$$

The last term in (41) can be rewritten as

$$2(w_{MV}' \Sigma \hat{w}_{MV} - w_{MV}' \Sigma w_{MV}).\tag{42}$$

The first term is the return covariance of a portfolio with the portfolio weights  $\hat{w}_{MV}$  and the global minimum variance portfolio  $w_{MV}$ . The second term is the return variance of the global minimum variance portfolio. Huang and Litzenberger (1988), p. 68, prove that the return covariance of an arbitrary stock portfolio and the global minimum variance portfolio is equal to the return variance of the global minimum variance portfolio. Therefore, the last term in (41) drops out. This completes the proof of Proposition 4.

## Appendix 2

In this appendix we prove Proposition 5. In Lemma 1 we show how to express the unconditional estimation risk  $\bar{R}(\cdot)$  of any unbiased weight estimator  $\check{w}_{MV}$  as a function of the estimator's unconditional variance  $\text{var}(\check{w}_{MV}^{ex})$ . In Lemma 2 we compute the unconditional variance of a specific unbiased weight estimator, the traditional weight estimator. Combining these two lemmata, we obtain the expression for the estimation risk  $\bar{R}(\hat{w}_{MV})$  as stated in Proposition 5.

### Lemma 1

*Let  $\check{w}_{MV}$  be any unbiased weight estimate. Then the unconditional out-of-sample return variance is*

$$\mathbb{E}(\text{var}(\check{r}_{T+1,MV}|r_1, \dots, r_T)) = \sigma_{MV}^2 + \bar{R}(\check{w}_{MV}) \quad (43)$$

with

$$\bar{R}(\check{w}_{MV}) = \text{tr}[\text{var}(\check{w}_{MV}^{ex})\Omega]. \quad (44)$$

Proof of Lemma 1: Using (14) we can rewrite the out-of-sample return as

$$\check{r}_{T+1,MV} = r_{T+1,N} - (\check{w}_{MV}^{ex})'(r_{T+1,N}\mathbf{1} - r_{T+1}^{ex}). \quad (45)$$

The unconditional out-of-sample variance is

$$\mathbb{E}(\text{var}(\check{r}_{T+1,MV}|r_1, \dots, r_T)) = \sigma_N^2 + \mathbb{E}((\check{w}_{MV}^{ex})'\Omega\check{w}_{MV}^{ex}) - 2E(\check{w}_{MV}^{ex})'\text{cov}(r_{T+1,N}\mathbf{1} - r_{T+1}^{ex}, r_{T+1,N}). \quad (46)$$

Setting  $E(\check{w}_{MV}^{ex}) = w_{MV}^{ex} + E(\check{w}_{MV}^{ex} - w_{MV}^{ex})$  we rewrite the expression  $\mathbb{E}((\check{w}_{MV}^{ex})'\Omega\check{w}_{MV}^{ex})$  as

$$\begin{aligned} \mathbb{E}((\check{w}_{MV}^{ex})'\Omega\check{w}_{MV}^{ex}) &= (w_{MV}^{ex})'\Omega w_{MV}^{ex} + \mathbb{E}((\check{w}_{MV}^{ex} - w_{MV}^{ex})'\Omega(\check{w}_{MV}^{ex} - w_{MV}^{ex})) \\ &\quad + 2E(\check{w}_{MV}^{ex} - w_{MV}^{ex})'\Omega w_{MV}^{ex}. \end{aligned} \quad (47)$$

Inserting (47) in (46) and using

$$\sigma_{MV}^2 = \sigma_N^2 + (w_{MV}^{ex})'\Omega w_{MV}^{ex} - 2(w_{MV}^{ex})'\text{cov}(r_{T+1,N}\mathbf{1} - r_{T+1}^{ex}, r_{T+1,N}) \quad (48)$$

we get

$$\mathbb{E}(\text{var}(\check{r}_{T+1,MV}|r_1, \dots, r_T)) = \sigma_{MV}^2 + \mathbb{E}((\check{w}_{MV}^{ex} - w_{MV}^{ex})' \Omega (\check{w}_{MV}^{ex} - w_{MV}^{ex})). \quad (49)$$

Finally we deal with the expression  $\mathbb{E}((\check{w}_{MV}^{ex} - w_{MV}^{ex})' \Omega (\check{w}_{MV}^{ex} - w_{MV}^{ex}))$ .

$$\begin{aligned} \mathbb{E}((\check{w}_{MV}^{ex} - w_{MV}^{ex})' \Omega (\check{w}_{MV}^{ex} - w_{MV}^{ex})) &= \mathbb{E}(\text{tr}((\check{w}_{MV}^{ex} - w_{MV}^{ex})' \Omega (\check{w}_{MV}^{ex} - w_{MV}^{ex}))) \\ &= \mathbb{E}(\text{tr}((\check{w}_{MV}^{ex} - w_{MV}^{ex})(\check{w}_{MV}^{ex} - w_{MV}^{ex})' \Omega)) \\ &= \text{tr}(\mathbb{E}((\check{w}_{MV}^{ex} - w_{MV}^{ex})(\check{w}_{MV}^{ex} - w_{MV}^{ex})') \Omega) \\ &= \text{tr}(\text{var}(\check{w}_{MV}^{ex}) \Omega) \end{aligned} \quad (50)$$

Lemma 1 results directly from (49) in combination with (50).

The estimation risk given by (44) depends on the estimator's variance  $\text{var}(\check{w}_{MV}^{ex})$ . For the traditional estimator we can state this variance explicitly. This is done in Lemma 2.

**Lemma 2**

*The unconditional variance of the traditional weight estimator  $\hat{w}_{MV}^{ex}$  is*

$$\text{var}(\hat{w}_{MV}^{ex}) = \sigma_{MV}^2 \frac{1}{T - N - 1} \Omega^{-1}. \quad (51)$$

Proof of Lemma 2: From the first statement of Proposition 2 in connection with the first statement of Proposition 3 we get the conditional variance:

$$\text{var}(\hat{w}_{MV}^{ex}|z) = \sigma_{MV}^2 (z'z - T\bar{z}\bar{z}')^{-1} \quad (52)$$

The variance decomposition theorem provides the relation between the unconditional and conditional variance:

$$\text{var}(\hat{w}_{MV}^{ex}) = \mathbb{E}(\text{var}(\hat{w}_{MV}^{ex}|z)) + \text{var}(\mathbb{E}(\hat{w}_{MV}^{ex}|z)) \quad (53)$$

As the estimator  $\hat{w}_{MV}^{ex}$  is unbiased, the second term on the right hand side of (53) is zero. Therefore, it remains to determine the expectation of  $(z'z - T\bar{z}\bar{z}')^{-1}$ . The matrix  $(z'z - T\bar{z}\bar{z}')$  is Wishart distributed, which follows from the assumption of

normally distributed returns:

$$z'z - T\bar{z}\bar{z}' = \sum_{t=1}^T (z_t - \bar{z})(z_t - \bar{z})' \sim W(\Omega, T - 1, N - 1) \quad (54)$$

The expectation of a random matrix whose inverse is Wishart distributed is shown in Press (1972), p. 112:

$$E \left( (z'z - T\bar{z}\bar{z}')^{-1} \right) = \frac{1}{T - N - 1} \Omega^{-1} \quad (55)$$

Lemma 2 follows immediately from (55).

Inserting (51) into (44) yields (39). This completes the Proof of Proposition 5.

## Appendix 3

Based on (43) of Lemma 1 we can state the difference in the unconditional estimation risk between using an arbitrary unbiased weight estimator  $\check{w}_{MV}$  and using the traditional estimator  $\hat{w}_{MV}$ , respectively:

$$\bar{R}(\check{w}_{MV}) - \bar{R}(\hat{w}_{MV}) = \text{tr}[\text{var}(\check{w}_{MV}^{ex})\Omega] - \text{tr}[\text{var}(\hat{w}_{MV}^{ex})\Omega] \quad (56)$$

$$= \text{tr}[\Delta\Omega] \quad (57)$$

with

$$\Delta = \text{var}(\check{w}_{MV}^{ex}) - \text{var}(\hat{w}_{MV}^{ex}) \quad (58)$$

As  $\hat{w}_{MV}^{ex}$  is the *best unbiased estimator*, the difference matrix  $\Delta$  is at least positive semi-definite. Since the trace of the matrix product of two semi-definite matrices is never negative, the expression  $\text{tr}[\Delta\Omega]$  in (57) is not negative, either.<sup>15</sup> Therefore, there is no unbiased weight estimator with lower unconditional estimation risk than that of the traditional estimator.<sup>16</sup>



## Appendix 4

In this appendix we explicitly give the test statistics used in Section 7. In the case of non-normally distributed returns these statistics are only asymptotically exact and the estimated covariance matrix has to be adjusted as pointed out in Section 5. Let  $q = (q_1, \dots, q_{N-1})'$  be an arbitrary non-stochastic vector. Then the following statistic is  $t$ -distributed:

$$\frac{q' \hat{w}_{MV}^{ex} - q' w_{MV}^{ex}}{\sqrt{\hat{\sigma}_\varepsilon^2 q' (z' z - \bar{z} \bar{z}')^{-1} q}} \sim t(T - N) \quad (59)$$

Since the estimated weight of asset  $N$  is a linear combination of the other weights, i.e.  $\hat{w}_{MV,N} = 1 - \underline{1}' \hat{w}_{MV}^{ex}$ , we can derive the distribution of  $\hat{w}_{MV,N}$  from (59) by setting  $q = \underline{1}$ :

$$\frac{\hat{w}_{MV,N} - w_{MV,N}}{\sqrt{\hat{\sigma}_\varepsilon^2 \underline{1}' (z' z - \bar{z} \bar{z}')^{-1} \underline{1}}} \sim t(T - N) \quad (60)$$

In the third column of Table 2 we report the  $t$ -statistic as computed by (59) for the weights  $i = Can, Fra, Ger, Ita, Jap, Uk$  and by (60) for the weight  $i = US$ .

Let  $SSR$  and  $SSR_R$  be the sum of the squared residuals in the unrestricted and restricted regression. Let  $m \leq N - 1$  be the number of linear independent restrictions. Then the following statistic is  $F$ -distributed:

$$F = \frac{T - N}{m} \left( \frac{SSR_R}{SSR} - 1 \right) \sim F(m, T - N) \quad (61)$$

This statistic is calculated for the hypotheses  $H_{0,1}$  to  $H_{0,3}$ .

## References

- Bawa, Vijay S., Stephen J. Brown, and Roger W. Klein, 1979, *Estimation risk and optimal portfolio choice* (North Holland: Amsterdam).
- Chopra, Vijay K., and William T. Ziemba, 1993, The effect of errors in means, variances, and covariances on optimal portfolio choice, *Journal of Portfolio Management* 19, 6–11.
- Dickinson, John P., 1974, The reliability of estimation procedures in portfolio analysis, *Journal of Financial and Quantitative Analysis* 9, 447–462.
- Gorman, Larry R., and Bjorn Jorgensen, 2002, Domestic versus international portfolio selection: A statistical exam of the home bias, *Multinational Finance Journal* 6, 131–166.
- Greene, William H., 2000, *Econometric Analysis* (Prentice Hall: New Jersey) fourth edn.
- Hayashi, Fumio, 2000, *Econometrics* (Princeton University Press: Princeton).
- Huang, Chi-Fu, and Robert H. Litzenberger, 1988, *Foundations for Financial Economics* (Prentice Hall: New Jersey).
- Jagannathan, Ravi, and Tongshu Ma, 2003, Risk reduction in large portfolios: Why imposing the wrong constraints helps, *Journal of Finance* forthcoming.
- Jorion, Philippe, 1985, International portfolio diversification with estimation risk, *Journal of Business* 58, 259–278.
- , 1991, Bayesian and CAPM estimators of the means: Implications for portfolio selection, *Journal of Banking and Finance* 15, 717–727.
- Kan, Raymond, and Guofu Zhou, 2001, Tests of mean-variance spanning, Working paper University of Toronto.
- Ledoit, Olivier, and Michael Wolf, 2003, Improved estimation of the covariance matrix of stock returns with an application to portfolio selection, *Journal of Empirical Finance* 10, 603–621.

- Lütkepohl, Helmut, 1996, *Handbook of Matrices* (John Wiley and Sons: New York).
- Merton, Robert C., 1980, On estimating the expected return on the market: An exploratory investigation, *Journal of Financial Economics* 8, 323–361.
- Newey, Withney K., and Kenneth D. West, 1987, A simple positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix, *Econometrica* 55, 703–708.
- Ohkrin, Yarema, and Wolfgang Schmid, 2004, Distributional properties of portfolio weights, Working Paper University of Frankfurt (Oder).
- Press, James S., 1972, *Applied Multivariate Analysis* (Holt, Rinehart and Winston: New York).
- White, Halbert, 1980, A heteroskedasticity-consistent covariance matrix estimator and direct test for heteroskedasticity, *Econometrica* 48, 817–838.

# Notes

<sup>1</sup>In this paper we deal only with estimation risk resulting from unknown return distribution parameters. In the more general definition of Bawa, Brown and Klein (1979), estimation risk also includes situations in which not only the parameters, but also the functional form of the distribution is unknown.

<sup>2</sup>For example, using  $\tau = 10$  years of monthly ( $n = 12$ ) data provides us with  $T = 120$  observations.

<sup>3</sup>See Merton (1980). The distribution of the estimated return standard deviation  $\hat{\sigma}_i$  is only known asymptotically.

<sup>4</sup>Note that the error term  $\varepsilon_t$  is by construction uncorrelated with all the return differences  $r_{t,N} - r_{t,i}$ . The absence of correlation allows us to apply the OLS estimation technique.

<sup>5</sup>We will show in Section 5 that one can apply the OLS approach even when the returns are not normal and the error term does not satisfy the standard assumption of linear regression.

<sup>6</sup>The superscript *ex* indicates that the vector has no entry for asset  $N$ .

<sup>7</sup>See Greene (2000), p. 34.

<sup>8</sup>See for the results to follow for instance Greene (2000), pp. 499-523.

<sup>9</sup>In case, there is not only heteroscedasticity, but also autocorrelation in the data, one has to use the correction of Newey and West (1987) instead.

<sup>10</sup>Without the assumption of normality, Proposition 4 remains unchanged. Proposition 6 is weakened: the traditional estimator is only the best *linear* unbiased estimator. Regarding Proposition 5: Without the normality assumption, we cannot determine the unconditional estimation risk because this determination requires the expectation of a quadratic form which is only known for normally distributed random variables.

<sup>11</sup>This result proves the claim of Jagannathan and Ma (2003).

<sup>12</sup>See Appendix 4 for the exact formula of the test statistic.

<sup>13</sup>Our test is a simplified version of a spanning test. The spanning tests suggested in the literature (see, e.g., Kan and Zhou (2001)) test whether the inclusion of an additional asset changes the minimum variance frontier. Our test focuses not on the whole frontier, but solely on one portfolio of the frontier, the global minimum variance portfolio. If we find a significant change in the global minimum variance portfolio we know that the minimum variance frontier has changed as well. Thus, our test is a sufficient test for spanning. Since the global minimum variance portfolio does not depend on expected returns, our test has a higher power than traditional spanning tests.

<sup>14</sup>Jorion (1985) develops an alternative test to address this question. He uses a maximum likelihood test to check whether a given portfolio is significantly different from the global minimum variance portfolio. While the distribution of the Jorion (1985) test is known only asymptotically, the distribution of our test is known even in small samples.

<sup>15</sup>See Lütkepohl (1996), p. 21.

<sup>16</sup>If we give up the assumption of normality, the traditional estimator is the best linear unbiased estimator. For the Gauss-Markov-Theorem see, e.g., Hayashi (2000), p. 27-29.

cfr working papers are available for download from [www.cfr-cologne.de](http://www.cfr-cologne.de).

hardcopies can be ordered from: centre for financial research (cfr),  
albertus magnus platz, 50923 koeln, germany.

**2011**

No.	Author(s)	Title
11-09	V. Agarwal, S. Ray	Determinants and Implications of Fee Changes in the Hedge Fund Industry
11-08	G. Cici, L.-F. Palacios	On the Use of Options by Mutual Funds: Do They Know What They Are Doing?
11-07	V. Agarwal, G. D. Gay, L. Ling	Window Dressing in Mutual Funds
11-06	N. Hautsch, D. Hess, D. Veredas	The Impact of Macroeconomic News on Quote Adjustments, Noise, and Informational Volatility
11-05	G. Cici	The Prevalence of the Disposition Effect in Mutual Funds' Trades
11-04	S. Jank	Mutual Fund Flows, Expected Returns and the Real Economy
11-03	G.Fellner, E.Theissen	Short Sale Constraints, Divergence of Opinion and Asset Value: Evidence from the Laboratory
11-02	S.Jank	Are There Disadvantaged Clienteles in Mutual Funds?
11-01	V. Agarwal, C. Meneghetti	The role of Hedge Funds as Primary Lenders

**2010**

No.	Author(s)	Title
10-20	G. Cici, S. Gibson, J.J. Merrick Jr.	Missing the Marks? Dispersion in Corporate Bond Valuations Across Mutual Funds
10-19	J. Hengelbrock, E. Theissen, Ch. Westheide	Market Response to Investor Sentiment
10-18	G. Cici, S. Gibson	The Performance of Corporate-Bond Mutual Funds: Evidence Based on Security-Level Holdings
10-17	D. Hess, D. Kreutzmann, O. Pucker	Projected Earnings Accuracy and the Profitability of Stock Recommendations
10-16	S. Jank, M. Wedow	Sturm und Drang in Money Market Funds: When Money Market Funds Cease to Be Narrow

10-15	G. Cici, A. Kempf, A. Puetz	Caught in the Act: How Hedge Funds Manipulate their Equity Positions
10-14	J. Grammig, S. Jank	Creative Destruction and Asset Prices
10-13	S. Jank, M. Wedow	Purchase and Redemption Decisions of Mutual Fund Investors and the Role of Fund Families
10-12	S. Artmann, P. Finter, A. Kempf, S. Koch, E. Theissen	The Cross-Section of German Stock Returns: New Data and New Evidence
10-11	M. Chesney, A. Kempf	The Value of Tradeability
10-10	S. Frey, P. Herbst	The Influence of Buy-side Analysts on Mutual Fund Trading
10-09	V. Agarwal, W. Jiang, Y. Tang, B. Yang	Uncovering Hedge Fund Skill from the Portfolio Holdings They Hide
10-08	V. Agarwal, V. Fos, W. Jiang	Inferring Reporting Biases in Hedge Fund Databases from Hedge Fund Equity Holdings
10-07	V. Agarwal, G. Bakshi, J. Huij	Do Higher-Moment Equity Risks Explain Hedge Fund Returns?
10-06	J. Grammig, F. J. Peter	Tell-Tale Tails
10-05	K. Drachter, A. Kempf	Höhe, Struktur und Determinanten der Managervergütung- Eine Analyse der Fondsbranche in Deutschland
10-04	J. Fang, A. Kempf, M. Trapp	Fund Manager Allocation
10-03	P. Finter, A. Niessen-Ruenzi, S. Ruenzi	The Impact of Investor Sentiment on the German Stock Market
10-02	D. Hunter, E. Kandel, S. Kandel, R. Wermers	Endogenous Benchmarks
10-01	S. Artmann, P. Finter, A. Kempf	Determinants of Expected Stock Returns: Large Sample Evidence from the German Market

## 2009

No.	Author(s)	Title
09-17	E. Theissen	Price Discovery in Spot and Futures Markets: A Reconsideration
09-16	M. Trapp	Trading the Bond-CDS Basis – The Role of Credit Risk and Liquidity
09-14	A. Kempf, O. Korn, M. Uhrig-Homburg	The Term Structure of Illiquidity Premia
09-13	W. Bühler, M. Trapp	Time-Varying Credit Risk and Liquidity Premia in Bond and CDS Markets
09-12	W. Bühler, M. Trapp	Explaining the Bond-CDS Basis – The Role of Credit Risk and Liquidity
09-11	S. J. Taylor, P. K. Yadav, Y. Zhang	Cross-sectional analysis of risk-neutral skewness
09-10	A. Kempf, C. Merkle, A. Niessen	Low Risk and High Return - How Emotions Shape Expectations on the Stock Market
09-09	V. Fotak, V. Raman, P. K. Yadav	Naked Short Selling: The Emperor's New Clothes?

09-08	F. Bardong, S.M. Bartram, P.K. Yadav	Informed Trading, Information Asymmetry and Pricing of Information Risk: Empirical Evidence from the NYSE
09-07	S. J. Taylor , P. K. Yadav, Y. Zhang	The information content of implied volatilities and model-free volatility expectations: Evidence from options written on individual stocks
09-06	S. Frey, P. Sandas	The Impact of Iceberg Orders in Limit Order Books
09-05	H. Beltran-Lopez, P. Giot, J. Grammig	Commonalities in the Order Book
09-04	J. Fang, S. Ruenzi	Rapid Trading bei deutschen Aktienfonds: Evidenz aus einer großen deutschen Fondsgesellschaft
09-03	A. Banegas, B. Gillen, A. Timmermann, R. Wermers	The Performance of European Equity Mutual Funds
09-02	J. Grammig, A. Schrimpf, M. Schuppli	Long-Horizon Consumption Risk and the Cross-Section of Returns: New Tests and International Evidence
09-01	O. Korn, P. Koziol	The Term Structure of Currency Hedge Ratios

## 2008

No.	Author(s)	Title
08-12	U. Bonenkamp, C. Homburg, A. Kempf	Fundamental Information in Technical Trading Strategies
08-11	O. Korn	Risk Management with Default-risky Forwards
08-10	J. Grammig, F.J. Peter	International Price Discovery in the Presence of Market Microstructure Effects
08-09	C. M. Kuhnen, A. Niessen	Public Opinion and Executive Compensation
08-08	A. Pütz, S. Ruenzi	Overconfidence among Professional Investors: Evidence from Mutual Fund Managers
08-07	P. Osthoff	What matters to SRI investors?
08-06	A. Betzer, E. Theissen	Sooner Or Later: Delays in Trade Reporting by Corporate Insiders
08-05	P. Linge, E. Theissen	Determinanten der Aktionärspräsenz auf Hauptversammlungen deutscher Aktiengesellschaften
08-04	N. Hautsch, D. Hess, C. Müller	Price Adjustment to News with Uncertain Precision
08-03	D. Hess, H. Huang, A. Niessen	How Do Commodity Futures Respond to Macroeconomic News?
08-02	R. Chakrabarti, W. Megginson, P. Yadav	Corporate Governance in India
08-01	C. Andres, E. Theissen	Setting a Fox to Keep the Geese - Does the Comply-or-Explain Principle Work?

## 2007

No.	Author(s)	Title
-----	-----------	-------



07-16	M. Bär, A. Niessen, S. Ruenzi	The Impact of Work Group Diversity on Performance: Large Sample Evidence from the Mutual Fund Industry
07-15	A. Niessen, S. Ruenzi	Political Connectedness and Firm Performance: Evidence From Germany
07-14	O. Korn	Hedging Price Risk when Payment Dates are Uncertain
07-13	A. Kempf, P. Osthoff	SRI Funds: Nomen est Omen
07-12	J. Grammig, E. Theissen, O. Wuensche	Time and Price Impact of a Trade: A Structural Approach
07-11	V. Agarwal, J. R. Kale	On the Relative Performance of Multi-Strategy and Funds of Hedge Funds
07-10	M. Kasch-Haroutounian, E. Theissen	Competition Between Exchanges: Euronext versus Xetra
07-09	V. Agarwal, N. D. Daniel, N. Y. Naik	Do hedge funds manage their reported returns?
07-08	N. C. Brown, K. D. Wei, R. Wermers	Analyst Recommendations, Mutual Fund Herding, and Overreaction in Stock Prices
07-07	A. Betzer, E. Theissen	Insider Trading and Corporate Governance: The Case of Germany
07-06	V. Agarwal, L. Wang	Transaction Costs and Value Premium
07-05	J. Grammig, A. Schrimpf	Asset Pricing with a Reference Level of Consumption: New Evidence from the Cross-Section of Stock Returns
07-04	V. Agarwal, N.M. Boyson, N.Y. Naik	Hedge Funds for retail investors? An examination of hedged mutual funds
07-03	D. Hess, A. Niessen	The Early News Catches the Attention: On the Relative Price Impact of Similar Economic Indicators
07-02	A. Kempf, S. Ruenzi, T. Thiele	Employment Risk, Compensation Incentives and Managerial Risk Taking - Evidence from the Mutual Fund Industry -
07-01	M. Hagemeister, A. Kempf	CAPM und erwartete Renditen: Eine Untersuchung auf Basis der Erwartung von Marktteilnehmern

## 2006

No.	Author(s)	Title
06-13	S. Čeljo-Hörhager, A. Niessen	How do Self-fulfilling Prophecies affect Financial Ratings? - An experimental study
06-12	R. Wermers, Y. Wu, J. Zechner	Portfolio Performance, Discount Dynamics, and the Turnover of Closed-End Fund Managers
06-11	U. v. Lilienfeld-Toal, S. Ruenzi	Why Managers Hold Shares of Their Firm: An Empirical Analysis
06-10	A. Kempf, P. Osthoff	The Effect of Socially Responsible Investing on Portfolio Performance
06-09	R. Wermers, T. Yao, J. Zhao	The Investment Value of Mutual Fund Portfolio Disclosure
06-08	M. Hoffmann, B. Kempa	The Poole Analysis in the New Open Economy Macroeconomic Framework
06-07	K. Drachter, A. Kempf, M. Wagner	Decision Processes in German Mutual Fund Companies: Evidence from a Telephone Survey

06-06	J.P. Krahnen, F.A. Schmid, E. Theissen	Investment Performance and Market Share: A Study of the German Mutual Fund Industry
06-05	S. Ber, S. Ruenzi	On the Usability of Synthetic Measures of Mutual Fund Net-Flows
06-04	A. Kempf, D. Mayston	Liquidity Commonality Beyond Best Prices
06-03	O. Korn, C. Koziol	Bond Portfolio Optimization: A Risk-Return Approach
06-02	O. Scaillet, L. Barras, R. Wermers	False Discoveries in Mutual Fund Performance: Measuring Luck in Estimated Alphas
06-01	A. Niessen, S. Ruenzi	Sex Matters: Gender Differences in a Professional Setting

## 2005

No.	Author(s)	Title
05-16	E. Theissen	An Analysis of Private Investors' Stock Market Return Forecasts
05-15	T. Foucault, S. Moinas, E. Theissen	Does Anonymity Matter in Electronic Limit Order Markets
05-14	R. Kosowski, A. Timmermann, R. Wermers, H. White	Can Mutual Fund „Stars“ Really Pick Stocks? New Evidence from a Bootstrap Analysis
05-13	D. Avramov, R. Wermers	Investing in Mutual Funds when Returns are Predictable
05-12	K. Giese, A. Kempf	Liquiditätsdynamik am deutschen Aktienmarkt
05-11	S. Ber, A. Kempf, S. Ruenzi	Determinanten der Mittelzuflüsse bei deutschen Aktienfonds
05-10	M. Bär, A. Kempf, S. Ruenzi	Is a Team Different From the Sum of Its Parts? Evidence from Mutual Fund Managers
05-09	M. Hoffmann	Saving, Investment and the Net Foreign Asset Position
05-08	S. Ruenzi	Mutual Fund Growth in Standard and Specialist Market Segments
05-07	A. Kempf, S. Ruenzi	Status Quo Bias and the Number of Alternatives - An Empirical Illustration from the Mutual Fund Industry –
05-06	J. Grammig, E. Theissen	Is Best Really Better? Internalization of Orders in an Open Limit Order Book
05-05	H. Beltran, J. Grammig, A.J. Menkveld	Understanding the Limit Order Book: Conditioning on Trade Informativeness
05-04	M. Hoffmann	Compensating Wages under different Exchange rate Regimes
05-03	M. Hoffmann	Fixed versus Flexible Exchange Rates: Evidence from Developing Countries
05-02	A. Kempf, C. Memmel	On the Estimation of the Global Minimum Variance Portfolio
05-01	S. Frey, J. Grammig	Liquidity supply and adverse selection in a pure limit order book market

**2004**

No.	Author(s)	Title
04-10	N. Hautsch, D. Hess	Bayesian Learning in Financial Markets – Testing for the Relevance of Information Precision in Price Discovery
04-09	A. Kempf, K. Kreuzberg	Portfolio Disclosure, Portfolio Selection and Mutual Fund Performance Evaluation
04-08	N.F. Carline, S.C. Linn, P.K. Yadav	Operating performance changes associated with corporate mergers and the role of corporate governance
04-07	J.J. Merrick, Jr., N.Y. Naik, P.K. Yadav	Strategic Trading Behavior and Price Distortion in a Manipulated Market: Anatomy of a Squeeze
04-06	N.Y. Naik, P.K. Yadav	Trading Costs of Public Investors with Obligatory and Voluntary Market-Making: Evidence from Market Reforms
04-05	A. Kempf, S. Ruenzi	Family Matters: Rankings Within Fund Families and Fund Inflows
04-04	V. Agarwal, N.D. Daniel, N.Y. Naik	Role of Managerial Incentives and Discretion in Hedge Fund Performance
04-03	V. Agarwal, W.H. Fung, J.C. Loon, N.Y. Naik	Risk and Return in Convertible Arbitrage: Evidence from the Convertible Bond Market
04-02	A. Kempf, S. Ruenzi	Tournaments in Mutual Fund Families
04-01	I. Chowdhury, M. Hoffmann, A. Schabert	Inflation Dynamics and the Cost Channel of Monetary Transmission



centre for financial research  
cfr/university of cologne  
albertus-magnus-platz  
d-50923 cologne  
fon +49(0)221-470-6995  
fax +49(0)221-470-3992  
kempf@cfr-cologne.de  
www.cfr-cologne.de