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SFB 649 Discussion Paper 2010-037

# Dynamical systems forced by shot noise as a new paradigm in the interest rate modeling

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# **Dynamical systems forced by shot noise as a new paradigm in the interest rate modeling**

Alexander L. Baranovski (WestLB AG)

Abstract. In this paper we give a generalized model of the interest rates term structure including Nelson-Siegel and Svensson structure. For that we introduce a continuous  $m$ -factor exponential-polynomial form of forward interest rates and demonstrate its considerably better performance in a fitting of the zero-coupon curves in comparison with the well known Nelson-Siegel and Svensson ones. In the sequel we transform the model into a dynamic model for interest rates by designing a switching dynamical system of the considerably reduced dimension  $n < m$  generating the forward rate curves in form a càdlàg function. A system is described by  $n$ -th order linear differential equation driven by a stochastic or chaotic shot noise. From fitted forward rates we specify the parameters of the switching system and discuss perspectives of our models to produce term-structure forecasts at both short and long horizons.

Keywords: forward interest rates, shot noise processes, switching dynamical systems, chaotic Brownian subordination, chaotic maps

JEL classification: C13, C20 and C22

Disclaimer: The ideas presented below reflect the personal view of the author and are not necessarily identical to the official methodology used at WestLB AG

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## **1. Exponential-polynomial models of interest rates**

We introduce an exponential-polynomial term structure model of interest rates by

$$f(t, Z) = c_l + f_0(t, Z), \quad f_0(t, Z) = \sum_{i=1}^{l-1} \varphi_i(t) \cdot \exp(-\gamma_i t) \quad (1)$$

where  $z := \{c_1^{(1)}, \dots, c_{p_{l-1}}^{(l-1)}, \gamma_1, \dots, \gamma_{l-1}\} \in Z \subset R^m$ ,  $t > 0$  denotes time to maturity  $T$ ,

$\varphi_i(t) = \sum_{k=1}^{p_i} c_k^{(i)} t^{k-1}$  are polynomials of degree  $p_i - 1$  with coefficients  $c_k^{(i)}$ ;  $k = 1, \dots, p_i$  ( $p_i \in \{1, 2, 3, \dots\}$ ) and  $\gamma_i$  are positive real numbers. The total number of parameters in (1) is  $m = \sum_{i=1}^{l-1} p_i + l$ .

We note that the widely used Nelson-Siegel (N-S) [Nelson 1987] and Svensson (SV) [Svensson 1994] families of  $f(t)$  can be easily derived from (1). Assuming  $l = 2$  and  $p_1 = 2, p_2 = 1$ , i.e. dimensionality  $m=4$ , (1) leads to N-S forward rate curve

$$f_{NS}(t) = (c_1^{(1)} + c_2^{(1)}t) \cdot \exp(-\gamma_1 t) + c_2 \quad (2)$$

as well as the SV curve is given by

$$f_{SV}(t) = (c_1^{(1)} + c_2^{(1)}t) \cdot \exp(-\gamma_1 t) + c_1^{(2)} \cdot \exp(-\gamma_2 t) + c_3 \quad (3)$$

with  $l = 3$ ,  $p_1 = 2$  and  $p_2 = 1, p_3 = 1$  i.e.  $m = 6$ .

Another special case of (1) is a curve of the exponentials mixture under  $p_i = 1, \forall i$ , i.e.

$$f_{EXP}(t) = \sum_{i=1}^{l-1} c_i \cdot \exp(-\gamma_i t) + c_l \quad (4)$$

We show that a performance of a new term structure (1) in a fitting of the yields is considerably higher of the well known Nelson-Siegel and Svensson ones. By other words a today's choice of the parameters in (1) is to be not limited by the state space  $Z \subset R^4$  as for the N-S curve or  $Z \subset R^6$  in the Svensson model, i.e.  $Z \subset R^n, n \geq 6$ .

The corresponding term structure of the bond prices will be then given by

$$B(t, Z) := \exp\left(-\int_0^t f(x, Z) dx\right) \text{ at } Z \subset R^n$$

## 2. ODE for the interest rate models

We establish that dynamics of the interest rates  $f(t)$  in a model (1) follows a  $n$ -th order ODE

$$f^{(n)} + \beta_{n-1} f^{(n-1)} + \beta_{n-2} f^{(n-2)} + \dots + \beta_1 f^{(1)} + \beta_0 f = 0 \quad (5)$$

where  $n = \sum_{i=1}^{l-1} p_i + 1$ ,  $p_i$  is the multiplicity of the root  $\gamma_i$ ,  $i = 1, \dots, l$

of the corresponding characteristic polynomial

$$D(\gamma) = \gamma^n + \sum_{i=1}^{n-1} \beta_{n-i} \gamma^{n-i} \quad (6)$$

The coefficients of the ODE are given by the Vieta formula

$$\beta_{n-j} = (-1)^j \sum_{i_1 < i_2 < \dots < i_j} \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_j} \quad (7)$$

with  $\gamma_1 = \gamma_2 = \dots = \gamma_{p_1} = \gamma_1$ ,  $\gamma_{p_1+1} = \gamma_{p_1+2} = \dots = \gamma_{p_1+p_2} = \gamma_2$  and so on.

*Example (Nelson-Siegel).*

Recall that the Nelson-Siegel model corresponds to the case  $l = 2$  and  $p_1 = 2$ . Then

$n = p_1 + 1 = 3$ ,  $\gamma_1 = \gamma_2 = \gamma_1$  and  $\gamma_3 = \gamma_2 = 0$ . It follows that

$$\begin{aligned} (j=1): \beta_2 &= (-1)^1 (\gamma_1 + \gamma_2 + \gamma_3) = -2\gamma_1 \\ (j=2): \beta_1 &= (-1)^2 (\gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3) = \gamma_1^2 \\ (j=3): \beta_0 &= (-1)^3 \gamma_1 \gamma_2 \gamma_3 = 0 \end{aligned}$$

Thus, the corresponding differential equation is

$$f''' - 2\gamma_1 f'' + \gamma_1^2 f' = 0.$$

It is easy to check that the Nelson-Siegel curve  $f(t) = c_1 + (c_2 + c_3 t) \cdot \exp(-\gamma_1 t)$

is a general solution of the ODE as a combination of the two obvious particular solutions of  $r' = 0$  and  $r'' - 2\gamma_1 r' + \gamma_1^2 r = 0$ , such that

$$c_1 = f(0) + \frac{2f'(0)}{\gamma_1} + \frac{f''(0)}{\gamma_1^2}, c_2 = -2 \frac{f'(0)}{\gamma_1} - \frac{f''(0)}{\gamma_1^2}, c_3 = -f'(0) - \frac{f''(0)}{\gamma_1}. \quad (8)$$

*Example (Svensson).*

By analogy to the previous example the SV curve follows a 4<sup>th</sup> order ODE in the form

$$f^{(IV)} - (2\gamma_1 + \gamma_2) f''' + (\gamma_1^2 + 2\gamma_1\gamma_2) f'' - \gamma_1^2 \gamma_2 f' = 0$$

Generally ODE (5) can be presented in a matrix form

$$\frac{d}{dt} \mathbf{f} = \mathbf{F}(\boldsymbol{\gamma}) \cdot \mathbf{f}, \quad 0 \leq t \leq T \quad (9)$$

$$\text{where } \mathbf{f} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_l \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ -\beta_0(\boldsymbol{\gamma}) & -\beta_1(\boldsymbol{\gamma}) & \dots & \dots & \dots & -\beta_{n-1}(\boldsymbol{\gamma}) \end{pmatrix} \text{ and}$$

$$f_i \equiv \frac{d^i}{dt^i} f(t).$$

Eq. (9) has a vector solution  $\mathbf{f}(t) = e^{\mathbf{F}(\boldsymbol{\gamma})t} \cdot \mathbf{f}(0)$ .

Define by

$$r(t, \tau) := f(t, Z_\tau) \quad (10)$$

the instantaneous forward rate at time  $t$  for date  $\tau$  and by

$$r(0, \tau) = \lim_{t \rightarrow 0} r(t, \tau) = f(0, Z_\tau) \quad (11)$$

the short rates, where stochastic process  $Z_\tau$  with values in  $Z \subset R^m$  contains two

groups of processes  $\{c_1^{(1)}(\tau), \dots, c_{p_{l-1}}^{(l-1)}(\tau)\}$  and  $\{\gamma_1(\tau), \dots, \gamma_{l-1}(\tau)\}$  such that the processes

$c_i^j(\tau)$  depend on the stochastic processes  $\{\gamma_1(\tau), \dots, \gamma_{l-1}(\tau)\}$  and the short rates (initial conditions) as shown by (8) for a Nelson-Siegel term structure.

Taking into account (9), (10) the evolution of the forward rates on the date  $\tau$  is given by

$$\frac{d}{dt} \mathbf{r} = \mathbf{F}(\boldsymbol{\gamma}_\tau) \cdot \mathbf{r} \quad (12)$$

with the given vector  $\boldsymbol{\gamma}_\tau$  and initial state vector  $\mathbf{r}(0, \tau)$ .

Assuming  $Z_\tau$  is given  $\tau$  – evolution of  $r(t, \tau)$  can be described by two different ways. First one presents  $\tau$  – evolution of  $r(t, \tau)$  as a train of curves (12) for dates  $\tau = 1, 2, \dots$  in a form (see Fig. 1)

$$\frac{d}{dt} \mathbf{r} = \mathbf{F}(\gamma_\tau) \cdot \mathbf{r} + \sum_i \xi_i \delta(t - i \cdot T), \quad 0 \leq t \leq \infty$$

or equivalently

$$\frac{d}{dt} \mathbf{r} = \mathbf{F}(\gamma_\tau) \cdot \mathbf{r} + \xi_\tau \delta(t - \tau \cdot T), \quad (13)$$

where  $\tau = \left\lfloor \frac{t}{T} \right\rfloor + 1$  is a counting process ( $\lfloor t \rfloor$  denotes the floor function (largest integer smaller or equal to  $t$ )) and

$$\xi_\tau = \mathbf{f}(0, \mathbf{Z}_{\tau+1}) - \mathbf{f}(T, \mathbf{Z}_\tau). \quad (14)$$

We call the vector  $\xi_i$  as the stochastic amplitude of the impulse perturbation, which acts on system (13) at times  $t = \tau T$ ,  $\tau = 1, 2, 3, \dots$  such that

$$r(\tau \cdot T + 0, \tau) = r(\tau \cdot T - 0, \tau) + \xi_\tau.$$

Thus  $r$  is a cadlag function, i.e. a right continuous function  $r(\tau \cdot T + 0, \tau) \equiv r(\tau \cdot T, \tau)$ , defined on  $R^n$  and has a left limit.

Between kicks  $iT$  and  $(i+1)T$  a state vector is governed by the homogeneous system of linear differential equations (12) at  $\tau = i+1$ .

Fig. 1 illustrates the system (13) with a jump  $\xi_1 \equiv f(0, Z_2) - f(T, Z_1)$  at  $t = T$

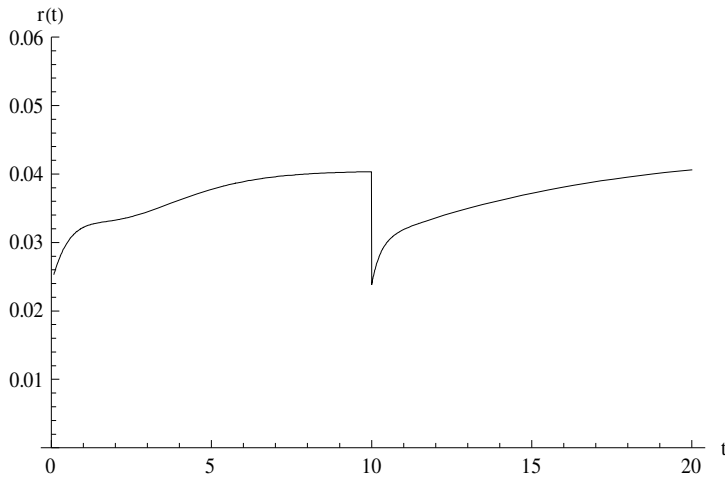


Fig.1 Train of the two first curves with  $T = 10$  years.

The solution to (13) is explicitly given by

$$r = e^{F(\gamma_\tau)(t-(\tau-1)T)} \cdot \left[ \sum_{i=1}^{\tau} \exp\left(T \cdot \sum_{j=i}^{\tau} F(\gamma_j)\right) \times \xi_{i-1} \right]$$

where we denote  $\xi_0 \equiv r(0)$ .

The second approach is based on a simple idea to express a random variable  $\mathbf{f}(t, \mathbf{Z}_\tau)$

by  $\mathbf{f}(t, \mathbf{Z}_{\tau_{fixed}}) + \zeta(t)$  in the interval  $0 \leq t \leq T$ . By other words we assume that dynamics of the interest rates  $r(t, \tau)$  can be modelled by

$$d\mathbf{r} = \mathbf{F} \cdot \mathbf{r} dt + d\zeta, \quad 0 \leq t \leq T, \quad \forall \tau \quad (15)$$

where  $\mathbf{F}$  is a  $n \times n$  matrix with the constant coefficients  $\beta_i \equiv \beta_i(\gamma_{fixed})$  for any  $\tau$ . The model (15) generates a predicted term structure, whose exponential-polynomial shape depends on the model parameters and the initial short rate. One can show that (15) is more general and includes a class of equilibrium models such as Vasicek, CIR, lognormal models.

### 3. Approach I – demonstrating example: from estimating the yield curve to its dynamic modelling

First we empirically estimate process  $Z_\tau$ . For that we are going to fit the default-free yield spreads, downloaded from the Reuters database. The observable time period is 23.02. 2006 – 14.01. 2008, i.e. contains  $q = 478$  dates. In framework of the above exponential-polynomial approach we introduce a term structure model of yields curves by

$$Y(t, Z) := \frac{1}{t} \int_0^t f(x, Z) dx \quad (16)$$

(16) can be easily done in a closed form. For this we need the relation [Prudnikov 1981]



$$\int_0^t x^k e^{-\gamma x} dx = -M_k(t) e^{\gamma t} + \frac{k!}{\gamma^{k+1}}, k = 0, 1, 2, \dots, \text{ where } M_k(t) = \sum_{i=0}^k \frac{k!}{(k-i)!} \frac{t^{k-i}}{\gamma^{i+1}}, \gamma < 0.$$

Substituting the model (1) into (16) leads to

$$Y(t, Z) = c_l + \frac{1}{t} \sum_{i=1}^{l-1} \sum_{k=1}^{p(i)} c_k^{(i)} \left[ -M_{k-1}(t) + \frac{k-1!}{\gamma^k} \right] \quad (17)$$

We introduce a minimization criterion as:

$$\rho(\tau) = \frac{1}{N} \sum_i (s_i(\tau) - Y_\tau(t_i, Z))^2 \rightarrow \min_Z \quad (18)$$

where  $\{s_i(\tau), i = 1, 2, \dots, N_\tau\}$  are quotes of the yields on the base date  $\tau$ . The cost function  $\rho(\tau)$  is to be minimized by the appropriate choice of the  $m = \sum_{i=1}^{l-1} p_i + l$  parameters of the state space  $Z$ . It is clear that in the fitting problem the following restriction

$$m = \dim(Z) = \sum_{i=1}^{l-1} p_i + l \leq \min(\{N_\tau, \tau = 1, \dots, q\}) \quad (19)$$

is to be provided.

Given the number  $l$  of the parameters  $\gamma_i, i = 1, \dots, l$  and distribution of their multiplicities  $p_i, i = 1, \dots, l$  such that the condition (19) holds the nonlinear regression technique for the least squares criterion (18) leads to the minimum of the cost function with the optimal parameters  $\{c_1^{(1)}, \dots, c_{p_{l-1}}^{(l-1)}, \gamma_1, \dots, \gamma_{l-1}\} \in Z \subset R^m$ .

We extend the criterion (7) by

$$\bar{\rho} = \frac{1}{q} \sum_{\tau=1}^q \rho(\tau) \rightarrow \min_{l, p_1, \dots, p_l} \quad (20)$$

with the obvious restrictions for the parameters

$$l, p_1, \dots, p_l \in Z^+ = \{1, 2, 3, \dots\}. \quad (21)$$

We note that (20) according to a law of large numbers/ Birkhoff ergodic theorem approaches the mean of the stochastic/ chaotic cost function with  $q \rightarrow \infty$ .

The problem (18)-(21) for euro swap rates has the following twofold solution

1)  $l = 5, p_l = 1, p_i = 3, i = 1, \dots, l-1$  and  $\{c_1^{(1)}, \dots, c_{p_{l-1}}^{(l-1)}, \gamma_1, \dots, \gamma_{l-1}\} \in Z \subset R^{17}$  ( $m = 17, N_\tau = 60, \forall \tau$ ) for a “laminar” period of the observed yields quotes on the bond market: 23.02.06 ( $\tau = 1$ ) – 17.07.07 ( $\tau = 355$ )

2)  $l = 5, p_l = 1, p_{l-1} = 1, p_i = 3, i = 1, \dots, l-2$  and  $\{c_1^{(1)}, \dots, c_{p_{l-1}}^{(l-1)}, \gamma_1, \dots, \gamma_{l-1}\} \in Z \subset R^{15}$  ( $m=15$ ) for a “turbulence” period 18.07.07 ( $\tau = 356$ ) – 14.01.08 ( $\tau = 478$ ) .

The above calculated  $m + l + 1$  parameters specify a general term structure model of interest rates by the exponential-quadratic curves (1) ( $p_l = 3$ ) as well as a general term structure model of yields by the exponential-cubic curves (17).

Fig. 2 demonstrates a performance of the general model for the Euro swap rates by comparison of its cost function (18) with the cost functions of the conventional Nelson-Siegel and Svensson models and exponential model (4) with  $l = 6$ .

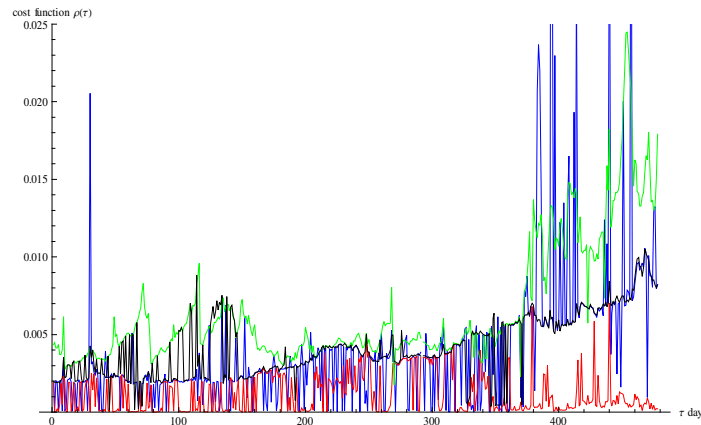


Fig. 2 Comparison of the cost functions for the observable time period: 23.02. 2006 – 14.01. 2008. N-S is a green curve, SV is a black, the exponential is a blue, and the general model is a red curve.

Moreover the ratios

$$\frac{\bar{\rho}_{NS}}{\bar{\rho}_{GEN}} = \frac{0.00653394}{0.00074} \approx 9, \quad \frac{\bar{\rho}_{SV}}{\bar{\rho}_{GEN}} = \frac{0.00419495}{0.00074} \approx 6 \text{ and } \frac{\bar{\rho}_{EXP}}{\bar{\rho}_{GEN}} = \frac{0.0039557}{0.00074} \approx 5 \quad (22)$$

quantify the performance of the general model with the derived optimal exponential-cubic curve. Thus our yield curve fitting is about 9 and 6 times better than conventional N-S and SV one, as well as 5 times better than an exponential curve (4)

with  $l = 6$ , respectively.

The mean of the cost function (20) for the exponential model (4) has a local minimum at  $l = 6$  in value  $\bar{\rho}_{EXP} = 0.0039$  and at  $l = 5$  in value  $\bar{\rho}_{GEN} = 0.00074$  for a general model as shown in Fig. 3.

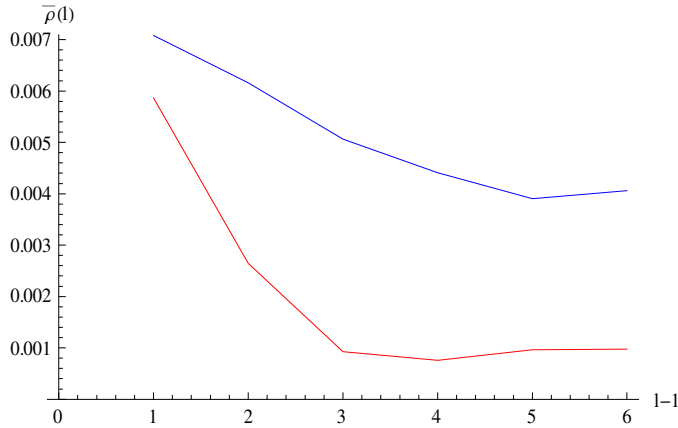


Fig. 3. The means of the cost functions: blue curve – exponential and red one – general model

Fig. 4 collects all base curves fitted to the available data on the date 25.03.08 .

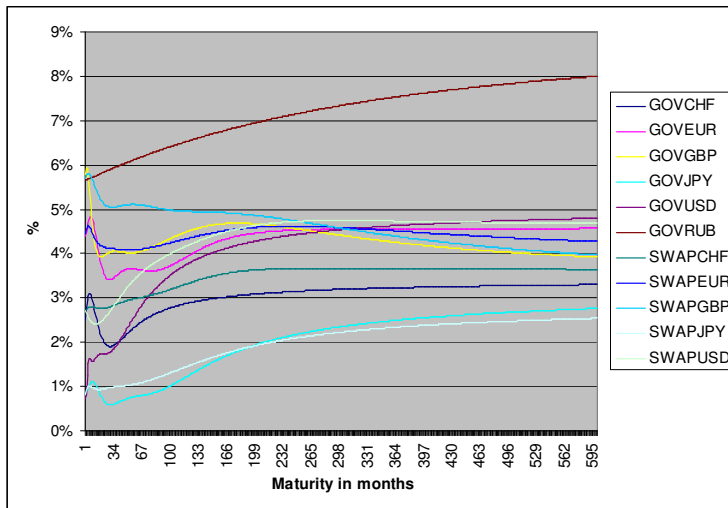


Fig. 4 Fitted zero curves on the spot date 25.03.08

Repeating a fitting procedure to the another date, let's say  $\tau + 1 = 26.03.08$ , we get a similar set of the base curves (16) which can be described by the system (13), where the impulse perturbation  $\xi_\tau$  (14) is to be predetermined.

As an example we design a dynamical system for N-S instantaneous forward rate curve  $r(t, \tau) = c_1(\tau) + (c_2(\tau) + c_3(\tau)t) \cdot \exp(-\gamma_\tau t)$  described by the following low-dimensional ODE

$$r''' - 2\gamma_\tau r'' + \gamma_\tau^2 r' = \xi_\tau \delta(t - \tau \cdot T) \quad (23)$$

where

$$\xi_\tau = c_1(\tau + 1) + c_2(\tau + 1) - c_1(\tau) - (c_2(\tau) + c_3(\tau)T) \cdot \exp(-\gamma_\tau T). \quad (24)$$

according to (10) and (14).

From the output of the above fitting procedure we retrieve the time series  $\{c_1(\tau), c_2(\tau), c_3(\tau), \gamma_\tau | \tau = 1, 2, \dots, q = 478\}$  for the observable time period.

Applying (24) we immediately get time series of stochastic perturbation.

Let us introduce the  $k$ -th order increments for both processes  $\xi_\tau, \gamma_\tau$  by

$$\Delta^{k+1} \xi_\tau = \Delta^k \xi_{\tau+1} - \Delta^k \xi_\tau \quad (25)$$

$$\Delta^{k+1} \gamma_\tau = \Delta^k \gamma_{\tau+1} - \Delta^k \gamma_\tau, \quad (26)$$

where  $\Delta^0 \equiv 1, k = 0, 1, 2, \dots$

We are now able to do an elementary statistical analysis of both processes  $\xi_\tau, \gamma_\tau$ .

Table 1 contains the histograms of the processes and Table 2 collects the histograms of the increments.

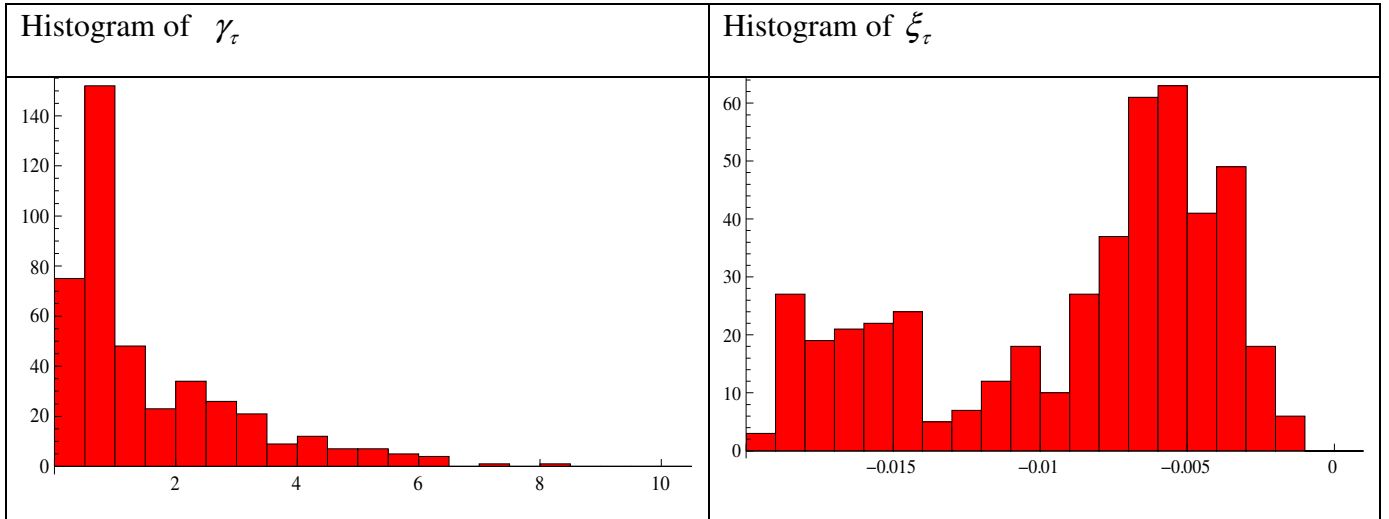


Table 1.

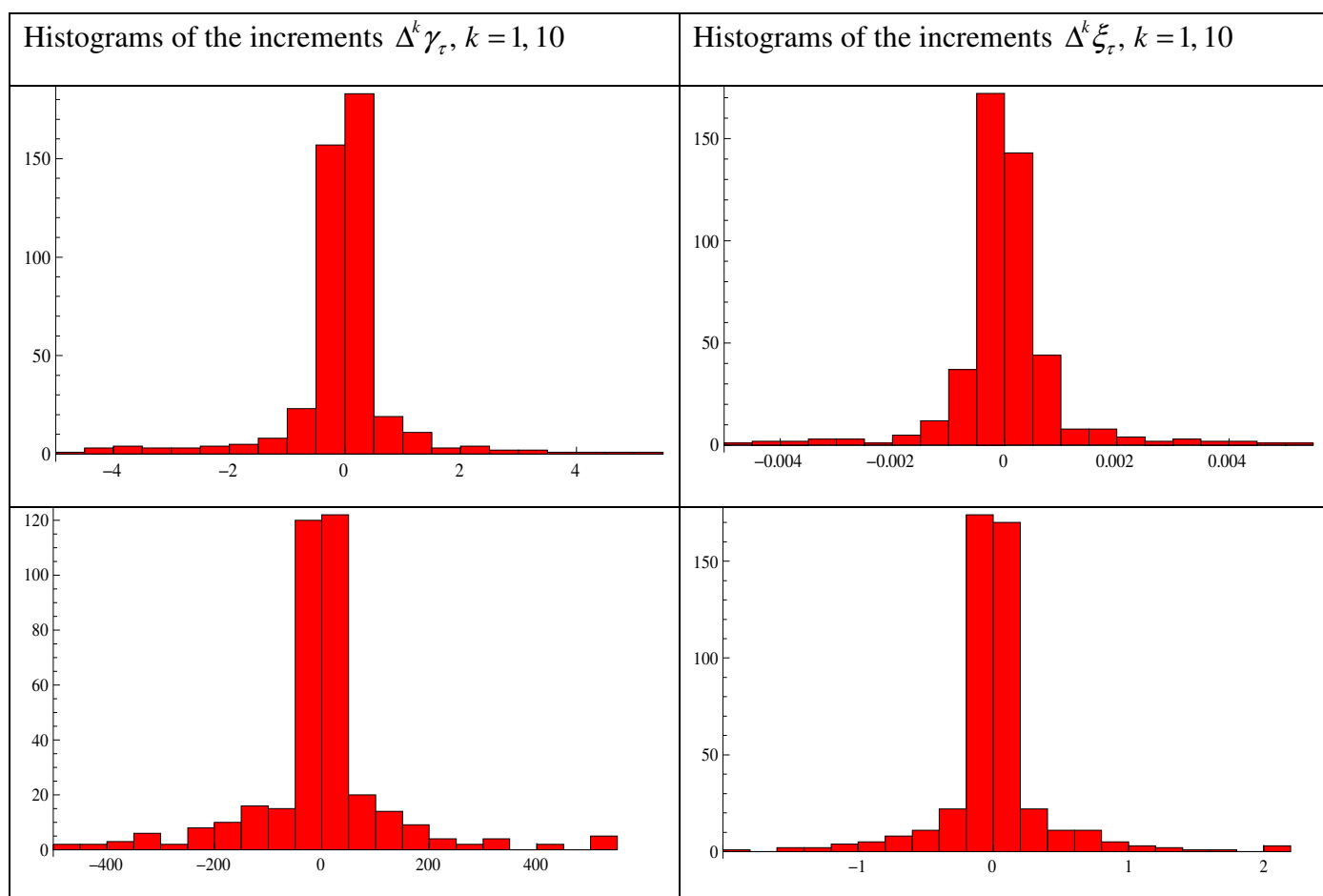


Table 2.

We note that the both processes are diffusion processes characterized by symmetrical bell-like but not-Gaussian distributions of their increments. A  $\chi^2$  Pearson's test with the confidence level 0.95 rejects a hypothesis of the independence of the increments for both forced signal  $\xi_\tau$  and  $\gamma_\tau$ .

The means of  $\Delta^k \gamma_\tau$  and  $\Delta^k \xi_\tau, \forall k \geq 1$  are zeros. The sample variances  $\sigma_{\Delta^k \gamma}^2, \sigma_{\Delta^k \xi}^2$  grow exponentially with the order  $k$  as shown in Fig. in a log scale

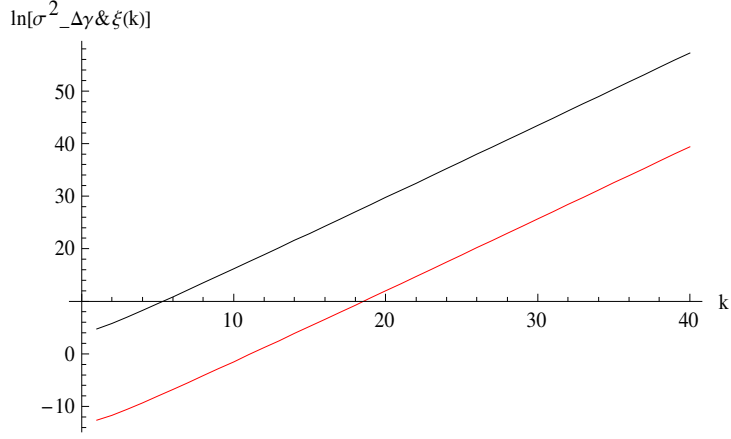


Fig.5. Red line is  $\ln \sigma_{\Delta^k \xi}^2$ , black line is  $\ln \sigma_{\Delta^k \gamma}^2$

A distance between the above two lines does not remain constant with  $k$  but grows slowly

with rate  $\frac{d}{dk} \ln \frac{\sigma_{\Delta^k \gamma}^2}{\sigma_{\Delta^k \xi}^2} \xrightarrow{k \rightarrow \infty} 0.00594878$ . It means that a variance of the  $\Delta^k \gamma_\tau$  grows a bit quicker

than a variance of  $\Delta^k \xi_\tau$ ,  $\forall k \geq 1$ .

The sample autocorrelation functions

$$C_{\Delta^k \gamma}(j) = \frac{(\sigma_{\Delta^k \gamma}^2)^{-1}}{q-k-j} \sum_{\tau=1}^{q-k-j} \Delta^k \gamma_\tau \Delta^k \gamma_{\tau+j}, \quad j = 0, 1, 2, \dots \quad (27)$$

$$C_{\Delta^k \xi}(j) = \frac{(\sigma_{\Delta^k \xi}^2)^{-1}}{q-k-j} \sum_{\tau=1}^{q-k-j} \Delta^k \xi_\tau \Delta^k \xi_{\tau+j}, \quad j = 0, 1, 2, \dots \quad (28)$$

are presented in Table 3.

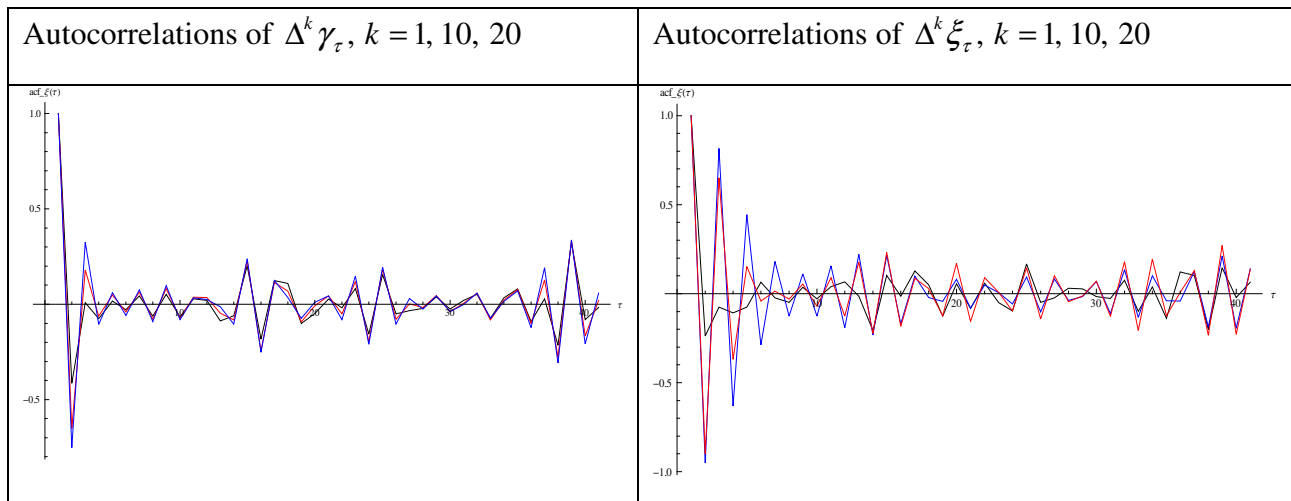


Table 3.

To estimate a mutual correlation of  $\xi_\tau$  and  $\gamma_\tau$  and their increments we introduce the Pearson product-moment correlation coefficient [Norman L. Johnson 1995]

$$R(j, k) = \frac{\sigma_{\Delta^k \gamma}^{-1} \sigma_{\Delta^k \xi}^{-1}}{q - k - j} \sum_{\tau=1}^{q-k-j} \Delta^k \xi_\tau \cdot \Delta^k \gamma_{\tau+j}; \quad j = 0, 1, \dots; \quad k = 0, 1, \dots \quad (29)$$

depicted in Figures 6 – 8.

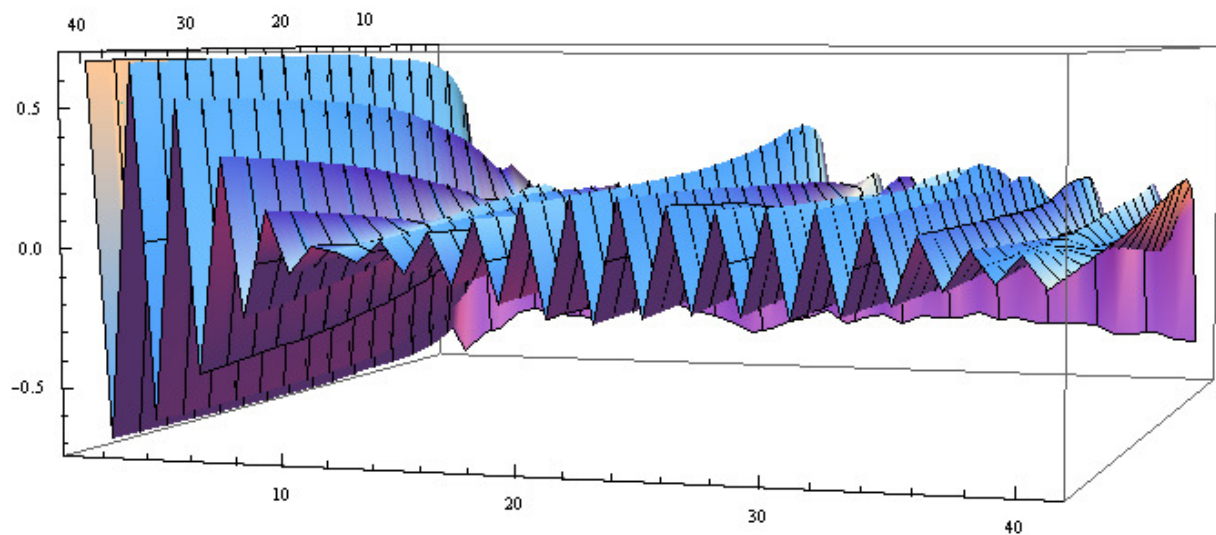


Fig. 6 . Geometric interpretation of the matrix R of dimension 40x40

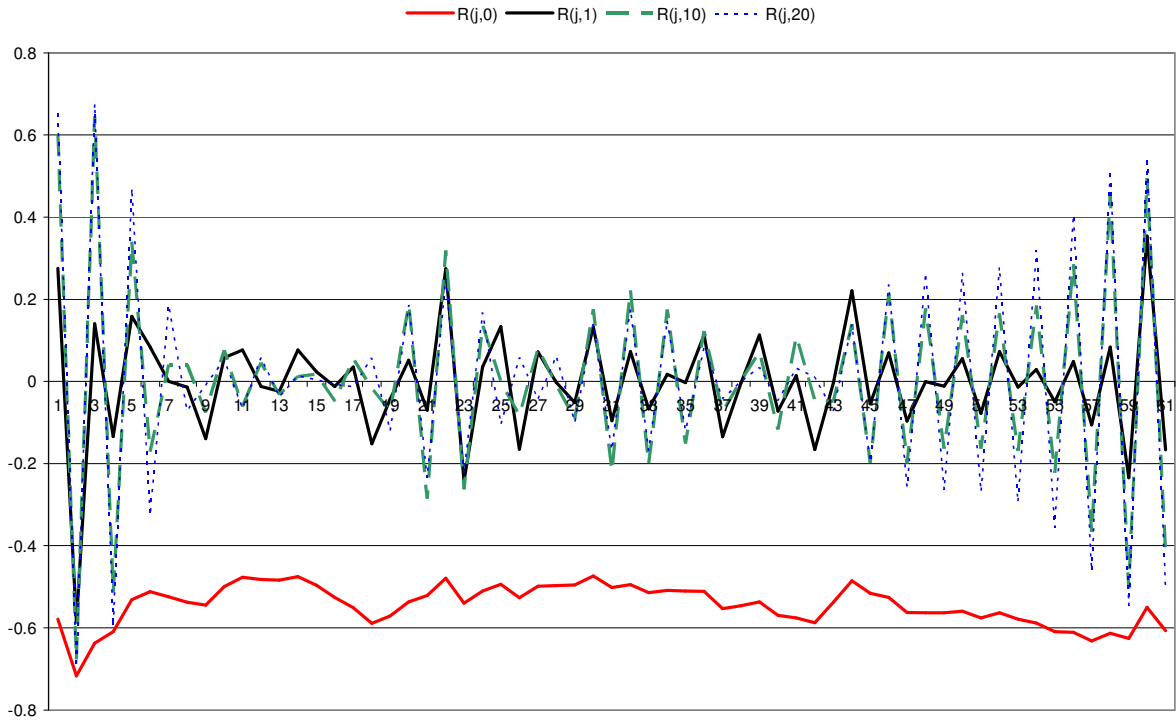


Fig. 7 The Pearson's correlation coefficient at  $j = 0, 1, \dots, 60; k = 0, 1, 10, 20$

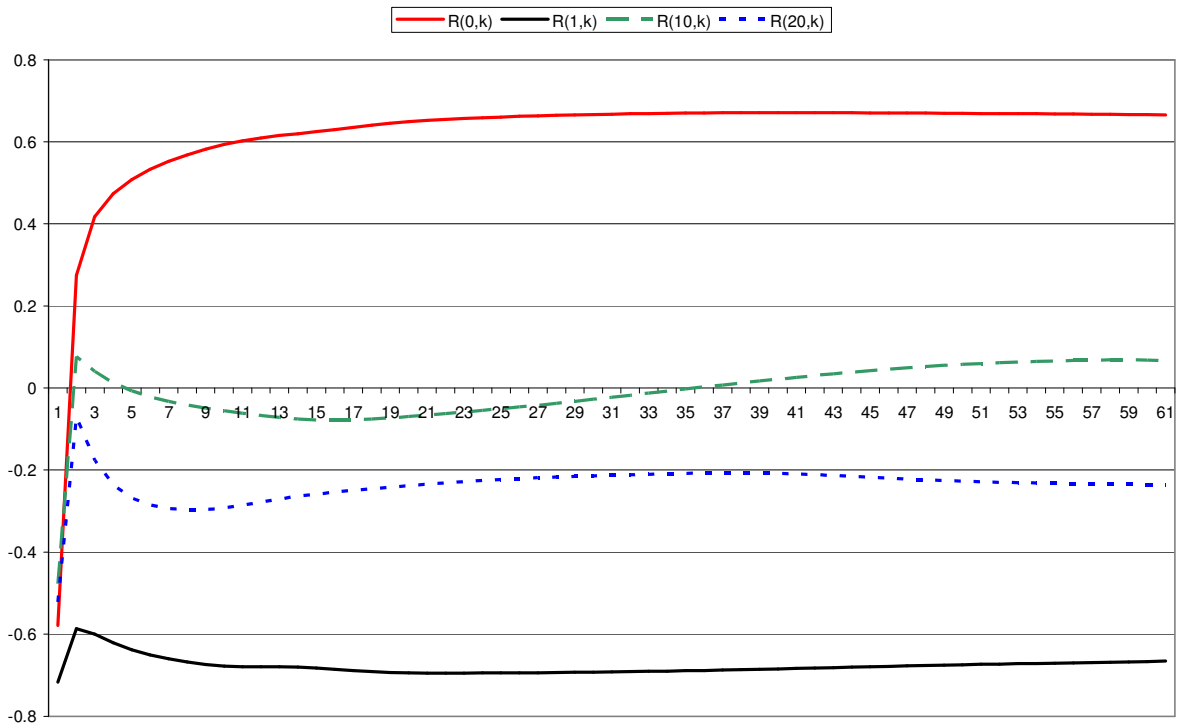


Fig. 8 The Pearson's correlation coefficient at  $j = 0, 1, 10, 20; k = 0, 1, \dots, 60$



#### 4. Approach II – forcing signal as a shot noise

We note that applying Vieta formula (7) at  $j = n$  the coefficient  $\beta_0$  is equal to zero since  $\gamma_l = 0$ . It implies that a general solution of the ODE (5) is a combination of the obvious solution of  $f' = 0$  and the particular solution  $\eta(t)$  of

$$f^{(n-1)} + \beta_{n-1}f^{(n-2)} + \beta_{n-2}f^{(n-3)} + \dots + \beta_1 f = 0 \quad (30)$$

i.e.  $f = r \equiv \eta + c$ .

We are specifically interested in the behaviour of the system (30) with a shot noise as a chaotic/stochastic perturbation, i.e.

$$\frac{d}{dt} \boldsymbol{\eta} = \mathbf{F} \cdot \boldsymbol{\eta} + \sum_i^{N(t)} \mathbf{A}_i \delta(t - t_i), \quad 0 \leq t \leq T \quad (31)$$

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_{n-2} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & & 0 & \\ \vdots & & \ddots & \ddots & \ddots \\ 0 & \dots & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ -\beta_1 & & \dots & \dots & -\beta_{n-2} & -\beta_{n-1} \end{pmatrix}.$$

where  $t_k$  are successive occurrence or arrival times of  $\delta$ -impulses,

$$t_0 (= 0) < t_1 < \dots < t_k < \dots < T$$

$N(t) = \max\{k : t_k \leq t\}$  is a counting process.

The impulse perturbation acts on system (31) at times  $t = t_k, k = 1, 2, 3, \dots$  such that

$$\boldsymbol{\eta}(t_k + 0) = \boldsymbol{\eta}(t_k - 0) + \mathbf{A}_k. \quad (32)$$

In sequel we assume that  $\eta$  is a càdlàg function.

We introduce the positive inter-arrival times  $T_k$  such that

$$t_k = t_{k-1} + T_k = \sum_{i=1}^k T_i$$

Between kicks a state vector is governed by the following homogeneous system of linear

differential equations

$$\frac{d}{dt}\boldsymbol{\eta} = \mathbf{F} \cdot \boldsymbol{\eta}$$

and the initial condition of the system  $\boldsymbol{\eta}_0 \equiv \boldsymbol{\eta}(t_0 + 0)$  defines an evolution of a state vector (Cauchy theorem).

#### 4.1. One-dimensional case

We consider a special case of (31) in a form of one-dimensional ODE:

$$\frac{d}{dt}\eta = \sum_i^{N(t)} A_i \delta(t - t_i), \quad 0 \leq t \leq T \quad (33)$$

##### 4.1.1. Response to a Shot Noise. The generalized Wiener process

Integrating Eq. (33) we immediately get a solution

$$\eta(t) = \sum_{k=1}^{N(t)} A_k \equiv \eta_{N(t)} \quad (\eta(0) = 0) \quad (34)$$

A plot of the process  $\eta$  is depicted in Fig.

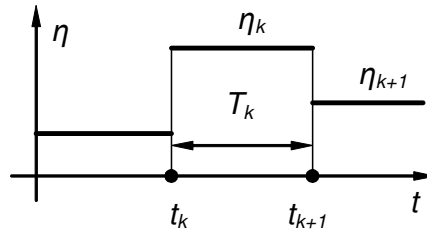


Fig. 9. A solution of Eq. (1)

Thus the process  $\eta(t)$  (Fig. 9) is a rectangular signal with step heights  $\eta_k$  satisfying the relation:

$$\eta_k = \eta_{k-1} + A_k \quad (35)$$

*Probability density function*

We first establish that the distribution function of  $\eta(t)$  is

$$P(\eta(t) \leq z) = \sum_{k=1}^{\infty} P_k(t) P(\eta_k \leq z),$$

where  $P_k(t) = P(t_{k-1} \leq t < t_k) = P(N(t) = k)$ .

We assume zero mean for the magnitudes  $A_k$ . It immediately implies that  $\eta(t)$  is a martingale with a zero mean

$$E(\eta) = E(N(t))E(A) = 0 \quad (\bar{A} = 0) \text{ as } E(\eta | N(t)) = N(t)E(A) \quad (36)$$

To calculate a variance of  $\eta(t)$  we use a law of total variance

$$D(\eta) = E[D(\eta | N(t))] + D[E(\eta | N(t))]$$

One can show that the conditional variance is

$$D(\eta | N(t)) = N(t)D(A) + 2 \sum_{n=1}^{N(t)-1} (N(t)-n)c_A(n) \quad (37)$$

where  $c_A(k) = E(A_i A_{i+k})$  is the autocorrelation function (acf) and  $\sigma_A^2 \equiv D(A)$  is the variance of the  $A_k$ .

Hence, for i.i.d. random or uncorrelated chaotic magnitudes  $A_k$  ( $c_A(k)=0$ ) we have

$$D(\eta) \Big|_{c_A(n)=0} = D(A)E(N(t)) + D(N(t))E^2(A) = D(A)\bar{N}_t \quad (38)$$

where  $\bar{N}_t \equiv H(t) = \sum_k k P\{N(t) = k\} = \sum_k P\{t_k < t\} \stackrel{t \gg \bar{t}}{\square} \frac{t}{E(t_k)}$  is the intensity function.

For correlated random/chaotic  $A_k$  we assume that the first moment of the autocorrelation function  $c_A(k)$  is finite

$$\sum_{k=1}^{\infty} k |c_A(k)| < \infty \quad (39)$$

and then the variance is given by

$$\sigma_{\eta_k}^2 = k\sigma_A^2 + o(1). \quad (40)$$

Let us introduce a new variable  $\varepsilon_k = \frac{\eta_k - \bar{\eta}_k}{\sigma_{\eta_k}}$  with  $E(\varepsilon_k) = 0$  and  $E(\varepsilon_k^2) = 1$ . It can be

shown that  $\varepsilon_k$  converges in distribution to the standard normal law, i.e. the central limit theorem holds both with i.i.d.random [Feller] and chaotic magnitudes  $A_k$  [Chernov 1995]. In [Baranovski 2003], authors have presented the analytical expressions for the characteristic functions of the chaotic partial sums  $\eta_k$  of the magnitudes  $A_k$  generated by PWL onto maps and shown their fast convergence to the limit  $\exp(-\omega^2 / 2)$ .

We consider a piecewise constant function  $W_k(t)$  on  $t \in [0, 1]$  such that

$$W_L(t) = \frac{\eta_{\lfloor kt \rfloor}}{\sqrt{D(A)}\sqrt{L}} = \frac{1}{\sqrt{D(A)}\sqrt{L}} \sum_{i=1}^{\lfloor kt \rfloor} A_i, \quad t \in [0, 1], k = 0, 1, \dots, L \quad (41)$$

where  $\lfloor x \rfloor$  is the floor function (it gives the greatest integer less than or equal to  $x$ ).

Then for any  $k$   $\{W_k\}$  induces a measure on the space of continuous functions on  $[0, 1]$ .

According to the invariance principle this measure converges weakly, as  $k \rightarrow \infty$ , to the

Wiener process  $W$  [Chernov 1995] Fig. 10 depicts examples of functions  $\{W_k\}$  for different

$k$  when the magnitudes  $A_k$  are chaotic variables generated by a tent map on  $[-1, 1]$ :

$$A_{n+1} = 1 - 2|A_n|, n = 1, 2, \dots \quad (42)$$

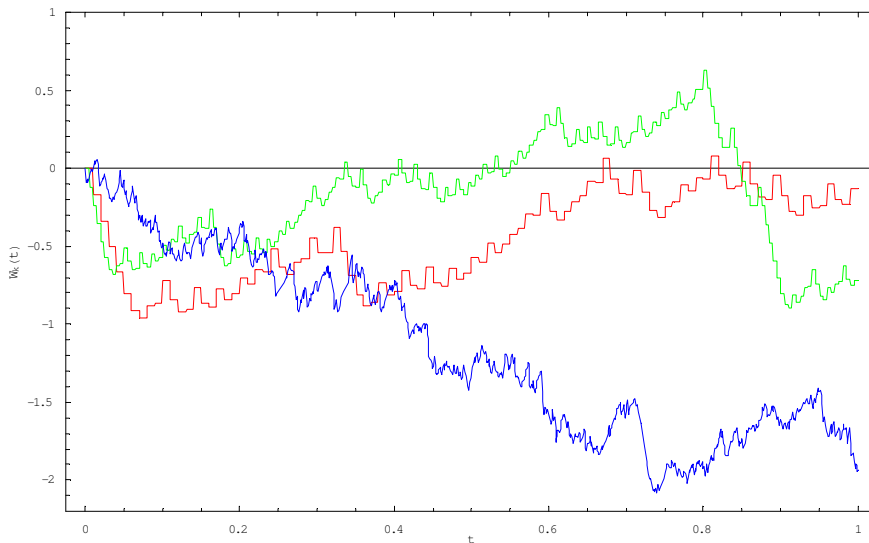


Fig. 10 Three realizations of the process  $W$  for  $k = 100, 300$  and  $10000$  (red, green and blue line)

The weak invariance principle known also as the functional central limit theorem provides an approximation deterministic dynamical systems by a Brownian motion on large space and time scales.

Thus the distribution of  $\eta(t)$  tends to the Gaussian law with the mean (36) and variance (38).

This confirms the diffusion character of  $\eta(t)$ . It follows that the Eq. (33) can be used for stochastic and chaotic modeling of the Wiener process.

### Example 1. Valuation of the European call option.

The underlying asset of the European option is assumed to grow at the constant risk-free rate  $r$  perturbed by a stochastic/chaotic marked point process  $\eta(t)$ . Thus an asset price is modeled as

$$\frac{dS}{S} = rdt + d\eta \quad (43)$$

Properties:

- 1) Markov property: the next asset price ( $S+dS$ ) depends solely on today's price
- 2) The next value for  $S$  is higher than the old by an amount

$$E(dS) = rSdt \text{ (as } E(d\eta) = 0\text{)}$$

- 3) Variance of  $dS$  is

$$D(dS) = E(dS^2) - E^2(dS) = E(S^2(d\eta)^2) = S^2D(d\eta) = S^2D(A)dH(t)$$

We want to price a call option, i.e.

$$C(t, K) = e^{-rt} E[(S - K)^+] = e^{-rt} \sum_k P(N(t) = k) \cdot E[(S - K)^+ | N(t) = k], \quad (44)$$

where  $K$  is a strike price. We calculate a conditional expectation

$$\begin{aligned} C_k(t, K) &= E[(S - K)^+ | N(t) = k] = \int_0^\infty (x - K)^+ \partial P(S \leq x | N(t) = k) = \\ &= \int_K^\infty (x - K) \cdot \partial_x P\left(S_0 e^{\tilde{r}t + \eta(t)} \leq x | N(t) = k\right) = \int_K^\infty (x - K) \cdot \partial_x P\left(\eta_k \leq \ln\left(\frac{x}{S_0}\right) - \tilde{r} \cdot t\right) = \\ &= \int_K^\infty (x - K) \cdot p_{\eta_k} \left(\ln\left(\frac{x}{S_0}\right) - \tilde{r} \cdot t\right) \frac{dx}{x} \end{aligned} \quad (45)$$

where  $\tilde{r} = r - \frac{\sigma^2}{2}$ ,  $\sigma^2 = \frac{D(A)}{\bar{T}}$ .

A pdf of  $\eta_k$  can be found via its characteristic function. We note that

$$\psi_{\eta_k}(\omega) = E\left(i\omega \sum_{p=1}^k A_p\right) = \Theta_k(\omega, \omega, \dots, \omega) \quad (46)$$

where

$$\Theta_k(\omega_1, \omega_2, \dots, \omega_k) = E\left(\exp\left(i \sum_{i=1}^k \omega_i A_i\right)\right) = \int \dots \int \exp\left(i \sum_{i=1}^k \omega_i \cdot x_i\right) \cdot p_A(x_1, \dots, x_k) dx_1 \dots dx_k \quad (47)$$

is a  $k$ -dimensional characteristic function of a sequence  $\{A_1, \dots, A_k\}$  having a joint pdf  $p_A(x_1, x_2, \dots, x_k)$ .

For a case of i.i.d. random values  $A_k$  (46) simplifies to

$$\psi_{\eta_k}(\omega) = \Theta_1^k(\omega) \quad (48)$$

where  $\Theta_1(\omega) = \int_X e^{i\omega x} p_A(x) dx$  is the characteristic function of the distribution of  $A_k$ .

Here we focus on a special case of (46) when the magnitudes  $A_k$  are generated by a chaotic mapping

$$A_k = \varphi(A_{k-1}) \quad (49)$$

in an interval  $X$ .

A joint pdf does not factorize in this case and calculates as

$$p_A(x_1, x_2, \dots, x_k) = p_A(x_1) \cdot \prod_{i=1}^{k-1} \delta(x_{i+1} - \varphi^{(i)}(x_1)), \quad (50)$$

where  $p_A(x)$  is the invariant density of the map  $\varphi$ .

The goal equation (46) simplifies for piece-wise linear onto maps

$$\varphi(x) = \{\varphi_i(x) = a_i x + b_i, x \in J_i, i = 1, 2, \dots, m\} \quad (51)$$

such that  $\forall i: \varphi: J_i \rightarrow X = (0,1)$ .

We collect their main probabilistic properties:

- The invariant density is uniform with  $\bar{A} = \frac{1}{2}$  and variance  $\sigma_A^2 = \frac{1}{12}$
- The autocorrelation function is

$$c_A(k) = \sigma_A^2 r^k, \quad -1 < r = \sum_{i=1}^m \frac{1}{|a_i| \cdot a_i} < 1 \quad (52)$$

A property (39)-(40) can be easily illustrated with the exponentially decaying acf.

We next substitute the acf (52) into (37) and get

$$\sigma_{\eta_k}^2 = k\sigma_A^2 \frac{1-r}{1+r} - \sigma_A^2 \frac{2r(1-r^k)}{(1-r)^2}$$

This confirms (39) at large  $k$  as  $r^k \rightarrow 0$ .

The characteristic function can be also calculated analytically. Substituting (50) into (47) for the inner integral we have

$$\begin{aligned} \int_{\mathbf{x}} e^{i \cdot \mathbf{v}_1 \cdot \mathbf{x}_1} \cdot \delta(x_2 - \varphi(x_1)) \cdot \dots \cdot \delta(x_k - \varphi^{(k-1)}(x_1)) dx_1 &= \sum_{l=1}^m \int_{J_l} e^{i \cdot \mathbf{v}_1 \cdot \mathbf{x}_1} \cdot \prod_{i=1}^{k-1} \delta(x_{i+1} - \varphi^{(i-1)}(\varphi_l(x_1))) dx_1 \\ &= \sum_{l=1}^m \frac{1}{|a_l|} \cdot \int_{\mathbf{x}} e^{i \cdot \mathbf{v}_1 \cdot \frac{z-b_l}{a_l}} \cdot \delta(x_2 - z) \cdot \dots \cdot \delta(x_k - \varphi^{(k-2)}(z)) dz \\ &= \sum_{l=1}^m \frac{1}{|a_l|} \cdot e^{i \cdot \mathbf{v}_1 \cdot \frac{x_2-b_l}{a_l}} \cdot \delta(x_3 - \varphi(x_2)) \cdot \dots \cdot \delta(x_k - \varphi^{(k-2)}(x_2)). \end{aligned}$$

Hence the following recurrence equation can be obtained

$$\Theta_k(\omega_1, \dots, \omega_k) = \sum_{l=1}^m \frac{1}{|a_l|} \cdot e^{-i \cdot \omega_1 \cdot \frac{b_l}{a_l}} \cdot \Theta_{k-1}\left(\omega_2 + \frac{\omega_1}{a_l}, \omega_3, \dots, \omega_k\right)$$

the solution of which is

$$\Theta_k(\omega_1, \dots, \omega_k) = \sum_{i_1, i_2, \dots, i_{k-1}=1}^m \prod_{n=1}^{k-1} \frac{1}{|a_{i_n}|} \cdot e^{-i \cdot \sum_{n=1}^{k-1} \omega_n \cdot \sum_{p=n}^{k-1} b_{i_p} \cdot \prod_{l=n}^p \frac{1}{a_{i_l}}} \Theta_1\left(\sum_{n=1}^{k-1} \omega_n \cdot \prod_{p=n}^{k-1} \frac{1}{a_{i_p}} + \omega_k\right), \quad (53)$$

where  $\Theta_1(\omega) = \int_{\mathbf{x}} e^{i\omega x} p_A(x) dx = \int_0^1 e^{i\omega x} dx = \frac{e^{i\omega} - 1}{i\omega}$  is the characteristic function of the uniform

distribution. Setting  $\omega_1 = \omega_2 = \dots = \omega_k = \omega$  in (53) and substituting the result into (46) we first get a characteristic function and then a required pdf of  $\eta_k$  by use an inverse Fourier transform. In [Baranovski 2003] authors have shown a fast convergence of the characteristic

function  $\Theta_k(\omega, \dots, \omega)$  of the cumulative sum  $\eta_k = \sum_{p=1}^k A_p$  to  $\cos(\bar{\eta} \omega) \exp\left(-\frac{\omega^2 \sigma_{\eta}^2}{2}\right)$ , which is

the characteristic function of a normal distribution with the mean  $\bar{\eta}$  and variance  $\sigma_{\eta}^2$ .

For example, a tent map on the unit interval has the following characteristic function [Baranovski 2003]

$$\Theta_k(\omega) = \frac{1}{2^{k-1}} \sum_{j=1}^{2^{k-1}} e^{i\omega f(k,j)} \Theta_1\left(\frac{\omega}{2^{k-1}}(2j-1)\right),$$

$$\text{where } f(k,j) = \begin{cases} f(k-1,j) + \frac{2j-1}{2^{k-1}}, & \text{for } j=1,2,\dots,2^{k-2} \\ f(k-1,j-2^{k-2}), & \text{for } j=2^{k-2}+1,\dots,2^{k-1} \end{cases}; f(1,1)=0.$$

Then we get a price for the European call option

$$C(t,K) = e^{-r \cdot t} \sum_k P(N(t)=k) \cdot C_k(t,K),$$

where

$$C_k(t,K) = \frac{1}{2a} \sum_{i=1}^{2^{k-1}} \frac{1}{2i-1} \begin{cases} S_0 e^{\tilde{r} \cdot t} (e^{B_2} - e^{B_1}) - K \cdot 2a \cdot \frac{2i-1}{2^{k-1}}, & \text{if } B(t) < B_1; \\ S_0 e^{\tilde{r} \cdot t} (e^{B_2} - e^{B(t)}) - K \cdot (B_2 - B(t)), & \text{if } B_1 < B(t) < B_2; \\ 0, & \text{if } B(t) > B_2. \end{cases}$$

$$B(t) = \ln\left(\frac{K}{S_0}\right) - \tilde{r} \cdot t, B_1 = 2a \cdot \left(f(k,i) - \frac{k}{2}\right), B_2 = B_1 + 2a \cdot \frac{2i-1}{2^{k-1}}$$

which converges to the Black-Scholes price as shown in Fig. 11.

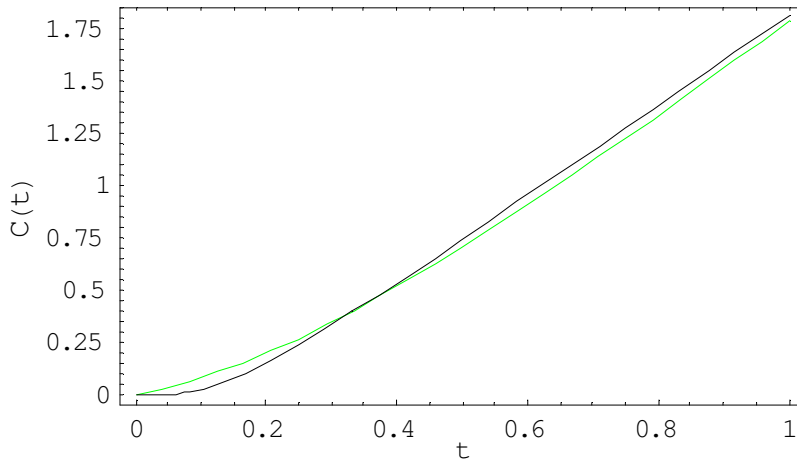


Fig.11. Chaotic price of the European call option (green curve) vs Black-Scholes price (black curve)

**Example 2. Building hybrid (stochastic/chaotic) processes by Brownian subordination.**



Here we calculate a price of the European call option if the underlying asset follows a Wiener process with a time driven by stochastic/chaotic marked point process

$$\eta(t) = \sum_{k=1}^{N(t)} A_k \quad (54)$$

Properties:

1) mean:  $E(W(\eta)) = E[E(W(\eta)|\eta)] = E(0) = 0$

2) variance:

$$\begin{aligned} D(W(\eta)) &= E[D(W(\eta)|\eta)] + D[E(W(\eta)|\eta)] = E[\eta(t)] \\ &= E[E(\eta|N(t))] = E[N(t) \cdot E(A)] = \bar{N}_t \cdot \bar{A} \Rightarrow \bar{A} > 0 \end{aligned}$$

3) distribution function:

$$P\{W(\eta) < y\} = \sum_k P(N(t) = k) \cdot P\{W(\eta) < y | N(t) = k\}$$

Price of the European call option:

$$\begin{aligned} C(t, K) &= e^{-r \cdot t} E \left[ \left( S_0 \cdot e^{\tilde{r}t + W\left(\sum_{i=1}^{N(t)} A_i\right)} - K \right)^+ \right] \\ &= e^{-r \cdot t} \sum_k P(N(t) = k) \cdot \left\{ S_0 e^{\tilde{r}t} \frac{1}{2} e^{\frac{k\bar{A}}{2}} \left[ 1 - \operatorname{erf} \left( \frac{B(t)}{\sqrt{2k\bar{A}}} \right) - \sqrt{\frac{k\bar{A}}{2}} \right] - K \cdot \frac{1}{2} \left[ 1 - \operatorname{erf} \left( \frac{B(t)}{\sqrt{2k\bar{A}}} \right) \right] \right\} \end{aligned} \quad (55)$$

Comparison with Black-Sholes price:

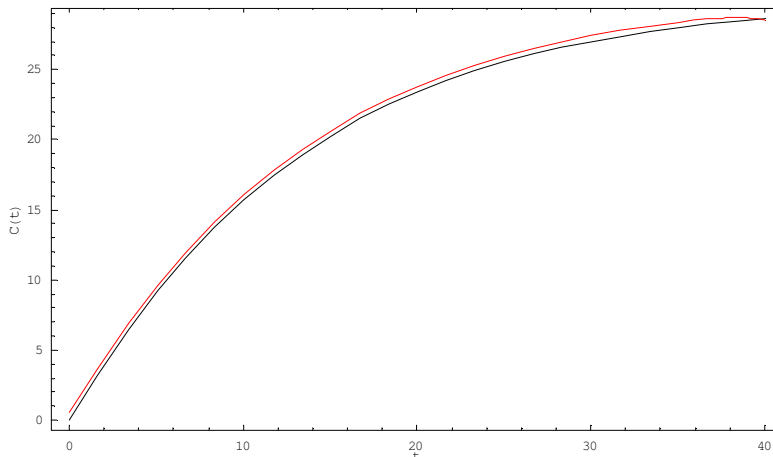


Fig. 12. Black curve is a Black-Sholes price; red one is  $C(t,k)$

#### 4.2. Case of simple real roots of a characteristic polynomial

We consider the case when the characteristic polynomial of the system (31) has simple real negative roots  $\gamma_i, i=1,2,\dots,n-1$ . The Routh-Hurwitz theorem provides necessary conditions for that. By introducing the following two matrices

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \gamma_1 & \gamma_2 & \dots & \gamma_{n-1} \\ \dots & \dots & \dots & \dots \\ \gamma_1^{n-2} & \gamma_2^{n-2} & \dots & \gamma_{n-1}^{n-2} \end{pmatrix}, \mathbf{E}(t) = \begin{pmatrix} e^{\gamma_1 t} & 0 & 0 & 0 \\ 0 & e^{\gamma_2 t} & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & e^{\gamma_{n-1} t} \end{pmatrix} \quad (56)$$

The solution of (31) is then given by:

$$\boldsymbol{\eta}(t) = \mathbf{\Lambda E}(t) \mathbf{D}_k, \text{ at } t_{k-1} \leq t < t_k \quad (57)$$

where the vector of constants  $\mathbf{D}_k = (D_1^{(k)}, \dots, D_{n-1}^{(k)})^T$  can be specified from the initial conditions  $\boldsymbol{\eta}(t_{k-1})$  by

$$D_k = E(-t_{k-1}) \cdot \mathbf{\Lambda}^{-1} \boldsymbol{\eta}(t_{k-1}) \quad (58)$$

As  $\boldsymbol{\eta}$  is a càdlàg we first get for  $t = 0$  ( $k = 1$ )

$$\boldsymbol{\eta}(+0) \equiv \boldsymbol{\eta}_0 = \mathbf{\Lambda} \cdot \mathbf{E}(0) \cdot \mathbf{D}_1 \Rightarrow D_1 = \mathbf{\Lambda}^{-1} \cdot \boldsymbol{\eta}_0$$

then from (32) one can show that for  $t = t_{k-1}$

$$\boldsymbol{\eta}(t_{k-1}) \equiv \boldsymbol{\eta}(t_{k-1} + 0) = \mathbf{\Lambda} \cdot E(t_{k-1}) \cdot D_k = \boldsymbol{\eta}(t_{k-1} - 0) + A_{k-1} \equiv \mathbf{\Lambda} \cdot E(t_{k-1}) \cdot D_{k-1} + A_{k-1} \quad (59)$$

It leads to a recurrent equation:

$$\mathbf{D}_k = \mathbf{D}_{k-1} + \mathbf{E}(-t_k) \cdot \mathbf{\Lambda}^{-1} \mathbf{A}_{k-1}$$

with a solution

$$\mathbf{D}_k = \Lambda^{-1} \boldsymbol{\eta}_0 + \sum_{\ell=1}^{k-1} \mathbf{E}(-\tau_\ell) \Lambda^{-1} \mathbf{A}_\ell, \quad (60)$$

where  $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}_0$ .

Substituting (60) into (57) leads to a general solution of (31) as a mixture of the magnitudes of the all previous kicks in the system:

$$\boldsymbol{\eta}(t) = \Lambda \mathbf{E}(t) \Lambda^{-1} \boldsymbol{\eta}_0 + \sum_{\ell=1}^{k-1} \Lambda \mathbf{E}(t-t_\ell) \Lambda^{-1} \mathbf{A}_\ell, \quad \text{at } t_{k-1} \leq t < t_k \quad (61)$$

From (59) and (58) we derive

$$\mathbf{A}_{k-1} = \boldsymbol{\eta}(t_{k-1}) - \Lambda \cdot \mathbf{E}(T_k) \cdot \Lambda^{-1} \cdot \boldsymbol{\eta}_{k-2} \quad (62)$$

Using (61) Eq. (62) transforms to

$$\mathbf{A}_{k-1} = \boldsymbol{\eta}(t_{k-1}) - \Lambda \mathbf{E}(t_{k-1}) \Lambda^{-1} \boldsymbol{\eta}_0 - \sum_{l=1}^{k-2} \Lambda \mathbf{E}(t_{k-1}-t_l) \Lambda^{-1} \cdot \mathbf{A}_l.$$

The inverse matrix  $\Lambda^{-1}$  exists as the Vandermonde determinant  $\det(\Lambda)$  does not equal to zero and for large time  $t$

$$\Lambda \begin{pmatrix} e^{\gamma_1 t} & & \\ & \dots & \\ & & e^{\gamma_{n-1} t} \end{pmatrix} \Lambda^{-1} \xrightarrow{t \rightarrow \infty} 0.$$

A stationary mode is then established by the second term in (61).

#### 4.2.1. Periodic perturbation: the inter-arrival times are equal to $\Omega$

Now we start to analyze the statistical properties of the stationary mode of process  $\boldsymbol{\eta}(t)$  of a system (31) forced by a periodic shot noise.

Let amplitudes  $A_i^{(p)}$  be independent zero mean random values with the probability density function:  $p_{A_i}(x, t)$ ,  $i=1, 2, \dots, n-1$  common for all  $p$ , i.e.  $t$ ;  $A_i^{(k)}$  and  $A_j^{(m)}$  are mutually

independent values  $\forall i \neq j, j \in \{1, \dots, n-1\}, \forall k, m$ .

Then from (58) the stationary mode is given by

$$\eta_j(t) = \sum_{p=1}^{k-1} \sum_{i=1}^{n-1} A_i^{(p)} \cdot \alpha_{j,i}^{(p)}(t), \quad j = 0, 1, \dots, n-2, \quad (63)$$

where

$$\alpha_{j,i}^{(p)}(t) = \sum_{m=1}^{n-1} s_{m,i} \cdot (\gamma_m)^j e^{\gamma_m \cdot (t-p \cdot \Omega)}, \quad (64)$$

$s_{m,i}$  being the elements of the inverse matrix  $\Lambda^{-1}$ .

Then a vector  $\bar{\eta}$  of the means and a vector  $\sigma_\eta^2$  of the variances of the process  $\eta(t)$  can be calculated from (63) as

$$\bar{\eta} = \mathbf{v} \cdot \bar{A} \quad (65)$$

$$\sigma_\eta^2 = \mathbf{\vartheta} \cdot \sigma_A^2 \quad (66)$$

where  $\bar{A} = (\bar{A}_1, \dots, \bar{A}_{n-1})^T$ ,  $\sigma_A^2 = (\sigma_{A_1}^2, \dots, \sigma_{A_{n-1}}^2)^T$ ,

$\mathbf{v} = (v_{j,i}(t)), j = 0, 1, \dots, n-2; i = 1, \dots, n-1$  is a matrix  $(n-1) \times (n-1)$  with the elements

$$v_{j,i}(t) = \sum_{p=1}^{k-1} \alpha_{j,i}^{(p)}(t), \quad j = 0, 1, \dots, n-2; \quad i = 1, \dots, n-1.$$

$\mathbf{\vartheta} = (v_{j,i}(t)), j = 0, 1, \dots, n-2; i = 1, \dots, n-1$

$$v_{j,i}(t) = \sum_{p=1}^{k-1} [\alpha_{j,i}^{(p)}(t)]^2, \quad j = 0, 1, \dots, n-2; \quad i = 1, \dots, n-1.$$

#### 4.2.2. Asymptotic properties of $\eta(t)$

We introduce the following notations: an exponential pulse shape  $g_i(z) = e^{\gamma_i \Omega z}$ , an amplitude

$$\tilde{A}_k^{(i)} = A_k^{(i)} e^{-\gamma_i \Omega (k-1)}.$$

Taking into account that  $k-1 = \left\lfloor \frac{t}{\Omega} \right\rfloor$  for an arbitrary interval  $(k-1)\Omega < t < k\Omega$ , the process  $\eta(t)$  can be rewritten in the following form

$$\eta(t) = \sum_{i=1}^{n-1} z_i(t),$$

where  $z_i(t) = \sum_{k=1}^{\tilde{A}_i^{(k)}} g_i \left( \frac{t}{\Omega} \bmod 1 \right)$  is a sequence of pulses adjoining to each other with given form  $g_i = g_i(z)$  at  $0 < z \leq 1$  and random amplitudes distributed on  $p_{A^{(i)}}(x), \forall i$  and fixed duration  $\Omega$ .

The characteristic function of the process  $\eta(t)$  factorizes:

$$\Psi(u, t) = E(e^{jx(t)u}) = \prod_{i=1}^n E(e^{jz_i(t)u}) = \prod_{i=1}^n \Psi_i(u, t)$$

and the distribution function:

$$F(y, t) = P(\eta(t) \leq y)$$

can be easily calculated by use of the inverse theorem.

The mean value is given by

$$\int_{-\infty}^{\infty} y dF(y, t) = E(\eta(t)) = \sum_{i=1}^n E(A_k^{(i)}) g_i \left( \frac{t}{T} \bmod 1 \right) = 0, \quad (k-1)\Omega < t < k\Omega,$$

as  $\frac{t - (k-1)\Omega}{\Omega} = \frac{t}{\Omega} \bmod 1$ . The variance of  $\eta(t)$  calculates as

$$E(\eta^2(t)) = E \left( \sum_{i=1}^{n-1} A_k^{(i)} g_i \left( \frac{t}{T} \bmod 1 \right) \right)^2 = \sum_{i=1}^{n-1} E \left( A_k^{(i)} g_i \left( \frac{t}{T} \bmod 1 \right) \right)^2 = \sum_{i=1}^{n-1} E(z_i^2(t)) = \sum_{i=1}^{n-1} b_i^2(t) = B_{n-1}^2(t)$$

We assume that the variance  $b_i^2(t)$  of the elementary process  $z_i(t)$  is finite entailing that for almost all  $t$ :

$$B_n^2(t) \xrightarrow[n \rightarrow \infty]{} \infty.$$

Then one can show that a Lindeberg condition [Feller 1968] is satisfied:

$$\lim_{n \rightarrow \infty} \frac{1}{B_{n-1}^2(t)} \sum_{i=1}^{n-1} \int_{|y| \geq \varepsilon B_{n-1}(t)} y^2 dF_i(y, t) = 0, \quad \forall \varepsilon > 0, \quad (67)$$

where  $F_i(y, t)$  is the distribution function of  $z_i(t)$ . Moreover

$$F_i(y, t) = P(z_i(t) < y) = F_{A_i} \left( \frac{y}{g_i \left( \frac{t}{T} \bmod 1 \right)} \right) \Rightarrow F_i(-y, t) = 1 - F_i(y, t)$$

hence  $\int_{|y| \geq \varepsilon B_n(t)} y^2 dF_i(y, t) = 0$ . But at  $n \rightarrow \infty$

$$\int_{|y| < \varepsilon B_n(t)} y^2 dF_i(y, t) \rightarrow b_i^2(t), \quad \varepsilon B_n(t) \rightarrow \infty.$$

Eq. (67) is a necessary and sufficient condition of convergence  $F(y, t)$  to normal distribution with parameters  $E(\eta(t))=0$  and  $B_{n-1}^2(t)$ , according to the Lindeberg and Feller theorem [Feller 1968].

#### 4.2.3. An empirical model

Let us consider an exponential term structure of interest rates (4) with  $l = 6$  which is characterized by a performance (22). For that we fix a dimension  $n = 6$  of a system (31) and set  $T_k = \Omega$  as well as an arbitrary time  $t$  in a  $k$ -th interval  $t \in [(k-1)\Omega, k\Omega)$ .

According to the second approach (read discussion in section 2 and equation (15) take a spectrum of the eigenvalues by

$$\gamma_1 = -4.21, \gamma_2 = -2.68, \gamma_3 = -1.94, \gamma_4 = -3.83, \gamma_5 = -1.59 \quad (68)$$

corresponding to the median fitted curve of the instantaneous forward rate on the date  $\tau_{fixed} = 255$  (22.02.2007). From the eigenvalues we calculate the matrix  $\mathbf{\Lambda}$  and its inverse  $\mathbf{\Lambda}^{-1}$ .

Above fitting procedure provides a sample of trajectories

$$\boldsymbol{\eta}_\tau(t) = \mathbf{f}(t, Z_\tau) - \mathbf{c}_{fix}, \quad \tau = 1, \dots, q, \quad (69)$$

where  $\mathbf{c}_{fix} = (c_l(\tau_{fix}), 0, \dots, 0)^T$ ,  $c_l(\tau_{fix}) \equiv c_6(255) \equiv 0.043$

We rewrite Eq. ( 62) in the form

$$\mathbf{A}_k(\tau) = \boldsymbol{\eta}_\tau(k \cdot \Omega) - \boldsymbol{\Lambda} \cdot \mathbf{E}(\Omega) \cdot \boldsymbol{\Lambda}^{-1} \cdot \boldsymbol{\eta}_\tau((k-1) \cdot \Omega) \quad (70)$$

and calculate the sample central moments of the magnitudes  $\mathbf{A}_k$

$$\boldsymbol{\mu}_i^{(k)} = \begin{bmatrix} \mu_{i,1}^{(k)} \\ \mu_{i,2}^{(k)} \\ \vdots \\ \mu_{i,n-1}^{(k)} \end{bmatrix} = \frac{1}{q} \sum_{\tau} (\mathbf{A}_k(\tau) - \bar{\mathbf{A}}_k)^i$$

where the vector  $\bar{\mathbf{A}}_k = (\bar{A}_1^{(k)}, \bar{A}_2^{(k)}, \dots, \bar{A}_{n-1}^{(k)})^T$  has the components  $\bar{A}_j^{(k)} = \frac{1}{q} \sum_{\tau=1}^q \bar{A}_j^{(k)}(\tau)$ .

The central moments of the magnitudes  $A_1^{(k)}$  are given by the following empirical relations:

$$\mu_{2i,1}^{(k)} = \sigma_{A_1^{(k)}}^{2i} \cdot e^{\beta(i-1)}, \quad i = 1, 2, \dots \quad (71)$$

and

$$\mu_{2i-1,1}^{(k)} = \left( \sigma_{A_1^{(k)}}^2 \right)^{i-\frac{1}{2}} \cdot \alpha \cdot e^{\beta(i-2)}, \quad i = 2, 3, \dots, \quad (72)$$

where  $\beta = 6.16751$ ,  $\alpha = 21.7945$ .

The central moments of the magnitudes  $A_j^{(k)}$ ,  $j = 2, 3, \dots, n-1$  demonstrate also the following patterns:

the even moments

$$\mu_{2i,j}^{(k)} = \sigma_{A_j^{(k)}}^{2i} \cdot \phi_i^{(k)}, \quad i = 1, 2, \dots \quad (73)$$

and the odd moments

$$\mu_{2i-1,j}^{(k)} = \left( \sigma_{A_j^{(k)}}^2 \right)^{2i-1} \cdot \iota_i^{(k)}, \quad i = 2, 3, \dots \quad (74)$$

where the constants  $\phi_i^{(k)}$ ,  $\iota_i^{(k)}$  can be tabulated.

#### 4.2.4. A chaotic model

Here we discuss an inverse problem: how to design a dynamical system (31) forming process  $\eta(t)$  with the given statistical properties. For that we need to provide a generator of the magnitudes with the prescribed properties discussed above.

We will consider a case when the vector of magnitudes  $\mathbf{A}_k$  is given by

$$\mathbf{A}_k = (0, 0, \dots, 0, A_k)^T, \quad (75)$$

where the magnitudes are chaotic variables. It means that a just  $(n-2)$ -th derivative of a solution  $\eta(t)$  changes by jump  $A_k$ , which is governed by a chaotic map  $A_k = \varphi(A_{k-1})$ .

The system coefficients  $\beta_k$  are coupled with the eigenvalues  $\{\gamma_1, \dots, \gamma_{n-1}\}$  by Vieta's formula.

Without loss of generality we fix an arbitrary time  $t$  in a  $k$ -th interval  $t \in [(k-1)\Omega, k\Omega)$ . Then from (61) the stationary mode is given by

$$\eta_j(t) = \frac{d^j}{dt^j} \eta(t) \equiv \eta^{(j)}(t) = \sum_{p=1}^{k-1} \alpha_{p,j}(t) A_p, \quad j = 0, 1, \dots, n-2 \quad (76)$$

where

$$\alpha_{p,j}(t) = \sum_{i=1}^{n-1} s_{i,n-1} \cdot \gamma_i^j e^{\gamma_i \left( \frac{t}{\Omega} - p \right)}, \quad (77)$$

$s_{i,n-1}$  being the elements of the inverse matrix  $\mathbf{\Lambda}^{-1}$ .

The mean and variance of the process  $\eta(t)$  and its derivatives are

$$\bar{\eta}_j = E(\eta_j(t)) = \bar{A} \sum_{p=1}^{k-1} \alpha_{p,j}(t) \quad (78)$$

$$\sigma_{\eta_j}^2 = D(\eta_j(t)) = \sigma_A^2 \sum_{i=1}^{k-1} \alpha_{i,j}^2(t) + 2 \sum_{l=1}^{k-2} c_A(l) \sum_{i=1}^{k-l-1} \alpha_{i,j}(t) \alpha_{i+l,j}(t) \quad (79)$$

The distribution function of  $\eta_j(t)$  is then defined by

$$F_j(y, t) = P(\eta_j(t) \leq y) = P\left( \sum_{p=1}^{k-1} \alpha_{p,j}(t) A_p \leq y \right) \quad (80)$$

and its characteristic function becomes



$$\psi_j(\omega, t) = E \left( i \omega \sum_{p=1}^{k-1} \alpha_{p,j}(t) A_p \right) = \Theta_{k-1}(\omega \alpha_{1,j}(t), \omega \alpha_{2,j}(t), \dots, \omega \alpha_{k-1,j}(t)) \quad (81)$$

where  $\Theta_k(\omega_1, \omega_2, \dots, \omega_k)$  is the  $k$ -dimensional characteristic function of the sequence  $\{A_p, p=1, \dots, k-1\}$ .

Note that for a case of i.i.d. random values  $A_k$  (68) simplifies to

$$\Psi_j(\omega, t) = \prod_{i=1}^{k-1} \Theta_1(\omega \alpha_{i,j}(t)) \quad (82)$$

where  $\Theta_1(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} p_A(x) dx$  is the characteristic function of the distribution of  $A_k$ .

As above shown a central limit theorem holds for  $\sum_{p=1}^{k-1} A_p$ . Then a characteristic function (81)

of  $\eta_j(t)$  approaches  $\cos(\bar{\eta}_j \cdot \omega) \exp(-2^{-1} \omega^2 \sigma_{\eta_j}^2) \forall j$  at large  $t$  or  $k$ . Thus the response of a linear system (31) forced by chaotic shot noise is normally distributed process.

From the eigenvalues (68) by use of Vieta's formulas we define the coefficients  $\beta_i, i = 0, 1, 2, 3, 4$  leading to the following 5th order differential equation

$$\eta^{(v)} + 14.24 \cdot \eta^{(iv)} + 82.25 \cdot \eta''' + 208.81 \cdot \eta'' + 267.94 \cdot \eta' + 132.77 \cdot \eta = \sum_i A_i \delta(t - i \cdot \Omega) \quad (83)$$

The equation has a solution (76) at  $j = 0$ .

We assume that the amplitudes  $A_k$  of the impulse perturbation are chaotic uncorrelated variables with zero mean. Then from (79) we get

$$\sigma_{\eta}^2 = \sigma_A^2 \cdot \mu(t) \quad (84)$$

where  $\mu(t) = \sum_{i=1}^{k-1} \alpha_{i,0}^2(t)$ .

Under the given spectrum of the eigenvalues  $\gamma_i, i = 1, \dots, 5$  (68) and the following coefficients  $s_{1,5} = 0.29, s_{2,5} = 0.7, s_{3,5} = -0.9, s_{4,5} = -0.54, s_{5,5} = 0.44$  and, for example,  $\Omega = 1/360$  (one day) a component  $\mu(t)$  can be easily calculated by (77) as shown in Fig. 13

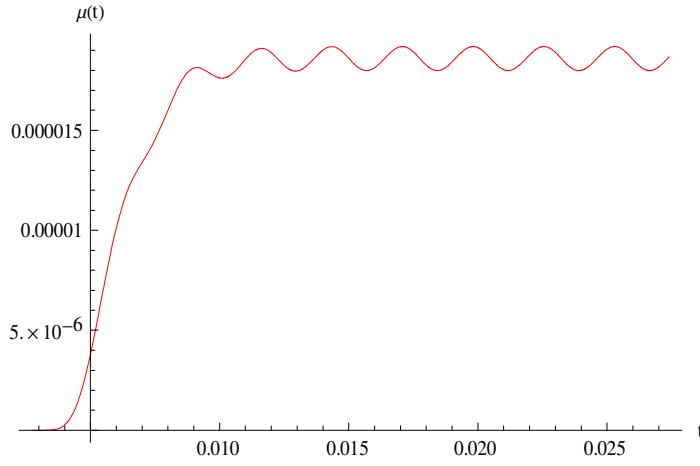


Fig. 13  $\mu(t)$  within  $\Omega < t < 10 \Omega$

The  $\mu(t)$  quickly becomes a periodic function with a period  $\Omega$ .

In Fig. 14 we plot a ratio  $\frac{\sigma_{\eta}^2(t)}{\mu(t)}$  calculated from a sample variance of  $\eta(t)$ .

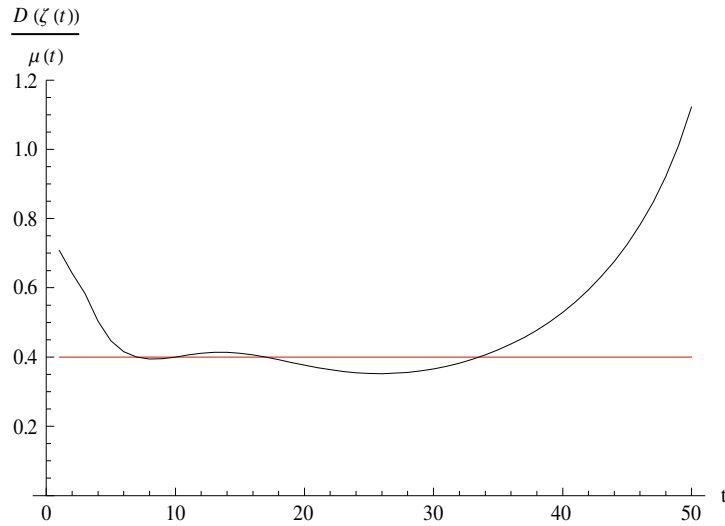


Fig. 14. A ratio  $\frac{\sigma_{\eta}^2(t)}{\mu(t)}$  within  $0 < t < 50$  years

From (84) and Fig. 18 we can approximate a variance of the magnitudes by

$$\sigma_A^2 = \frac{\sigma_{\eta}^2}{\mu(t)} \approx 0.4, \quad 5 \leq t \leq 35 \quad (85)$$

For a special case  $t = 7$  (years) we calculate  $\sigma_{\eta}^2 \approx 7.459 \cdot 10^{-6}$ ,  $\mu(t) \approx 1.867 \cdot 10^{-5}$  and then

$$\frac{\sigma_{\eta}^2(t)}{\mu(t)} = 0.3995.$$

On the base of an approach [Baranovski&Daems 1995] we design a piece-wise linear map

$$A_{i+1} = \begin{cases} 2A_i + b, & -b \leq A_i < 0 \\ -2A_i + b, & 0 \leq A_i \leq b \end{cases} \quad (86)$$

which is characterized by an uniform probabilistic measure on the interval  $[-b, b = 1.095]$  with zero mean, the variance 0.4 and zero acf  $c_A(n) = 0, \forall n \geq 1$ .

Taking into account an analysis in section 4.1. and having a chaotic sequence  $\{A_1, A_2, \dots, A_{k-1}\}$  one can compute a solution  $\eta(t)$  and its fourth derivative  $\eta_4(t)$  on the time interval  $[0, k\Omega]$  as shown in Figs. 15 - 17.

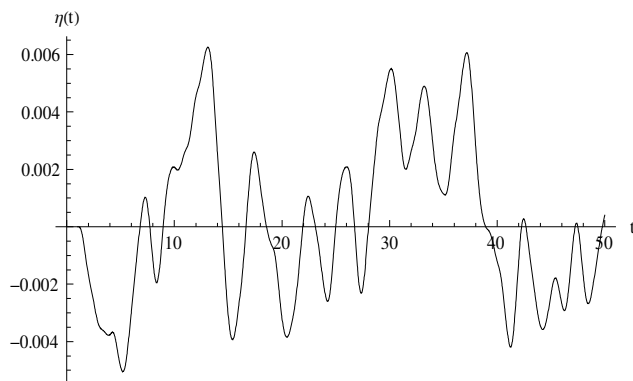


Fig. 15 a solution  $\eta(t)$  on the time interval  $[\Omega, 50 \text{ years}]$ ;  $\Omega = 1$  (one year)

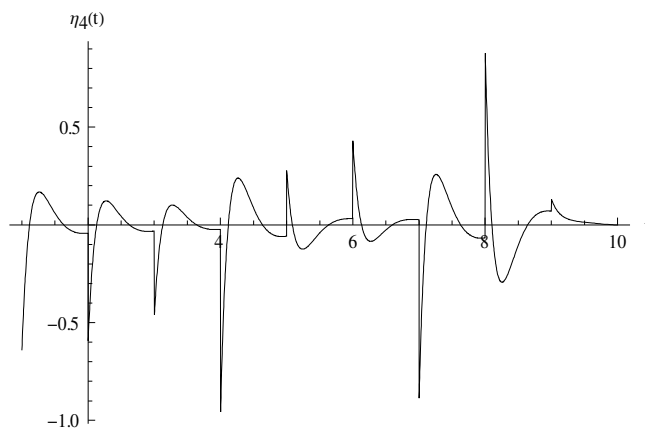


Fig. 16 a solution  $\eta_4(t)$  on the time interval  $[\Omega, 10 \text{ years}]$ ;  $\Omega = 1$  (one year)

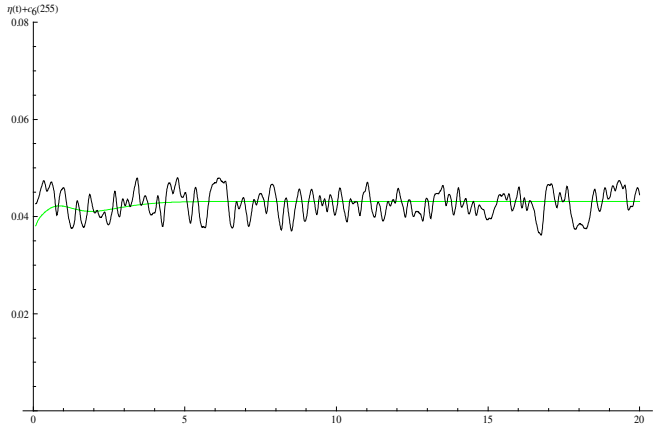


Fig. 17 . A path (black trajectory) of a solution of dynamical system with amplitudes  $A_k$  generated by tent map (70) in comparison to  $f(t, Z_{255})$  (green curve): the instantaneous forward rate at time  $t$  for date 22.02.2007 ;  $\Omega = 1/12$  (one month)

Fig. 16 demonstrates a jump character of the last component of the vector solution  $\eta(t)$  and confirms that the magnitudes of the jumps are  $\eta_4(k\Omega + 0) - \eta_4(k\Omega - 0) \equiv A_k$ .

A phase portrait on Fig. 18 represents an attractor of the dynamical system.

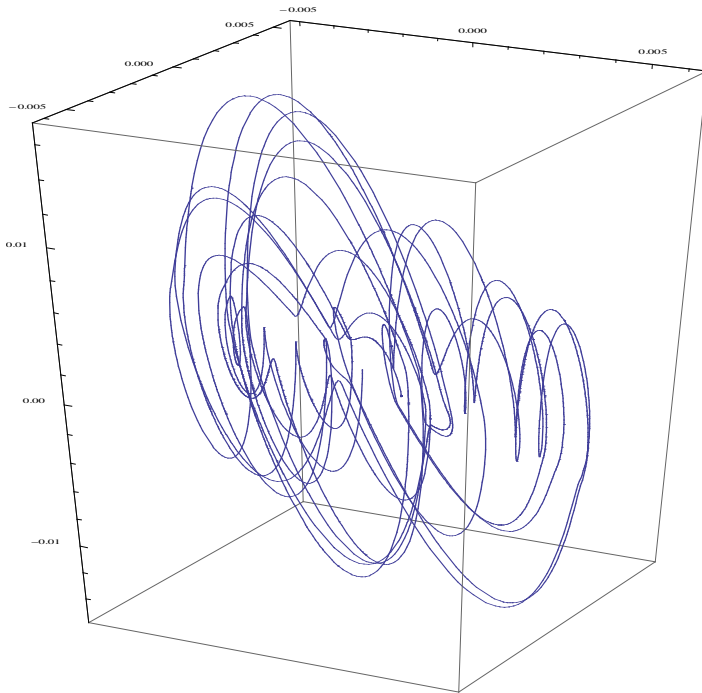


Fig. 18. Parametric plot of the vector  $[\eta_0(t), \eta_1(t), \eta_2(t)]$  at  $0 \leq t \leq 50$  and  $\Omega = 1$  (one year)

Fig. 19 shows plots  $[\eta_0(t), \eta_1(t), \eta_2(t)]$  and  $[\eta_1(t), \eta_2(t), \eta_3(t)]$  together.

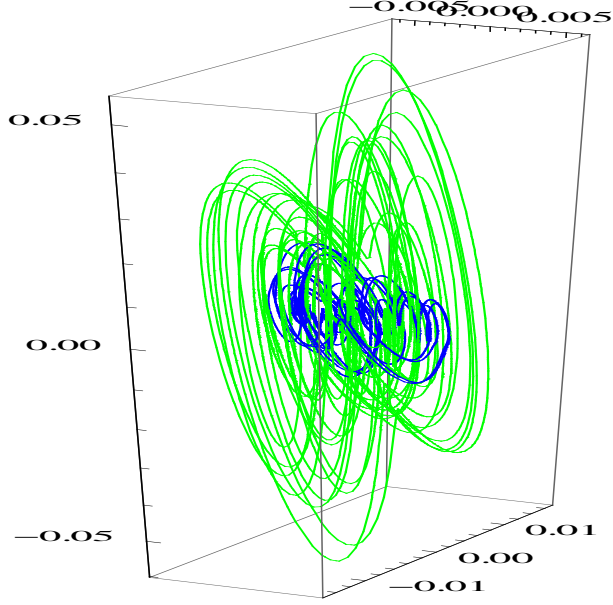


Fig. 19 Parametric plots  $[\eta_0(t), \eta_1(t), \eta_2(t)]$  (blue curve) and  $[\eta_1(t), \eta_2(t), \eta_3(t)]$  (green curve)

It is clear from Fig.23 that a dynamical system (70) demonstrates “expansion”, characterized by exponential growth of the variances  $\sigma_{\eta_j}^2$  with  $j$ , order of derivative of the solution  $\eta(t)$ .

A characteristic function of  $\eta_j(t)$  can be analytically obtained. We first establish that the map (87) is topologically equivalent to a tent map (51) with the parameters  $\{(a_1 = 2, b_1 = 0), (a_2 = -2, b_1 = 2)\}$ . A corresponding homeomorphism is the following linear function  $2b \cdot x - b$ ;  $b = 1.095$  [Baranovski&Daems 1995] . It implies that a characteristic function (81) of the solution  $\eta_j(t)$  in a  $k$ -th interval  $t \in [(k-1)\Omega, k\Omega)$  is given by

$$\psi_j(\omega, t) = \Theta_{k-1} \left( 2b\omega\alpha_{1,j}(t), 2b\omega\alpha_{2,j}(t), \dots, 2b\omega\alpha_{k-1,j}(t) \right) \cdot \text{Exp} \left( -i \cdot b\omega \sum_{p=1}^{k-1} \alpha_{p,j}(t) \right), \quad (87)$$

where the  $(k-1)$ -dimensional characteristic function (66) of a tent map.

Fig. 20 shows the characteristic function (87) at  $t = 7$  (years), i.e.  $t = (k-1) \cdot \Omega$ ,  $k = 8$ ,  $\Omega = 1$  in

comparison with  $\text{Exp} \left( -\frac{\omega^2 \sigma_{\eta}^2}{2} \right)$  as the characteristic function of a Gaussian distribution with

zero mean and a variance  $\sigma_\eta^2 = \sigma_A^2 \sum_{i=1}^7 \alpha_{i,0}^2(t) \approx 7.467 \cdot 10^{-6}$  calculated from (79) at  $c_A(n) = 0, \forall n \geq 1$ .

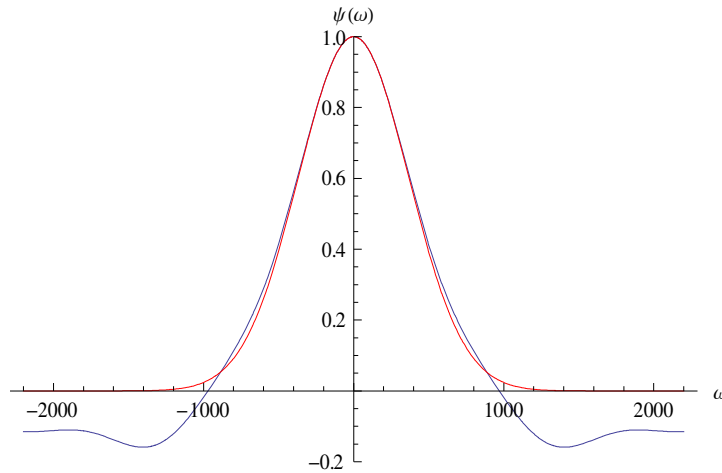


Fig. 20. Characteristic function of  $\eta(t)$  at  $t = 7$  (years) (black curve) and Gaussian process (red curve)

A histogram of the 1000 paths of  $\eta(t)$  at  $t = 7$  is presented in Fig. 21

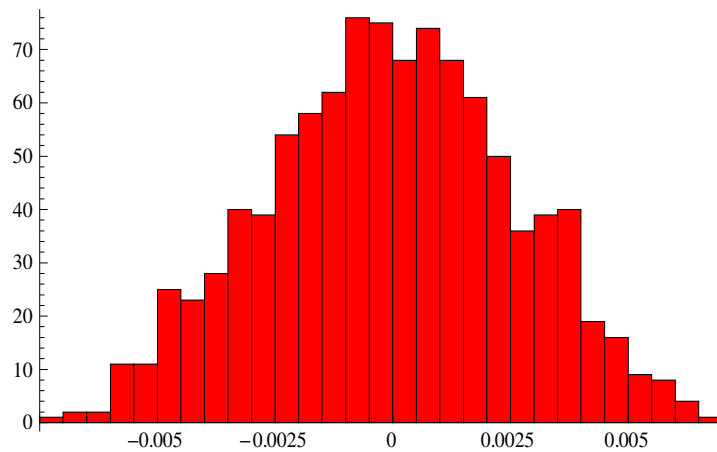


Fig. 21 Histogram of the solution  $\eta(t)$  at  $t=7$

A  $\chi^2$  Pearson's test with the confidence level 0.95 accepts a hypothesis on normal distribution of the solution  $\eta(t)$  with zero mean and the variance  $7.46 \cdot 10^{-6}$ .

In Table 6 we collect all remaining characteristic functions of  $\eta_j(t)$ ,  $j = 1, 2, 3, 4$  at  $t = 7$  (years), i.e.  $t = (k-1) \cdot \Omega$ ,  $k = 8, \Omega = 1$  in comparison with the corresponding asymptotical

curves  $Exp\left(-\frac{\omega^2 \sigma_{\eta_j}^2}{2}\right)$ ,  $j = 1, 2, 3, 4$ , the characteristic functions of a Gaussian distribution with

zero mean and a variance  $\sigma_{\eta_j}^2 = \sigma_A^2 \sum_{i=1}^7 \alpha_{i,j}^2(t)$ ,  $j = 1, 2, 3, 4$  (Fig. 26)

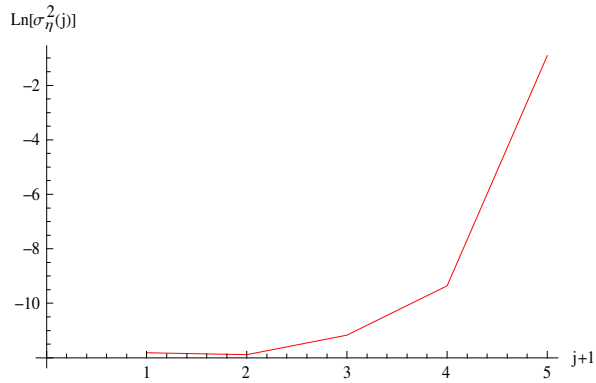
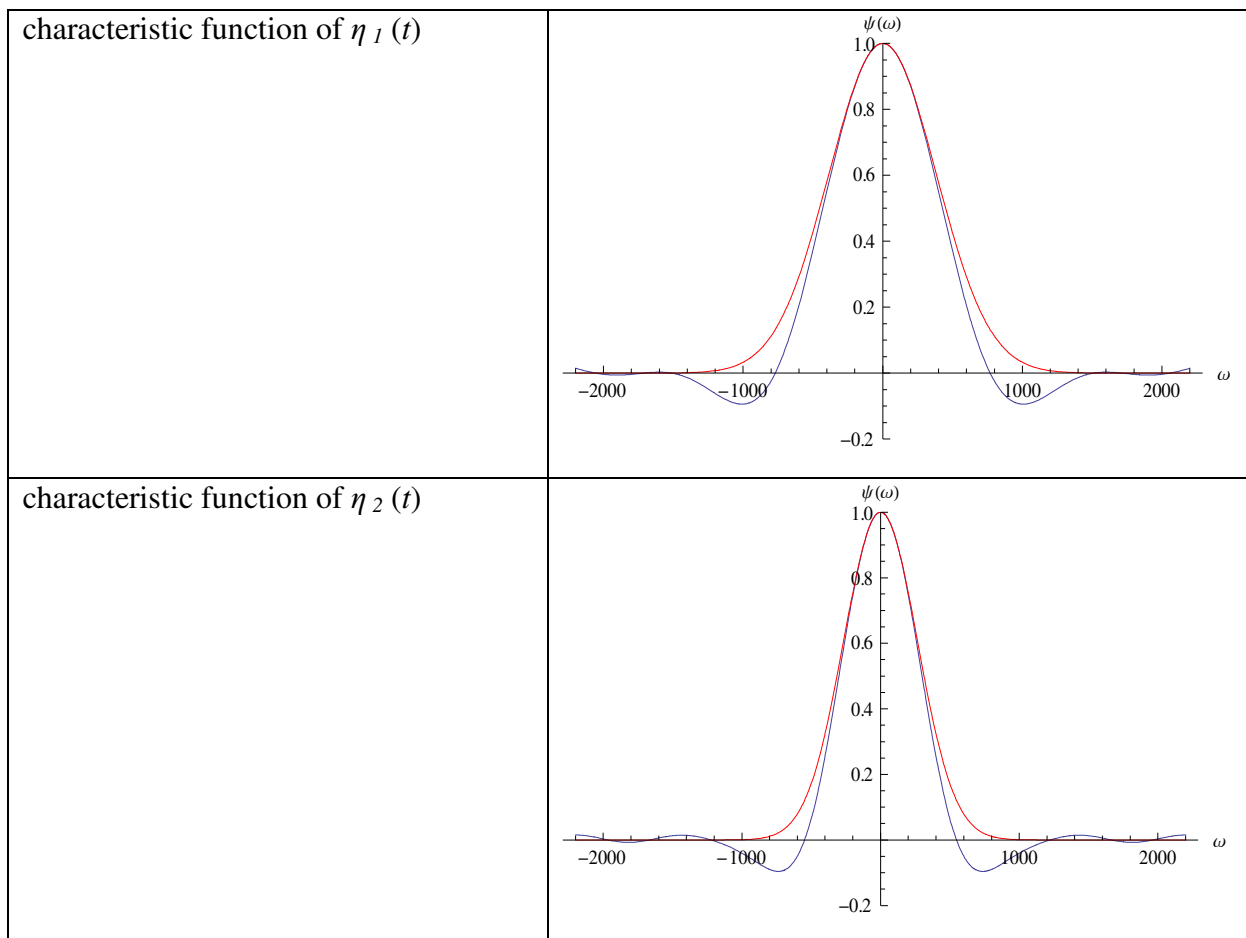


Fig. 22. Log of the variance



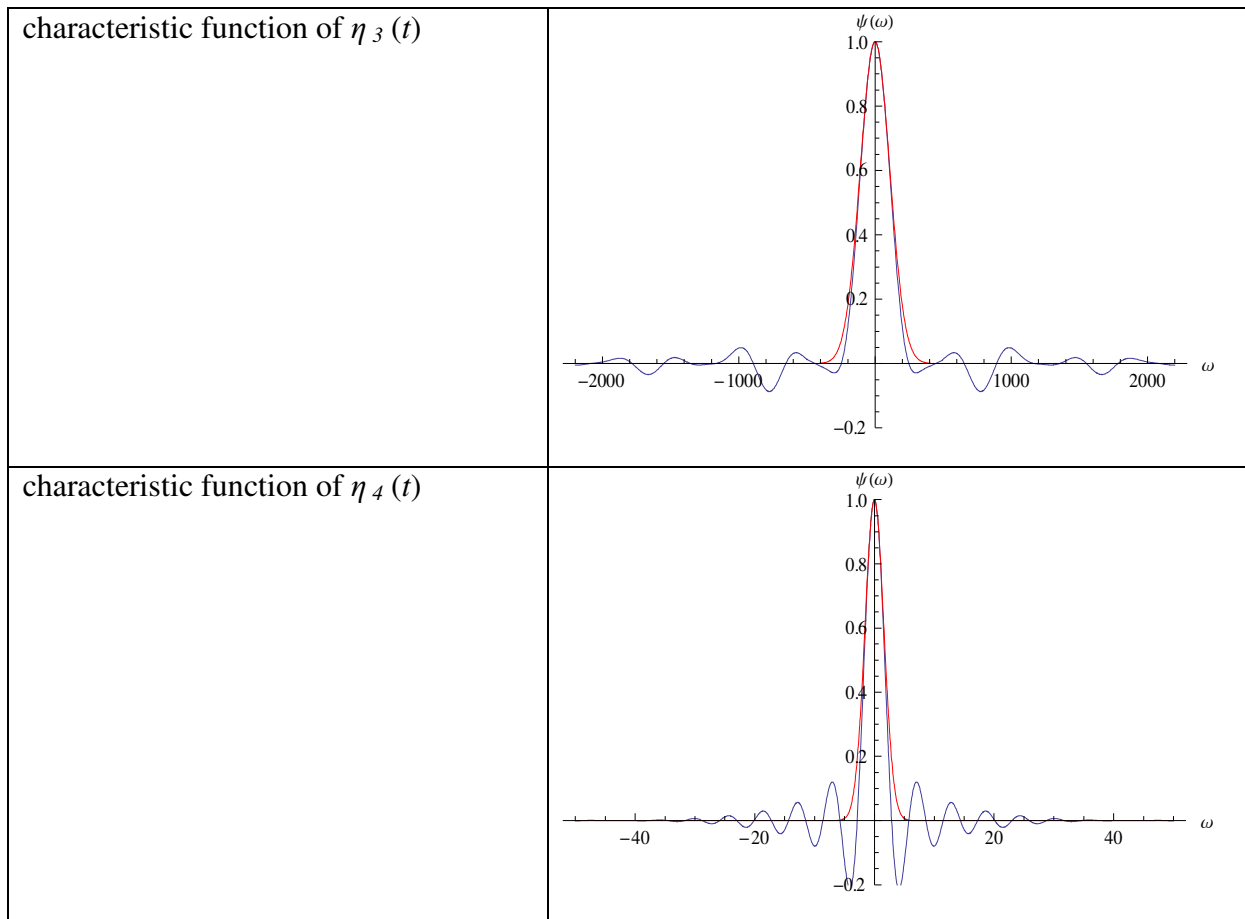


Table 6. Characteristic functions of the components of a solution  $\eta$  in comparison with the corresponding asymptotical characteristic functions of a Gaussian distribution

## Conclusions

In this paper we have first proposed an exponential-polynomial model of the interest rates and then demonstrated its performance in a fitting of the zero-coupon curves. Capturing dynamic dependencies in the fitted curves we have in a second step designed a dynamical system forced by shot noise with chaotic/stochastic jumps. In our proposed class the mean-reversion speed of the diffusive and the jump part can be adjusted separately or jointly by a suitable design of chaotic maps with prescribed probabilistic properties.

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