

Enders, Zeno; Hakenes, Hendrik

**Working Paper**

## On the existence and prevention of asset price bubbles

Preprints of the Max Planck Institute for Research on Collective Goods, No. 2010,44

**Provided in Cooperation with:**

Max Planck Institute for Research on Collective Goods

Suggested Citation: Enders, Zeno; Hakenes, Hendrik (2010) : On the existence and prevention of asset price bubbles, Preprints of the Max Planck Institute for Research on Collective Goods, No. 2010,44, Max Planck Inst. for Research on Collective Goods, Bonn

This Version is available at:

<http://hdl.handle.net/10419/57494>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*



On the Existence and  
Prevention of Asset Price  
Bubbles

Zeno Enders  
Hendrik Hakenes





# **On the Existence and Prevention of Asset Price Bubbles**

Zeno Enders / Hendrik Hakenes

November 2010

# On the Existence and Prevention of Asset Price Bubbles\*

Zeno Enders<sup>†</sup>  
University of Bonn

Hendrik Hakenes<sup>‡</sup>  
University of Hannover  
Max Planck Institute, Bonn

November 3, 2010

## Abstract

We develop a model of rational bubbles based on the assumptions of unknown market liquidity and limited liability of traders. In a bubble, the price of an asset rises dynamically above its steady-state value, justified by rational expectations about future price developments. The larger the expected future price increase, the more likely it is that the bubble will burst because market liquidity becomes exhausted. Depending on the interactions between uncertainty about market liquidity, fundamental riskiness of the asset, the compensation scheme of the fund manager, and the risk-free interest rate, we give a condition for whether rational bubbles are possible. Based on this analysis, we discuss several widely-discussed policy measures with respect to their effectiveness in preventing bubbles. A reduction of manager bonuses or a Tobin tax can create or eliminate the possibility of bubbles, depending on their implementation. Monetary policy and long-term compensation schemes can prevent bubbles.

**Keywords:** Bubbles, Rational Expectations, Bonuses, Compensation Schemes, Financial Crises, Financial Policy.

**JEL-Codes:** G01, G12, E44, E1.

---

\*We thank Felix Bierbrauer, John Boyd, Christoph Engel, Martin Hellwig, Andreas Irmen, and participants of the econ workshop at the Max Planck Institute Bonn, the research seminar of the Deutsche Bundesbank, the Austrian National Bank, the German Economic Association, the German Finance Association, and the Universities of Bonn, Hannover, Heidelberg, Innsbruck, Karlsruhe, Mannheim, and Tübingen for discussion. A former version of this paper was called “The Birth and Burst of Asset Price Bubbles.” The usual disclaimer applies.

<sup>†</sup>University of Bonn, BGSE, Adenauerallee 24-42, D-53113 Bonn, zeno.enders@uni-bonn.de

<sup>‡</sup>University of Hannover, Königsworther Platz 1, D-30167 Hannover, hakenes@fnt.uni-hannover.de

# 1 Introduction

Under what conditions are asset price bubbles possible? Which policies can prevent them? In the light of recent economic experience, these questions seem important and topical. The last 15 years have seen at least two important market developments that are now considered as bubbles. Both the so-called dot-com bubble in the late 1990s and the recent housing bubble in the United States and elsewhere produced large reallocations of wealth during their buildup and after their crashes, thereby affecting real macroeconomic variables. Although the phenomenon of bubbles has long been recognized, economic policy has not been able to prevent their repeated occurrence. Neither does a commonly accepted model of bubbles exist that can be generally used to generate policy advice. Our paper contributes to the development of such an understanding, which might eventually help in guiding policymakers. To this end, we develop a theoretical model, show circumstances under which bubbles can occur, and discuss which policy measures can help in their prevention.

We construct a simple workhorse model of a bubble based on the crucial assumption that the potential amount of liquidity in the market is not precisely known. As financial markets become more complex and opaque, the assumption of imprecise information about market size seems very natural. Within a bubble, traders are only willing to invest if they believe that there might be another market participant in the future to whom they can sell the asset at a higher price. As observed by Tirole (1982), if the maximum market liquidity were known, the highest possible price of an asset could be computed by the traders, and by backward induction, no bubble could emerge in the first place.<sup>1</sup>

The second important feature of our model is limited liability. In particular, we consider investors who delegate investment to fund managers.<sup>2</sup> The model applies, however, directly to more general intermediated finance, such as through banks, investment banks, insurance companies, and private equity firms, as well as to non-intermediated, debt-financed investments. In the absence of a bubble, we find that the risk appetite induced by the limited liability of fund managers pushes asset

---

<sup>1</sup>Also, Santos and Woodford (1997) show that the conditions for the existence of bubbles are very restrictive if one assumes a fixed number of households that participate in the asset market and own finite aggregate endowments. Tirole (1985) analyzes “rational” bubbles that are possible in overlapping-generations models if the real interest rate is smaller than the growth rate of the economy. The present paper does not require this assumption. Note that a stochastic growth rate would induce an uncertain future market size as is assumed here.

<sup>2</sup>According to the OECD database on institutional investors’ assets, in 2007 institutional investors in the U.S. managed assets worth 211.2% of GDP, showing their prominent role in investment decisions. Furthermore, their size has grown steadily over the last decade, with a yearly average growth rate of 6.6% from 1995-2005 within the OECD(17) area (see Gonnard, Kim, and Ynesta, 2008).

prices above their fundamental values (as already shown by Allen and Gale, 2000); because of limited liability in case of a low or zero return, the fund manager can increase her expected payoff by engaging in riskier assets. Equilibrium asset prices are therefore driven above fundamentals, but in a static way. These price deviations are not induced by expectations and are not subject to sudden corrections (bursts).

Adding the assumption of uncertain market liquidity extends the space of possible price paths drastically. Combined with a high-powered incentive scheme, an expectations-induced bubble with a dynamic price path may emerge. In such a bubble, traders are aware that they are in a bubble.<sup>3</sup> A high price reduces the probability that current asset holders will find future buyers at a yet higher price. Given this increased risk, they demand a higher expected gain from the asset. This accelerator mechanism drives prices up over time until the bubble collapses because either the previously unknown maximum market liquidity is reached or the underlying fundamental breaks down (e. g., due to bankruptcy of the issuing firm).<sup>4</sup>

Importantly, the model allows for bubbles in some parameter ranges, but not in others. The above-explained feedback between higher prices and an increased fragility of the bubble may not have a fixed point right away or at some future date. In either case, there is no possible price path, and the bubble is unfeasible in the first place. Depending on the interaction of limited liability, uncertainty about the market size, riskiness of the asset, and the risk-free interest rate, the prerequisites for bubbles can be fulfilled or not.<sup>5</sup> We can thus discuss policies to prevent bubbles. This is

---

<sup>3</sup>Referring to the dot-com bubble, Brunnermeier and Nagel (2004) provide evidence that hedge funds were riding the bubble, a result similar to a previous finding by Wermers (1999). They relate this to, among others, a short-term horizon of the managers. This is in line with our model.

<sup>4</sup>Note that, in this model, the existence of a bubble requires neither heterogeneous traders nor asymmetric information among them. This stands in contrast to a series of existing papers on bubbles. In Allen, Morris, and Postlewaite (1993), private information can drive a price above its fundamental value. Scheinkman and Xiong (2003) and Bolton, Scheinkman, and Xiong (2006) assume that buyers of an asset hope to sell it to overoptimistic agents in the next period. This is only possible in the case of heterogeneous beliefs. In Hong, Scheinkman, and Xiong (2006), the presence of overoptimistic agents and short-sale constraints create bubbles, where prices drop after an increase in asset float. Allen and Gorton (1993) show that asymmetry of information between investors and heterogeneous managers can lead to deviations of prices from fundamentals if liquidity needs are stochastic. The model of Brunnermeier and Abreu (2003) relies on dispersed opinions that, in combination with coordination failure, can trigger bubbles. Also, in Pástor and Veronesi (2006, 2009), there is initial noisy information and learning over time, leading to stock price behavior that can be confused with a bubble. Froot, Scharfstein, and Stein (1992) analyze which information can influence trading, potentially leading to herding equilibria. In DeLong, Shleifer, Summers, and Waldmann (1990), rational traders' behavior is influenced by noise traders, who follow positive-feedback strategies. Allen, Morris, and Shin (2006) highlight the role of higher-order expectations if traders have asymmetric information. Liquidity needs of firms that are not fully met by uninformed investors can lead to bubbles in Farhi and Tirole (2010). In Plantin and Shin (2010), not all traders have market access at a given point in time.

<sup>5</sup>Taking the housing bubble as an example, we find that all conditions that are favorable for the

particularly important because, in the model, bubbles harm the welfare of market participants. One of the widely discussed possible policy measures is a reduction of bonuses. We find that a system that reduces bonus payments can actually backfire and increase the parameter range where bubbles are feasible. Only a cap on bonuses, if it is small enough, can effectively prevent the emergence of bubbles. Similarly, a financial transaction (Tobin) tax enlarges the parameter range where bubbles are possible, if levied on all forms of financial assets. On the other hand, if imposed on risky assets only, it prevents the emergence of bubbles. Also, a monetary policy rule that takes asset price inflation into account can render bubbles impossible. Finally, mandatory long-term compensation and/or capital requirements fulfill the same purpose.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 2.2 develops a steady-state (rational expectations) equilibrium price process. Section 3 constructs a simple example of a non-steady-state (rational expectations) equilibrium price process, which we call a bubble. We give a necessary and sufficient condition for the existence of such example bubbles. In section 4, we show that this very condition is necessary and sufficient for the existence of bubbles in general. The condition lends itself to basic policy analysis, which is done in section 5 by discussing several policy measures. In section 6, we show that bubbles reduce welfare. While all other sections take the managers' compensation scheme as given, we consider one (of possibly many) ways to endogenize bonus payments in section 7. Section 8 concludes. All proofs are in the appendix.

## 2 The Model

### 2.1 Setup

Consider an infinite horizon economy with overlapping generations of two types of agents, investors and fund managers.<sup>6</sup> In period  $t$ , a continuum of measure  $N$  investors is born, each with an initial endowment of 1 dollar.  $N$  is stochastic but fixed over time. Market participants know the distribution  $F(N)$  and density  $f(N)$ , but not the actual realization. Investors consume and die in the next period,

---

emergence of bubbles were fulfilled. Low interest rates prevailed for a long period, while increasingly international financial flows and more complex financial instruments obscured the potential market liquidity. Furthermore, the 2004 decision of the Securities and Exchange Commission to allow the large investment banks to take on more debt increased their limited liability and increased uncertainty about the maximum market liquidity even further.

<sup>6</sup>Like Allen and Gale (1997), we use the OLG structure as a metaphor for other sources of non-modeled market imperfections, such as heterogeneous liquidity preferences.

$t + 1$ . They cannot directly participate in the financial market. However, there is a continuum of fund managers (short: managers) whom investors employ to invest in bonds or stocks. Because the number of managers is assumed to be unlimited, an investor will always find a manager to handle her wealth. Each manager can handle the funds of one investor only. The manager is compensated by a linear scheme with limited liability. Her compensation can consist of a success-dependent bonus and a base salary  $S$ . For earning a gross yield  $y$ , she receives  $\max\{\alpha(y - \beta); 0\} + S$ , with  $S \geq 0$ ,  $\alpha \in [0; 1]$ , and  $\beta \geq 0$ .<sup>7</sup> The parameters are treated as exogenous for now and will be endogenized in section 7. Note that this formulation encompasses different setups. For example, debt-financed, non-intermediated investments would correspond to  $\alpha = 1$  and  $\beta$  set to the debt repayment.<sup>8</sup>

There are two assets, a safe asset (short: storage) of unlimited supply and a single risky asset (short: asset) of volume 1. The safe asset bears a net interest of  $r$ . The risky asset can be interpreted as shares of a firm. This firm pays dividends of  $d$  in each period.<sup>9</sup> However, in each period, there is a probability  $1 - q$  that the firm will go bankrupt and cease to pay dividends forever. Hence, the time of bankruptcy is determined by a Poisson process. The risky asset is traded on a competitive market in each period. Its price follows an endogenous time-discrete stochastic process  $\{\tilde{p}_t\}_{t \geq 0}$ .

## 2.2 The Steady-State Price

Consider the following simple stochastic process  $\{\tilde{p}_t\}_{t \geq 0}$ . The price of the asset is a constant,  $\tilde{p}_t = \bar{p}$ . Only if the underlying firm goes bankrupt (with probability  $1 - q$ ) and cash ceases to flow, the price drops to  $\tilde{p}_t = 0$ . Hence, the price follows a very simple binomial process with  $\Pr_t\{\tilde{p}_{t+1} = \bar{p} | p_t = \bar{p}\} = q$ . Zero is an absorbing state. Let us derive the price  $\bar{p}$  for which this process is a rational expectations equilibrium.

In a market equilibrium, prices must be such that the managers' compensation is the same for storage and for the risky asset. If the manager opts for storage, her compensation is  $\max\{0; \alpha(1 + r - \beta)\} + S = \alpha(1 + r - \beta) + S$ .<sup>10</sup> If the manager buys

---

<sup>7</sup>As a limiting case of the model,  $\beta = 0$  coincides with unlimited liability.

<sup>8</sup>Contracts in which a manager has to surpass a hurdle rate to receive a bonus in addition to her base salary are frequently used in the literature. E. g., using the return to a benchmark portfolio as the hurdle rate, Ou-Yang (2003) and Dybvig, Farnsworth, and Carpenter (2010) show that this kind of contract is optimal. Bernanke, Gertler, and Gilchrist (1999) use a similar contract with  $\alpha = 1$  and  $S = 0$ . They build on Townsend (1979), who derives the functional form of this contract as optimal in a setting with costly state verification.

<sup>9</sup>One may also interpret the asset as real estate. If the house is used as rental property,  $d$  is the rent per period.

<sup>10</sup>We assume for now that  $\beta \leq 1 + r$ . In section 7, this assumption is endogenized.



shares of the firm at a price  $p_t = \bar{p}$ , she benefits from the dividend with probability  $q$ . She thus earns  $d/p_t$  with probability  $q$ . If the firm does not pay a dividend, the price drops to zero. Otherwise, the price remains at  $\tilde{p}_{t+1} = \bar{p}$ , and the manager additionally gets  $p_{t+1}/p_t = \bar{p}/\bar{p} = 1$  from selling the asset. This stochastic process is depicted in figure 1.

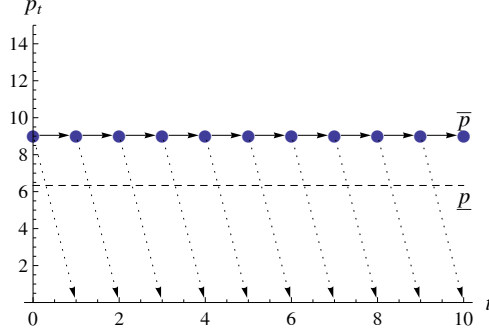


Figure 1: The Price Process in Steady State.

Parameters are  $\gamma = 2$ ,  $\beta = 0.9$ ,  $q = 95\%$ ,  $d = 1$ , and  $r = 10\%$ .

In steady state, a manager's expected compensation on date  $t$  is

$$E_t \max \left\{ 0; \alpha \left( \frac{\tilde{p}_{t+1}}{p_t} + \frac{d}{p_t} - \beta \right) \right\} + S = q \alpha \left( \frac{\bar{p} + d}{\bar{p}} - \beta \right) + S \quad (1)$$

The market must clear, so managers must be indifferent between storage and the risky investment,

$$\alpha (1 + r - \beta) + S = q \alpha \left( \frac{\bar{p} + d}{\bar{p}} - \beta \right) + S, \quad (2)$$

$$\Rightarrow \bar{p} = \frac{d q}{(1 - \beta)(1 - q) + r}. \quad (3)$$

The *steady-state price*  $\bar{p}$  price depends on the compensation scheme ( $\beta$ ). The *fundamental value* obtains as the market equilibrium, in the absence of delegation,

$$\underline{p} := \frac{d q}{1 - q + r}. \quad (4)$$

Hence, only if  $\beta = 0$  (no delegation or unlimited liability) or if  $q = 1$  (no risk), fundamental value and steady-state price are equal,  $\underline{p} = \bar{p}$ . The effect that managers with limited liability push prices of risky assets above their fundamental levels has been analyzed before by Allen and Gale (2000). We get the following comparative statics.

**Remark 1** *The ratio between steady-state price and fundamental value  $\bar{p}/\underline{p}$  is higher for a low risk-free rate  $r$ , high fundamental risk (large  $q$ ), and high-powered compensation packages (high  $\beta$ ).*

Let us make one important clarification. In the numerical example for figure 1, the fundamental value is  $\underline{p} = 6.33$ , but the steady-state price is  $\bar{p} = 9.05$ . This price deviation is due to the limited liability of managers. However, it is a *static* deviation, which is driven by fundamentals ( $q$ ,  $d$ , and  $r$ ) and the managers' compensation package ( $\beta$  and  $\alpha$ , where  $\alpha$  is irrelevant). The price deviation is hence driven by future risk ( $q$ ) and dividends ( $d$ ), but *not* by fund managers' expectations about future price developments. The deviation is constant over time and cannot burst, so its existence is less interesting from a financial stability perspective. Nevertheless, this deviation can magnify price movements resulting from, e. g., changing dividend payments. By contrast, the bubble described in the following section is dynamic by nature. It can be sustained only if the price is expected to continue increasing in the future. Increasingly large price deviations will be fueled by the *expectation* that future managers will buy at an even higher price. A bubble grows dynamically, and it can burst at any time.

### 3 An Example of a Bubble

Consider now a situation where the price  $p_t$  is above the steady-state price  $\bar{p}$  at some date  $t$ . The only conceivable reason to buy is that managers expect the price to rise even further, at least with some probability. Otherwise, it would be a dominant strategy for managers to store rather than to invest in the asset. We will call this expectations-driven price deviation a *bubble*. In this section, we discuss the existence of bubbles under two specific assumptions, one concerning possible price paths  $\{\tilde{p}_t\}_t$  and one concerning the distribution of liquidity  $F(N)$ . Both assumptions will be generalized in section 4.

First, let us concentrate on a *trinomial* process with

$$\tilde{p}_{t+1} = \begin{cases} 0, & \text{with probability } 1 - q \\ \bar{p}, & \text{with probability } q - Q_t \\ p_{t+1}, & \text{with probability } Q_t \end{cases} \quad (5)$$

with  $Q_t \leq q$ .<sup>11</sup> All variables  $\{p_t, Q_t\}_t$  will be determined endogenously. For our purposes, trinomial processes are the simplest ones, allowing for a fundamental default of the firm (case one), a bursting of the bubble (case two), and the continuation of the bubble (case three).

---

<sup>11</sup>Note the notational difference between  $\tilde{p}_{t+1}$  and  $p_{t+1}$ .  $\tilde{p}_{t+1}$  is the stochastic price at date  $t + 1$  that can assume three different values.  $p_{t+1} > p_t$  is the largest of these realizations.

**Assumption 1** For now, consider only trinomial price processes, as in (5).

Second, let us concentrate on a parameterized version of  $F(N)$ . To be concrete, we assume that  $\ln N$  is exponentially distributed; thus,  $F(N) = 1 - e^{-\gamma(\ln N - \ln N_0)}$  for some positive constant  $N_0$ , and  $F(N) = 1 - (N/N_0)^{-\gamma}$ . Here,  $N_0$  is a lower bound on the number of investors. Remember that  $N$  is fixed over time.

**Assumption 2** For now, assume that  $\ln N$  is exponentially distributed.

Because each investor possesses 1 dollar, an asset's price  $p$  can never exceed  $N$ . Hence,  $N$  can be interpreted as an unknown upper bound on liquidity in the market. The smaller the value of  $\gamma$ , the more uncertainty exists about the number of investors, i.e., the more uncertain is the liquidity potential of the market. For  $\gamma \rightarrow \infty$ , we get the limiting case of a known number of investors. The following figure 2 shows the distributions and density functions for  $N_0 = 20$  and shape parameters  $\gamma = 2$  (solid) and  $\gamma = 4$  (dashed).

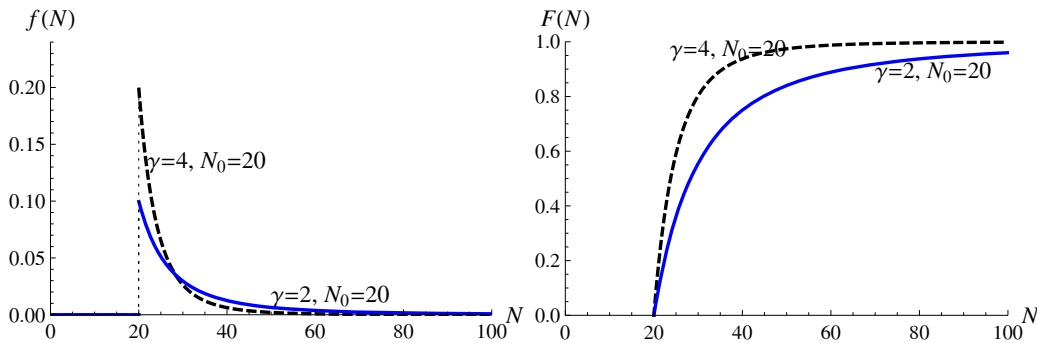


Figure 2: Examples of Density and Distribution Functions

Now we turn to the discussion of a bubble path. A possible price process is depicted in figure 3. The process starts at some price  $p_0 > \bar{p}$ ; the resulting bubble can then grow further and further,  $p_0 < p_1 < p_2 < \dots$ . For a given price increase from  $p_t$  to  $p_{t+1}$ , more liquidity will be absorbed by the market in  $t + 1$  than in  $t$ . As a consequence,  $p_{t+1}$  may exceed the liquidity potential  $N$  at some date. In this case, the price hits a ceiling and no more price increases are expected; i.e., managers cannot expect to sell the asset at a higher price in the future. Hence, the bubble collapses back to the steady-state price  $\tilde{p}_{t+1} = \bar{p}$ . This ceiling  $N$  is not pictured in the figure as it is unknown. The date at which the bubble bursts is (and must be) unknown, but with certainty the ceiling will be hit at some date.

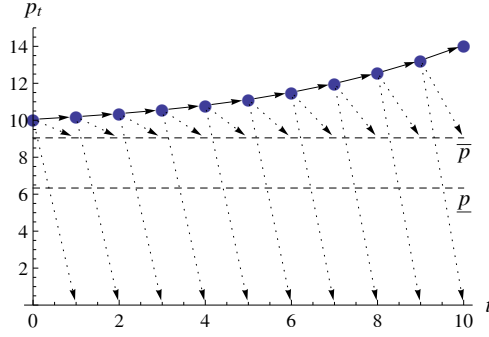


Figure 3: A Trinomial Price Process with a Bubble

As in figure 1, parameters are  $\gamma = 2$ ,  $\beta = 0.9$ ,  $q = 95\%$ ,  $d = 1$ , and  $r = 10\%$ .

Alternatively, if the underlying firm goes bankrupt, the price will drop to  $\tilde{p}_{t+1} = 0$ . Consequently, the conditional probability of a continuation (*non-collapse*) of the bubble is

$$\begin{aligned} Q_t &= q \Pr\{p_{t+1} \leq N | p_t \leq N\} = q \frac{1 - F(p_{t+1})}{1 - F(p_t)} \\ &= q \frac{N_0^\gamma / p_{t+1}^\gamma}{N_0^\gamma / p_t^\gamma} = q p_t^\gamma / p_{t+1}^\gamma, \end{aligned} \quad (6)$$

where  $q$  is the probability that a firm continues to operate and  $Q_t$  is the probability that the firm's asset price continues to rise. The probability that the bubble bursts although the firm is still solvent is thus  $1 - Q_t - (1 - q) = q - Q_t = q(1 - p_t^\gamma / p_{t+1}^\gamma)$ .

If the share price falls because the firm is insolvent, we assume that the price will drop to zero and no dividends will be paid. The bonus to the manager is then  $\alpha \max\{0/p_t + 0/p_t - \beta; 0\} = 0$ . If, alternatively, the share price falls because a bubble bursts, the price will drop to  $\bar{p}$ . For now, let us simply assume that there is no bonus if a bubble bursts. We give a condition and analyze the alternative in the proof of proposition 1 in the appendix. The asset market can only be in equilibrium if a modified version of (2) holds, taking into account the probability of a burst and the increased bonus if the bubble does not burst,

$$\begin{aligned} \alpha(1 + r - \beta) + S &= Q_t \alpha \left( \frac{p_{t+1} + d}{p_t} - \beta \right) + S, \\ &= q \left( \frac{p_t}{p_{t+1}} \right)^\gamma \alpha \left( \frac{p_{t+1} + d}{p_t} - \beta \right) + S, \\ \frac{1 + r - \beta}{q} &= \left( \frac{p_t}{p_{t+1}} \right)^\gamma \left( \frac{p_{t+1}}{p_t} + \frac{d}{p_t} - \beta \right). \end{aligned} \quad (7)$$

Equation (7) implicitly determines a price process in a rational expectations equilibrium. For any given  $p_0 > \bar{p}$ , (7) implicitly defines  $p_1$ , and (6) defines the according

$Q_0$ , so all variables for  $\tilde{p}_1$  in (5) are defined. Then, starting from  $p_1$  in a next step, (7) and (6) define  $p_2$  and  $Q_1$ , so  $\tilde{p}_2$  is defined. Following this procedure gives the complete process recursively. One such process is shown in figure 3.

However, equation (7) does not necessarily yield a solution for any set of parameters. As discussed above, the higher the potential future price  $p_{t+1}$ , the more likely it is that the maximum liquidity  $N$  will be hit and the bubble will burst. The more likely a bursting of the bubble is, however, the larger the expected price increase must be to compensate managers for the risk they face. A feedback multiplier effect evolves, which does not necessarily reach an equilibrium price  $p_{t+1}$  for all  $p_t$ . This is the case if there is a  $\hat{p}$  above which potential future price increases cannot compensate for the accompanying higher risk of a burst. Because all market participants can calculate the date  $t$  at which this  $\hat{p}$  is reached, if it exists, a bubble will burst with certainty at some date  $t + 1$ , i. e.,  $Q_t = 0$ . If the bubble cannot be sustained at a date  $t + 1$ , managers will anticipate this, and a backward induction argument shows that the bubble will not be sustainable right from the start. An example is given in figure 4. At date 7, the price has risen too high, i. e., above the dashed line representing  $\hat{p}$ , and the hypothetical bubble can no longer be sustained. Consequently, the according initial price  $p_0$  cannot be part of a rational expectations equilibrium process in the first place.

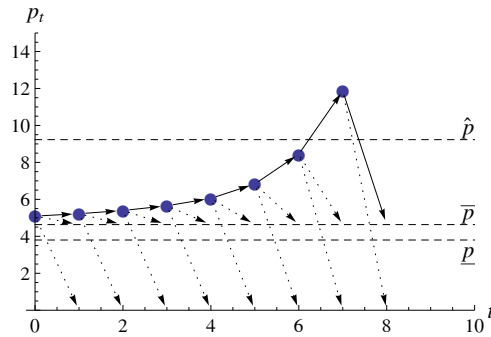


Figure 4: A Trinomial Price Process with a Non-sustainable Bubble

Parameters are  $\gamma = 2$ ,  $\beta = 0.9$ ,  $q = 95\%$ , and  $d = 1$ , but now  $r = 20\%$ . Note that this hypothetical bubble bursts with certainty after date 7, so it cannot emerge in the first place.

We are interested in conditions under which a bubble can or cannot be sustained. In order to be sustainable, the implicit equation (7) must have a solution at any date  $t$ , or equivalently, for any initial price  $p_t$ . Rewriting (7) by defining the auxiliary variable  $\phi_t = p_{t+1}/p_t$  as the relative price increase yields

$$\phi_t^\gamma \frac{1 + r - \beta}{q} = \phi_t + \frac{d}{p_t} - \beta \quad (8)$$

The left-hand side of the equation is independent of  $p_t$  (it only depends on its ratio to  $p_{t+1}$ ), but the right-hand side depends on the starting point  $p_t$ . Figure 5 shows the left-hand side (thick blue curve) and the right-hand side for two starting prices,  $p_t = \bar{p}$  (dashed black line) and  $p_t = \infty$  (thin black line), where both sides were multiplied by  $q$ . First, consider  $p_t = \bar{p}$ . From the figure, one can see that the intersection with the thick curve is at  $\phi_t = 1$ , which implies that  $p_{t+1} = \phi_t p_t = p_t$ , so there is no price increase. Starting with  $p_t = \bar{p}$ , we are of course in steady state, and the price does not change over time. This steady state is a limiting case.

If the initial price is slightly above  $\bar{p}$  due to higher expectations, the straight line shifts downward, implying that it intersects with the curve at some  $\phi_t > 1$ . In the next period, the price will be higher still, so the intersection  $\phi_{t+1}$  will be even higher. A bubble emerges, and the growth rate  $\phi_t = p_{t+1}/p_t$  increases with time. For  $\lim p_t \rightarrow \infty$ , the limiting line  $q(\phi - \beta)$  is reached (solid line). Because the intersection point moves right as  $p_t$  increases, the bubble becomes less and less stable; the probability of a burst,  $1 - Q_t = 1 - q/\phi_t$  increases.

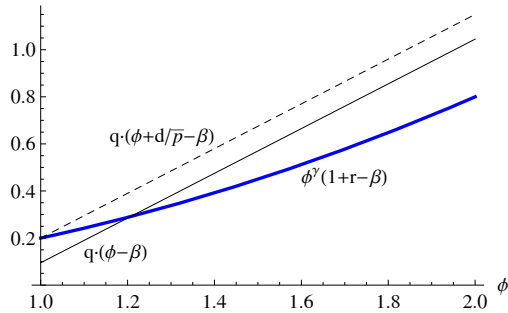


Figure 5: Possibility of a Bubble

As in figure 3, parameters are  $\gamma = 2$ ,  $\beta = 0.9$ ,  $q = 95\%$ ,  $d = 1$ , and  $r = 10\%$ .

**Remark 2** *In a bubble process, the relative price increase  $\phi_t = p_{t+1}/p_t$  grows over time and  $Q_t$  falls over time, so the bubble becomes less stable.*

In figure 4, we saw a hypothetical example of a bubble that was not feasible. Plotting the left and right sides of equation (8) for this set of parameters yields figure 6. Here, as the asset price increases and the straight line moves downward, at some point an intersection between the line and the bold curve ceases to exist. As a consequence, the asset price cannot rise without bounds. However, because of the upper bound, a backward induction argument applies, and no rational price deviation can exist in the first place.

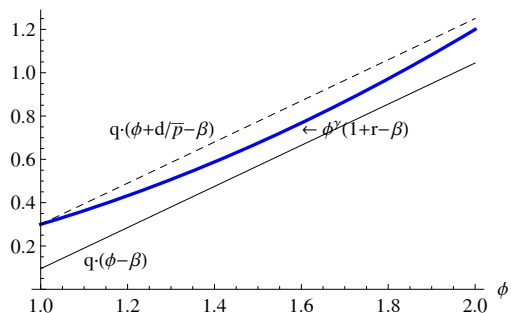


Figure 6: Non-Existence of Bubbles

Like in figure 4, parameters are  $\gamma = 2$ ,  $\beta = 0.9$ ,  $q = 95\%$ ,  $d = 1$ , and  $r = 20\%$ .

As a consequence, in order to show that a bubble can be sustained in a market, it suffices to consider large prices  $p_t$ . In the limit  $p_t \rightarrow \infty$ , equation (8) simplifies to

$$\phi^\gamma (1 + r - \beta) = q(\phi - \beta). \quad (9)$$

The equation does not depend on time, so we have dropped the index  $t$ . If (9) has a solution for  $\phi > 1$ , the corresponding market can sustain a bubble.<sup>12</sup> This implies that for arbitrarily high prices  $p_t$ , there is always a price  $p_{t+1}$  that is high enough to make fund managers buy at date  $t$ . If (9) does not have a solution for  $\phi$ , then there exists a price  $\hat{p}$  beyond which no further increase is impossible. Nobody will buy, and the bubble will burst. Hence, using backward induction, the bubble cannot begin to form at date  $t = 0$ . The only possible initial price is then  $p_0 = \bar{p}$ .

**Existence of Bubble Processes.** The above numerical examples in figures 3 and 4 seem to suggest that lower interest rate levels support bubbles, whereas higher interest rates can puncture a bubble. Reassuringly, this is in line with traditional intuitions about bubbles.

Let us now analyze more generally conditions under which bubble processes can exist. In figure 5, one can see that the solution ceases to exist if the thin black line no longer intersects with the thick blue curve, like in figure 6. A general condition is given in the following proposition.

---

<sup>12</sup>There are at most two solutions to (9) with  $\phi \geq 1$  (values of  $\phi < 1$  would stand for bubbles with falling prices and, formally, negative probabilities of a burst). We do not consider the high solution in the following because the corresponding equilibrium is unstable.

**Proposition 1** *Under assumptions 1 and 2, in a rational expectations equilibrium, a price process can exhibit a trinomial bubble if and only if  $\gamma < q/(1 + r - \beta)$  and*

$$\gamma^\gamma \left( \frac{\beta}{\gamma - 1} \right)^{\gamma-1} \leq \frac{q}{1 + r - \beta}. \quad (10)$$

Figure 7 illustrates the proposition for the case  $\gamma = 2$ .

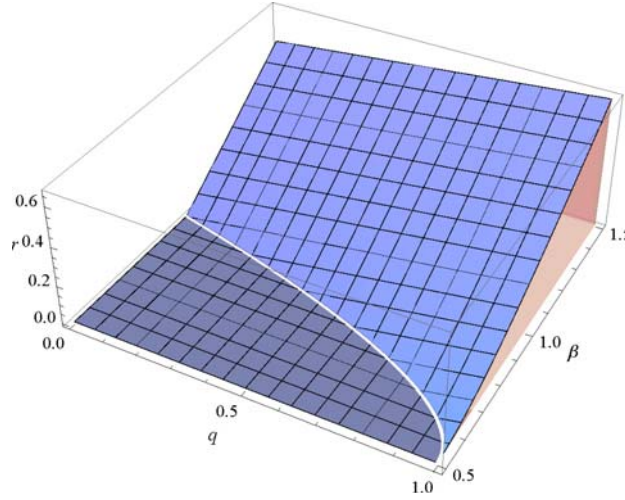


Figure 7: Parameter Range where Bubbles are Possible

Here,  $\gamma = 2$ . For parameters *below* the surface, bubbles are possible.

**Remark 3** *Bubbles tend to be possible for a low risk-free rate  $r$ , low fundamental risk (large  $q$ ), large uncertainty about liquidity  $\gamma$ , and high-powered compensation packages (high  $\beta$ ).*

Remember that a higher risk of a bubble bursting implies a larger potential price increase to compensate managers and that a larger potential price increase makes bursting more likely. If and only if this problem has a fixed point at all times, a bubble can emerge. If risk-free rates are higher, storage becomes more attractive to managers; they need to be compensated by a larger potential price increase of the risky asset in order to hold it. But this makes a burst yet more likely, which inhibits the convergence to a new fixed point. Hence, for a larger risk-free rate  $r$ , bubbles may cease to be possible. This is in line with the intuitions that central banks can puncture bubbles by increasing interest rates and that bubbles are especially likely to exist if interest rates are low. Furthermore, bubbles can exist especially if  $q$  is high; that is, if the underlying asset is rather safe, which decreases the likelihood of a burst. The parameter  $\gamma$  captures the uncertainty in the market. The smaller the



value of  $\gamma$ , the larger are the mean and variance of the distribution, and the more uncertain is the potential market size. The parameter  $N_0$  does not appear in the analysis, which shows that, for the existence of a bubble, only the shape of the upper tail matters; bubbles tend to exist for smaller values of  $\gamma$ . In the extreme case of  $\gamma \rightarrow 1$ , the expected market size becomes infinite, and  $\gamma^\gamma (\beta/(\gamma-1))^{\gamma-1} \rightarrow 1$ . Hence, a bubble can emerge if  $q > 1 + r - \beta$ . On the other hand, if  $\gamma \rightarrow \infty$ , the market size is almost certainly  $N_0$ , and a bubble can never be sustained independent of the sizes of other parameters. This is the traditional backward induction argument of Tirole (1982). Finally, the parameter  $\beta$  describes how steep the incentive schemes of managers are. The larger the value of  $\beta$ , the later the bonus payments to the manager are activated, the stronger the power of the contract is, and the more prominent the effect of the limited liability of the manager becomes. Hence, we have the result that the emergence of bubbles may become possible in the context of a high-powered compensation scheme for managers.

Note again the difference between the *dynamic* price deviation from the steady-state price in this section and the *static* deviation of the steady-state price from the fundamental value of section 2.2. Table 1 summarizes comparative statics for both. An upward arrow means that the *static* deviation (left column) becomes larger or that the necessary condition for a bubble to emerge is met for a larger range of all other parameters (right column).

	Static deviation (Remark 1)	Bubble condition (Remark 3)
$\gamma$	—	↘
$r$	↘	↘
$\beta$	↗	↗
$q$	↘	↗
$d$	— (but multiplier)	—

Table 1: Effects of Variations of Parameters

There is one major difference between the two columns. A static deviation is larger for inherently risky assets, but bubbles tend to emerge for inherently safe assets. Note one subtle but important difference between the inherent and the financial risk of an asset. To give an example, building a house may be inherently safer than buying stock in a firm. But taking into account financial risk, a house may be a riskier investment, especially if it is built during a bubble. Our model distinguishes between these notions of risk. Inherent risk is captured by  $1 - q$ , the risk of failure of the underlying asset. Additional financial risk can occur if movements in fundamentals are strongly magnified because the asset price deviates from the fundamental value in a static way or because condition (10) holds and a bubble can form. Interestingly,

the two sources of financial risk react similarly to most parameter changes, but their reactions with respect to the underlying risk  $1 - q$  are directly opposed.

This fine differentiation suggests different explanations for the two most prominent bubbles in recent decades. Real estate and mortgages are inherently safe, so according to proposition 1, a speculative bubble on these assets should be feasible. In this sense, the theory matches the recent real estate bubble.<sup>13</sup> On the other hand, dot-com firms are inherently risky. According to proposition 1, a speculative bubble may be impossible, but the static deviation above potentially changing fundamentals will be especially large. As a consequence, following our theory, the bursting of the dot-com “bubble” may have been the correction of expectations, bloated by a large multiplier.<sup>14</sup>

**The Half Life of a Bubble.** Given all of the above discussion, we can calculate the half life of a trinomial bubble. Note first that the firm itself can default. In each period, it survives with probability  $q$ ; hence, after  $t$  periods, it survives with probability  $q^t$ . The half life  $T$  is the period after which firms have collapsed with probability 50%, so  $q^T = 1/2$  and thus  $T = -\ln 2 / \ln q$ . Given that a bubble can burst for two reasons, one fundamental (the firm defaults) and one financial (market liquidity is exhausted), the half life of a bubble will always fall short of  $T$ . The conditional bursting probability increases over time, and it has no closed-form solution. However, we can derive comparative statics for a bubble that is fully developed, thus for large  $p_t$ . For this limit,  $\phi = p_{t+1}/p_t$  is implicitly defined by (9), and  $Q = q/\phi^\gamma$ , so the half time  $T'$  of the bubble is given by

$$T' = -\frac{\ln 2}{\ln Q} = -\frac{\ln 2}{\ln q - \gamma \ln \phi}. \quad (11)$$

Ceteris paribus, as  $\phi$  increases, the half time decreases. In calculating the effects of parameter changes on the half life of a bubble, we find that the same changes that can impede a bubble’s formation also shorten the half life of a bubble.

**Remark 4**  $dT'/dr < 0$ ;  $dT'/d\beta > 0$ ;  $dT'/dq > 0$ ; and  $dT'/d\gamma < 0$ .

---

<sup>13</sup>As argued in footnote 5, the development of the housing bubble was also promoted by the constellation of the remaining parameters, i. e., low interest rates, opaque financial markets, and the 2004 decision of the Securities and Exchange Commission to allow the large investment banks to take on more debt, which corresponds to a higher  $\beta$  (see section 5.4). Allen and Gale (2000), among others, also point to an empirical connection between financial liberalization, credit expansion, and bubble emergence.

<sup>14</sup>Pástor and Veronesi (2009) model the learning process about the productivity of new technologies and apply it to the introduction of railroad and internet technologies. The mentioned multiplier is then an amplification of movements generated by learning.

## 4 Bubbles in General

This section delivers the main results of the paper. We have already shown that if liquidity  $\ln N$  is exponentially distributed (assumption 2), a trinomial process (assumption 1) can develop a bubble if and only if condition (10) holds. More formally, if (10) holds, there are multiple rational expectations equilibria within the class of trinomial processes; otherwise, the equilibrium is unique.

This section shows that the condition is more general. First, and most importantly, we can generalize assumption 1. Hence, if (10) is not satisfied, the steady-state equilibrium is unique, without further qualifications. No bubbly equilibria are possible, whether trinomial or of any other shape. Second, we can also generalize assumption 2 about the probability distribution of liquidity.

### 4.1 Generalizing Assumption 1

We have argued that a very special kind of bubble process, the trinomial bubble, exists if and only if (10) holds. We now make this result more general by showing that if (10) fails to hold, the only rational expectations equilibrium process is the non-bubble process with price  $\bar{p}$ . The argument is intuitive. A trinomial bubble does not exist if the thick curve and the thin straight line in figure 6 do not intersect, i. e., if there is no solution for  $\phi$ , and hence prices in the bubble eventually increase too quickly to be sustainable. The point in the graph where the line and the curve are closest corresponds to the price increase with the lowest ratio of risk (of a bubble burst, related to  $\phi^\gamma$ ) to potential gains (of a price increase, related to  $\phi$ ). Concentrating all probability mass on this point maximizes the attractiveness of an investment in the risky asset, thereby creating favorable conditions for the emergence of bubbles. Distributing probability mass to other price increases, i. e., deviating from the assumption of a trinomial price process, reduces the willingness of managers to invest in the risky asset and can therefore eliminate the possibility of bubbles. Thus if, for a given set of parameters, not even a trinomial bubble path exists, no bubble can exist at all.

**Proposition 2** *Under assumption 2, in a rational expectations equilibrium, a price process can exhibit a general bubble if and only if  $\gamma < q/(1 + r - \beta)$  and*

$$\gamma^\gamma \left( \frac{\beta}{\gamma - 1} \right)^{\gamma-1} \leq \frac{q}{1 + r - \beta}, \quad (12)$$

*hence if (10) holds.*

## 4.2 Generalizing Assumption 2

Up to now, we have assumed that  $\ln N$  is exponentially distributed. In this subsection, we will derive a bubble condition for general distributions of the liquidity constraint  $N$ . However, the bubble condition (10) used the distribution-specific parameter  $\gamma$ . Hence, we need to find out what the essence of  $\gamma$  is, i. e., which property of the distribution is captured by  $\gamma$ . Most importantly,  $\gamma$  describes the fatness of the tail of the distribution, thereby measuring the uncertainty about the maximum market liquidity.

**Proposition 3** *For a general distribution  $F(N)$  with density  $f(N)$ , define*

$$\gamma := \min \left\{ n : \lim_{x \rightarrow \infty} x^n f(x) \neq 0 \right\} - 1. \quad (13)$$

*Then, in a rational expectations equilibrium, a price process can exhibit a general bubble if and only if  $\gamma < q/(1+r-\beta)$  and*

$$\gamma^\gamma \left( \frac{\beta}{\gamma-1} \right)^{\gamma-1} \leq \frac{q}{1+r-\beta}, \quad (14)$$

*hence, if (10) holds. If  $\gamma = \infty$ , bubbles are impossible.*

We now discuss some intuitions for equation (13). We already know that the possibility of a bubble depends on the shape of the distribution  $f(N)$  for large  $N$ , thus for large prices. When  $\ln N$  is exponentially distributed, the parameter  $\gamma$  actually gives the order of convergence of  $F(N) \rightarrow 1$  for  $N \rightarrow \infty$ . It also gives the order if  $f(N) \rightarrow 0$  for  $N \rightarrow \infty$ , less one. Intuitively, for a density function  $f(N)$  of any other shape, the relevant statistic for the potential existence of a bubble is also the order of convergence towards zero, as defined by (13). If  $f(N)$  decays very quickly, the information about an upper liquidity ceiling is rather precise, so a bubble cannot exist. If  $f(N)$  decays slowly, there is always room for further price increases without hitting the liquidity ceiling with high probability, which makes the emergence of bubbles possible. The proof of the proposition in the appendix is a more accurate version of this intuition and also gives a concrete example.

## 5 Policy Measures

In this section, we examine whether certain policy measures that have been suggested in the public debate can prevent the creation of bubbles in our model. Specifically, we look at an asset-price augmented Taylor rule, caps on bonuses, mandatory long-term compensation, and a financial transaction (Tobin) tax.

## 5.1 Monetary Policy

We have already seen that a central bank can puncture a bubble by increasing interest rates. Let us now analyze the impact of an automatic, pre-announced interest rate increase in the case of a bubble, following a Taylor rule that takes asset price inflation into account. Specifically, assume a version of the rule used in Bernanke and Gertler (1999, 2001),

$$r_t = \bar{r} + \psi_\pi (\pi_t - \bar{\pi}) + \psi (p_t/p_{t-1} - \bar{\pi}), \quad (15)$$

where  $\pi_t$  is gross consumer price index (CPI) inflation and  $p_t/p_{t-1}$  asset price inflation of the only risky asset in the economy, as defined above. For simplicity, we neglect the influence of asset price inflation on CPI inflation by setting CPI inflation equal to its target rate  $\bar{\pi}$ , which is itself set to unity.<sup>15</sup> This does not influence our conclusions below. As in the above analysis, in a bubble,  $p_{t+1}/p_t$  converges towards a constant  $\phi$ . Inserting (15) into (9) yields

$$\phi^\gamma (1 + \bar{r} + \psi (\phi - 1) - \beta) = q (\phi - \beta). \quad (16)$$

As for (10), we can derive a condition for parameters  $\bar{r}$ ,  $\psi$ ,  $\beta$ ,  $\gamma$  and  $q$  to determine whether (16) has a solution for  $\phi > 1$ . Unfortunately, the condition is algebraically complex. A bubble equilibrium exists if and only if

$$\begin{aligned} q(\phi - \beta) &\geq \phi^\gamma (1 + \bar{r} + \psi (\phi - 1) - \beta) \text{ with} \\ \phi &= \frac{1}{2\psi\gamma} \left( (1 - \beta - \gamma)(1 - \psi) + \bar{r} + \beta\gamma - \bar{r}\gamma + \beta\gamma\psi \right. \\ &\quad \left. + \sqrt{(\bar{r} + (1 - \beta)(1 - \psi))((1 - \gamma)^2(1 + \bar{r} - \psi) - \beta((1 + \gamma^2)(1 - \psi) - 2\gamma(1 + \psi)))} \right) > 1. \end{aligned}$$

Figure 8 shows parameters  $\bar{r}$  and  $\psi$  for which bubbles can exist for  $\gamma = 2$ ,  $\beta = 0.9$  and  $r = 10\%$ . The figure shows that, to prevent the emergence of bubbles, the central bank can either raise the steady-state interest rate  $\bar{r}$  or threaten to raise interest rates in the future if a bubble should occur by committing to a Taylor rule with a positive  $\psi$ . If the central bank opts for the latter option, it never actually needs to raise interest rates: interest rate increases occur only as a consequence of asset price movements, but because of the credible announcement of this policy (with a sufficiently large  $\psi$ ), asset prices do not rise and bubbles are prevented.<sup>16</sup>

**Remark 5** *Monetary policy that systematically reacts to asset price increases reduces the range of parameters under which bubbles are possible.*

<sup>15</sup>Hence,  $r_t$  is the real interest rate. We implicitly assume that the central bank has at least some impact on short-term real rates, as it is standard in monetary economic theory.

<sup>16</sup>In this respect, the model differs from Bernanke and Gertler (1999, 2001), who show that monetary policy should not react to asset prices based on the assumption of *exogenous* bubbles.

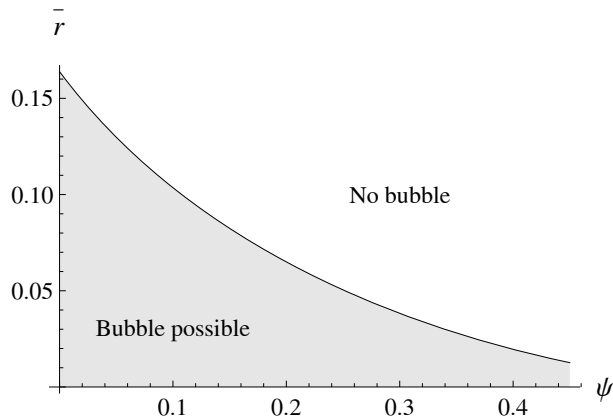


Figure 8: Effects of the Taylor Rule

This argument shows that an augmented Taylor rule could cause fewer distortions than discretionary interest-rate policies do. However, if the central bank cannot differentiate between price movements due to bubbles and changes in the underlying fundamentals (such as the probability of bankruptcy,  $1 - q$ ) or if it is uncertain which assets to monitor, it faces a tradeoff between preventing bubbles and the risk of unnecessarily changing the interest rate in times without bubbles. A thorough examination of this trade-off would require a fully specified DSGE (dynamic stochastic general equilibrium) model, which is beyond the scope of this paper.

## 5.2 Caps on Bonuses

The bonus payment to a manager is  $B = \alpha (\phi_t + d/p_t - \beta)$  if the underlying asset continues to pay off (probability  $q$ ) and if any bubble that exists does not burst (probability  $1 - Q$ ). Absent a bubble, this bonus payment is a constant. Let us first ask whether a potential cap on this bonus would bind early or late in the life of a bubble. In both cases, the bubble would have to burst with probability 1 at some date  $\bar{t}$ , so a backward induction argument would show that the bubble cannot exist in the first place. In terms of the bonus payment,  $\phi_t$  increases over time, but  $d/p_t$  decreases. In the aggregate, due to (8), we have

$$B_t = \alpha (\phi_t + d/p_t - \beta) = \alpha \phi_t^\gamma (1 + r - \beta)/q.$$

Hence, bonuses increase over time in a bubble, and caps on bonus payments become binding in later stages of a bubble. As a consequence, we can concentrate on large prices  $p_t$  so that  $\phi_t$  approaches a constant and the maximum bonus is

$$B = \alpha (\phi - \beta) = \alpha \phi^\gamma (1 + r - \beta)/q.$$

Now assume that the regulator puts a cap  $\bar{B}$  on the bonus. There are two ways in which the regulation can be implemented. *First*, the compensation scheme could be adjusted such that bonuses above  $\bar{B}$  cannot occur, for example by reducing  $\alpha$  or increasing  $\beta$ . However,  $\alpha$  does not have an effect on the existence of bubbles, and an increase in  $\beta$  would favor the emergence of bubbles. Hence, this policy could backfire and render bubbles possible.

*Second*, one could put a cap on bonuses and adjust it to  $\min\{\max\{\alpha((p_{t+1} + d)/p_t - \beta); 0\}; \bar{B}\}$ . Then, the bubble will burst with certainty at some specific point if  $\alpha(\phi - \beta) > \bar{B}$ , hence if  $\phi > \bar{B}/\alpha + \beta$ . Economically speaking, from a certain point onwards, managers' bonuses cannot rise further to compensate them for the still-increasing risk of a bursting bubble. Consequently, for a given compensation scheme with parameters  $\alpha$  and  $\beta$ , a cap on bonus payments  $\bar{B}$  will render a bubble unfeasible if  $\bar{B}/\alpha + \beta < \phi$ , with  $\phi$  implicitly defined by (9).

**Remark 6** *To prohibit the emergence of a bubble by cutting bonuses, raising  $\beta$  is counterproductive, reducing  $\alpha$  is irrelevant, and putting a cap  $\bar{B}$  on bonuses is effective if the cap is low enough.*

The model can indicate which types of assets might need a cap on bonus payments and for which assets the cap must be lowered. First, relatively safe assets (high  $q$ ) tend to develop bubbles, and the price increase  $\phi$  is especially low for safe assets. This implies that managers trading in markets with relatively safe assets (e.g., mortgages, bond markets) should have a ceiling in their bonus contracts that should be relatively low for safer assets. Second, if interest rates  $r$  are high, bubbles can be prevented. The opposite is true for low levels of  $r$ , calling for lower caps in times of low interest rates. Third, bubbles can emerge especially for high benchmarks  $\beta$ , and the limit price increase  $\phi$  is lower for large  $\beta$ . Because  $\bar{B} < \alpha(\phi - \beta)$ , the minimal effective cap  $\bar{B}$  depends negatively on  $\beta$ . With a higher benchmark, the cap must be stricter.

### 5.3 Long-term Compensation

In the recent political discussion, it has often been argued that managers' incentives should be made more sustainable such that they concentrate more on long-term goals and avoid short-termism. The same argument might apply to the fund managers in our model. To analyze this question, let us assume that the manager receives  $\max\{0; \alpha(y - \beta)\}$  as before, but that she is liable with her compensation for potential

future losses. Hence, she will get nothing if the accumulated yield is negative in the next period. In steady state, the market price will then be

$$\alpha(1+r-\beta) = q^2 \alpha((p_t + d)/p_t - \beta),$$

$$p_t = \bar{p} := \frac{dq^2}{(1-\beta)(1-q^2) + r},$$

i. e., smaller than without long-term liability. If a bubble exists, the probability that the bubble does not burst after two periods is

$$Q = q^2 p_t / p_{t+2} = q^2 / \phi^{2\gamma}.$$

As a consequence, the one-period price increase  $\phi$  is determined by

$$\alpha(1+r-\beta) = Q \alpha(\phi - \beta) = q^2 / \phi^{2\gamma} \alpha(\phi - \beta),$$

$$\phi^{2\gamma}(1+r-\beta) = q^2(\phi - \beta).$$

The equation is similar to (9), but  $\gamma$  is substituted by  $2\gamma$ , and  $q$  is substituted by  $q^2$ . Because bubbles exist especially for small  $\gamma$  and large  $q$ , according to proposition 1, we find that long-term liability prevents the formation of bubbles. For an even longer liability period, the effect would be larger.

**Remark 7** *If fund managers are liable for future developments with their bonuses, bubbles can be prevented.*

## 5.4 Capital Requirements

We have already argued that our “fund managers” can be many kinds of financial intermediaries, for example banks. In this case, capital regulation would be the most prominent policy tool. Our model suggests that capital requirements have the effect, among other things, of preventing bubbles. The reason for this effect is straightforward. Assume that our fund manager is a bank that invests a total of one dollar such that aggregate market liquidity remains at  $N\$$ . The bank finances its investment by either own capital or debt, where  $\kappa$  denotes the share of own capital. Previous to investment, the perfectly competitive bank agrees to repay its debtors a specified rate of return of  $\bar{\beta}$ . In a stylized setup, the bank is liable to its debtors only with its total return  $y$  generated in a given period. Hence, the bank’s profit is

$$\max \{0; \kappa y + (1 - \kappa)(y - \bar{\beta})\} = \max \{0; y - (1 - \kappa)\bar{\beta}\},$$

such that  $(1 - \kappa)\bar{\beta}$  can be interpreted as the original parameter  $\beta$  used in the previous sections, setting  $\alpha$  to one and  $S$  to zero. A capital requirement regulation forces the bank to reduce its level of debt, i. e., to increase  $\kappa$ , thereby lowering  $\beta$  of the original model. As we have seen, this can prevent bubbles. The same would be true for margin requirements or ceilings on leverage for investors.

**Remark 8** *Bubbles can be prevented by stricter capital requirements.*



## 5.5 Financial Transaction Tax

There are different possible ways to implement a so-called Tobin tax. In the following, we assume that the tax must be paid by the buyer of an asset. We denote the tax on transactions of the safe asset  $\tau$  and the potentially different tax on the risky asset  $\tau'$ . Under such a tax regime, the no-arbitrage condition (9) changes to

$$\phi^\gamma (1 + r - \beta - \tau) = q(\phi - \beta - \tau'). \quad (17)$$

The modified conditions for the existence of bubbles are then  $\gamma < [q(1 - \tau')]/[1 + r - \beta - \tau]$  and

$$\gamma^\gamma \left( \frac{\beta + \tau'}{\gamma - 1} \right)^{\gamma-1} \leq \frac{q}{1 + r - \beta - \tau}. \quad (18)$$

The derivative of the left-hand side of the above expression with respect to  $\tau'$  is positive, i. e., increasing the tax on transactions of the risky asset can make bubbles impossible. However, the way in which the tax is implemented is important. If it is levied on all financial assets, including the safe one,  $\tau$  equals  $\tau'$  and the derivative of the right-hand side w. r. t. the common tax rate is larger than the derivative of the left-hand side. Hence, in such a case, the possibility of bubbles can actually be created by the Tobin tax.

**Remark 9** *If the financial transaction tax is levied on the risky asset only, bubbles can be prevented. However, placing the tax on the safe and the risky assets alike can make bubbles possible.*

## 6 Welfare

In order to justify any policy measure for the prevention of bubbles, it is necessary to analyze the welfare effect of bubbles. We assume that all agents are risk neutral and have identical utility functions,  $u_t^i = c_{t-1}^i + \rho c_t^i$  for agent  $i$ , who is born on date  $t - 1$ . Consumption of this agent on date  $t$  is denoted by  $c_t^i$ . The discount factor  $\rho$  must satisfy  $1/\rho \leq 1 + r$ ; otherwise, agents would have no incentive to invest in the risk-free asset. As a consequence,  $c_{t-1}^i = 0$ . Taking  $\rho$  also as the inter-generational discount factor, we can write

$$E_0 W = E_0 \sum_{t=0}^{\infty} \sum_i \rho^t u_t^i = \sum_{t=0}^{\infty} \rho^t E_0 C_t,$$

where  $C_t$  is aggregate expected consumption on date  $t$ . Payments between managers and investors in the same generation are mere transfers and do not directly enter

the welfare function. Now, absent a bubble, the price of the asset is always  $\bar{p}$ . Hence, the generation that consumes on date 0 earns  $C_0 = \bar{p}$  from selling the asset. Generation 1 pays  $\bar{p}$  for the asset. Because there are  $N$  investors, each owning one dollar, the aggregate endowment of generation 1 is  $N$ . The investment into the risk-free asset is  $N - \bar{p}$  because  $\bar{p}$  is already spent on the risky asset. With probability  $q$ , generation 1 also gets  $\bar{p}$  from selling the asset plus the dividend  $d$ . Hence, the aggregate expected consumption of generation 1 is

$$E_0 C_1 = q(d + \bar{p}) + (N - \bar{p})(1 + r).$$

Generation 2 buys the asset only with probability  $q$ ; with probability  $1 - q$ , the firm is bankrupt and there is nothing to buy. Hence,

$$E_0 C_2 = q^2(d + \bar{p}) + (N - q\bar{p})(1 + r).$$

The equations for the following generations are similar. Let us now look at the expected consumption in a bubble. For concreteness, consider the trinomial ‘example’ bubble process of section 3. Generation 0 gets  $C'_0 = p_0 > \bar{p}$  from selling the asset. Generation 1 buys the asset at price  $p_0$ , but expects the price to rise to  $p_1$  with probability  $Q_0$ , to fall to  $\bar{p}$  with probability  $q - Q_0$ , and to fall to 0 with probability  $1 - q$ . Hence,

$$E_0 C'_1 = Q_0(d + p_1) + (q - Q_0)\bar{p} + (N - p_0)(1 + r),$$

and so on. Now consider welfare differences,

$$\begin{aligned} C'_0 - C_0 &= p_0 - \bar{p}, \\ E_0(C'_1 - C_1) &= Q_0(p_1 - \bar{p}) - (1 + r)(p_0 - \bar{p}), \\ E_0(C'_2 - C_2) &= Q_1 Q_0(p_2 - \bar{p}) - Q_0(1 + r)(p_1 - \bar{p}), \end{aligned}$$

and so forth. Hence, the aggregate welfare difference amounts to

$$\begin{aligned} E_0 \Delta W &= (p_0 - \bar{p}) + \sum_{t=1}^{\infty} \rho^t \prod_{t'=0}^{t-2} Q_{t'} \left( Q_{t-1}(p_t - \bar{p}) - (1 + r)(p_{t-1} - \bar{p}) \right) \\ &= \sum_{t=0}^{\infty} \rho^t (p_t - \bar{p}) (1 - \rho(1 + r)) \prod_{t'=0}^{t-1} Q_{t'}, \end{aligned}$$

which is non-positive if  $1 + r \geq 1/\rho$ . Consequently, the welfare effect of a bubble is always negative and is zero only in the limiting case of  $1 + r = 1/\rho$ . Besides the effect of shifting consumption across generations, total resources are reduced in a bubble because of the reduced investment in the safe asset—representing a foreign bond, productive constant-returns-to-scale capital or the like.

Alternatively, one can argue in the following way. The payments of the risky asset are not affected if there is a bubble. However, in a bubble, on date  $t$ , the young

generation pays a price  $p_t$  higher than  $\bar{p}$  to the old generation born on date  $t - 1$ . This is simply a transfer of wealth between generations, with two consequences. Due to the higher price  $p_t > \bar{p}$ , the young generation invests less into the safe asset, at an opportunity cost of  $(1 + r)(p_t - \bar{p})$ . The transfer is carried one period forward, so it is discounted. Because the risk-free rate  $1 + r$  exceeds the inverse discount factor  $1/\rho$ , the aggregate welfare effect is negative. Because bubbles always involve prices above  $\bar{p}$ , this argument proves the following proposition.

**Proposition 4** *Assume that (10) holds and  $1 + r > 1/\rho$ . Then, of all equilibria, the steady-state equilibrium is strictly welfare optimal.*

A social planner would set the price of the risky asset to zero. However, this is not a feasible solution in a decentralized equilibrium. Note furthermore that, c. p., risk aversion of the agents would reinforce the negative welfare effects of bubbles due to higher intergenerational consumption volatility. Hence, our analysis with risk-neutral agents gives a lower bound of the welfare effects of bubbles.

## 7 Endogenizing the Compensation Scheme

In the above analysis, the parameters of the compensation scheme for the managers,  $\alpha$ ,  $\beta$ , and  $S$ , are taken as exogenous. These parameters can be endogenized in many different ways, for example by embedding the above bubble model into a model with costly state verification or model hazard. In this section, we demonstrate one very simple way of endogenizing the compensation scheme: by simply assuming that investors are risk averse but managers are not. The parameters  $\alpha$ ,  $\beta$ , and  $S$  are then the solution of an optimal risk-sharing problem. There is one conceptual problem. Bubbles are possible for multiple price paths, all of which are rational expectations equilibria. So, in parameter ranges where bubbles can exist, investors cannot build ex ante expectations about their emergence. One way to bypass this problem is to assume that the investor treats bubbles as zero-probability events. We will find that investors, because of their risk aversion, set a high  $\beta$ , which facilitates the emergence of bubbles and increases the risk for investors. Hence, financial contracting involves an externality. An individually rational contract design can lead to the destabilization of financial markets by creating the prerequisites for bubbles.

The remaining setup is as described in the previous section; i. e., an investor delegates the investment decision to a manager, whose actions she cannot observe. Because there are more managers than investors in the economy, the investor can make take-it-or-leave-it offers to the managers, which maximize the expected profit of the

investor. In doing so, she has to consider the manager's participation constraint. Letting  $y$  again denote the revenue generated by the manager, the expected gross profit of the investor is then<sup>17</sup>

$$E_t \Pi = E_t y - E_t \max \left\{ 0; \alpha (y - \beta) \right\} - S.$$

We restrict the parameter  $\alpha$  to be less than or equal to unity because, in the opposite case, a higher  $y$  can lead to a lower profit of the investor. Put differently, in the extreme, a very high realization of  $y$  could lead to bankruptcy of the investor under  $\alpha > 1$ . The manager will only accept the contract if it fulfills

$$E_t \max \left\{ 0; \alpha (y - \beta) \right\} + S \geq A, \quad (19)$$

where  $A > 0$  is the outside option of the manager (such as academia). Because there is a continuum of managers, the investor will choose  $\alpha$ ,  $\beta$ , and  $S$  such that the manager will be at the limit of her participation constraint. This implies that equation (19) will hold with equality. Inserting this into the above profit function yields  $E_t \Pi = E_t y - A$ . Hence, the investor maximizes her profit by reaping the complete surplus of the manager. The relation between  $S$ ,  $\alpha$ , and  $\beta$  can be seen by rewriting (19) as

$$\begin{aligned} S &= A + \hat{Q} \alpha \beta - \alpha \hat{\phi} \quad \text{with} & (20) \\ \hat{Q} &= \int_{\beta}^{\infty} f(y) dy \quad \text{and} \quad \hat{\phi} = \int_{\beta}^{\infty} y f(y) dy, \end{aligned}$$

where the probability distribution of  $y$  is denoted by  $f(y)$ . The risk-neutral manager is indifferent between values of  $S$ ,  $\alpha$ , and  $\beta$  as long as this equation is fulfilled. The risk-averse investor, however, has an incentive to minimize the variance of her profits in the different states of the economy. To this end, let us rewrite the expected utility of the investor as

$$E_t U(\Pi) = \int_0^{\beta} U(y - S) f(y) dy + \int_{\beta}^{\infty} U(y [1 - \alpha] + \alpha \beta - S) f(y) dy.$$

The investor maximizes this expression subject to (20),  $S \geq 0$ , and  $\alpha \leq 1$ . Because of her risk aversion, she tries to increase the profit in states with a low realization of  $y$  relative to states with a high  $y$ . Therefore, she chooses  $\alpha = 1$ ,  $S = 0$ , and from equation (20), we obtain

$$\beta = \frac{\hat{\phi} - A}{\hat{Q}}. \quad (21)$$

---

<sup>17</sup>By the standard risk-sharing argument, as, e. g., in Allen and Gale (1994), the investment returns must be tranced, and the less risk-averse agent (here, the manager) retains the junior tranche, whereas the more risk-averse agent (here, the investor) takes the senior tranche.

The right-hand side decreases, starting from a large number for  $\beta = 0$ , to minus infinity for  $\beta$  approaching infinity. Hence, a fixed point can be found with  $\hat{Q} = q$  and  $\hat{\phi} = q(1 + r)$ . Equation (21) changes to

$$\beta = \frac{(1 + r)q - A}{q} < 1 + r.$$

Hence, we obtain the following remark.

**Remark 10** *A risk-averse investor chooses a loan contract with  $\beta < 1 + r$ .*

Importantly, this condition does not contradict (10). Consequently, with endogenous compensation contracts, the propositions immediately applies.

## 8 Conclusion

Our model endogenizes two reasons why the price of an asset may deviate from its fundamental value. *First*, as also analyzed by Allen and Gale (2000), fund managers may drive up the price of risky assets because of their limited liability. This effect is larger for riskier assets. The price deviation is *not* driven by expectations and is constant over time; it involves no dynamics. *Second*, a fund manager may be willing to spend more than the fundamental value on an asset because she expects to earn even more when she sells the asset. This price deviation is completely driven by *expectations* and is dynamic, typically involving large, unpredictable abnormal returns until the bubble bursts.

These two stories are in line with anecdotal evidence. During the dot-com bubble (1998–2001), fantasies about the potential of internet firms were exuberant. The asset prices of these firms may have been even more exaggerated due to the limited liability of traders. Hence, the traders’ limited liability let the exuberance appear magnified. When expectations became more realistic, asset prices collapsed because the correction of expectations was again magnified. This complete argument follows from the *first* story because it is especially reasonable for risky assets, like the stock of dot-com firms.

Following the “as long as the music is playing, you’ve got to get up and dance” explanation for the recent U. S. housing bubble, managers bought securities because they assumed they could sell them at a higher price later, driving up prices. This argument follows the *second* story because it is especially reasonable for fundamentally safe assets, like real estate. In both episodes, the circumstances that are, according to our model, favorable for both types of price deviations were fulfilled.

Our model suggests some possible ways to avoid such bubbles. One can increase interest rates, implement a Taylor rule that reacts to asset-price developments, or put a ceiling on bonus payments to fund managers. By virtue of its relative simplicity, the model lends itself to further discussions. For example, one could consider several assets and discuss whether the collapse of a bubble in one market can spread to other markets. One could also plug bubbles into macro models and investigate business cycle and growth effects. Especially after the recent burst of the housing bubble, possible applications seem numerous and relevant.

## Appendix

**Proof of remark 1.** We have

$$\frac{\bar{p}}{\underline{p}} = \frac{(1-q) + r}{(1-\beta)(1-q) + r}.$$

Simple analysis proves the remark. ■

**Proof of proposition 1.** In the exposition in the main text, we have treated only the case in which no bonus is paid if a bubble bursts. Hence, we start the proof of the proposition by giving a condition for this case and analyzing the alternative.

If a bubble bursts, the firm still pays the dividend, so the bonus payment to the manager is

$$\alpha \max \left\{ \frac{d}{p_t} + \frac{\bar{p}}{p_t} - \beta; 0 \right\} = \alpha \max \left\{ \frac{d + \frac{dq}{(1-\beta)(1-q)+r}}{p_t} - \beta; 0 \right\}. \quad (22)$$

This implies that, if the price is only slightly above the steady-state price  $\bar{p}$  (i. e., the bubble is small), the manager will earn a bonus even when the bubble bursts. The corresponding condition is

$$p_t < \hat{p} := \left( d + \frac{dq}{(1-\beta)(1-q)+r} \right) / \beta. \quad (23)$$

Now, if  $p_t$  is less than  $\hat{p}$  such that (23) is satisfied, a modified version of (7) applies. In market equilibrium,

$$\begin{aligned} \alpha(1+r-\beta) + S &= Q_t \alpha \left( (p_{t+1} + d)/p_t - \beta \right) + (q - Q_t) \alpha \left( (\bar{p} + d)/p_t - \beta \right) + S, \\ \frac{1+r-\beta}{q} &= \left( \frac{p_t}{p_{t+1}} \right)^\gamma \frac{p_{t+1}}{p_t} + \left( 1 - \left( \frac{p_t}{p_{t+1}} \right)^\gamma \right) \frac{\bar{p}}{p_t} + \frac{d}{p_t} - \beta. \end{aligned} \quad (24)$$

Again, starting from  $p_t$ , we have an implicit equation for  $p_{t+1}$  in a rational expectations equilibrium. Substituting  $p_{t+1} = \phi_t p_t$ , we obtain

$$\phi_t^\gamma \frac{1+r-\beta}{q} = \phi_t + (\phi_t^\gamma - 1) \frac{\bar{p}}{p_t} + \phi_t^\gamma \left( \frac{d}{p_t} - \beta \right). \quad (25)$$

However, in a bubble, the price  $p_t$  increases over time and eventually exceeds the threshold  $\hat{p}$ . Therefore, to find out whether bubbles are feasible, it suffices to consider the case  $p_t > \hat{p}$ , as we have done in the main text.

We have already argued that the probability that a bubble bursts increases with  $p_t$ . However, because  $p_t$  is an increasing function of  $t$ , a bubble is sustainable if and only if it is sustainable for  $p_t \rightarrow \infty$ . Hence, if (9) has a solution for  $\phi$ , the bubble is sustainable. Now consider the limiting case in which the line  $q(\phi - \beta)$  and the curve  $\phi^\gamma(1+r-\beta)$  only just touch. At the point of contact, the slopes must be equal, so

$$(1+r-\beta)\gamma\phi^{\gamma-1} = q,$$

which implies that the point of contact is  $\phi = \beta\gamma/(\gamma-1)$ . Substituting this solution into (9), we find that the limiting case is reached at

$$\left( \frac{\beta\gamma}{\gamma-1} \right)^\gamma (1+r-\beta) = q \left( \frac{\beta\gamma}{\gamma-1} - \beta \right).$$

Some algebra yields (10). We have checked the conditions under which (9) has a solution. However, in a bubble, prices must increase, so  $\phi > 1$ . Considering the geometry of the problem, this is the case if the *right*-hand side of (9) is steeper in  $\phi$  than the left-hand side is at the point  $\phi = 1$ . Otherwise, the curves would intersect for  $\phi < 1$ . Evaluating the derivatives of the left and right-hand sides of (9) at the point  $\phi = 1$ , we obtain the condition  $\gamma < q/(1+r-\beta)$ . ■

**Proof of remark 3.** There are two conditions that may become increasingly strict or lax. First, condition (10) is satisfied iff

$$(1+r-\beta)\gamma^\gamma \left( \frac{\beta}{\gamma-1} \right)^{\gamma-1} - q \leq 0.$$

The derivative of this term with respect to  $q$  is negative, so the condition is more likely to be satisfied for large  $q$ . The derivative with respect to  $r$  is positive; hence, (10) holds for small  $r$ . The derivative with respect to  $\gamma$  is

$$(1+r-\beta)\gamma^\gamma \left( \frac{\beta}{\gamma-1} \right)^{\gamma-1} \ln \frac{\beta\gamma}{\gamma-1}.$$

Now remember that the point at which the curves touch is  $\phi = \beta \gamma / (\gamma - 1)$ . The above logarithm is therefore positive, and the complete derivative with respect to  $\gamma$  is positive. A larger  $\gamma$  makes bubbles less likely. Finally, the derivative with respect to  $\beta$  is

$$\gamma^\gamma \left( \frac{\beta}{\gamma - 1} \right)^{\gamma-1} \frac{(1+r)(\gamma-1) - \beta\gamma}{\beta}.$$

Again, at the touching point,  $\phi$  must exceed 1, so  $\beta \geq (\gamma - 1)/\gamma$ . For the limiting  $\beta = (\gamma - 1)/\gamma$ , the numerator of the above fraction becomes  $(1+r)(\gamma-1) - \beta\gamma = -r(\gamma-1) < 0$ . Hence, for any  $\beta$  larger than the limiting  $(\gamma - 1)/\gamma$ , the numerator must be negative. Thus, the whole derivative is negative, and a larger  $\beta$  makes bubbles more likely. The second condition,  $\gamma < q/(1+r-\beta)$ , has the same comparative statics. ■

**Proof of remark 4.** We start by discussing  $dT'/dr$ . Because  $\phi$  is implicitly given by (9), let us define  $\Psi = \phi^\gamma (1+r-\beta) - q(\phi-\beta)$ , such that  $\Psi = 0$  at the equilibrium  $\phi$ . Then, the implicit function theorem yields

$$\frac{d\phi}{dr} = - \frac{\partial\Psi/\partial r}{\partial\Psi/\partial\phi} = - \frac{\phi^\gamma}{\gamma\phi^{\gamma-1}(1+r-\beta) - q}.$$

Because  $\partial\Psi/\partial\phi < 0$  and  $\phi^\gamma > 0$ , we know that  $d\phi/dr > 0$ . Now, because in (11),  $dT'/d\phi < 0$ , the half life of a bubble drops as interest rates increase,  $dT'/dr < 0$ .

Now consider  $dT'/d\beta$ . Following the same procedure as above, note that  $\partial\Psi/\partial\beta = q - \phi^\gamma$ . Because  $q \leq 1$  but  $\phi > 1$ , the term is negative. Consequently,  $d\phi/d\beta < 0$ , and thus  $dT'/d\beta > 0$ .

Now let us turn to  $dT'/dq$ . There are two effects. As a direct effect, considering (11),  $T'$  increases as  $q$  increases. A real default of the firm would also make the bubble burst. Hence, as the real half life of the firm increases, the half life of the bubble is also lengthened. The second effect is indirect.  $\partial\Psi/\partial q = \beta - \phi < 0$ , which implies  $d\phi/dq < 0$ , implying a longer half life. Thus, both effects go in the same direction, which implies that  $dT'/dq > 0$ .

Finally, consider  $dT'/d\gamma$ . The total derivative of  $Q$  is

$$\begin{aligned} \frac{dQ}{d\gamma} &= \frac{\partial Q}{\partial \gamma} + \frac{\partial Q}{\partial \phi} \cdot \frac{d\phi}{d\gamma} = \frac{\partial Q}{\partial \gamma} - \frac{\partial Q}{\partial \phi} \cdot \frac{\partial\Psi/\partial\gamma}{\partial\Psi/\partial\phi} \\ &= \frac{q^2 \phi^{-\gamma} \ln \phi}{(1+r-\beta)\gamma\phi^{\gamma-1} - q}. \end{aligned}$$

The numerator is negative, so the whole fraction  $dQ/d\gamma$  is negative. As a result, the half life decreases as  $\gamma$  increases,  $dT'/d\gamma < 0$ . Remembering that the variance of the distribution is inversely related to  $\gamma$ , we can state that the half life of a bubble increases with rising uncertainty about aggregate liquidity. ■



**Proof of proposition 2.** Assume that a price process exhibits a bubble, that  $p_t > \bar{p}$  at a date  $t$ , and that  $\tilde{p}_{t+1}$  is distributed with distribution  $F(\tilde{p}_{t+1})$ . Then, in a rational expectations equilibrium,

$$\begin{aligned} \alpha(1+r-\beta) + S &= \int_0^\infty Q_t \alpha \max\left\{\frac{\tilde{p}_{t+1} + d}{p_t} - \beta; 0\right\} df(\tilde{p}_{t+1}) + S, \\ \frac{1+r-\beta}{q} &= \int_0^\infty h(\tilde{p}_{t+1}) df(\tilde{p}_{t+1}), \text{ where} \\ h(\tilde{p}_{t+1}) &= \max\left\{\frac{p_t^\gamma}{\tilde{p}_{t+1}^\gamma} \left(\frac{\tilde{p}_{t+1} + d}{p_t} - \beta\right); 0\right\} \end{aligned} \quad (26)$$

is an auxiliary function. The  $p_{t+1}$  implicitly defined by (7) solves this equation for a distribution that has probability mass only at one point  $p_{t+1}$  (and zero and  $\bar{p}$ ). The question is, from this three-point distribution, can we shift probability mass to other prices such that the above (26) still holds? The answer depends on the shape of  $h(\tilde{p}_{t+1})$ . Some straightforward analysis shows that  $h(\tilde{p}_{t+1})$  is zero up to  $\tilde{p}_{t+1} = \beta p_t - d$ , then increases and decreases again. For  $\tilde{p}_{t+1} \rightarrow \infty$ , it again approaches zero asymptotically. The maximum of the integral is reached if all probability mass is located at

$$\tilde{p}_{t+1}^* = \gamma \frac{\beta p_t - d}{\gamma - 1} > \beta p_t - d.$$

Hence, a trinomial process with the possible events  $p_{t+1}^*$ ,  $\bar{p}$ , and 0 maximizes the right-hand side of (26). Shifting probability mass to other parts of  $h(\tilde{p}_{t+1})$  reduces the value of the integral. Note that no bubble can emerge if the left-hand side of (26) is larger than the right side for any price path. We can therefore conclude that if no trinomial bubble process exists, no other bubble process can exist either. On the other hand, if a trinomial bubble process exists, it is an example of a general bubble process. As a consequence, (10) is the general condition for the existence of bubble processes in rational expectations equilibrium.  $\blacksquare$

**Proof of proposition 3.** For general  $F(N)$  with infinite support, equation (9) reads

$$\lim_{p \rightarrow \infty} \frac{1 - F(p)}{1 - F(\phi p)} (1 + r - \beta) = q(\phi - \beta).$$

Define the first term as

$$G(\phi) := \lim_{p \rightarrow \infty} \frac{1 - F(p)}{1 - F(\phi p)} = q \frac{\phi - \beta}{1 + r - \beta}.$$

If this equation has a solution for  $\phi > 1$ , bubbles are feasible. We know that for  $\phi = 1$ , the left-hand side exceeds the right-hand side,

$$G(1) = 1 > q \frac{1 - \beta}{1 + r - \beta} \quad (27)$$

because  $1+r-\beta > q(1-\beta)$ . Therefore, bubbles are feasible if and only if  $G(\phi)$  crosses  $q \frac{\phi-\beta}{1+r-\beta}$  from above for some  $\phi$ . A necessary condition would be  $G'(\phi) < \frac{q}{1+r-\beta}$  for some  $\phi > 1$ .

Let us delve deeper into the possible structure of  $G(\phi)$ . L'Hôpital's rule yields

$$G(\phi) = \lim_{p \rightarrow \infty} \frac{1 - F(p)}{1 - F(\phi p)} = \lim_{p \rightarrow \infty} \frac{f(p)}{\phi f(\phi p)} = \frac{1}{\phi} \lim_{p \rightarrow \infty} \frac{f(p)}{f(\phi p)}. \quad (28)$$

In the original example, under assumption 2,  $f(p) = \gamma N_0^\gamma p^{-(1+\gamma)}$ , so  $G(\phi) = \phi^\gamma$ . Now what shapes can  $G(\phi)$  potentially take if assumption 2 is dropped? Because of (13),  $\gamma + 1$  is the smallest number such that  $x^\gamma f(x)$  does not converge to zero. In Landau notation,  $f(N) = c_1 x^{-\gamma-1} + O(|x|^{-c_2})$ , where  $c_1$  is a constant, and  $c_2 > \gamma + 1$  such that the error term decays faster than the main component  $x^{-\gamma-1}$ . Consequently, with this approximation,

$$G(\phi) = \lim_{p \rightarrow \infty} \frac{1}{\phi} \lim_{p \rightarrow \infty} \frac{c_1 p^{-\gamma-1} + O(|p|^{-c_2})}{c_1 (\phi p)^{-\gamma-1} + O(|\phi p|^{-c_2})} = \phi^\gamma. \quad (29)$$

Thus, the behavior of  $G(\phi)$  only depends on the order of convergence of  $f(p)$  towards zero for large  $p$ .

Let us, in addition to this proof, give an example for the procedure. Assume that  $N$  follows the Levy distribution,  $N \sim \text{Levy}(1, 1)$ , thus

$$f(N) = \frac{e^{1/(1-x)}}{\sqrt{\pi} (x-1)^{3/2}}.$$

For this distribution,  $\lim_{x \rightarrow 0} f(1/x) x^{-n}$  is zero for  $n < 3/2$ , infinite for  $n > 3/2$ , and  $1/\sqrt{\pi}$  for  $n = 3/2$ . Therefore,  $f(1/x)$  can be approximated by  $x^{3/2}/\sqrt{\pi}$  for small  $x$ , and  $f(N)$  can be approximated by  $N^{-3/2}/\sqrt{\pi}$  for large  $N$ . In this example,  $n = 3/2$  and  $\gamma = n - 1 = 1/2$ . ■

**Proof of remark 9.** Let a tax on transactions of the safe asset be denoted by  $\tau$ , a tax on selling the risky asset by  $\tau''$ , and a tax on buying the risky asset by  $\tau'$ . Hence,  $\tau'' = 0$  means that the buyer of an asset has to pay the tax (the case analyzed in the main text), while  $\tau'$  equals zero if the seller pays the tax. The equilibrium condition (9) is then

$$\phi^\gamma (1 + r - \beta - \tau) = q [\phi (1 - \tau'') - \beta - \tau'].$$

Following the steps taken in the proof of proposition 1 yields  $\phi = \gamma(\beta + \tau') / [(\gamma - 1)(1 - \tau'')]$  as the touching point of both sides of the above equation. The condition for  $\phi > 1$  turns into

$$\gamma < \frac{q(1 - \tau'')}{(1 + r) - \beta - \tau}. \quad (30)$$

The modified condition for the existence of bubbles is then

$$\left(\frac{\gamma}{1-\tau''}\right)^\gamma \left(\frac{\beta+\tau'}{\gamma-1}\right)^{\gamma-1} \leq \frac{q}{1+r-\beta-\tau}.$$

The right-hand side of this equation increases in  $\tau$ , while the left-hand side increases in  $\tau''$  and  $\tau'$ . Considering equal tax rates on the risky and the safe assets, one can use equation (30) to show that the derivative of the right-hand side w. r. t. the common tax rate is larger than the derivative of the left-hand side for both cases,  $\tau = \tau''$  and  $\tau' = 0$ , or alternatively that  $\tau = \tau'$  and  $\tau'' = 0$ . ■

## References

- ALLEN, F., AND D. GALE (1994): *Financial Innovation and Risk Sharing*. MIT Press, Cambridge, MA.
- (1997): “Financial Markets, Intermediaries, and Intertemporal Smoothing,” *The Journal of Political Economy*, 105(3), 523–546.
- (2000): “Bubbles and Crises,” *Economic Journal*, 110(1), 236–255.
- ALLEN, F., AND G. GORTON (1993): “Churning Bubbles,” *Review of Economic Studies*, 60(4), 813–836.
- ALLEN, F., S. MORRIS, AND A. POSTLEWAITE (1993): “Finite Bubbles with Short Sale Constraints and Asymmetric Information,” *Journal of Economic Theory*, 61(2), 206–229.
- ALLEN, F., S. MORRIS, AND H. S. SHIN (2006): “Beauty Contests and Iterated Expectations in Asset Markets,” *Review of Financial Studies*, 19(3), 719–752.
- BERNANKE, B., AND M. GERTLER (1999): “Monetary Policy and Asset Price Volatility,” *Federal Reserve Bank of Kansas City Economic Review*, 84(4), 17–52.
- (2001): “Should Central Banks Respond to Movements in Asset Prices?,” *American Economic Review: Papers and Proceedings*, 91(2), 253–257.
- BERNANKE, B. S., M. GERTLER, AND S. GILCHRIST (1999): “The Financial Accelerator in a Quantitative Business Cycle Framework,” in *Handbook of Macroeconomics*, vol. 3, ed. by J. B. Taylor, and M. Woodford, pp. 1341 – 1393. Elsevier North-Holland.
- BOLTON, P., J. SCHEINKMAN, AND W. XIONG (2006): “Executive Compensation and Short-Termist Behaviour in Speculative Markets,” *Review of Economic Studies*, 73(3), 577–610.

- BRUNNERMEIER, M. K., AND D. ABREU (2003): “Bubbles and Crashes,” *Econometrica*, 71(1), 173–204.
- BRUNNERMEIER, M. K., AND S. NAGEL (2004): “Hedge Funds and the Technology Bubble,” *Journal of Finance*, 59(5), 2013–2040.
- DELONG, J. B., A. SHLEIFER, L. H. SUMMERS, AND R. J. WALDMANN (1990): “Positive Feedback Investment Strategies and Destabilizing Rational Speculation,” *Journal of Finance*, 45(2), 379–395.
- DYBVIIG, P. H., H. K. FARNSWORTH, AND J. N. CARPENTER (2010): “Portfolio Performance and Agency,” *Review of Financial Studies*, 23(1), 1–23.
- FARHI, E., AND J. TIROLE (2010): “Bubbly Liquidity,” Discussion Paper, Harvard.
- FROOT, K. A., D. S. SCHARFSTEIN, AND J. C. STEIN (1992): “Herd on the Street: Informational Inefficiencies in a Market with Short-Term Speculation,” *Journal of Finance*, 47(4), 1461–1484.
- GONNARD, E., E. J. KIM, AND I. YNESTA (2008): “Recent Trends in Institutional Investors Statistics,” *OECD Financial Market Trends*.
- HONG, H., J. SCHEINKMAN, AND W. XIONG (2006): “Asset Float and Speculative Bubbles,” *Journal of Finance*, 61(3), 1073–1117.
- OU-YANG, H. (2003): “Optimal Contracts in a Continuous-Time Delegated Portfolio Management Problem,” *Review of Financial Studies*, 16(1), 173–208.
- PÁSTOR, Ľ., AND P. VERONESI (2006): “Was there a Nasdaq Bubble in the Late 1990s?,” *Journal of Financial Economics*, 81(1), 61–100.
- (2009): “Technological Revolutions and Stock Prices,” *American Economic Review*, 99(4), 1451–1483.
- PLANTIN, G., AND H. S. SHIN (2010): “Carry Trades and Speculative Dynamics,” Discussion Paper, Princeton.
- SANTOS, M. S., AND M. WOODFORD (1997): “Rational Asset Pricing Bubbles,” *Econometrica*, 65(1), 19–57.
- SCHEINKMAN, J., AND W. XIONG (2003): “Overconfidence and Speculative Bubbles,” *Review of Political Economy*, 111(6), 1183–1219.
- TIROLE, J. (1982): “On the Possibility of Speculation under Rational Expectations,” *Econometrica*, 50(5), 1163–1181.
- (1985): “Asset Bubbles and Overlapping Generations,” *Econometrica*, 53(6), 1499–1528.

TOWNSEND, R. M. (1979): “Optimal Contracts and Competitive Markets with Costly State Verification,” *Journal of Economic Theory*, 21, 265–293.

WERMERS, R. (1999): “Mutual Fund Herding and the Impact on Stock Prices,” *Journal of Finance*, 54(2), 581–622.