

Gürtler, Marc; Heithecker, Dirk

Working Paper

Multi-period defaults and maturity effects on economic capital in a ratings-based default-mode model

Working papers // Institut für Finanzwirtschaft, Technische Universität Braunschweig, No. FW19V2

Provided in Cooperation with:

Technische Universität Braunschweig, Institute of Finance

Suggested Citation: Gürtler, Marc; Heithecker, Dirk (2005) : Multi-period defaults and maturity effects on economic capital in a ratings-based default-mode model, Working papers // Institut für Finanzwirtschaft, Technische Universität Braunschweig, No. FW19V2, Institut für Finanzwirtschaft, Technische Universität Braunschweig, Braunschweig

This Version is available at:

<http://hdl.handle.net/10419/55248>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

Working Paper Series



Multi-Period Defaults and Maturity Effects on Economic Capital in a Ratings-Based Default-Mode Model

by Marc Gürtler and Dirk Heithecker

No.: FW19V2/05

First Draft: 2005-12-22

This Version: 2005-04-19

Braunschweig University of Technology
Institute for Economics and Business Administration
Department of Finance
Abt-Jerusalem-Str. 7
D-38106 Braunschweig

Multi-Period Defaults and Maturity Effects on Economic Capital in a Ratings-Based Default-Mode Model^{***}

by Marc Gürtler* and Dirk Heithecker**

Abstract. In the last decade, portfolio credit risk measurement has improved significantly. The current state-of-the-art models analyze the value of the portfolio at a certain risk horizon, e.g. one year. Most popular has become the Merton-type one-factor model of Vasicek, that builds the fundament of the new capital adequacy framework (Basel II) finally adopted by the Basel Committee On Banking Supervision in June 2004. Due to this approach credit risk only arises from defaults, and the model provides an analytical solution for the risk measures Value at Risk and Expected Loss. One of the less examined questions in this field of research is, how the time to maturity of loans affects the portfolio credit risk. In practice there is common agreement that credit risk rises with the maturity of a loan, but only few solutions considering different maturities are discussed. We present two new approaches, how to cope with the problem of the maturity in the Vasicek-model. We focus on the influence of the maturity in the theoretical framework of Merton and show solutions from empirical data of four rating agencies. Our results are close to the parameters, that are used in the maturity adjustment of Basel II and may help to get a better understanding on economic capital allocation of long-term loans.

Keywords: Basel II, Capital Adequacy Requirements, Probability of Default, Default Mode Models, Maturity Adjustment, Time Horizon

JEL classification: G21, G28

* **Professor Dr. Marc Gürtler**
Braunschweig University of Technology
Department of Finance
Abt-Jerusalem-Str. 7, 38106 Braunschweig, Germany
Phone: +49 531 3912895 - Fax: 3912899
eMail: marc.guertler@tu-bs.de

****Dipl.-Wirtsch.-Ing. Dirk Heithecker**
Braunschweig University of Technology
Department of Finance
Abt-Jerusalem-Str. 7, 38106 Braunschweig, Germany
Phone: +49 531 3912894 - Fax: 3912899
eMail: d.heithecker@tu-bs.de

^{***} We gratefully thank Andreas Huber from the CREDITREFORM Rating AG, Department of Credit Risk and Portfolio Analysis for providing data and helpful support.

1. Introduction

During the recent ten years credit risk evaluation has been one of the most challenging topics in the field of banking and finance. Best effort has been put into the development of portfolio models that specify the distribution of a credit portfolio on a certain risk horizon. From this analysis arises the possibility to quantify the portfolio risk due to risk measures like Value at Risk (VaR) and Expected Loss (EL), or the so-called unexpected loss ($UL=VaR-EL$).¹ In risk management these measures are used to identify provisions and economic capital that should be raised by a bank to protect itself against future losses from credit defaults.²

Well known (commercial) models are CreditPortfolioViewTM, CreditRisk+TM, CreditPortfolioManagerTM, and CreditMetricsTM, that all have influenced academic research and credit risk applications in banks.³ After all, a one-factor version of CreditMetricsTM, that has its origin in the seminal model due to Merton (1974, 1977) and Black and Scholes (1973), builds the fundament of the capital adequacy formula for credit risk in the new capital standards (Basel II) finalized by the Basel Committee On Banking Supervision (2004).⁴

All of these commercial and supervisory models are two state discrete-time portfolio models and they mainly differ in the assumptions of the distribution of the input factor⁵ and their attitude towards credit risk arising from changes in the portfolio value. CreditMetricsTM and CreditPortfolioManagerTM evaluate altering portfolio values due to changes in the credit spreads of loans, that are mainly caused by changes of the rating grade (rating migrations) and therefore of the debtor's probability of default. These models make use of a mark-to-market (MTM) paradigm.⁶ Other models like CreditPortfolioViewTM or CreditRisk+TM only account for losses due to the defaults of loans. These models follow the so-called default mode (DM) paradigm.

However, all considered models agree in the fact, that the value of the credit portfolio is only observed with respect to a predefined time horizon (typically one year) that is consequently

¹ See Basel Committee On Banking Supervision (1999), p. 14.

² See e.g. Marisson (2002), pp. 19-23, and Chorafas (2005), pp. 32-35.

³ See Crosbie and Bohn (2003), Credit Suisse Financial Products (1997), Wilson (1997a,b) and Gupton, Finger, and Bathia (1997) for details about this models.

⁴ See Basel Committee On Banking Supervision (2005) as well as Finger (1999, 2001), Wilde (2001) and Gordy (2003).

⁵ However, the results of the models are quite similar if parameterised from a single data source. See Koyluoglu, and Hickmann (1998a,b) or Hamerle and Rösch (2004).

⁶ See Basel Committee On Banking Supervision (1999) p. 22.

equal to the time horizon in Basel II. Obviously, this time horizon generally does not correspond with the actual maturity of the loans in a credit portfolio.⁷ However, for such models, neither a well described theoretic approach nor an empirical implementation exists, that discusses a possible impact of the time horizon and maturity of loans on the risk measure of credit portfolios. Particularly, this is valid for DM-models like the one of Basel II - although the credit risk quantification in Basel II contains a maturity adjustment for long-term loans.

Presently, few literature analyse the effect of the maturity of loans in the MTM paradigm. Kalkbrenner and Overbeck (2001, 2002) estimate the distributions of the market values of credit portfolios with different maturities at the end of a one-year planning horizon. Their analysis is based on the one-year migration matrix and the maturity varying credit spreads.⁸ They conclude, that the maturity adjustment of Basel II is conservative in comparison to their results. While Kalkbrenner and Overbeck use Monte Carlo simulations, Barco (2004) analyses this maturity effect in a similar model analytically.⁹ He states, that the capital curves for the maturity adjustment are much flatter than the one of Basel II. A similar approach is introduced by Grundke (2003), who uses different spread curves that are generated from the Merton (1974) model. He concludes, that the Basel II adjustment is at least explainable for reasonable spread curves.¹⁰ However, all these approaches suffer from the problem, that they measure the effect of maturity of loans in the MTM-paradigm, whereas Basel II as well as most portfolio models in banks (like CreditPortfolioViewTM or CreditRisk+TM) are DM-models for which no analysis exists, that deals with the influence of longer maturities on credit risk.¹¹ In addition, the concrete derivation of the Basel II maturity adjustment is not disclosed in detail so far.¹²

In order to account for maturity effects when calculating economic capital in a DM-model the default probability is one of the most important parameter. Thus, in first step we have to

⁷ Especially capital investment loans of companies have longer maturities than one year.

⁸ The authors use data series from Standard & Poor's, KMV German Corporates for the migration matrices as well as CreditMetricsTM and Gordy and Heitfield (2001) for credit spreads.

⁹ He uses data from Standard & Poor's for the migration matrices and LI (1998) as well as Kiesel, Perraudin, and Taylor (2003) for the credit spreads.

¹⁰ Especially, this is valid for a "CCC"-rated bond. For the two other rating classes investigated ("AAA" and "BBB") his maturity adjustment still differs from the Basel II adjustment.

¹¹ Li, Song, and Ong (1999) discuss the effect of the maturity in a default mode model but with respect to shorter time horizons than one year.

¹² Some notes are made in Basel Committee On Banking Supervision (2005), pp. 9-10. Nevertheless, these explanations are far from being applicable in a straight forward manner.

concretize the term default especially with respect to the maturity of a loan. There are two possibilities to understand default until maturity of a loan in a discrete-time framework.

Firstly, because the market value of a loan with respect to its current rating often is not observable and¹³ the loan possibly is intended to be held to maturity (buy and hold)¹⁴ the bank only perceives default if a loan is not repaid.¹⁵ Thus, in such a situation – considering the discrete nature of credit portfolio models - default occurs if and only if the value of the borrowers assets falls below a certain barrier at maturity. To calculate the economic capital on the basis of this type of default we develop a model approach which we call “Capital to Maturity”-approach. Secondly we take into account that banks should analyse (perhaps by the use of ratings) the loans of their credit portfolio continuously. A bank at least should rate a borrower at certain points in time, e.g. each year. In this context default occurs if the borrower is overindebted at one of the specified rating dates. Since economic capital has to be adjusted according to the corresponding default probabilities at each points in time the development of a model to determine the economic capital on the basis of such a meaning of default we denote the “Capital for One Period”-approach. To conclude, with our distinction between a “Capital to Maturity”-approach and a “Capital for One Period”-approach we are in line with the Committee On Banking Supervision that differentiates between a “liquidation period”-approach and a “constant time horizon for all asset classes”-approach with basically the same meaning as the approaches mentioned above.¹⁶

Thus, with this article we make a contribution to the ongoing scarce research on the influence of different maturities of loans on credit risk based on the framework of Merton (1974) and Vasicek (1987, 1991) which in turn is the bottom of the Basel II-model. We develop two possible solutions how the economic capital requirements of a individual loan (quantified by its UL contribution the portfolio UL) rises with its maturity. This theoretically derived maturity adjustments are implemented and analysed empirically based on the average default rates from four rating agencies. In addition, both adjustments are compared with the one’s of

¹³ See Basel Committee On Banking Supervision (1999), p. 17.

¹⁴ Perhaps due to limited markets, the credit could not be traded before maturity.

¹⁵ A similar proceeding is used in the IAS 39, where loans and receivables shall be “measured at amortised cost” if the fair value is not observable (IAS 39.46/47) and losses only occur, if there is “objective evidence that an impairment loss has been incurred” (IAS 39.63). This especially is valid for so called “held-to-maturity” investment (IAS 39.9). For details see European Union (2004).

¹⁶ See Basel Committee On Banking Supervision (1999), p. 16.

Basel II since understanding differences between supervisory capital requirements and own, internal approaches may be one of the key Iss.s in bank's credit risk management in future.¹⁷

The rest of the paper is organized as follows: In section 2 we briefly review the one-factor credit portfolio model and derive the “Capital to Maturity”- and the “Capital for One Period”- approach on the basis of Merton's model. Section 3 describes the empirical implementation of the two paradigms and presents concrete parameters of a maturity adjustment formula. We finalize the paper with a conclusion in section 4.

2. Modelling Maturity Effects in Ratings-Based Capital Rules

In the following subsection 2.1 we highlight the main results of the model of Merton (1974) and Vasicek (1987, 1991, 2002).¹⁸ According to the framework, we derive the maturity effect on a single exposure, especially on the default probability, in subsection 2.2 and 2.3. We discuss the mentioned two different approaches.

2.1 The Model Outline of Basel II and Vasicek

We use a simple two state discrete-time portfolio model based on a DM paradigm with an ability-to-pay process.¹⁹ According to this approach, in $t = 0$ the possible loss of a credit portfolio at time horizon $t = T$ is evaluated, and losses only result from defaulted loans. The state of default of a single credit is fixed if its asset return does not reach a given default point at $t = T$.²⁰ A one-factor model for the asset returns of different credits considers default correlations within the portfolio. Concretely, with reference to Merton (1974) and Vasicek (1987), we assume that²¹

- (a) the observed bank holds loans of various borrowers i ($i \in \{1, \dots, n\}$) with respective exposure L_i ,
- (b) the logarithm $\ln(\tilde{A}_i)$ of the value \tilde{A}_i of the firms assets of each borrower i follows a Wiener process with constant drift μ_i and volatility σ_i ,

¹⁷ See e.g. Chorafas (2005), pp. 32, 61-66.

¹⁸ This model often called the one-factor-model of CreditMetrics, see Gupton, Finger, and Bhatia (1997) and Finger (1999).

¹⁹ For definitions of a “two state discrete-time portfolio model” see Rösch (2004), p. 311, and Liebig, Hamerle, and Rösch (2003), p. 3, of the “DM paradigm” see Basel Committee On Banking Supervision (1999), pp. 17-18, and of the “ability-to-pay process” see Bluhm and Overbeck (2003), p. 35, and Crosbie (1999).

²⁰ In this paper T represents a standardised time horizon, e.g. $T = 1$ year. Modified time horizons are treated in the following sections and denoted with M .

²¹ To keep track of the model, stochastic variables are marked with a tilde “ $\tilde{\cdot}$ ”.

(c) default of borrower i occurs at time $t = T$, if the value $\tilde{A}_i^{(T)}$ of the firms assets falls below a barrier $B_i^{(T)} = B_i^{(0)} \cdot \exp(r_i \cdot T)$ in which r_i stands for the (continuous) contractual interest rate.

As a result of assumption (b) the value $\tilde{A}_i^{(T)}$ of the firms assets at $t = T$ can be written as

$$\ln(\tilde{A}_i^{(T)}) = \ln(\tilde{A}_i^{(0)}) + \mu_i^{(T)} + \sigma_i^{(T)} \cdot \tilde{a}_i^{(T)} \quad (1)$$

with mean $\mu_i^{(T)} = \mu_i \cdot T$ and standard deviation $\sigma_i^{(T)} = \sigma_i \cdot \sqrt{T}$. The variable $\tilde{a}_i^{(T)} \sim N(0,1)$ is standard normal distributed and can be interpreted as the normalized (continuous) return of \tilde{A}_i in the period from $t = 0$ to $t = T$. Due to assumption (c) the default probability of a single borrower i at $t = 0$ can be quantified as

$$PD_i^{(0,T)} = P(\tilde{A}_i^{(T)} < B_i^{(T)}) = N(b_i^{(0,T)})$$

$$\text{with } b_i^{(0,T)} = \left[\ln(B_i^{(0)}/A_i^{(0)}) - \mu_{i,\text{eff}}^{(T)} \right] / \sigma_i^{(T)} \quad \text{and } \mu_{i,\text{eff}}^{(T)} = (\mu_i - r_i) \cdot T. \quad (2)$$

In the literature $b_i^{(0,T)}$ is denoted as the default point.²² Since the default probability $PD_i^{(0,T)}$ is assumed to be exogenously given as a credit rating and not calculated on the basis of equation (2), these models are often called “ratings-based”.²³ Precisely, borrowers are divided into different risk buckets (rating grades) and for each risk bucket the probability of default is determined empirically over a time period of length T .²⁴ However, for a model based investigation of the maturity effect it seems to be a good choice to take equation (2) as a basis, since the variables addressed (financial leverage ($B_i^{(0)}/A_i^{(0)}$), expected return after cost of outside capital ($\mu_{i,\text{eff}}^{(T)}$), and risk of return $\sigma_i^{(T)}$) mainly influence the credit rating.²⁵

The potential gross loss rate $\tilde{\Lambda}_n^{(0,T)}$ of the credit portfolio (before recovery) is²⁶

²² See Crosbie and Bohn (2003), p. 7, Crouhy, Galai, and Mark (2000), pp. 373-374, or Ong (1999), p. 85.

²³ See e.g. Gordy (2003).

²⁴ Due to Basel II, at least seven rating grades for non-defaulted loans are compulsory, and the probability of default is the long-run average of the one-year default rates. See Basel Committee On Banking Supervision (2004), paragraphs 404 and 447. Due to this approach, all loans in one risk bucket are treated equally. In the context under consideration, the index i can be used for a specific risk bucket as well as for an individual loan of this bucket.

²⁵ In practice, credit ratings and predictions of default probabilities are often done via discriminate analysis or binary regressions, see Altmann (2001) or Blochwitz, Liebig, and Nyberg (2000). Models based on the original Merton framework are implemented by Moody's KMV (EDFTM RiskCalcTM), see Dwyer, Kocagil, and Stein (2004), and by RiskMetrics Group (CreditGradesTM), see Finger (2002).

²⁶ Like Vasicek (2002) we only consider the gross loss rate before recovery, i.e. the loss (rate) given default (LGD) is equal to one. This proceeding may be satisfied by the fact, that the IRB-model of Basel II only refers to the default rate of the portfolio rather than the loss rate. Precisely, the LGD is treated as a constant in the

$$\tilde{\Lambda}_n^{(0,T)} = \sum_{i=1}^n \omega_i \cdot \mathbf{I}(\tilde{A}_i^{(T)} < B_i^{(T)}) := \sum_{i=1}^n \omega_i \cdot \tilde{\ell}_i^{(0,T)} \quad \text{with } \omega_i = L_i / \sum_{i=1}^n L_i, \quad (3)$$

where $\tilde{\ell}_i^{(0,T)}$ is the loss rate of each individual loan. In order to derive an analytical solution, we make some additional assumptions. According to Vasicek (1991) and Gordy (2003) we accept that

- (d) the normalised returns $\tilde{a}_i^{(T)}$ of all borrowers $i = \{1, \dots, n\}$ follow a classical one factor model with only non negative correlations $\text{corr}(\tilde{a}_i^{(T)}, \tilde{a}_j^{(T)}) = \sqrt{\rho_i \cdot \rho_j} \geq 0$, with $i, j \in \{1, \dots, n\}$ and $i \neq j$,
- (e) the portfolio is “infinitely granular” or “asymptotic invariant”.

With these assumptions the loss rate $\lim_{n \rightarrow \infty} \tilde{\Lambda}_n^{(0,T)} =: \tilde{\Lambda}_\infty^{(0,T)}$ of the (infinitely granular) portfolio can be analysed with respect to the risk measure Value at Risk (VaR), Expected Loss (EL) and Unexpected Loss (UL).²⁸ Precisely, we may write the UL of the loss rate $\tilde{\Lambda}_\infty^{(0,T)}$ as a sum of the UL of each individual portfolio loss rate $\tilde{\ell}_i^{(0,T)}$

$$\text{UL}(\tilde{\Lambda}_n^{(0,T)}) := \sum_{i=1}^n \omega_i \cdot \text{UL}(\tilde{\ell}_i^{(0,T)}) \quad (4)$$

with the borrowers individual risk contribution to the portfolio-UL of

$$\text{UL}(\tilde{\ell}_i^{(0,T)}) := \text{VaR}_z(\tilde{\ell}_i^{(0,T)}) - \mathbb{E}(\tilde{\ell}_i^{(0,T)}) \quad (5)$$

$$\text{with } \text{VaR}_z(\tilde{\ell}_i^{(T)}) = \mathbf{N}\left(\frac{\mathbf{N}^{-1}(\text{PD}_i^{(0,T)}) - \sqrt{\rho_i} \cdot \mathbf{x}_{q_{1-z}}^{(T)}}{\sqrt{1-\rho_i}}\right) \quad \text{and } \mathbb{E}(\tilde{\ell}_i^{(0,T)}) = \text{PD}_i^{(0,T)}. \quad (6)$$

Here $\mathbf{x}_{q_{1-z}}^{(T)} = q_{1-z}(\tilde{\mathbf{x}}^{(T)})$ is the $(1-z)$ -quantile of $\tilde{\mathbf{x}}^{(T)}$.²⁹ If a bank plans to meet a given target probability of default of (only) α , a bank should hold economic capital for each debtor i amounting to the respective UL contribution (times the exposure L_i of the loan) in order to

foundation approach of the IRB model. In the advanced approach own models have to built up. Therefore, we keep at analysing the maturity effect on the default rate or gross loss rate rather than LGD.

²⁷ $\mathbf{I}(S)$ indicates the indicator function with $\mathbf{I}(S) = 1$, if statement S is true, and $\mathbf{I}(S) = 0$, otherwise.

²⁸ See appendix A.1 for details.

²⁹ The equations (5) and (6) are similar to the capital requirements, like they are formulated in the IRB approach of Basel II, considering a maturity of one year ($M = 1$). Additionally, $\text{PD}_i^{(T)}$ is the expected probability of default (PD) at time horizon of one year ($T = 1$ year) and ρ_i is the correlation (R), that is a function of $\text{PD}_i^{(T)}$, accepting values between 0.12 and 0.24. For details, see Basel Committee On Banking Supervision (2004), paragraphs 272 and 285. Like discussed in footnote 26, the LGD is set to 1.

protect itself from potential future losses.³⁰ Since $UL(\tilde{\ell}_i^{(0,T)})$ only depends on the individual parameters $PD_i^{(0,T)}$ and ρ_i but not on the other exposures in the portfolio, these capital charges are called “portfolio invariant”.³¹ The UL contribution is shown in Figure 1. It has a concave characteristic in $PD_i^{(0,T)}$ for $PD_i^{(0,T)} < 0.5$.³² Thus, the UL contribution is more sensitive to changes in $PD_i^{(0,T)}$ if the probability of default is low than if it is high.

- Figure 1 about here -

This well known derivation of the portfolio invariant UL contribution builds the basic framework for following proceeding to develop economic capital. In credit risk management, the concrete time horizon $\Delta t = T$ of one year is often used, but we discuss the effect of longer maturities in the next sections 2.2 and 2.3.³³ Since the economic capital requirement will increase with rising probability of default, we analyse the effect of longer maturity on the probability of default of an individual borrower in the context of the MERTON-model and how this circumstance influences the portfolio credit due to the VASICEK-model. As already mentioned, we consider two paradigms: the so called “Capital to Maturity”-approach, that considers a time horizon equal to the maturity of the loan, and the “Capital to Maturity”-approach, that accounts for a time horizon shorter than the maturity of the loan in a multi-period framework.

2.2 The “Capital to Maturity”-Approach

As stated in the introduction we presume in this section, that the capital requirement for a loan hold by the bank amounts to the UL contribution until time to maturity of the loan. Theoretically, in the VASICEK-Model altering maturities t of (all) loans $i = \{1, \dots, n\}$ in the portfolio just lead to modified probabilities of default $PD_i^{(0,M)}$ in equation (2). Therefore, we firstly discuss the modification of the probability of default $PD_i^{(0,M)}$ with respect to the

³⁰ Here we additionally assume, that expected losses are covered by provisions, that reduce the economic capital.

³¹ See Gordy (2003), p. 208.

³² For $PD > 0.5$ the density function of the portfolio losses changes the characteristic, e.g. the median will be higher than PD with $PD < 0.5$, but lower if $PD > 0.5$. This leads to a declining UL contribution if $PD > 0.5$. For details see Vasicek (2002).

³³ Due to Basel II, the maturity M in the IRB foundation approach is set to $M = 2.5$. In the advanced IRB approach the values vary between one and five, i.e. $M \in [1, 5]$.

variable M in the context of MERTON (1974). After summing up the main results, we secondly give a conclusion how these results can be integrated in a capital formula.

Precisely, we compare the probability of default of a loan with maturity $t = T$ to one with maturity $M = m \cdot T$ and $m \geq 1$. The probability of default at $t = m \cdot T$ becomes³⁴

$$PD_i^{(0,m \cdot T)} = N(b_i^{(m \cdot T)}) \text{ with } b_i^{(m \cdot T)} = \frac{N^{-1}(PD_i^{(0,T)})}{\sqrt{m}} - k_i \cdot \frac{m-1}{\sqrt{m}} \text{ and } k_i = \mu_{i,\text{eff}}^{(T)} / \sigma_i^{(T)}. \quad (7)$$

Thus, the probability of default $PD_i^{(0,m \cdot T)}$ at $t = m \cdot T$ depends on the probability of default $PD_i^{(0,T)}$ at $t = T$ and a parameter k_i , that can be retrieved from the return after cost of outside capital ($\mu_{i,\text{eff}}^{(T)}$) and its standard deviation ($\sigma_i^{(T)}$). Furthermore, we want to analyse in which way an extension of maturity (from T to $m \cdot T$) influences the probability of default. Concretely, we get for $m > 1$

$$PD_i^{(0,m \cdot T)} > PD_i^{(0,T)} \Leftrightarrow \frac{N^{-1}(PD_i^{(0,T)})}{\sqrt{m}} - k_i \cdot \frac{m-1}{\sqrt{m}} > N^{-1}(PD_i^{(0,T)}) \Leftrightarrow k_i < -\frac{N^{-1}(PD_i^{(0,T)})}{\sqrt{m+1}} \quad (8)$$

and thus, the probability of default $PD_i^{(0,m \cdot T)}$ will rise in comparison to $PD_i^{(0,T)}$ if k_i or $PD_i^{(0,T)}$ are low. Consequentially, the probability of default $PD_i^{(0,m \cdot T)}$ is more likely to rise compared to $PD_i^{(0,T)}$ for low probabilities of default $PD_i^{(0,T)}$ than for high values of $PD_i^{(0,T)}$.

We receive a similar result if we consider situations with m nearly equal to 1 since³⁵

$$\partial PD_i^{(0,m \cdot T)} / \partial m \Big|_{m=1} = -\frac{N^{-1}(PD_i^{(0,T)})}{2 \cdot \sqrt{m^3}} - k_i \cdot \left(\frac{m+1}{2 \cdot \sqrt{m^3}} \right) \Big|_{m=1} > 0 \Leftrightarrow k_i < -0.5 \cdot N^{-1}(PD_i^{(0,T)}). \quad (9)$$

To sum up, an extension of maturity (from $M = T$ to $M = m \cdot T$) for low probabilities $PD_i^{(0,T)}$ leads to a stronger increase of the probability of default than for high probabilities $PD_i^{(0,T)}$. This impact is expected to strengthen when examining the UL contribution, since the function UL subject to PD has a concave characteristic. This connection will be analysed next.

For the UL contribution of a loan of borrower i with maturity $m \cdot T$ we receive

$$UL^{(CtM)}(\tilde{\ell}_i^{(0,m \cdot T)}) := \text{VaR}_z^{(CtM)}(\tilde{\ell}_i^{(0,m \cdot T)}) - E^{(CtM)}(\tilde{\ell}_i^{(0,m \cdot T)}) \quad (10)$$

³⁴ See appendix A.2 for a derivation of (7).

³⁵ For an analysis of the derivations of the probability of default in the Merton-model also see Farnen, Fleten, Westgaard, and Van Der Wijst (2004).

$$\text{with } \text{VaR}_z^{(\text{CtM})}(\tilde{\ell}_i^{(0,m,T)}) = N\left(\frac{N^{-1}(\text{PD}_i^{(0,m,T)}) - \sqrt{\rho_i} \cdot \mathbf{x}_{q_{1-z}}^{(m,T)}}{\sqrt{1-\rho_i}}\right) \text{ and } E^{(\text{CtM})}(\tilde{\ell}_i^{(0,m,T)}) = \text{PD}_i^{(0,m,T)} \quad (11)$$

$$\text{and } \text{PD}_i^{(0,m,T)} = N\left(\frac{N^{-1}(\text{PD}_i^{(0,T)})}{\sqrt{m}} - k_i \cdot \frac{m-1}{\sqrt{m}}\right). \quad (12)$$

Obviously, only the parameters $\text{PD}_i^{(0,T)}$, ρ_i , m , and k_i have to be estimated to determine the “maturity adjusted” $\text{UL}^{(\text{CtM})}(\tilde{\ell}_i^{(0,m,T)})$. Since the first two parameters³⁶ are ascertainable from the one period ratings based model³⁷ and the parameter m is given, it is only necessary to estimate $k_i = \mu_{i,\text{eff}}^{(T)} / \sigma_{A,i}^{(T)}$. Because $\mu_{i,\text{eff}}^{(T)}$ und $\sigma_{A,i}^{(T)}$ are not available the estimation of k_i has to be based on empirical data using equation (12) and empirical cumulative default rates (to estimate $\text{PD}_i^{(0,m,T)}$). The concrete (“matching”) procedure for such an implicit model based estimation of $k_i =: k_i^{(\text{CtM})}$ will be explained in the empirical section 3.³⁸

In contrast to this, an alternative approach in order to calculate the UL at maturity $m \cdot T$ is to determine the UL at maturity T and adjust it by a function, that depends on m . Precisely, from (10) and (12) we receive the UL contribution of the loan via an equation

$$\text{UL}^{(\text{CtM})}(\tilde{\ell}_i^{(0,m,T)}) = \text{UL}^{(\text{CtM})}(\tilde{\ell}_i^{(0,T)}) \cdot g^{(\text{CtM})}(\text{PD}_i^{(0,T)}, k_i, m, \rho_i) \quad (13)$$

$$\text{with } g^{(\text{CtM})}(\text{PD}_i^{(0,T)}, k_i, m, \rho_i) = \frac{\text{UL}(\tilde{\ell}_i^{(0,m,T)})}{\text{UL}(\tilde{\ell}_i^{(0,T)})} = \frac{N\left(\frac{N^{-1}(\text{PD}_i^{(0,m,T)}) - \sqrt{\rho_i} \cdot \mathbf{x}_{q_{1-z}}^{(T)}}{\sqrt{1-\rho_i}}\right) - \text{PD}_i^{(0,m,T)}}{N\left(\frac{N^{-1}(\text{PD}_i^{(0,T)}) - \sqrt{\rho_i} \cdot \mathbf{x}_{q_{1-z}}^{(T)}}{\sqrt{1-\rho_i}}\right) - \text{PD}_i^{(0,T)}}. \quad (14)$$

The function $g^{(\text{CtM})}$ can be called “maturity adjustment” under the “Capital to Maturity”-Paradigm. It could be determined, if the parameters $\text{PD}_i^{(0,T)}$, ρ_i , m , and k_i are known.

If the parameters are not available, we have to approximate $g^{(\text{CtM})}$ by a (simple) function which is fitted to the data. Such a function would allow for a capital formula (only depending

³⁶ Due to the assumptions of the Vasicek-model correlation parameter ρ_i remains unchanged with different maturities $m \cdot T$.

³⁷ For the determination of $\text{PD}_i^{(0,T)}$ and ρ_i see e.g. Gordy and Heitfield (2002), p. 7, Rösch (2003), p. 312, Demey, Jouanin, Roget, and Roncalli (2004), p. 105, Düllmann and Trapp (2004), p. 8, Hamerle, Liebig, and Rösch (2003), p. 78 (for a Maximum Likelihood estimator) as well as Gordy (2000), p. 133, and Bluhm and Overbeck (2003), p. 40 (for a Method of Moments estimator).

³⁸ We use $k_i^{(\text{CtM})}$ with the index “CtM” since the concrete value depends on the described specific model. Of course, $\text{PD}_i^{(0,m,T)}$ could be used directly to obtain $\text{UL}^{(\text{CtM})}(\tilde{\ell}_i^{(0,m,T)})$.

on the probability of default $PD_i^{(0,T)}$) that accounts for the maturity of a loan under the “Capital to Maturity”-approach.

In the next section, we analyse the corresponding effect in the context of the “Capital for One Period” paradigm.

2.3 The “Capital for One Period” approach

Taking into account, that credit engagements are observed over time, i.e. the borrower is re-rated after a certain period, say one year, a bank may only be interested in the risk contribution of a loan until the revaluation. According to the introduction we call this approach “Capital for One Period”. Thus, the time horizon for determining credit risk does not change with maturity of the loan as in the “Capital to Maturity”-approach. The economic capital is only specified for the next period, e.g. until the next annual financial statement.

If the bank is not able to cancel a credit contract until maturity, it faces the problem, that the probability of default may increase over several periods until maturity and therefore the capital requirements do. If the bank is not able to raise additional economic capital for this credit, possibly it would violate the capital rules. To avoid such a situation, an adjustment of the Vasicek-formula is needed. Since obviously the change in the individual probability of default of the borrower is of interest, we firstly investigate its change over time in the Merton-framework in a j -period context. Secondly, we suggest an adjustment based on the received results.

The maturity of the loan is $M = m \cdot T$ with $m > 1$ and the debtor will be re-evaluated $j-1$ times (with $j \in \mathbb{N}^+$) until time to maturity. Thus, the relevant points in time are $t_h = n_h \cdot T$ with $h = \{0, 1, 2, 3, \dots, j\}$ and n_h a real number with $0 = n_0 < n_1 < n_2 < \dots < n_j = m$. Without loss of generality we may set $n_1 = 1$. Especially, we are interested in the expected probability of default until maturity $t = M = m \cdot T$ evaluated at $t = 0$. We get

$$PD_i^{(0,m \cdot T)} = PD_i^{(0,T)} + \Delta PD_i^{(t_1,t_2)} + \Delta PD_i^{(t_2,t_3)} + \dots + \Delta PD_i^{(t_{j-1},t_j)} = PD_i^{(0,t_1)} + \sum_{s=2}^j \Delta PD_i^{(t_{s-1},t_s)} \quad (15)$$

with

$$\Delta PD_i^{(t_{s-1},t_s)} := PD_i^{(t_{s-1},t_s)} \cdot \prod_{k=1}^{s-1} (1 - PD_i^{(t_{k-1},t_k)}). \quad (16)$$

The (conditional) probability of default $PD_i^{(t_{k-1}, t_k)}$ represents the probability of default at t_k evaluated at $t = 0$, considering (only) the case, that the borrower has survived until t_{k-1} . Thus, the value $PD_i^{(0, m \cdot T)}$ depends on the probability of default $PD_i^{(0, T)}$ in the first period and the marginal values $\Delta PD_i^{(t_s, t_{s-1})}$, that can be interpreted as the increase of the probability of default for each additional period. These “marginal” probabilities of default can be calculated as³⁹

$$\Delta PD_i^{(t_h, t_{h-1})} = N_{h+1} \left(-N^{-1} \left(PD_i^{(0, T)} \right), \dots, -N^{-1} \left(PD_i^{(0, t_{h-1})} \right), N^{-1} \left(PD_i^{(0, t_h)} \right); (\rho_{op})_{o,p=1, \dots, h} \right) \quad (17)$$

with the (unconditional) default probabilities

$$PD_i^{(0, t_h)} = N \left(b_i^{(0, t_h)} \right) \text{ and } b_i^{(0, t_h)} = \frac{N^{-1} \left(PD_i^{(0, T)} \right)}{\sqrt{n_h}} - k_i \cdot \frac{n_h - 1}{\sqrt{n_h}} \quad (18)$$

in which $N_\eta(x_1, \dots, x_\eta; (\rho_{op})_{o,p=1, \dots, \eta})$ describes the η -dimensional normal distribution at (x_1, \dots, x_η) with correlation matrix $(\rho_{op})_{o,p=1, \dots, \eta}$. The latter matrix can be determined by the calculation rule $(\rho_{op})_{o,p=1, \dots, \eta} = T \cdot T^T$ with $T = (\tau_{op})_{o,p=1, \dots, \eta}$ using $\tau_{op} = \sqrt{n_p / n_o} \cdot I(p \leq o)$ for $o, p = 1, \dots, \eta$.

Concerning the capital requirements of the specified loan, in the first period the UL contribution $UL(\tilde{\ell}_i^{(0, T)})$ is derived under consideration of the probability $PD_i^{(0, T)}$. If the loan survives at $t = T$ we expect a UL contribution $UL(\tilde{\ell}_i^{(T, t_2)})$ using $PD_i^{(T, t_2)}$ for the second period. Similar, if the loan would survive until t_{k-1} , for the following period a UL contribution $UL(\tilde{\ell}_i^{(t_{k-1}, t_k)})$ using $PD_i^{(t_{k-1}, t_k)}$ will be expected. If $PD_i^{(t_{k-1}, t_k)} > PD_i^{(0, T)}$ is valid for any k , the capital requirement for this credit will raise over the time with reference to the first period (as long as the debtor survives and $PD_i^{(0, T)}, PD_i^{(t_{k-1}, t_k)} < 0,5$). If no “buffer” for this expected increase would have been hold in $t = 0$, the bank would have to raise additional capital. In order to prevent this, the maximum expected default probability should be used when determining the economic capital via the UL contribution.

Considering the simple two period case with $n_h \in \{0, 1, m\}$, the UL contribution $UL_i^{(T, m \cdot T)}$ will increase in comparison to $UL_i^{(0, T)}$ of the first period if the following statement holds:⁴⁰

$$PD_i^{(0, T)} < PD_i^{(T, m \cdot T)} (< 0,5) \text{ if } PD_i^{(0, T)} < 0,5 - \sqrt{0,25 - \Delta PD_i^{(T, m \cdot T)}} \text{ and } \Delta PD_i^{(T, m \cdot T)} < 0,25. \quad (19)$$

³⁹ See appendix A.3 for details.

⁴⁰ See appendix A.4 for details.

Obviously, with low values for $PD_i^{(0,T)}$ it will be more likely to fulfill the inequality than with high values. Additionally, low probabilities of default incorporate a high possibility of survival during the first period. Therefore, $\Delta PD_i^{(T,m\cdot T)}$ is expected to be (relatively) high, which leads to an increase of the right hand side of inequality (19). Summing up, for declining $PD_i^{(0,T)}$ the chance of an increase of the UL contribution is expected to grow and a capital buffer is more likely to be needed.

Finally, we determine such a capital buffer for the UL contribution. If the loan has a time to maturity of $m\cdot T$ and if the loan is revaluated we receive the capital rule

$$UL^{(CoP)}(\tilde{\ell}_i^{(0,m\cdot T)}) = VaR_z^{(CoP)}(\tilde{\ell}_i^{(0,m\cdot T)}) - E^{(CoP)}(\tilde{\ell}_i^{(0,m\cdot T)}) \quad (20)$$

$$\text{with } VaR_z^{(CoP)}(\tilde{\ell}_i^{(0,m\cdot T)}) = N\left(\frac{N^{-1}(\widehat{PD}_i^{(0,m\cdot T)}) - \sqrt{\rho_i} \cdot x_{q_{1-z}}^{(m\cdot T)}}{\sqrt{1-\rho_i}}\right), \quad E^{(CoP)}(\tilde{\ell}_i^{(m\cdot T)}) = \widehat{PD}_i^{(0,m\cdot T)} \quad (21)$$

$$\text{and } \widehat{PD}_i^{(0,m\cdot T)} = \max\left(PD_i^{(0,T)}, PD_i^{(t_1,t_2)}, PD_i^{(t_2,t_3)}, \dots, PD_i^{(t_{j-1},m\cdot T)}\right), \quad (22)$$

in which $PD_i^{(t_2,t_3)}, PD_i^{(t_3,t_4)}, \dots$ are obtained from equations (17) and (18). As presented in the ‘‘Capital to Maturity’’ approach,⁴¹ only the parameters $PD_i^{(0,T)}$, ρ_i , m , and k_i are of interest when determining $UL^{(CoP)}(\tilde{\ell}_i^{(0,m\cdot T)})$. Again, the first three parameters are known from the one period model or are exogenously given. It is only necessary to specify $k_i = \mu_{i,\text{eff}}^{(T)} / \sigma_{A,i}^{(T)}$ from empirical data. In ratings-based models, we use equation (17) and (18) and empirical default rates for the estimation (‘‘matching’’) of an implicit, model based $k_i =: k_i^{(CoP)}$, since the relevant parameter k_i is not observable directly. This will be done in the empirical section 3.

Again, it is possible to calculate the UL at maturity $M = m\cdot T$ on the basis of an adjusted UL at maturity T . Precisely, from equations (15) to (18) as well as (20) and (22) we get the UL contribution of the loan via the equation

$$UL^{(CoP)}(\tilde{\ell}_i^{(0,m\cdot T)}) = UL^{(CoP)}(\tilde{\ell}_i^{(0,T)}) \cdot g^{(CoP)}\left(PD_i^{(0,T)}, k_i, m, \rho_i\right) \quad (23)$$

⁴¹ See subsection 2.2 and especially the footnotes 36 to 38.

$$\text{with } g^{(\text{CoP})}(\text{PD}_i^{(0,T)}, k_i, m, \rho_i) = \frac{\text{UL}^{(\text{CoP})}(\tilde{\ell}_i^{(0,m,T)})}{\text{UL}^{(\text{CoP})}(\tilde{\ell}_i^{(0,T)})} = \frac{N\left(\frac{N^{-1}(\widehat{\text{PD}}_i^{(0,m,T)}) - \sqrt{\rho_i} \cdot x_{q_{1-z}}^{(T)}}{\sqrt{1-\rho_i}}\right) - \widehat{\text{PD}}_i^{(0,m,T)}}{N\left(\frac{N^{-1}(\text{PD}_i^{(0,T)}) - \sqrt{\rho_i} \cdot x_{q_{1-z}}^{(T)}}{\sqrt{1-\rho_i}}\right) - \text{PD}_i^{(0,T)}}. \quad (24)$$

The function $g^{(\text{CoP})}$ can be called ‘‘maturity adjustment’’ under the ‘‘Capital for one Period’’-paradigm. Once more, this adjustment function could be determined, if the parameters $\text{PD}_i^{(0,T)}$, ρ_i , m , and k_i are know. Alternatively (and analogously to the ‘‘Capital to Maturity’’-paradigm) a simple function may be used to fit $g^{(\text{CoP})}$ that only requires the knowledge of the probability of default $\text{PD}_i^{(0,T)}$ and the maturity of a loan.

After discussing theoretically the effect of the maturity of a loan with respect to the model of Vasicek (1987, 1991) and Merton (1974) and after presenting two different approaches for determining a maturity adjustment factor, in the next section 3 we implement both models under consideration of empirical data.

3. Estimation of the Maturity Adjustment Factor

In this section the maturity adjustment factor based on the framework of the model of section 2 is derived. We start with a description of the data and the general outline of the analysis in Subsection 3.1. Subsection 3.2 deals with the maturity adjustment under the ‘‘Capital to Maturity’’-paradigm, whereas in subsection 3.3 the maturity adjustment according to the ‘‘Capital to Maturity’’-paradigm is analyzed.

3.1 Outline of the Empirical Analysis and Data Description

In order to appraise the effect of longer time to maturity on risk capital in the Vasicek-model, we use long-term default rates as estimators for the probabilities of default. We take empirical data from worldwide corporate default studies of three rating agencies (Moody’s Investors Service, Standard & Poor’s, and Fitch) and from default studies of German corporates of the rating agency Creditreform Rating AG, that merely deals with small and medium sized companies.⁴² Concretely, we use the average cumulative default rates $\overline{\text{DR}}_i^{(0,m,T)}$ for each

⁴² See Hamilton, Varma, Ou, and Cantor (2005) for Moody’s, Brady and Vazza (2004) for Standard & Poor’s, and Needham, Verde, and Mah (2005) for Fitch. The data of the Creditreform Rating AG were obtained from the

rating grade $i \in \{1, \dots, I\}$ and period $T = 1$ year with $m \in \{1, 2, 3, 4, 5\}$.⁴³ The data contains default rates for up to $I = 7$ rating classes. The time series are described in Table 1.

- Table 1 about here -

For the “Capital to Maturity”-approach as well as “Capital for One Period”-approach we do the following analyses in order to evaluate the capability of the model to explain empirical relationships:

- (1) Estimating (“Matching”) of the model implied parameters $\hat{k}_i^{(CtM/CoP)}$ using empirical default rates $\overline{DR}_i^{(0,m \cdot T)}$.
- (2) Comparison of the model based unexpected losses ($UL_{mb}^{(CtM/CoP)}$) using the estimations $\hat{k}_i^{(CtM/CoP)}$ with the corresponding empirical parameters $UL_{emp.}^{(CtM/CoP)}$ derived from the empirical default rates $\overline{DR}_i^{(0,m \cdot T)}$.
- (3) Comparison of the model based maturity adjustment $g_{mb}^{(CtM/CoP)}$ with the empirical maturity adjustment $g_{emp.}^{(CtM/CoP)}$ using empirical default rates $\overline{DR}_i^{(0,m \cdot T)}$.
- (4) Estimating (“Matching”) of a simple (Basel II alike) function $g_{simple}^{(CtM/CoP)}$ using the $UL_{emp.}^{(CtM/CoP)}$ of the empirical default rates $\overline{DR}_i^{(0,m \cdot T)}$.
- (5) Comparison of the simple maturity adjustment $g_{simple}^{(CtM/CoP)}$ with the model based maturity adjustment $g_{mb}^{(CtM/CoP)}$ and with the empirical maturity adjustment $g_{emp.}^{(CtM/CoP)}$ using empirical default rates $\overline{DR}_i^{(0,m \cdot T)}$.

We use the cumulative default rate to estimate the probability of default at time horizon $t = m \cdot T$ in the “Capital to Maturity”-approach (see equation (12)) and in the “Capital for One Period”-approach (see equation (17)), i.e.⁴⁴

$$\widehat{PD}_i^{(0,m \cdot T)} = \overline{DR}_i^{(0,m \cdot T)}. \quad (25)$$

Department of Credit Risk and Portfolio Analysis, Creditreform Rating AG. We claim, that this data might best represent observations of a credit portfolio of a medium sized (German) bank.

⁴³ In order to keep the empirical part tractable, we only account for a maturity of at most 5 years like in Basel II. See Basel Committee On Banking Supervision (2004), paragraph 320.

⁴⁴ In this context, we are aware of the fact, that cumulative default rates over a period do not represent the probabilities of default at a certain point in time as it is modelled in the original framework of Merton (1974) as well as Vasicek (1987). However, all common credit portfolio models like CreditPortfolioView, CreditRisk+, CreditPortfolioManager, and CreditMetrics, and its derivatives suffer from this fact when implemented on empirical data. Since these models are used with great success we accept this inaccuracy.

In order to calculate the empirical UL-contribution in the “Capital for One Period”-approach we use the conditional default rate as an estimator for the conditional probability of default like it is presented in equation (16), i.e.

$$\widehat{\text{PD}}_i^{(n_h-1 \cdot T, n_h \cdot T)} = \left(\overline{\text{DR}}_i^{(0, n_h \cdot T)} - \overline{\text{DR}}_i^{(0, n_h-1 \cdot T)} \right) / \left(1 - \overline{\text{DR}}_i^{(0, n_h-1 \cdot T)} \right) =: \overline{\text{DR}}_i^{(n_h-1 \cdot T, n_h \cdot T)} \quad (26)$$

with $n_h \in \{1, 2, 3, 4, 5\}$.

For the concrete calculation of the UL contribution as well as the simplified maturity adjustment we stick were close to the guidelines of the Basel Committee On Banking Supervision. Firstly, in order to derive the UL contribution of each rating grade i the asset correlation ρ_i is needed. We get from the proposal of Basel II, that this is a function of the estimated probability of default, i.e.⁴⁵

$$\hat{\rho}_i = f\left(\widehat{\text{PD}}_i^{(0, T)}\right) = 0.12 \cdot \frac{1 - \exp\left(-50 \cdot \widehat{\text{PD}}_i^{(0, T)}\right)}{1 - \exp(-50)} + 0.24 \cdot \left(1 - \frac{1 - \exp\left(-50 \cdot \widehat{\text{PD}}_i^{(0, T)}\right)}{1 - \exp(-50)} \right). \quad (27)$$

with $T = 1$ year.

Secondly, for the simple maturity adjustment we use mapping functions $g_{\text{simple}}^{(\text{CtM}/\text{CoP})}$, that are of the same structure as the one implemented in Basel II, that is⁴⁶

$$g_{\text{simple}, q}^{(\text{CtM}/\text{CoP})}\left(\widehat{\text{PD}}_i^{(0, T)}, m\right) = \frac{1 + (m - 2.5) \cdot \left(a_q^{(\text{CtM}/\text{CoP})} - b_q^{(\text{CtM}/\text{CoP})} \cdot \ln\left(\widehat{\text{PD}}_i^{(0, T)}\right) \right)^2}{1 - 1.5 \cdot \left(a_q^{(\text{CtM}/\text{CoP})} - b_q^{(\text{CtM}/\text{CoP})} \cdot \ln\left(\widehat{\text{PD}}_i^{(0, T)}\right) \right)^2} \quad (28)$$

with $q \in \{\text{invest (ment grades)}; \text{spec (ulative grades)}; \text{all (grades)}\}$ (29)

and m measuring the time to maturity (in years). Additionally, since this function is specified for more than one risk grade, the index q of credit quality of the borrower indicates the range of grades that is covered by the function. Therefore, the advantage of this function compared to the model based adjustment is the need of only two parameters for each q whereas $g_{\text{mb}}^{(\text{CtM}/\text{CoP})}$ needs as much parameters as rating grades under consideration.

- Figure 2 about here -

⁴⁵ See Basel Committee On Banking Supervision (2004), paragraph 272.

⁴⁶ See Basel Committee On Banking Supervision (2004), paragraph 272, in which $a = 0.11852$ and $b = 0.05478$.

In Figure 2 the maturity adjustment is plotted with respect to m and $\widehat{PD}_i^{(0,T)}$ using three different values for a and b (for all grades). The function can be characterized as follows:

- (i) the adjustment is (linear) increasing in m ,
- (ii) the adjustment is (convex) decreasing in PD ,
- (iii) for $m = 1$ the adjustment is “neutral” ($g = 1$).

Therefore, function (28) especially meets the requirements how they are deduced from the theoretical framework in section 2, i.e. that the maturity adjustment is expected to be high for low PD 's and increasing in m .⁴⁷ The impact of the characteristics (i) and (ii) can be influenced by varying a and b , that is shown in Figure 2 as well. The dotted surface (\cdots) displays the maturity adjustment for the values taken from Basel II. The magnitude of the adjustment with respect to the maturity (m) especially can be influenced by the parameter b , as it is shown by the dashed surface (\cdots) in Figure 2. Parameter a affects the sensitivity with respect to the probability of default (PD) as it can be seen from the straight line surface ($-$).

In order to map all functions numerically for getting $\hat{k}_i^{(CtM/CoP)}$ as well as $\hat{a}_q^{(CtM/CoP)}$ and $\hat{b}_q^{(CtM/CoP)}$, we mostly use an algorithm based on a Newton method.⁴⁸ For each parameter we compute confidence intervals of level 0.99 if possible. Additionally, we calculate the TSS (Total Sum of Squares) and ESS (Error Sum of Squares). Here, the TSS quantifies the sum of quadratic errors of a so-called naive model, in which the parameters are estimated by the mean of the empirical realizations. The ESS specifies the sum of quadratic errors of the considered model with respect to the empirical values. The results are reported in sections 3.2 for the “Capital to Maturity”-approach and in 3.3 for the “Capital for One Period”-approach.

3.2 Maturity Adjustment Factor for the “Capital to Maturity”-Approach

For the “Capital to Maturity”-approach only the probability of default at time horizon $t = m \cdot T$ with $T = 1$ year and $m \in \{1, \dots, 5\}$ is of interest. This probability is estimated from the average

⁴⁷ We deal with the qualitative as well as the quantitative evaluation of function (28) in detail in the following sections.

⁴⁸ The procedure is mainly based on an analytical derived Jacobian matrix. In some cases numerical procedures were needed to estimate this matrix. See Mathworks (2004) and Coleman and Li (1996, 1994) for details.

cumulative default rate $\overline{DR}_i^{(0,m:T)}$. The empirical observed cumulative default rates from four time series are presented in Figure 3 on a yearly period for 5 years.

Obviously, the characteristics of the curves are similar. For rating/risk grades with a good credit-worthiness (investment grade)⁴⁹ the default rate increases slightly and nearly linear over the years. However, the default rates of the Creditreform obviously are higher (at a time horizon over 1 year) in comparison to the data for investment grades of the other rating agencies. For rating/risk grades with a doubtful credit-worthiness (speculative grade)⁵⁰ the default rate increases with a decreasing slope (concave characteristics). Except for rating grades B and CCC/C of Fitch, the default rate rises with shifting to longer time horizons in each grade. The latter is caused by the limited sample in that rating categories and the short period that is used for averaging.⁵¹ Additionally, the default rate of the worst rating/risk grade from Fitch is substantially lower than the ones reported from the other rating agencies (e.g. the 5-years default rate varies between 40.60% (C_MM_r) and 60.40% (S_TM_r), but is only 31.63% (F_CDR_r) for FITCH).

- Figure 3 about here -

Despite the problems of using the default rate as an estimator for the probability of default at time horizon $t = m \cdot T$,⁵² the observed characteristics can be explained qualitatively with the Merton-model due to equation (8), since normally the default probability is expected to rise with time to maturity and that this increase will be flattened for lower rating grades.

As a next step we validate the Merton-based approach quantitatively by matching the model based multi-period defaults from equation (7) with the empirical data in order to check its goodness of fit. In fact, we estimated the model implied $\hat{k}_i^{(CTM)}$ for each rating class and each times series. The results are presented in Table 2.

- Table 2 about here -

⁴⁹ These are the Grades Aaa, Aa, A (Moody's), AAA, AA, A (Standard & Poor's, Fitch) and 1, 2, 3 (Creditreform).

⁵⁰ These are the Grades Baa, Ba, B, Caa/C (Moody's), BBB, BB, B, CCC/C (Standard & Poor's, Fitch) and 4, 5, 6 (Creditreform).

⁵¹ For details see Mah and Verde (2003), p. 8.

⁵² See footnote 44 for details.

Obviously, for low rating grades (i.e. high probabilities of default in the first period) the parameter $\hat{k}_i^{(CtM)}$ can be matched nearly perfectly (coefficient of determination $R^2 > 0.9$). For high borrowers quality the model seems not be able to explain the default rates since the coefficient of determination R^2 in most cases (see M_CDR_1, S_CDR_r, and F_CDR_r) is below zero. However, this could be explained by two reasons.

Firstly, in good rating grades more or less no defaults occur over the first three years so it is difficult anyway to fit any model to the data. Secondly, the negative R^2 is caused by the following technical reason. Our model is able to explain the cumulative default rates to some extent, but the naive model that just assumes identical cumulative default rates for all points in time (amounting to the mean of the default rates from 1 year to 5 years) leads to a lower error sum of squares (TSS) than error sum of squares (ESS) of our model. But the naive model assumes, that the maturity has no effect on the default rate, which definitely not fits with the empirical data. Thus, the naive model causes low error sum of squares for high rating grades, but lacks any economic sense.

Additionally, from the “Capital to Maturity” model we expect the parameter $\hat{k}_i^{(CtM)}$ to decrease when shifting to low rating grades. This results from the fact that $\hat{k}_i^{(CtM)}$ stands for the return-risk rate, whereas the return of the borrower is expected to be high for high quality borrowers and low for low quality borrowers. For the risk (measure by the variance) we would expect a vice versa relationship. Table 2 exactly shows this expected formation for each series. Thus, when implementing our model on empirical data the results for $\hat{k}_i^{(CtM)}$ seemingly fit with the a possible economic interpretation.

After examine the multi-period defaults we have analyzed the UL contribution of each rating grade. In order to have a first guess how the UL contribution looks like, we calculate the empirical UL contribution for each rating grade and for each maturity using the average default rate as an estimator for the probability of default (see equation (25)). The results for the four mentioned data series are presented in Figure 4. The diagrams show, that especially for the investment grades and lower speculative grades (representing lower probabilities of default $\widehat{PD}_i^{(0,T)}$) the capital requirements grow rapidly whereas they are stable for the two lowest grades (with high probabilities of default $\widehat{PD}_i^{(0,T)}$). In case of Moody’s (M_CDR_1)

and Standard & Poor's (S_CDR_r) it declines for the highest grade at a stretch. This is caused by the fact that the default rate exceeds 50 %.

- Figure 4 about here -

Next, we compare the UL contributions of the “Capital to Maturity” paradigm defined by equations (10) and (11) by using the empirical cumulative default rates for $\widehat{PD}_i^{(0,m,T)}$ and by using equation (12) with the $\hat{k}_i^{(CtM)}$ and $\widehat{PD}_i^{(0,T)}$ for each rating grade. Thus, we are able to compare the empirical unexpected loss $UL_{emp}^{(CtM)}$ with the model based unexpected loss $UL_{mb}^{(CtM)}$.

- Table 3 about here -

As it is shown in Table 3, the model based measures are capable to explain the empirical derived parameter $UL_{emp}^{(CtM)}$ for nearly all rating grades. Just for the best rating grades the model seems not to support the empirical data. However, even for this grade the ESS is very small and the model is close to the empirical values. The negative value of R^2 just explains that the naive “model” outperforms the developed solution, but still lags economic explanation for multi-period defaults.

Finally, we plot the “Capital to Maturity” adjustment $g_{mb}^{(CtM)}$ that is calculated according to equation (14) by using the model based UL estimation $UL_{mb}^{(CtM)}$ in comparison to the empirical derived UL. The graphs are presented in Figure 5. As expected from the previous analysis especially for the speculative grades the model based maturity adjustment $g_{mb}^{(CtM)}$ is very close to the empirical derived values. For the high-quality grades the fit is not as convincing: for shorter maturities (two and three years) the model based adjustment overestimates the effect, whereas for long maturities (four and five years) the adjustment is too small. However, the general characteristics of $g_{mb}^{(CtM)}$ is even for the investment grades close to the empirical values.

- Figure 5 about here -

As a next step, we give estimations for the parameters $\hat{a}_i^{(CtM)}$ and $\hat{b}_i^{(CtM)}$ to determine the simple maturity adjustment $g_{\text{simple}}^{(CtM)}$. First of all it has to be stated, that the empirical UL contribution (see Figure 4) for all rating grades is increasing in m with a declining slope in $\widehat{PD}_i^{(0,T)}$. Thus, $g_{\text{simple}}^{(CtM)}$ according to (28) seems to be a good estimator for the maturity adjustment (see characteristics (i) and (ii) in section 3.1) for all rating grades.

- Table 4 about here -

We fitted the maturity adjustment by estimating the parameters $\hat{a}_i^{(CtM)}$ and $\hat{b}_i^{(CtM)}$ to the data of all four data series, using all investment grades and all speculative grades as well as all grades for a time horizon of five years in each case. The results are reported in Table 4. For all time series the TSS can be reduced significantly to the ESS. The coefficient of determination R^2 varies between 0.34 (A-Grades for F_CDR_r) and 0.94 (Grades 1-3 for C_MM_r). Except for the data from Creditreform the coefficient of determination is less for the investment grades (varying from 0.34 to 0.94) in comparison to the speculative grades (varying from 0.78 to 0.92). Considering all grades, the parameter R^2 is reliable high with values from 0.82 to 0.91.

A comparison of our values for $\hat{a}_{\text{all}}^{(CtM)}$, varying from 0.06 to 0.22, and $\hat{b}_{\text{all}}^{(CtM)}$, ranging from 0.06 to 0.10, with the corresponding values a_{BII} and b_{BII} of Basel II confirms a high approximation quality. However, our approach overestimates the increase of the UL contribution with respect to the time to maturity as well as with respect to the probability of default, since both $\hat{a}_{\text{all}}^{(CtM)}$ and $\hat{b}_{\text{all}}^{(CtM)}$ are higher in general.

- Figure 6 about here -

This result also can be visualized by a surface plot of the maturity adjustment with respect to the time to maturity m and rating grades (i.e. one year probabilities of default). The plot for a time horizon of five years are shown in Figure 6. The straight line surface (-) shows the maturity adjustment from the empirical data of default rates. Our fitted maturity adjustment results in the dashed surface (- -), whereas the values from Basel II lead to the dotted surface (••). Obviously, our results for simple maturity adjustment overestimate the empirical values

especially for the upper investment grades, but fits well with shifting to higher probabilities of default. This is caused by the fact, that for the upper investment grades the UL contribution is low, whereas it is high for speculative grades. Since the maturity adjustment is optimized with respect to the ESS of the UL contribution, derivations of the fitted maturity adjustment from the empirical maturity adjustment of the investment grades are less weighted than derivations of the speculative grades. However, the maturity adjustment of Basel II is lower than our values.

Finally, we compare our results for the model-based, the simple maturity and the Basel II adjustment in Table 5.

- Table 5 about here -

As it can be seen from the ESS figures, the model based adjustment $g_{mb}^{(CtM)}$ outperforms the simple method $g_{simple}^{(CtM)}$ in general except for the medium-quality borrowers (grade A for Moody's, Standard & Poor's, and Fitch). For $g_{mb}^{(CtM)}$ the coefficient of determination is reliable high except for the high investment grades. However, since the interesting parameter is the UL contribution of each grade and since this value is very low for loans with such a good grade – even long term loans – the imprecise adjustments might not be of interest there. Additionally, as it is shown in the last column, our “Capital to Maturity” approach does not fit at all with the Basel II adjustment.

To conclude, we have estimated the parameters of the maturity adjustment using both the model based solution as well as an simple, “Basel II formula” using the “Capital to Maturity”-approach. We get better results for the model based fit, but in total both formulas work well. With respect to the simple adjustment, our estimated individual parameters are in line with the supervisory values when the data up to a five years time horizon is used. Nevertheless, our parameters lead to a significant higher adjustment than the one used in Basel II.

3.3 Maturity Adjustment Factor for the “Capital for One Period”-approach

Under the “Capital to One Period”-approach the conditional probabilities of default $\widehat{PD}_i^{(n_{h-1} \cdot T, n_h \cdot T)}$ at a time horizon of $\Delta t = 1$ year are used for calculating the UL contribution,

where the conditional default rates $\overline{DR}_i^{(n_{h-1} \cdot T, n_h \cdot T)}$ are taken as estimators. These empirical derived values are determined in order to have a first guess.

- Figure 7 about here -

The characteristics of all four plots do not differ tremendously. For good creditworthiness, the conditional default rates only slightly change over time. This means, that the one-year probability of default of a firm with an investment grade is likely to stay over time as long as the debtor does not default. However, for bad creditworthiness the conditional default rates decline substantially. For the lowest credit/risk grade the conditional default rate drops in the first year by 5 percentage points for Moody's and Creditreform (from 22 % to 17 % and from 19 % to 14 %), by 18 percentage points for Standard & Poor's (from 31 % to 13 %), and by 14 percentage points for Fitch (from 24 % to 10 %). Thus, the creditworthiness of a firm with speculative grade is expected to rise as long as the firm will survive on the short time horizon. This result is also expected from the Merton-model (in the two-period-case) due to our analysis from equation (19) (with (17)), meaning, that the conditional one year probability of default is more likely to decline for firms with a high probability of default than for firms with a low probability of default in the first year. Besides this qualitative evaluation, we are also able to examine if the model fits quantitatively by determining the model based parameter $\hat{k}_i^{(CoP)}$. The results are presented in Table 6.

- Table 6 about here -

Since the numerical function that has to be matched on the empirical data is rather complicated, in most cases no confidence intervals could be evaluated. On the one hand, this is a disadvantage of this model. On the other hand it fits with the empirical data quite good. In contrast to the "Capital to Maturity" model, the sum of squares can be reduced even for the high quality grades (see the low values of the ESS) and thus the parameter R^2 varies from 0.08 to 1.00.

- Figure 8 about here -

The model implied parameter $\hat{k}_i^{(CoP)}$ has a U-shaped form, meaning that it firstly declines with shifting to lower credit grades but increases for the very low credit qualities. We are of the

opinion, that this characteristic is surprising on the first sight but economically explainable. Since $\hat{k}_i^{(\text{CoP})}$ is a return-risk parameter, it is relatively high for investment grades. Additionally, it has to be high for speculative grades, since at least the borrowers of very low creditworthiness with default in the late periods must incorporate a high return-risk parameter. Otherwise for low rated debtors no long-term defaults could be observed, since the chance of survival over the first period is very small. Thus, the model implied parameter $\hat{k}_i^{(\text{CoP})}$ is positively biased for such rating grades.

After discussing the multi-period defaults we examine the UL contribution of each rating grade under the “Capital for one Period” paradigm. Since the effect of the maximisation function according to (20) to (22) is not clear, we calculate the empirical UL contribution for each rating grade and for each maturity. The results for the four mentioned data series are shown in Figure 8. As expected, the UL contribution does not change when shifting to higher maturity for the lowest rating/risk grade, since the highest one-year probability of default is expected in the first year. The maturity mostly effects the lower investment grades and upper speculative grades. At a stretch the UL contribution rises for $m = 5$ years in comparison to $m = 1$ year for rating grade “Baa” (Moody’s) by (a factor of) 3.14, for rating grade “A” (Standard & Poor’s) by 2.81, for rating grade “A” (Fitch) by 2.97, and for rating grade “1” (Creditreform) by 2.39.

- Table 7 about here –

As a next step we calculate the model based UL contribution $UL^{(\text{CoP})}$ (Table 7). The outcomes vary tremendously: on the one hand the parameter R^2 is below zero since the naive model generates a better output than the multi-period model. On the other hand the sum of squared errors is very small in any case. Although the PD-adjustment could be matched well to the empirical cumulative default rates we detect this does not hold for the maturity adjustment $g_{mb}^{(\text{CoP})}$ due to the complicated UL-function for longer times to maturity.⁵³

- Figure 9 about here –

⁵³ These results form function (22).

This conclusion can also be drawn from the graphical examination of the maturity adjustment $g_{mb}^{(CoP)}$ which is calculated on the basis of equation (24) by using the model based UL estimation $UL_{mb}^{(CoP)}$ in comparison with the empirical derived unexpected loss. The plots are presented in Figure 9. We observe, that especially for the medium credit qualities the model based adjustment overestimates the empirical values by a factor of up to three. However, for the lower credit grades and the high credit grades the fit still is very good.

Due to this result, the use of the simple version might be a good choice. We argue, that function (28) for the maturity adjustment is suitable since the UL contribution is increasing with rising time to maturity, but with declining slope for shifting to high one-year probability of default.

- Table 8 about here –

Thus, we estimate the parameters $\hat{a}_i^{(CoP)}$ and $\hat{b}_i^{(CoP)}$ in order to define the maturity adjustment function $g_{simple}^{(CoP)}$. The results are displayed in Table 8. For the three analyses of each time series (considering the investment grades and the speculative grades separately as well as all grade as a whole), the TSS is reduced in comparison to the ESS significantly. The latter statement especially is valid for the speculative grades. Here the coefficient of determination R^2 varies from 0.63 (B-Grades or F_CDR_r) up to 0.99 (Grades 4-6 for C_MM_r). For the investment grades the results are less convenient, since R^2 realizes between 0.06 (Grades 1-3 for C_MM_r) and 0.71 (A-Grades for S_CDR_r). This might be due to the fact, that in the “Capital for one Period”-approach the UL contribution does not change in a great manner when shifting to longer time to maturity. Therefore, an adjustment function has not a great influence. Nevertheless, taking all rating grades into account, the coefficient of determination is high and ranging from 0.97 to 0.99.

Under consideration of the values a_{BII} and b_{BII} of Basel II, we find, that – on the one hand - our estimations $\hat{a}_{all}^{(CoP)}$ for all grades, ranging from 0 to 0.04, are much lower than the parameter 0.12 suggested by Basel II. On the other hand, the values for $\hat{b}_{all}^{(CoP)}$, varying from 0.06 to 0.08, are higher than the corresponding supervisory parameter 0.05. Since both effects may neutralize each other, we investigate the result visually.

- Figure 10 about here –

Therefore, in Figure 10 the surface plots of the maturity adjustment with respect to the rating/risk grades and time to maturity of up to 5 years are presented. Obviously, the straight line surface (-) with the maturity adjustment from empirical data only differs from our estimated maturity adjustment (dashed surface (-·-)) for high rating grades. Again we point out, that significant deviation of the estimated maturity adjustment from the empirical one is not problematic for low probabilities of default, since their UL contribution is low anyway. In addition we have to mention, that in the “Capital for one Period”-approach the dotted surface (··) of the supervisory maturity adjustment is nearly equal to our estimation.

- Table 9 about here –

For an overlook of the results of the adjustments using the “Capital for one Period” approach, we refer to Table 9. Obviously, for all maturity adjustments the ESS are small for low credit qualities, that correspond to the most interesting region of the UL contribution function since the values are reliable high. Considering the investment grades, the model based adjustments works best for the very high rating grades, whereas the $g_{\text{simple}}^{(\text{CoP})}$ provides good results for the lower grades of this segment. The parameter R^2 varies tremendously and is mostly close to unit or is negative. Again, the reason for this result is the quite complex function for the unexpected loss. However, it is worth mentioning, that the Basel II adjustment formula produces similar results as our simple adjustment.

To summarize, when applying the “Capital for One Period”-approach to empirical data, the maturity adjustment in a default mode model drops in value compared to the solution of the “Capital to Maturity”-approach. Taking into account time series with up to 5 years of time to maturity, our estimations for each parameter of the maturity adjustment, like it is implemented in Basel II, differ widely from the supervisory values. However, they nearly lead to the same adjustments than suggested by Basel II when used jointly. However, the model based approach does not improve the simple adjustment formula, although the model works fine in order to match empirical cumulative default rates.

4. Conclusion

One of the questions in credit portfolio modelling nearly not discussed, is how the time to maturity of a loan affects the appropriate economic capital, that the bank should hold against future losses. Especially, in DM models like the one of Basel II no solutions are examined. Due to the lack of intensive research, our paper focused on three topics: firstly, we stated some key Iss.s in order to motivate the influence of time to maturity on economic capital in a default mode model and we suggested two frameworks, that we called “Capital to Maturity”-approach and “Capital for One Period”- approach. Secondly, we analysed these approaches in a simple credit risk model, based on the framework of Vasicek (2002) and Merton (1974), in order to derive theoretically the maturity effects on economic capital of these paradigms. Thirdly, we implemented our approaches on empirical data from four rating agencies using a model based approach as well as a simpler form according to the maturity adjustment formula of Basel II. For both paradigms, our estimated values for the maturity adjustment formula are close to the parameters of Basel II when using short time horizons of up to five years. Particularly, the “Capital for One Period”-approach leads to a similar maturity adjustment. We claim, that our results might be a sophisticated contribution for understanding maturity effects on economic capital especially when such adjustments are required for practical use.

Appendix

A.1 Derivation of Equations (5) and (6)

With respect to assumption (d) in section 2.1 we retrieve the following factor model for the normalised returns $\tilde{a}_i^{(T)}$ of each borrower i taking out principal component analysis

$$\tilde{a}_i^{(T)} = \sqrt{\rho_i} \cdot \tilde{x}^{(T)} + \sqrt{1-\rho_i} \cdot \tilde{\epsilon}_i^{(T)} \text{ with } \tilde{x}^{(T)}, \tilde{\epsilon}_i^{(T)} \sim N(0,1) \text{ being i.i.d.}^{54} \text{ for } i = \{1, \dots, n\}, \quad (\text{A1})$$

whereas $\tilde{x}^{(T)}$ is the systematic factors and the $\tilde{\epsilon}_i^{(T)}$'s are the idiosyncratic factors.

Assumption (e) means, that with the number of borrowers shrinks to infinity the exposure share of each obligor in the portfolio tends to zero. Precisely, the following holds

$$\sum_{i=1}^n L_i \uparrow \infty \text{ and } \sum_{i=1}^n \left(L_i / \sum_{i=1}^n L_i \right)^2 < \infty \text{ with } n \rightarrow \infty.^{55} \quad (\text{A2})$$

Given a realisation of the systematic factor $\tilde{x}^{(T)}$, all asset return are independent distributed, and because of assumption (2) the Law of large Number is valid. The loss rate $\tilde{\Lambda}_\infty^{(T)}$ of the portfolio becomes⁵⁶

$$\tilde{\Lambda}_\infty^{(0,T)} | \tilde{x}^{(T)} := \lim_{n \rightarrow \infty} \tilde{\Lambda}_n^{(0,T)} | \tilde{x}^{(T)} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \omega_i \cdot \tilde{\ell}_i^{(0,T)} | \tilde{x}^{(T)} \stackrel{\text{a.s.}}{=} \sum_{i=1}^{\infty} \omega_i \cdot p_i(\tilde{x}^{(T)}) \quad (\text{A3})$$

$$\text{with } \omega_i = L_i / \sum_{i=1}^n L_i \text{ and } p_i(\tilde{x}^{(T)}) = E\left(\tilde{\ell}_i^{(0,T)} | \tilde{x}^{(T)}\right) = N\left[\left(N^{-1}(\text{PD}_i^{(0,T)}) - \sqrt{\rho_i} \cdot \tilde{x}^{(T)}\right) / \sqrt{1-\rho_i}\right]. \quad (\text{A4})$$

Furthermore, the VaR_z at confidence level z is quantified as the z -quantile q_z of the loss rate $\tilde{\Lambda}_\infty^{(0,T)}$ of the portfolio, i.e.

$$\text{VaR}_z\left(\tilde{\Lambda}_n^{(0,T)}\right) := q_z\left(\tilde{\Lambda}_n^{(0,T)}\right) \text{ with } P\left(\tilde{\Lambda}_n^{(0,T)} < q_z\left(\tilde{\Lambda}_n^{(0,T)}\right)\right) = z, \quad (\text{A5})$$

and because $p_i(\tilde{x}^{(T)})$ is strictly decreasing in $\tilde{x}^{(T)}$, we retrieve⁵⁷

$$\text{VaR}_z\left(\tilde{\Lambda}_\infty^{(0,T)}\right) := \lim_{n \rightarrow \infty} \text{VaR}_z\left(\tilde{\Lambda}_n^{(0,T)}\right) \stackrel{\text{a.s.}}{=} \sum_{i=1}^{\infty} \omega_i \cdot p_i(x_{q_{1-z}}^{(T)}) \text{ with } x_{q_{1-z}}^{(T)} = q_{1-z}\left(\tilde{x}^{(T)}\right). \quad (\text{A6})$$

Obviously, the expected value of the loss rate $\tilde{\Lambda}_\infty^{(0,T)}$ of the portfolio is

$$E\left(\tilde{\Lambda}_\infty^{(0,T)}\right) := \lim_{n \rightarrow \infty} E\left(\tilde{\Lambda}_n^{(0,T)}\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \omega_i \cdot E\left(I\left(\tilde{A}_i^{(T)} < B_i^{(T)}\right)\right) = \sum_{i=1}^{\infty} \omega_i \cdot \text{PD}_i^{(0,T)}. \quad (\text{A7})$$

Finally, the UL of the loss rate $\tilde{\Lambda}_\infty^{(0,T)}$ is just defined by the difference between the VaR and the expected value, i.e.

⁵⁴ We use ‘‘i.i.d.’’ as shortcut for ‘‘independently and identically distributed’’.

⁵⁵ See Bluhm, Overbeck, and Wagner (2003), p. 87 and Gordy (2003).

⁵⁶ See Bluhm, Overbeck, and Wagner (2003), p. 89, Gordy (2003), p. 208 and Vasicek (2002), p.p.160 for details.

⁵⁷ See Bluhm, Overbeck, and Wagner (2003), p. 91.

$$\text{UL}\left(\tilde{\Lambda}_\infty^{(0,T)}\right) := \lim_{n \rightarrow \infty} \text{UL}\left(\tilde{\Lambda}_n^{(0,T)}\right) = \text{VaR}_z\left(\tilde{\Lambda}_\infty^{(0,T)}\right) - \text{E}\left(\tilde{\Lambda}_\infty^{(0,T)}\right) \quad (\text{A8})$$

Since both the VaR (see equation (A6)) as well as the expected value (see equation (A7)) are linear in w_i for each $i \in \{1, \dots, n\}$, we may write the UL of the loss rate $\tilde{\Lambda}_\infty^{(0,T)}$ as a sum of the UL of each individual portfolio loss rate $\tilde{\ell}_i^{(0,T)}$

$$\text{UL}\left(\tilde{\Lambda}_\infty^{(0,T)}\right) := \sum_{i=1}^{\infty} \omega_i \cdot \text{UL}\left(\tilde{\ell}_i^{(0,T)}\right) \quad (\text{A9})$$

with the risk contribution to the unexpected loss

$$\text{UL}\left(\tilde{\ell}_i^{(0,T)}\right) := \text{VaR}_z\left(\tilde{\ell}_i^{(0,T)} \mid \tilde{\mathbf{x}}^{(T)}\right) - \text{E}\left(\tilde{\ell}_i^{(0,T)}\right) \quad (\text{A10})$$

$$\text{with } \text{VaR}_z\left(\tilde{\ell}_i^{(0,T)} \mid \tilde{\mathbf{x}}^{(T)}\right) = \text{N}\left(\frac{\text{N}^{-1}\left(\text{PD}_i^{(0,T)}\right) - \sqrt{\rho_i} \cdot \mathbf{x}_{q_{1-z}}^{(T)}}{\sqrt{1-\rho_i}}\right) \text{ and } \text{E}\left(\tilde{\ell}_i^{(0,T)}\right) = \text{PD}_i^{(0,T)}. \quad (\text{A11})$$

A.2 Derivation of Equations (7) and (8)

For the default probability $\text{PD}_i^{(0,m,T)}$ we get

$$\text{PD}_i^{(0,m,T)} = \text{N}\left(b_i^{(m,T)}\right) \text{ with } b_i^{(m,T)} = \frac{\ln\left(B_i^{(0)}/A_i^{(0)}\right) - \mu_{i,\text{eff}}^{(m,T)}}{\sigma_{A,i}^{(m,T)}} = \frac{b_i^{(T)}}{\sqrt{m}} - \frac{\mu_{i,\text{eff}}^{(T)}}{\sigma_{A,i}^{(T)}} \cdot \frac{m-1}{\sqrt{m}}. \quad (\text{A12})$$

The default probability rises, if

$$b_i^{(m,T)} > b_i^{(T)} \Leftrightarrow \frac{b_i^{(T)}}{\sqrt{m}} - \frac{\mu_{i,\text{eff}}^{(T)}}{\sigma_{A,i}^{(T)}} \cdot \frac{m-1}{\sqrt{m}} > b_i^{(T)} \Leftrightarrow \frac{\mu_{i,\text{eff}}^{(T)}}{\sigma_{A,i}^{(T)}} < -\frac{\sqrt{m}-1}{m-1} \cdot b_i^{(T)}. \quad (\text{A13})$$

For $m = 2$ we receive the former results.

For the derivation we get

$$\frac{\partial \text{PD}_i^{(0,m,T)}}{\partial m} = \frac{\partial \text{N}\left(b_i^{(m,T)}\right)}{\partial m} = -n\left(b_i^{(m,T)}\right) \cdot \frac{1}{2} \cdot \left(\frac{b_i^{(T)}}{\sqrt{m^3}} + \frac{\mu_{i,\text{eff}}^{(T)}}{\sigma_{A,i}^{(T)}} \cdot \frac{m+1}{\sqrt{m^3}}\right). \quad (\text{A14})$$

It is positive if

$$\frac{\mu_{i,\text{eff}}^{(T)}}{\sigma_{A,i}^{(T)}} \cdot (m+1) < -b_i^{(T)} \text{ for } b_i^{(T)} < 0 \Leftrightarrow \text{PD}_i^{(T)} < 0.5. \quad (\text{A15})$$

A.3 Derivation of Equation (17)

We write for the conditional value at $t_h = n_h \cdot T$ with $h = \{0, 1, 2, 3, \dots, j\}$ and $0 = n_0 \leq n_1 < n_2 < \dots < n_h = m$ of the firms assets

$$\tilde{A}_i^{(t_h)} \mid (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{h-1})}) = A_i^{(0)} \cdot \exp\left(\mu_{i,\text{eff}}^{(T)} \cdot h + \sigma_i^{(T)} \cdot \left[\sum_{j=1}^{h-1} \sqrt{n_s - n_{s-1}} \cdot \tilde{a}_i^{(t_j)}\right]\right) \quad (\text{A16})$$

Thus, for the conditional probability we get

$$\left[\text{PD}_i^{(t_0, M)} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{j-1})}) = \text{N} \left(\mathbf{b}_i^{(M)} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{j-1})}) \right) \right] | (\tilde{a}_i^{(t_1)} > \mathbf{b}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{j-1})} > \mathbf{b}_i^{(t_{j-1})}) \quad (\text{A17})$$

with the conditional default point

$$\mathbf{b}_i^{(t_h)} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{h-1})}) = \frac{\mathbf{b}_i^{(t_h)}}{\sqrt{\mathbf{n}_h - \mathbf{n}_{h-1}} / \sqrt{\mathbf{n}_h}} - \frac{\sum_{j=1}^{h-1} \sqrt{\frac{\mathbf{n}_j - \mathbf{n}_{j-1}}{\mathbf{n}_h}} \cdot \tilde{a}_i^{(t_j)}}{\sqrt{\mathbf{n}_h - \mathbf{n}_{h-1}} / \sqrt{\mathbf{n}_h}}. \quad (\text{A18})$$

using the (unconditional) default points

$$\mathbf{b}_i^{(t_h)} = \frac{\ln(\mathbf{B}/\mathbf{A}_i^{(0)}) - \mu_{i, \text{eff}}^{(T)} \cdot \mathbf{n}_h}{\sigma_i^{(T)} \cdot \sqrt{\mathbf{n}_h}} = \mathbf{b}_i^{(t_1)} \cdot \frac{\sqrt{\mathbf{n}_1}}{\sqrt{\mathbf{n}_h}} - k_i \cdot \frac{(\mathbf{n}_h - \mathbf{n}_1)}{\sqrt{\mathbf{n}_h}} \quad \text{with } k_i = \frac{\mu_{i, \text{eff}}^{(T)}}{\sigma_i^{(T)}}$$

Next, we use the definition

$$\Delta \text{PD}_i^{(t_{s-1}, t_s)} := \text{PD}_i^{(t_{s-1}, t_s)} \cdot \prod_{k=1}^{s-1} (1 - \text{PD}_i^{(t_{k-1}, t_k)}) \quad (\text{A19})$$

and get for

$$\Delta \text{PD}_i^{(t_{j-1}, m-T)} = \left(\prod_{h=1}^{j-1} \int_{\mathbf{b}_i^{(t_h)} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{h-1})})}^{\infty} \right) \mathbf{b}_i^{(M)} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{j-1})}) \int_{-\infty}^{\infty} \mathbf{n}(\mathbf{z}) \cdot \mathbf{n}(\mathbf{a}_i^{(t_{j-1})}) \cdot \dots \cdot \mathbf{n}(\mathbf{a}_i^{(t_1)}) \cdot d\mathbf{z} d\mathbf{a}_i^{(t_{j-1})} \dots d\mathbf{a}_i^{(t_1)} \quad (\text{A20})$$

where we write

$$\prod_{h=1}^{j-1} \int_{\mathbf{b}_i^{(t_h)} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{h-1})})}^{\infty} = \int_{\mathbf{b}_i^{(t_1)}}^{\infty} \dots \int_{\mathbf{b}_i^{(t_{j-1})} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{j-2})})}^{\infty} \cdot \quad (\text{A21})$$

Now we make the following transformation:

$$\mathbf{y}_i^{(t_h)} := f(\mathbf{a}_i^{(t_h)}) = -\mathbf{a}_i^{(t_h)} \Leftrightarrow \mathbf{a}_i^{(t_h)} = -\mathbf{y}_i^{(t_h)} =: -\mathbf{a}_i^{(t_h)} \quad \text{for } h = \{1, \dots, j-1\}. \quad (\text{A22})$$

Considering that

$$f(\infty) = -\infty, \quad f(\mathbf{b}_i^{(t_h)} | (\tilde{a}_i^{(t_s)})_{s=1,2,\dots,h-1}) = -\mathbf{b}_i^{(t_h)} | (\tilde{a}_i^{(t_s)})_{s=1,2,\dots,h-1}, \quad \mathbf{n}(-\mathbf{y}_i^{(t_h)}) = \mathbf{n}(\mathbf{y}_i^{(t_h)}) =: \mathbf{n}(\mathbf{a}_i^{(t_h)}) \quad \text{and} \\ d\mathbf{a}_i^{(t_h)} / d\mathbf{y}_i^{(t_h)} = -1 \Leftrightarrow d\mathbf{a}_i^{(t_h)} = -d\mathbf{y}_i^{(t_h)} =: -d\mathbf{a}_i^{(t_h)} \quad (\text{A23})$$

we receive

$$\Delta \text{PD}_i^{(n-T, m-T)} = \left(\prod_{h=1}^{j-1} \int_{-\overline{\mathbf{b}_i^{(t_h)}} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{h-1})})}^{-\infty} \right) \mathbf{b}_i^{(M)} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{j-1})}) \int_{-\infty}^{\infty} \mathbf{n}(\mathbf{z}) \cdot \mathbf{n}(\mathbf{a}_i^{(t_{j-1})}) \cdot \dots \cdot \mathbf{n}(\mathbf{a}_i^{(t_1)}) \cdot d\mathbf{z} d\mathbf{a}_i^{(t_{j-1})} \dots d\mathbf{a}_i^{(t_1)} \quad (\text{A24}) \\ = \left(\prod_{h=1}^{j-1} \int_{-\overline{\mathbf{b}_i^{(t_h)}} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{h-1})})}^{-\infty} \right) \mathbf{b}_i^{(M)} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{j-1})}) \int_{-\infty}^{\infty} \mathbf{n}(\mathbf{z}) \cdot \mathbf{n}(\mathbf{a}_i^{(t_{j-1})}) \cdot \dots \cdot \mathbf{n}(\mathbf{a}_i^{(t_1)}) \cdot d\mathbf{z} d\mathbf{a}_i^{(t_{j-1})} \dots d\mathbf{a}_i^{(t_1)}.$$

$$\text{where } \overline{\mathbf{b}_i^{(t_h)}} | (\tilde{a}_i^{(t_1)}, \dots, \tilde{a}_i^{(t_{h-1})}) = \frac{\mathbf{b}_i^{(n_h)}}{\sqrt{\mathbf{n}_h - \mathbf{n}_{h-1}} / \sqrt{\mathbf{n}_h}} + \frac{\sum_{j=1}^{h-1} \sqrt{\frac{\mathbf{n}_j - \mathbf{n}_{j-1}}{\mathbf{n}_h}} \cdot \tilde{a}_i^{(t_j)}}{\sqrt{\mathbf{n}_h - \mathbf{n}_{h-1}} / \sqrt{\mathbf{n}_h}}.$$

In order to solve the integral, we use the following expression⁵⁸

⁵⁸ The following integral follows from Tong (1990), p. 184.

$$N_{\eta}(x_1, \dots, x_{\eta}; (\rho_{op})_{o,p=1, \dots, \eta}) = \left(\prod_{i=1}^{\eta} \int_{-\infty}^{\tau_{op}} \frac{x_o - \sum_{p=1}^{o-1} \tau_{op} \cdot z_p}{\tau_{op}} \right) n(z_1) \cdot \dots \cdot n(z_{\eta}) \quad dz_{\eta} \dots dz_1 \quad (\text{A25})$$

where $N_{\eta}(x_1, \dots, x_{\eta}; (\rho_{op})_{o,p=1, \dots, \eta})$ describes the η -dimensional normal distribution at (x_1, \dots, x_{η}) with the correlation matrix $(\rho_{op})_{o,p=1, \dots, \eta} = T \cdot T^T$ whereas $T = (\tau_{op})_{o,p=1, \dots, \eta}$ is the lower triangular matrix. We get for the integral (A25)

$$\Delta PD_i^{(t_h, t_{h-1})} = N_{n+1} \left(-N^{-1} \left(PD_i^{(0,T)} \right), \dots, -N^{-1} \left(PD_i^{(0, t_{h-1})} \right), N^{-1} \left(PD_i^{(0, t_h)} \right); (\rho_{op})_{o,p=1, \dots, h} \right) \quad (\text{A26})$$

$$\text{with } PD_i^{(0, t_h)} = N \left(b_i^{(0, t_h)} \right) \text{ and } b_i^{(0, t_h)} = \frac{N^{-1} \left(PD_i^{(0, T)} \right)}{\sqrt{n_h}} - k_i \cdot \frac{n_h - 1}{\sqrt{n_h}} \quad (\text{A27})$$

$$\text{and } T = (\tau_{op})_{o,p=1, \dots, \eta} \text{ using } \tau_{op} = \sqrt{n_p / n_o} \cdot I(p \leq o) \text{ for } o, p = 1, \dots, \eta \quad (\text{A28})$$

A.4 Derivation of Equations (19)

The probability of default $PD_i^{(0,T)}$ of the first period will be lower than $PD_i^{(T, m-T)}$ if

$$PD_i^{(T, m-T)} = \frac{\Delta PD_i^{(T, m-T)}}{1 - PD_i^{(0,T)}} > PD_i^{(0,T)} \quad (\text{A29})$$

is valid. This can be written as

$$\left(PD_i^{(0,T)} - 0,5 \right)^2 > 0,25 - \Delta PD_i^{(T, m-T)}, \quad (\text{A30})$$

that is always true, as long as

$$\Delta PD_i^{(T, m-T)} > 0,25 \quad (\text{A31})$$

$$\text{or } PD_i^{(0,T)} > 0,5 + \sqrt{0,25 - \Delta PD_i^{(T, m-T)}}$$

$$\text{or } PD_i^{(0,T)} < 0,5 - \sqrt{0,25 - \Delta PD_i^{(T, m-T)}} \text{ if } \Delta PD_i^{(T, 2T)} < 0,25. \quad (\text{A32})$$

Since the cases of an high marginal increase of the probability of default, $\Delta PD_i^{(T, 2T)} > 0,25$, as well as a high probability of default in the first period, $PD_i^{(0,T)} > 0,5$, may not be relevant in practice, we focus on the second term in equation (A32).

B. References

- Altman, E. I. (2001): Distress Prediction Models and Some Applications: Revisiting the Z-Score and ZETA[®] Models, in: Altman, Edward I. (2001): Bankruptcy, Credit Risk and High Yield Junk Bonds, Blackwell Publisher, 7-36.
- Barco, M. (2004): Bringing Credit Portfolio Modelling to Maturity, *Risk* **17**(1), 86-90.
- Basel Committee on Banking Supervision (1999): Credit Risk Modelling: Current Practise and Applications, Bank for International Settlements.
- Basel Committee on Banking Supervision (2004): International Convergence of Capital Measurement and Capital Standards, Bank for International Settlements.
- Basel Committee on Banking Supervision (2005): An Explanatory Note on the Basel II IRB Risk Weight Function, Bank for International Settlements.
- Black, F., and Scholes, M. (1973): The pricing of options and corporate liabilities, *The journal of political economy*, **81**(3), 637-654.
- Blochwitz, S., Liebig, T., and Nyberg, M. (2000): Benchmarking Deutsche Bundesbank's Default Risk Model, the KMV Private Firm Model and Common Financial Ratios for German Corporations, Working Paper.
- Bluhm, C., and Overbeck, L. (2003): Systematic Risk in Homogeneous Credit Portfolios, *Credit risk: Measurement, Evaluation and Management*, Heidelberg.
- Bluhm, C., and Overbeck, L. (2003): Systematic risk in homogeneous credit portfolios, *Credit Risk*, 35-48.
- Bluhm, C., Overbeck, L., and Wagner, C. (2003): An Introduction to Credit Risk Modelling.
- Brady, B., and Vazza, Diane (2004): Corporate Defaults in 2003 Recede From Recent Highs, Standard & Poor's.
- Chorafas, D. N. (2005): Economic Capital Allocation with Basel II, Elsevier.
- Coleman, T. F., and Li, Y. (1994): On the Convergence of Reflective Newton Methods for Large-Scale Nonlinear Minimization Subject to Bounds, *Mathematical Programming* **67**, (2), 189-224.
- Coleman, T. F., and Li, Y. (1996): An Interior, Trust Region Approach for Nonlinear Minimization Subject to Bounds, *SIAM Journal on Optimization* **6**, 418-445.
- Credit Suisse Financial Products (1997): CreditRisk+ - A Credit Risk Management Framework, London.
- Crosbie, P. (1999): Modelling Default Risk, KMV Corporations.
- Crosbie, P., and Bohn, J. (2003): Modeling Default Risk, Moody's KMV Company.
- Crouhy, M., Galai, D., and Mark, R. (2000b): Risk Management.
- Demey, P., Louanin, J.-F., Roget, C., and Roncalli, T. (2004): Maximum likelihood estimate of default correlations, *Risk* **17**(11), 104-108.
- Düllmann, K., and Trapp, M. (2004): Systematic Risk in Recovery Rates – An Empirical Analysis of U.S. Corporate Credit Exposures, Working Paper.
- Dwyer, D. W., Kocagil, A. E., and Stein, R. M. (2004): MOODY'S KMV RiskCalc™ v3.1 Model: Next-Generation Technology for predicting private firm credit Risk, Moody's KMV.

- European Union (2004): International Accounting Standard 39, Financial Instruments: Recognition and Measurement, The Commission of the European Communities, 19.11.2004.
- Farmen, T. E. S., Fleten, S.-E., Westgaard, S., and van der Wijst, N. (2004): Default Greeks under an objective probability measure, Working Paper.
- Finger, C. C. (1999): Conditional Approaches for CreditMetrics Portfolio Distributions, *CreditMetrics Monitor* (4), 14–33.
- Finger, C. C. (2001): The One-Factor CreditMetrics Model In The New Basel Capital Accord, *RiskMetrics Journal* **2**(1), 8-18.
- Finger, C. C. (2002): CreditGradesTM – Technical Document, RiskMetrics Group.
- Gordy, M. (2000): A Comparative Anatomy of Credit Risk Models, in: *Journal of Banking and Finance* **24**, 119-149.
- Gordy, M.I, and Heitfield, E. (2002): Estimating Default Correlations from Short Panels of Credit Rating Performance Data, Working Paper, Federal Reserve Board.
- Gordy, M., and Heitfield, E. (2001): Maturity effects in a class of multi-period default mode models, Board of Governors of the Federal Reserve System, Washington DC.
- Gordy, M. (2003): A Risk-Faktor Model Foundation for Ratings-Based Bank Capital Rules, in: *Journal of Financial Intermediation* **12**, 199-232.
- Grundke, P. (2003): The Term Structure of Credit Spreads as a Determinant of the Maturity Effect on Credit Risk Capital, *Finance Letters* **1**(6), S. 4-9.
- Gupton, G. M., and Finger, C. C., and Bhatia, M. (1997): CreditMetrics – Technical Document, Morgan Guarantee Trust Co.
- Hamerle, A., Liebig, T., and Rösch, D. (2003): Benchmarking Asset Correlation, *Risk* **14**(11) 77-81.
- Hamerle, A., Liebig, T., and Rösch, D. (2003): Credit Risk Factor Modelling and the Basel II IRB Approach, Deutsche Bundesbank Discussion Paper.
- Hamerle, A., and Rösch, D. (2004): Parameterizing Credit Risk Models, Working Paper, www.defaultrisk.com.
- Hamilton, D. T., Varma, P., Ou, S., and Cantor, R. (2005): Default and Recovery Rates of Corporate Bond Issuers, 1920-2004, Moody's Investor Service.
- Kalkbrener, M., and Overbeck, L. (2001): Maturity as a factor for credit risk capital, Technical document, CIB/CRM/RAI Risk Research & Development, Deutsche Bank, Frankfurt.
- Kalkbrener, M., and Overbeck, L. (2002): The maturity effect on credit risk capital, *Risk* **14**(7), p. 59-63.
- Kiesel, R., and Perraudin, W., and Taylor, A. P. (2003): The Structure of Credit Risk: Spread Volatility and Rating Transitions, *Journal of Risk* **6**, 1 - 36.
- Koyluoglu, U., and Hickmann, A. (1998a): A Generalized Framework for Credit Portfolio Models, Working Paper, www.defaultrisk.com.
- Koyluoglu, U., and Hickman, A. (1998b): Reconcilable Differences, *Risk* **11**(10), 56-62.
- Li, D. X. (1998): Constructing a Credit Curve, *Risk Credit Risk Special Report* (11), 40-44.
- Li, W., Song, Y., and Ong, M. (1999): Maturity Mismatch, *Risk* **12**(11), 69-71.
- Mah, S. K., and Verde, M. (2003): Fitch Ratings Corporate Finance 2003 Transition and Default Study, Fitch Ratings Ltd.

-
- Marrison, C. (2002): *The Fundamentals of Risk Measurement*, McGraw-Hill.
- Mathworks (2004): *Optimization Toolbox for Use with MATLAB® – User’s Guide*, Version 3, Natick.
- Merton, R. C. (1974): On the Pricing of Corporate Debt: the Risk Structure of Interest Rates, in: *Journal of Finance* **29**, 449-470.
- Merton, R. C. (1977): On the pricing of contingent claims and the Modigliani-Miller theorem, *Journal of financial economics* **5**(2), 241-249.
- Needham, C. L., Verde, M., and Mah, S. K. (2005): *Fitch Ratings Global Corporate Finance 2004 Transition and Default Study*, Fitch Ratings Ltd.
- Ong, M. K. (1999): *Internal Credit Risk Models – Capital Allocation and Performance Measurement*.
- Rösch, D. (2003): Correlations and Business Cycles of Credit Risk: Evidence from Bankruptcies in Germany, *Financial Markets and Portfolio Management* **17**, 309-331.
- Tong, Y. L. (1990): *The multivariate normal distribution*, Springer.
- Vasicek, O. A. (1987): *Probability of Loss on Loan Portfolio*, KMV Corporation.
- Vasicek, O. A. (1991): *Limiting Loan Loss Probability Distribution*, KMV Corporation.
- Vasicek, O. A. (2002): Loan Portfolio Value, *Risk* **15**(12), 160-162.
- Wilde, T. (2001): IRB Approach Explained, *Risk* **14**(5), 87-90.
- Wilson, T. (1997a): Portfolio Credit Risk I, *Risk* **10**(9) 111-119.
- Wilson, T. (1997b): Portfolio Credit Risk II, *Risk* **10**(10), 56-62.

C. Tables

Table 1: Description of the Data Set Used in the Analysis

Dataseries	Source	Type	Number of Rating Grades	Observed Period	Time Horizon of Default Rates
M_CDR_1	MOODY'S	Average Cumulative Default Rates	7 Letter Rating	1970 to 2004	1 to 5 years yearly
S_CDR_r	STANDARD & POOR'S	Cumulative Default Rates	7 Rating	1981 to 2003	1 to 5 years yearly
F_CDR_r	FITCH	Average Cumulative Default Rates:	7 Rating	1990 to 2004	1 to 5 years yearly
C_MM_r	CREDITREFORM Rating AG	Average Rating Migrations Matrix	6 Risk Grade	1999 to 2004	1 to 5 years yearly

Table 2: Parameter Estimates for the multi-period PD-Adjustment under the “Capital to Maturity” approach over a five years period

Series	Rating Grade	$\hat{k}_i^{(CM)}$			Error		
		Lower Bound	Expected Value	Upper Bound	ESS ($\cdot 10^{-6}$)	TSS ($\cdot 10^{-6}$)	R ²
Moody's M_CDR_1	Aaa	0.53	1.05	1.56	1.16	0.62	< 0
	Aa	0.46	0.91	1.35	2.94	2.35	< 0
	A	0.47	0.63	0.79	4.24	15.21	0.721
	Baa	0.32	0.46	0.60	42.28	231.39	0.817
	Ba	0.06	0.16	0.26	330.02	5731.23	0.942
	B	-0.16	-0.08	-0.01	699.06	38276.37	0.982
	Caa-C	-0.36	-0.32	-0.27	405.29	88800.91	0.995
Standard & Poor's S_CDR_r	AAA	0.61	1.05	1.48	0.85	0.38	< 0
	AA	0.49	0.77	1.04	4.00	5.49	0.271
	A	0.45	0.62	0.79	7.25	22.91	0.684
	BBB	0.26	0.39	0.53	91.99	581.09	0.842
	BB	0.04	0.11	0.19	228.04	7,816.72	0.971
	B	-0.07	-0.03	0.00	123.31	27,517.75	0.996
	CCC/C	-0.18	-0.16	-0.14	95.25	30,425.55	0.997
Fitch F_CDR_r	AAA	0.85	1.13	1.40	0.17	0.00	< 0
	AA	0.70	1.10	1.49	0.45	0.13	< 0
	A	0.50	0.59	0.68	2.50	21.39	0.883
	BBB	0.27	0.35	0.43	34.12	670.25	0.949
	BB	0.21	0.24	0.26	14.48	3,050.13	0.995
	B	0.24	0.33	0.42	168.24	1,563.06	0.892
	CCC/C	-0.04	0.11	0.26	3781.91	5,640.49	0.330
Creditreform C_MM_r	Risk Grade 1	0.23	0.34	0.45	70.91	786.12	0.910
	Risk Grade 2	0.25	0.35	0.44	86.21	1,065.15	0.919
	Risk Grade 3	0.25	0.32	0.39	88.56	1,607.26	0.945
	Risk Grade 4	0.15	0.21	0.27	141.28	4,600.15	0.969
	Risk Grade 5	0.03	0.04	0.05	15.77	1,6315.05	0.999
	Risk Grade 6	-0.18	-0.11	-0.05	866.80	32,194.21	0.973

Table 3: Comparison of the empirical UL with the model based UL using the (estimated) multi-period PD-Adjustment under the “Capital to Maturity” approach over a five years period

Series	Rating Grade	Error			Series	Rating Grade	Error		
		ESS	TSS	R ²			ESS	TSS	R ²
Moody's M_CDR_l	Aaa	0.001	0.000	< 0	Fitch F_CDR_r	AAA	0.000	0.000	< 0
	Aa	0.002	0.002	< 0		AA	0.000	0.000	< 0
	A	0.002	0.006	0.738		A	0.001	0.007	0.914
	Baa	0.004	0.023	0.846		BBB	0.002	0.038	0.954
	Ba	0.002	0.054	0.954		BB	0.000	0.034	0.996
	B	0.000	0.023	0.987		B	0.001	0.019	0.944
	Caa-C	0.000	0.006	0.995		CCC/C	0.000	0.000	0.642
Standard & Poor's S_CDR_r	AAA	0.001	0.000	< 0	Credit- reform C_MM_r	R. Grade 1	0.004	0.042	0.906
	AA	0.002	0.003	0.297		R. Grade 2	0.002	0.032	0.936
	A	0.002	0.007	0.731		R. Grade 3	0.001	0.026	0.958
	BBB	0.004	0.031	0.874		R. Grade 4	0.001	0.026	0.976
	BB	0.001	0.057	0.975		R. Grade 5	0.000	0.010	1.000
	B	0.000	0.021	0.996		R. Grade 6	0.000	0.000	0.897
	CCC/C	0.000	0.002	0.995					

Table 4: Parameters From Own Estimates of the Simple Maturity Adjustment using “Capital to Maturity”-Paradigm over a Five Years Period

Series	Rating-Grades	a			b			Error		
		Lower Bound	Ex-pected Value	Upper Bound	Lower Bound	Ex-pected Value	Upper Bound	ESS	TSS	R ²
Moody's M_CDR_l	A-Grades	n.a.	0.0107	n.a.	n.a.	0.0743	n.a.	0.007	0.012	0.44
	B-Grades	0.1598	0.2551	0.3503	0.0418	0.0616	0.0814	0.018	0.231	0.92
	All Grades	0.0167	0.1477	0.2787	0.0459	0.0673	0.0887	0.163	0.903	0.82
Standard & Poor's S_CDR_r	A-Grades	-0.7913	0.6317	2.0547	2.0547	0.0000	2.0547	0.004	0.017	0.75
	B-Grades	0.1098	0.2336	0.3575	0.0403	0.0674	0.0945	0.024	0.191	0.87
	All Grades	0.0570	0.1718	0.2866	0.0460	0.0652	0.0845	0.154	0.910	0.83
Fitch F_CDR_r	A-Grades	-6.2528	0.5979	7.4486	7.4486	0.0000	7.4486	0.013	0.019	0.34
	B-Grades	-0.0470	0.1375	0.3221	0.0480	0.0846	0.1213	0.023	0.113	0.79
	All Grades	0.0577	0.1713	0.2849	0.0439	0.0637	0.0835	0.122	0.830	0.85
Credit- reform C_MM_r	Grades 1-3	-0.0305	0.1083	0.2471	0.0631	0.0896	0.1161	0.006	0.109	0.94
	Grades 4-6	-0.1469	0.0000	0.1469	0.0785	0.1252	0.1720	0.013	0.062	0.78
	All Grades	-0.0097	0.0605	0.1307	0.0848	0.0992	0.1137	0.027	0.292	0.91
Basel II	All Grades	0.11852			0.05478					

Table 5: Comparison of the Maturity Adjustments: the Error Sum of Squares and R^2 for the Model Based Adjustment (using different values k_i for Each Rating Class), the Simple Adjustment (using own Estimates for the Parameters a and b), and the Simple Adjustment (using Basel II Parameters for a and b).

Series	Rating Grade	Model Based Adjustment			Simple Adjustment		Basel II Adjustment	
		TSS	ESS	R^2	ESS	R^2	ESS	R^2
Moody's M_CDR_l	Aaa	2.700	5.396	< 0	64.526	< 0	4.533	< 0
	Aa	8.513	10.787	< 0	39.783	< 0	2.406	< 0
	A	32.094	8.395	0.738	0.371	0.988	37.060	< 0
	Baa	8.631	1.331	0.846	3.225	0.626	12.451	< 0
	Ba	2.671	0.123	0.954	2.652	0.007	5.147	< 0
	B	0.370	0.005	0.987	0.332	0.101	0.701	< 0
	Caa-C	0.036	0.000	0.995	0.376	< 0	0.227	< 0
Standard & Poor's S_CDR_r	AAA	1.822	4.193	< 0	76.954	< 0	3.829	< 0
	AA	15.444	10.850	0.297	22.068	< 0	6.809	< 0
	A	17.241	4.634	0.731	0.058	0.997	19.323	< 0
	BBB	5.044	0.634	0.874	1.799	0.643	7.585	< 0
	BB	2.624	0.066	0.975	2.575	0.019	5.514	< 0
	B	0.320	0.001	0.996	0.192	0.401	0.585	< 0
	CCC/C	0.014	0.000	0.995	0.321	< 0	0.150	< 0
Fitch F_CDR_r	AAA	0.000	0.894	< 0	80.230	< 0	10.937	< 0
	AA	0.688	2.156	-2.133	68.957	< 0	7.317	< 0
	A	24.170	2.069	0.914	3.746	0.845	39.652	< 0
	BBB	7.938	0.366	0.954	6.302	0.206	15.066	< 0
	BB	1.422	0.006	0.996	1.117	0.214	2.776	< 0
	B	0.682	0.038	0.944	0.440	0.355	1.127	< 0
	CCC/C	0.000	0.000	0.642	0.150	< 0	0.038	< 0
Credit-reform C_MM_r	R. Grade 1	8.619	0.810	0.906	0.348	0.960	14.754	< 0
	R. Grade 2	2.637	0.170	0.936	0.165	0.937	4.194	< 0
	R. Grade 3	1.228	0.052	0.958	0.146	0.881	1.841	< 0
	R. Grade 4	0.751	0.018	0.976	0.231	0.692	1.205	< 0
	R. Grade 5	0.111	0.000	1.000	0.054	0.511	0.135	< 0
	R. Grade 6	0.003	0.000	0.897	0.052	< 0	0.035	< 0

Table 6: Parameter Estimates for the multi-period PD-Adjustment under the “Capital for one Period” approach over a five years period

Series	Rating Grade	k_i			Error		
		Lower Bound	Expected Value	Upper Bound	ESS ($\cdot 10^{-6}$)	TSS ($\cdot 10^{-6}$)	R ²
Moody's M_CDR_l	Aaa	n.a.	1.490	n.a.	0.570	0.620	0.081
	Aa	0.170	1.000	1.830	2.015	2.348	0.142
	A	-0.096	0.516	1.129	9.808	15.208	0.355
	Baa	n.a.	0.630	n.a.	69.436	231.388	0.700
	Ba	n.a.	0.590	n.a.	78.022	5,731.228	0.986
	B	n.a.	0.540	n.a.	48.988	38,276.368	0.999
	Caa-C	n.a.	0.830	n.a.	228.669	88,800.908	0.997
Standard & Poor's S_CDR_r	AAA	n.a.	1.490	n.a.	0.330	0.380	0.132
	AA	-0.312	1.000	2.312	4.142	5.492	0.246
	A	-0.078	0.543	1.163	14.049	22.912	0.387
	BBB	n.a.	0.530	n.a.	261.568	581.088	0.550
	BB	n.a.	0.430	n.a.	712.052	7,816.720	0.909
	B	n.a.	0.560	n.a.	525.231	27,517.752	0.981
	CCC/C	n.a.	1.260	n.a.	672.757	30,425.548	0.978
Fitch F_CDR_r	AAA	10.534	10.534	10.534	0.000	0.000	1.000
	AA	-0.216	1.962	4.140	0.120	0.128	0.062
	A	-0.064	0.438	0.940	10.286	21.388	0.519
	BBB	n.a.	0.440	n.a.	99.568	670.248	0.851
	BB	n.a.	0.660	n.a.	49.815	3,050.128	0.984
	B	n.a.	0.720	n.a.	89.580	1,563.060	0.943
	CCC/C	n.a.	1.500	n.a.	4,391.864	5,640.492	0.221
Creditreform C_MM_r	Risk Grade 1	n.a.	0.640	n.a.	54.326	786.124	0.931
	Risk Grade 2	n.a.	0.600	n.a.	153.985	1,065.146	0.855
	Risk Grade 3	n.a.	0.750	n.a.	40.132	1,607.258	0.975
	Risk Grade 4	n.a.	0.690	n.a.	117.079	4,600.148	0.975
	Risk Grade 5	0.665	0.781	0.898	644.302	16,315.046	0.961
	Risk Grade 6	n.a.	0.970	n.a.	1,137.454	32,194.210	0.965

Table 7: Comparison of the empirical UL with the model based UL using the (estimated) multi-period PD-Adjustment under the “Capital for one Period” approach over a five years period

Series	Rating Grade	Error			Series	Rating Grade	Error		
		ESS	TSS	R ²			ESS	TSS	R ²
Moody's M_CDR_1	Aaa	0.000	0.000	< 0	Fitch F_CDR_r	AAA	0.000	0.000	1.000
	Aa	0.001	0.000	< 0		AA	0.000	0.000	< 0
	A	0.004	0.001	< 0		A	0.022	0.001	< 0
	Baa	0.004	0.002	< 0		BBB	0.018	0.005	< 0
	Ba	0.000	0.005	0.973		BB	0.001	0.003	0.666
	B	0.000	0.001	0.953		B	0.000	0.001	1.000
	Caa-C	0.000	0.000	1.000		CCC/C	0.000	0.000	1.000
Standard & Poor's S_CDR_r	AAA	0.000	0.000	< 0	Credit- reform C_MM_r	R. Grade 1	0.003	0.008	0.630
	AA	0.000	0.000	0.171		R. Grade 2	0.005	0.001	< 0
	A	0.015	0.001	< 0		R. Grade 3	0.000	0.000	0.598
	BBB	0.015	0.003	< 0		R. Grade 4	0.000	0.000	< 0
	BB	0.005	0.007	0.275		R. Grade 5	0.000	0.000	1.000
	B	0.000	0.001	0.870		R. Grade 6	0.000	0.000	1.000
	CCC/C	0.000	0.000	1.000					

Table 8: Parameters From Own Estimates of the Simple Maturity Adjustment using “Capital for one Period”-Paradigm over a Five Years Period

Series	Rating-Grades	a			b			Error		
		Lower Bound	Ex-pected Value	Upper Bound	Lower Bound	Ex-pected Value	Upper Bound	ESS	TSS	R ²
Moody's M_CDR_1	A-Grades	0.0000	0.0087	0.0000	0.0000	0.0603	0.0000	0.001	0.002	0.45
	B-Grades	-0.1317	0.0140	0.1597	0.0454	0.0769	0.1084	0.004	0.098	0.96
	All Grades	-0.0539	0.0207	0.0954	0.0535	0.0684	0.0834	0.009	0.691	0.99
Standard & Poor's S_CDR_r	A-Grades	-1.7031	0.4810	2.6652	2.6652	0.0004	2.6652	0.001	0.003	0.71
	B-Grades	-0.2050	0.0172	0.2393	0.0276	0.0779	0.1282	0.009	0.077	0.88
	All Grades	-0.0435	0.0427	0.1289	0.0475	0.0655	0.0835	0.014	0.718	0.98
Fitch F_CDR_r	A-Grades	-8.4547	0.4514	9.3576	9.3576	0.0000	9.3576	0.002	0.004	0.31
	B-Grades	-0.3604	0.0000	0.3604	0.0003	0.0775	0.1547	0.009	0.024	0.63
	All Grades	-0.1179	0.0040	0.1260	0.0448	0.0688	0.0927	0.016	0.617	0.97
Credit- reform C_MM_r	Grades 1-3	-0.6790	0.0000	0.6790	-0.0624	0.0734	0.2093	0.012	0.013	0.06
	Grades 4-6	-0.1362	0.0000	0.1362	-0.0131	0.0363	0.0857	0.001	0.095	0.99
	All Grades	-0.1545	0.0000	0.1545	0.0284	0.0666	0.1048	0.017	0.273	0.94
Basel II		0.11852			0.05478					

Table 9: Comparison of the Maturity Adjustments for the “Capital for One Period”
 Approach: the Error Sum of Squares and R^2 of
 the Model Based Adjustment (using different values k_i for Each Rating Class),
 the Simple Adjustment (using own Estimates for the Parameters a and b),
 and the Simple Adjustment (using Basel II Parameters for a and b).

Series	Rating Grade	Model Based Adjustment			Simple Adjustment		Basel II Adjustment	
		TSS	ESS	R^2	ESS	R^2	ESS	R^2
Moody's M_CDR_l	Aaa	1.026	1.283	< 0	8.374	< 0	6.748	< 0
	Aa	2.122	3.177	< 0	4.320	< 0	3.233	< 0
	A	4.867	22.643	< 0	1.085	0.777	1.270	0.739
	Baa	0.918	1.573	< 0	0.185	0.799	0.134	0.854
	Ba	0.249	0.007	0.973	0.228	0.086	0.125	0.500
	B	0.016	0.001	0.953	0.014	0.152	0.056	< 0
	Caa-C	0.000	0.000	1.000	0.007	< 0	0.055	< 0
Standard & Poor's S_CDR_r	AAA	0.051	0.064	< 0	11.480	< 0	9.775	< 0
	AA	2.752	2.281	0.171	1.677	0.391	1.116	0.594
	A	2.007	38.855	< 0	0.106	0.947	0.076	0.962
	BBB	0.414	2.519	< 0	0.069	0.834	0.077	0.813
	BB	0.326	0.236	0.275	0.416	< 0	0.283	0.129
	B	0.012	0.002	0.870	0.016	< 0	0.062	< 0
	CCC/C	0.000	0.000	1.000	0.006	< 0	0.037	< 0
Fitch F_CDR_r	AAA	0.000	0.000	1.000	10.809	< 0	10.937	< 0
	AA	0.051	0.064	< 0	9.656	< 0	9.777	< 0
	A	2.955	78.284	< 0	2.418	0.182	2.300	0.222
	BBB	1.036	3.831	< 0	1.289	< 0	0.862	0.168
	BB	0.124	0.041	0.666	0.136	< 0	0.112	0.095
	B	0.024	0.000	1.000	0.048	< 0	0.210	< 0
	CCC/C	0.000	0.000	1.000	0.003	< 0	0.050	< 0
Credit- reform C_MM_r	R. Grade 1	1.739	0.644	0.630	2.196	< 0	1.274	0.268
	R. Grade 2	0.100	0.405	< 0	0.081	0.193	0.264	< 0
	R. Grade 3	0.023	0.009	0.598	0.065	< 0	0.324	< 0
	R. Grade 4	0.006	0.008	< 0	0.044	< 0	0.252	< 0
	R. Grade 5	0.000	0.000	1.000	0.022	< 0	0.147	< 0
	R. Grade 6	0.000	0.000	1.000	0.005	< 0	0.065	< 0

D. Figures

Figure 1: The value for the UL contribution in the Vasicek-model with respect to the probability of default ($PD_i^{(0,T)}$) for correlation parameter $\rho_i=0.20$ and $\alpha = 0.001$

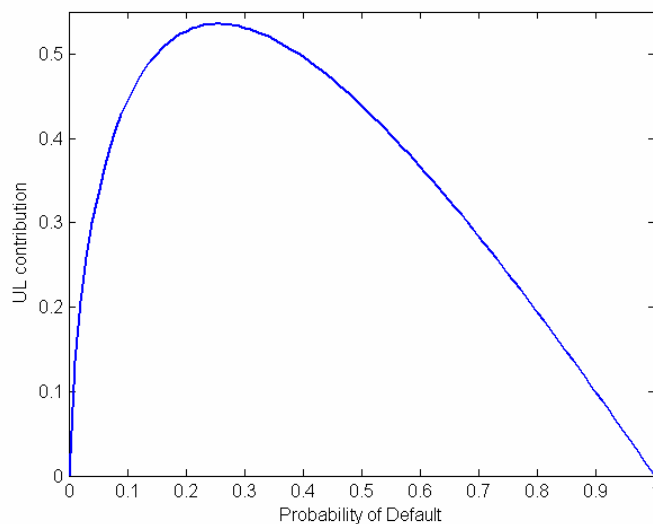


Figure 2: The function value of the maturity adjustment with respect to number of years ($m = 1$ to 5) and Probability of Default ($\widehat{PD}_i^{(0,1 \text{ year})} = 0.01$ to 0.3) using different parameters for a and b, that are $[a, b] = [0.11852, 0.05478]$ (\cdots), $[a, b] = [0.03951, 0.05478]$ ($-$), and $[a, b] = [0.11852, 0.01826]$ (\dashv)

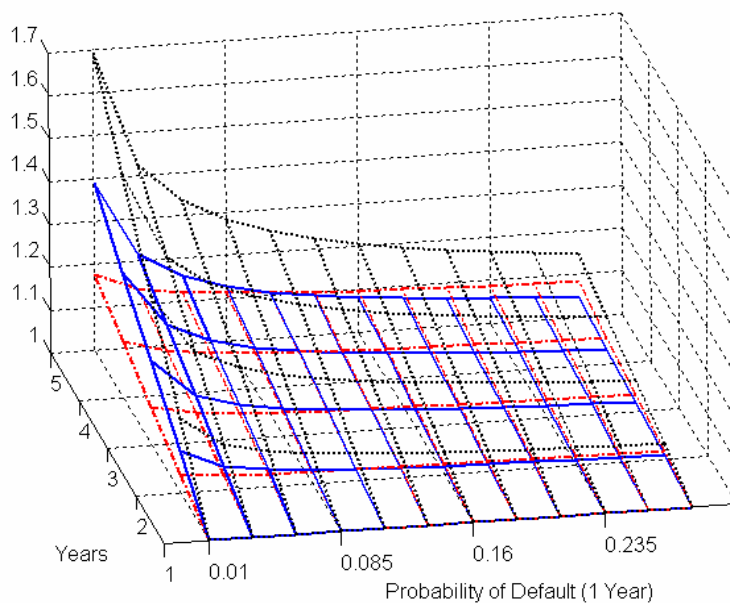


Figure 3: Average Cumulative Default Rates by Rating Grades for Moody's (M_CDR_l), Standard & Poor's (S_CDR_r), Fitch (F_CDR_r), and Creditreform (C_MM_r)

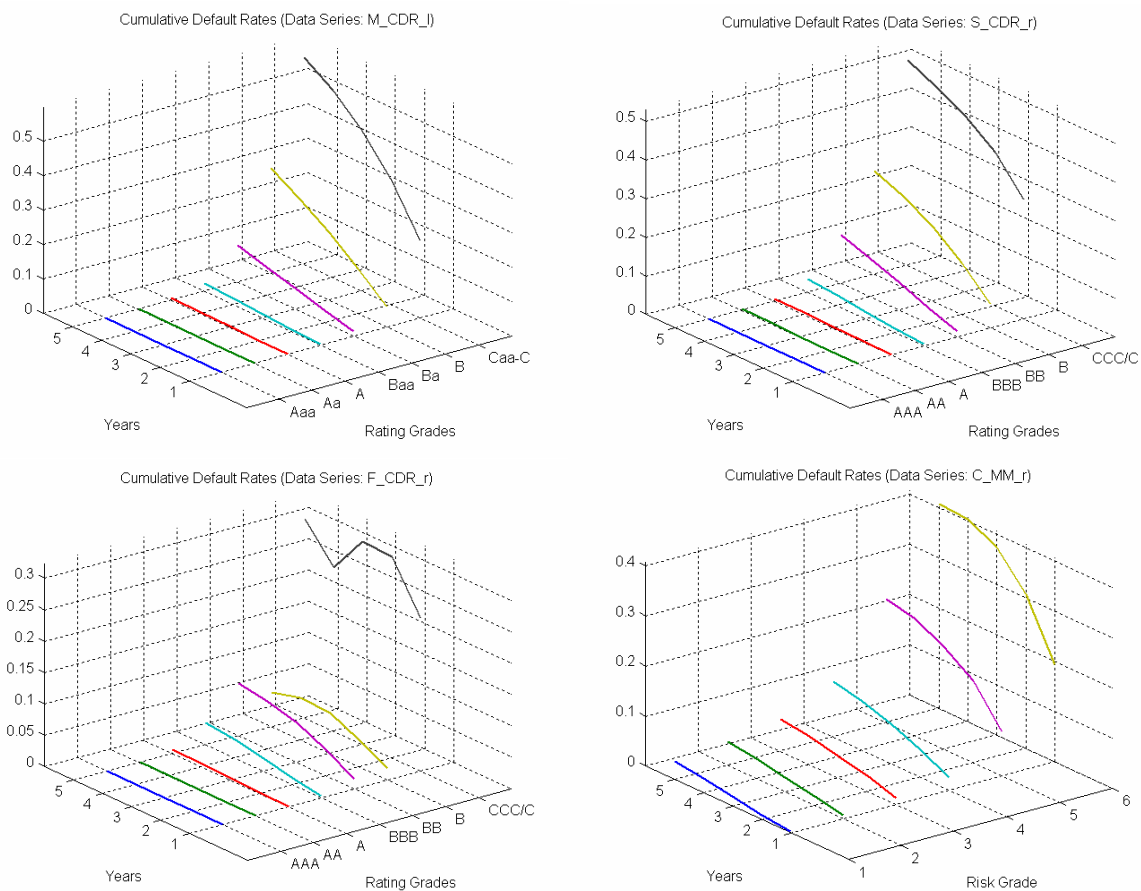


Figure 4: UL using Average Cumulative Default Rates by Rating Grades for Moody's (M_CDR_1), Standard & Poor's (S_CDR_r), Fitch (F_CDR_r), and Creditreform (C_MM_r)

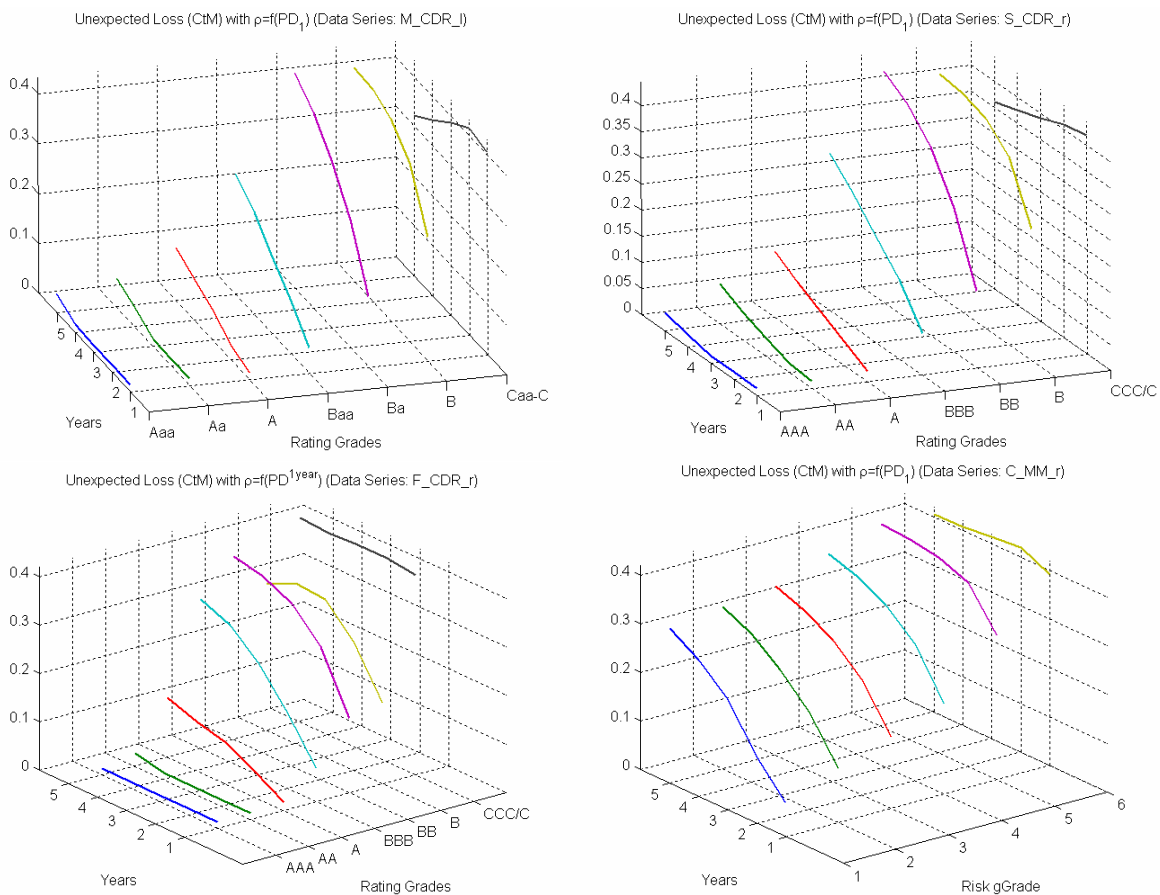


Figure 5: Comparison of the Maturity Adjustment using “Capital to Maturity”-Paradigm: Empirical Adjustment (-), and Model Based Adjustment (-) from Cumulative Default Rates over a Five Years Period from Moody’s (M_CDR_l), Standard & Poor’s (S_CDR_r), Fitch (F_CDR_r), and Creditreform (C_MM_r)

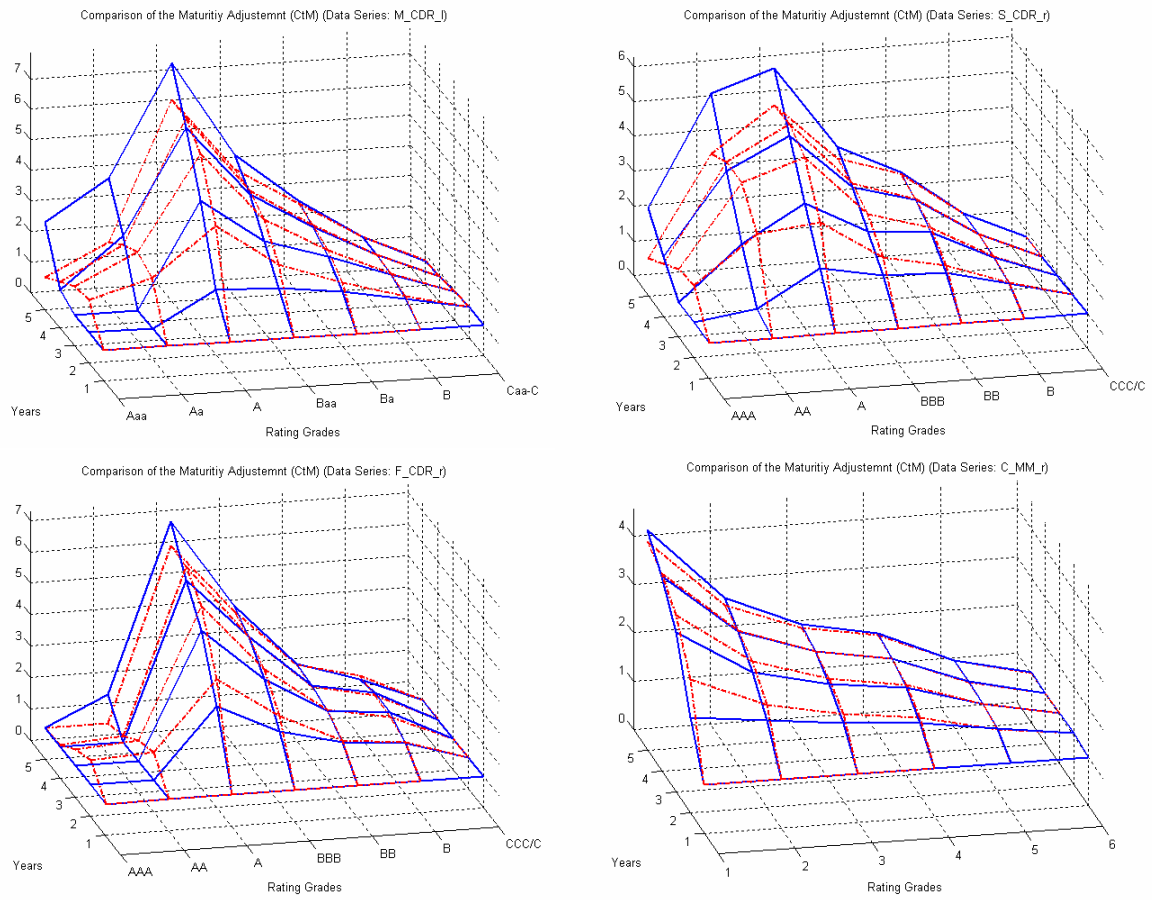


Figure 6: Comparison of the Maturity Adjustment using “Capital to Maturity”-Paradigm: Empirical Adjustment (-), Own Estimates (-·-) and from Basel II (••) using Cumulative Default Rates over a Five Years Period from Moody’s (M_CDR_l), Standard & Poor’s (S_CDR_r), Fitch (F_CDR_r), and Creditreform (C_MM_r)

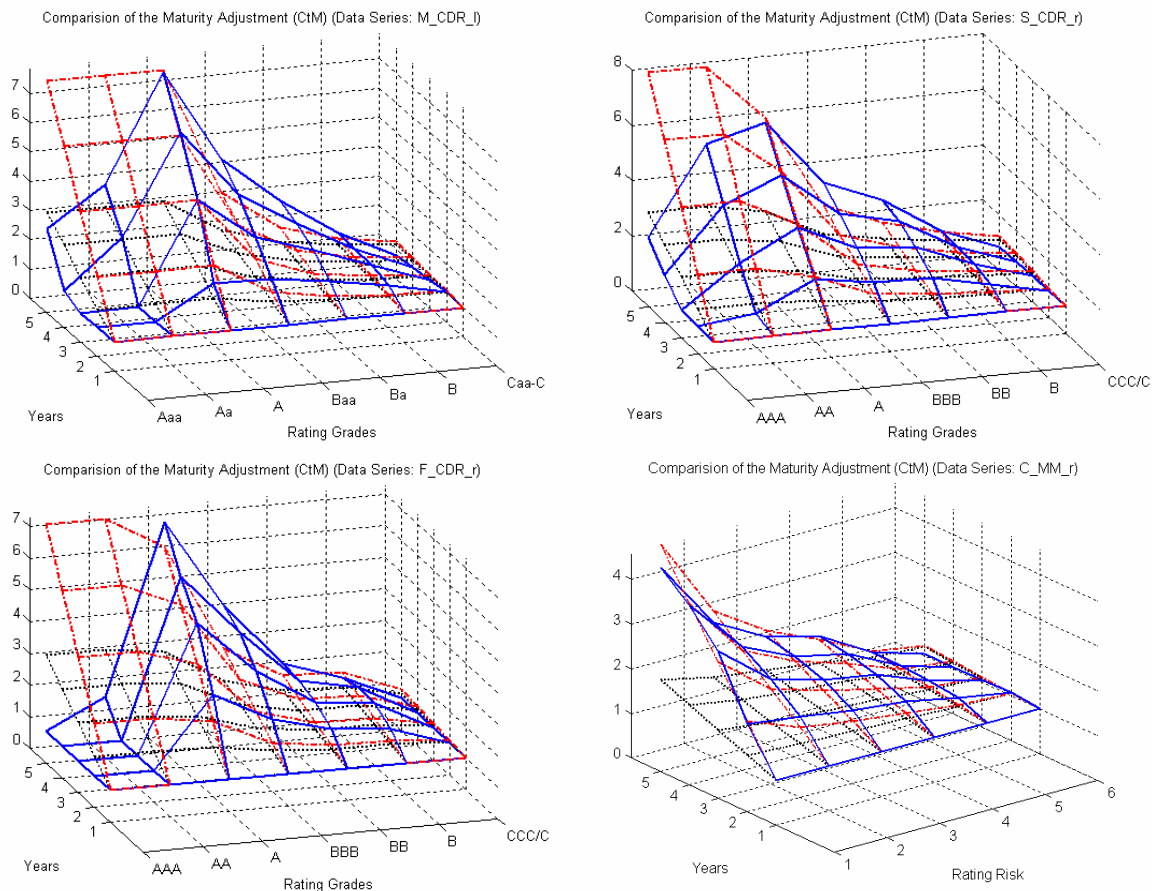


Figure 7: Average Conditional Default Rates be Rating Grades for Moody's (M_CDR_l), Standard & Poor's (S_CDR_r), Fitch (F_CDR_r), and Creditreform (C_MM_r)

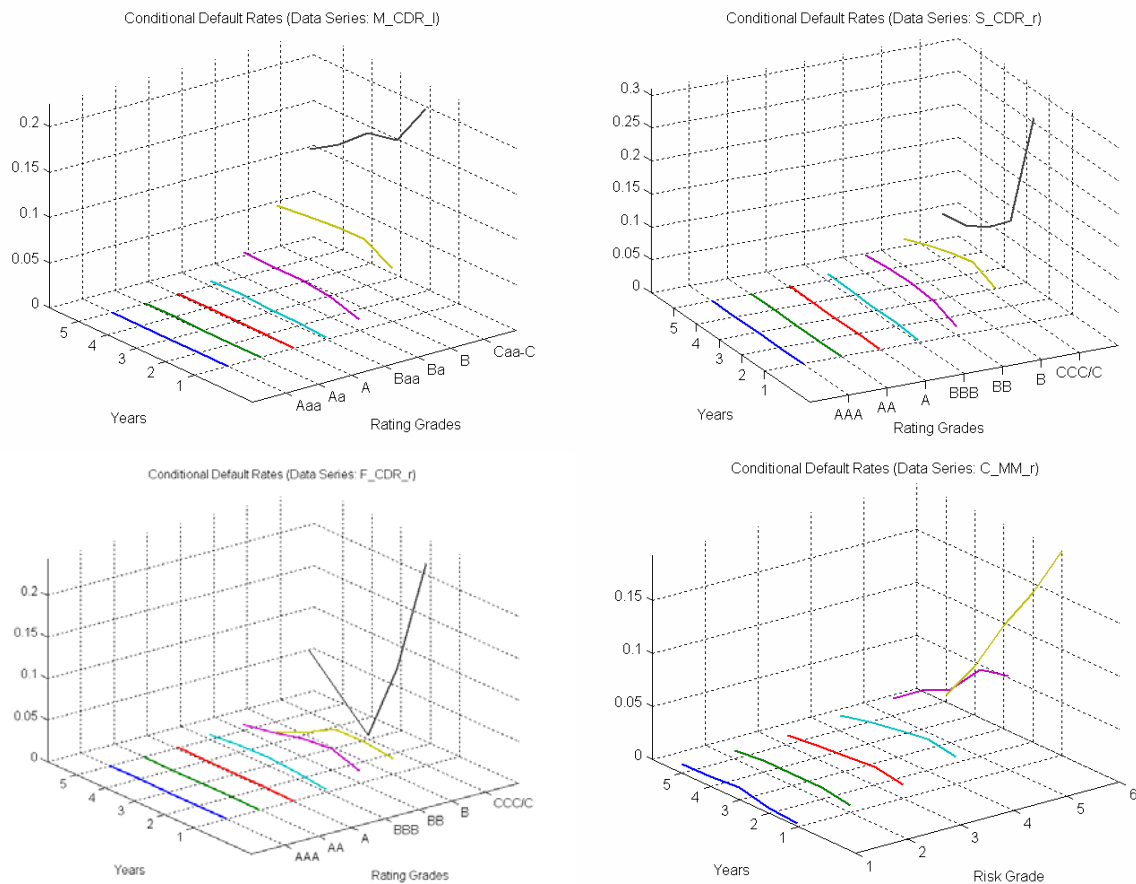


Figure 8: UL under the “Capital for one Period”-Paradigm using Average Conditional Default Rates by Rating Grades for Moody’s (M_CDR_l), Standard & Poor’s (S_CDR_r), Fitch (F_CDR_r), and Creditreform (C_MM_r)

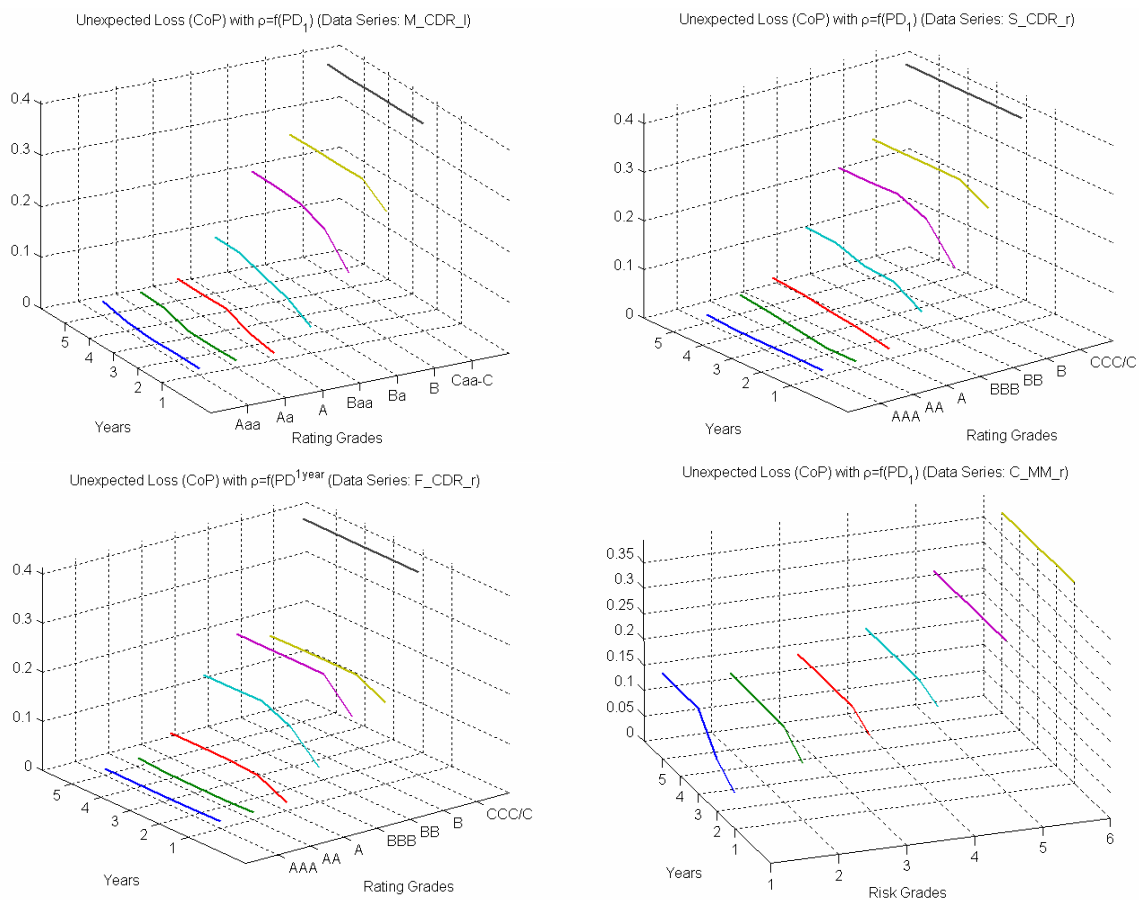


Figure 9: Comparison of the Maturity Adjustment using “Capital to Maturity”-Paradigm: Empirical Adjustment (-), and Model Based Adjustment (-·-) from Cumulative Default Rates over a Five Years Period from Moody’s (M_CDR_l), Standard & Poor’s (S_CDR_r), Fitch (F_CDR_r), and Creditreform (C_MM_r)

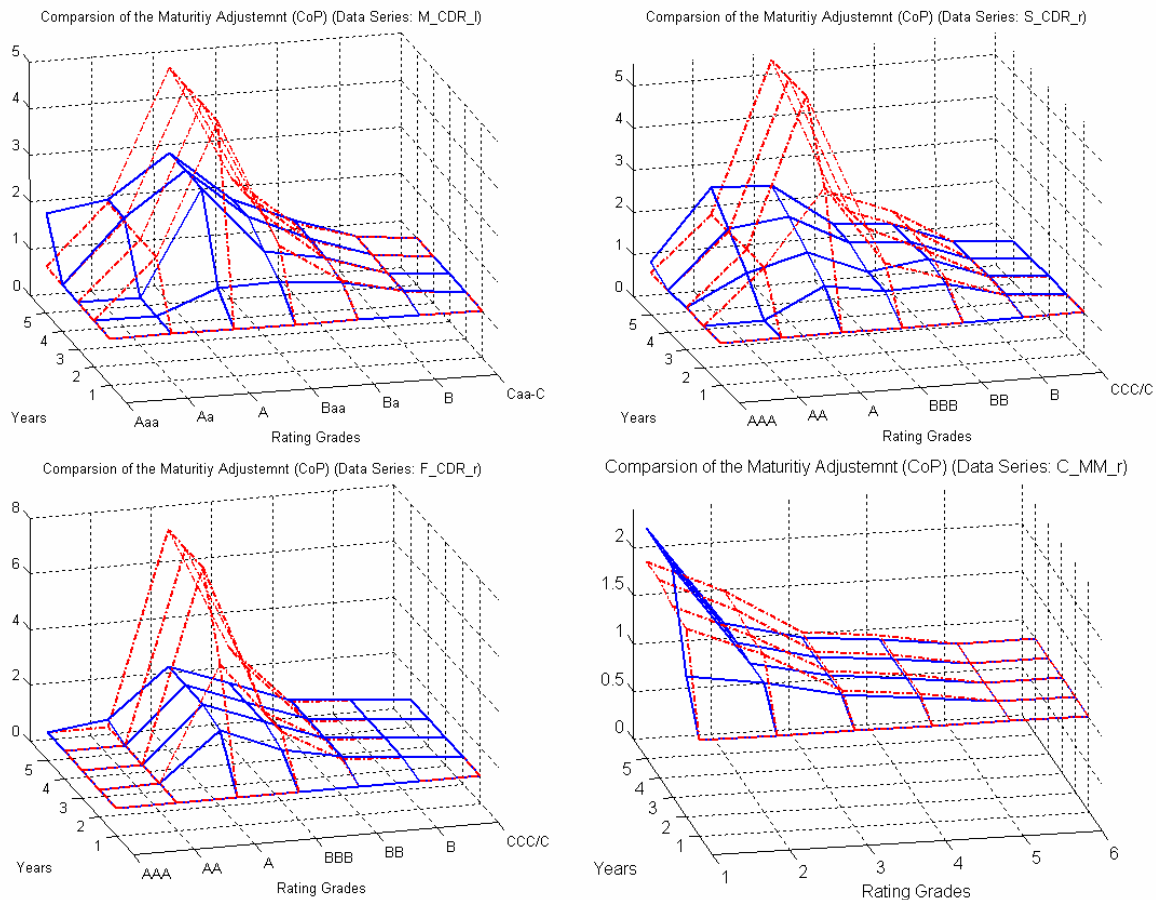


Figure 10: Comparison of the Maturity Adjustment using “Capital for One Period”-Paradigm: the Empirical Adjustment (-), Own Estimates (-.-) and from Basel II (••) using Cumulative Default Rates over a Five Years Period from Moody’s (M_CDR_l), Standard & Poor’s (S_CDR_r), Fitch (F_CDR_r), and Creditreform (C_MM_r)

