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## **Abstract**

This paper deals with a core-equilibrium equivalence in an economy with public goods where preferences of consumers display warm glow effects. We demonstrate that provided that each consumer becomes satiated to other consumers provision, it holds that, for a sufficiently large economy, the set of Edgeworth allocations is non-empty. Moreover, we show that an Edgeworth allocation could be decentralized as a warm glow equilibrium.

*Keywords:* competitive equilibrium, warm glow, public goods, Edgeworth, core, decentralization.

*JEL Classification Numbers:* H41, C71, D64.

# 1 Introduction

The seminal contribution of Debreu and Scarf (1963) shows that, in a pure exchange economy, the core allocations shrink to the Walrasian equilibrium as the set of consumers is appropriately enlarged. The core consists of allocations of resources such that no coalition could achieve a preferred outcome for its members by seceding from the grand coalition and proposing another allocation that is reachable within its own resources. The Debreu and Scarf contribution, which is a lucid formulation of an earlier conjecture by Edgeworth (1881), has gained its prominence as a justification of the emergence of competitive behavior as a result of social stability.

In economies with public goods, various attempts have been made to investigate whether a similar core-equilibrium convergence holds for the Lindahl price based mechanism. This is of a paramount importance, as it would provide Lindahl equilibrium with the same solid foundation of competitive equilibrium. Unfortunately, it turns out that this type of convergence is the exception rather than the rule. The literature is furnished with either robust examples of non convergence (for example, see Muench (1972), Milleron (1972) and Champsaur, Roberts and Rosenthal (1975)) or few context specific convergences. This is hardly surprising since intuitively, unlike the case of a pure exchange economy, potential blocking coalitions are more likely to fall short of the resources available to the grand coalition to produce public goods. Hence, unless the benefit of an ever-increasing public goods provision is limited when the economy grows large, the core equilibrium convergence is deemed to fail. The work of Wooders (1983) and Conley (1992) provide valuable insights into economies with public goods where the above limitation may hold. The game theoretic approach of Wooders (1983) imposes the assumption of per capita boundedness on the equal treatment payoffs of replica games to ensure the nonemptiness of an approximate limit core. Conley (1992) evokes the possibility of asymptotic satiation in public goods consumption due to the resulting colossal magnitude of the aggregate supply of public goods in large replica economies.

More recently, Florenzano and del Mercato (2006) show a subtle convergence of non standard core allocations to Lindahl equilibrium. Given the fact that the set of core allocations is usually bigger than the Lindahl allocations, Florenzano and del Mercato (2006) bypass this difficulty by constructing a sequence of artificial replica economies where public goods are normalized by the size of the coalition providing it. More generally, as in the clubs/local

public goods literature, where the public goods are subject to crowding and congestion, the social stability underlying the formation of communities and groups providing these goods will eventually settle the economy in a competitive equilibrium (for example, see Wooders (1989, 1997) and Conley and Smith (2005) for a survey).

The existence of Lindahl equilibrium in economies with public goods was first formalized by Foley (1967, 1970) (see also Fabre-Sender (1969), Milleron (1972), Roberts(1974), Bergstrom (1976) and Myles(1995)). Foely's approach consists of embedding the public goods economy into a larger private goods economy wherein each consumer is the only buyer of his own copy of each public good. The existence of Lindahl is then established by resorting to standard existence results for private goods economy. In the public goods literature, lately, the warm glow model, where consumers receive a direct utility for their own public goods provision, has been put forward by Andreoni (1989, 1990) (see also Becker (1974), Cornes and Sandler (1984)) as a possible candid description of public goods provision. In a recent paper, Allouch (2008) introduces a Lindahl-like competitive equilibrium for a warm glow economy and provides the three fundamental theorems of general equilibrium (the two Welfare Theorems and existence). It is worth noticing that the warm glow equilibrium coincides with the Lindahl equilibrium if we consider a standard formalization of utility functions. Then a natural question arises, which is, "*Under what circumstances the core allocations converge to the warm glow equilibrium?*"

Fortunately, one possible answer for the above question comes from the literature on warm glow giving itself:

"For example, as the size of the population increases, choosing a contribution level becomes more and more like picking the level of consumption for any conventional good. In the limit, the contributor simply weighs the relative merits of spending money on two different private goods,  $x_i$  and  $g_i$ ; the effect on his well-being through  $G$  becomes negligible." (p. 58, Bernheim and Rangel (2005) "Behavioral public economics: welfare and policy analysis with non-standard decision makers" *NBER Working Paper*, 11518.)

And,

"Another way to see this intuitively is that, as the size of the charity grows, all giving due to altruism will be crowded out,

leaving only giving due to warm-glow. This accords naturally with the observation that giving 100 dollars to an organization that collects millions is motivated more by an admiration for the organization than for any measurable effect of the marginal donation.” (p. 1223, Andreoni (2006) “Philanthropy,” in *the Handbook of Giving, Reciprocity and Altruism*.)

In this paper we attempt to articulate the above observations into an economic assumption in order to establish that core allocations converge to the warm glow equilibrium. Namely, we emphasize the fact that consumers charitable giving, at some point, is more driven by the act of giving itself rather than the aggregate provision of public goods, and thus increased public goods provision by other consumers has no effect on a particular consumer welfare. Stated Formally, our main economic assumption stipulates that, beyond a threshold of other consumers’ public goods provision, each consumer benefits only from own public goods provision and private goods consumption.

In the following, Section 2 is devoted to the model, Section 3 defines Edgeworth allocations and Section 4 shows the non-vacuity of the set of Edgeworth allocations. In Section 5 we introduce the concept of warm glow equilibrium and demonstrate our core-equilibrium equivalence result.

## 2 The model

We consider a public goods economy with  $i = 1, \dots, N$  consumers,  $l = 1, \dots, L$  private goods and  $k = 1, \dots, K$  public goods. A consumption bundle of private goods is denoted by  $x = (x^1, \dots, x^L) \in \mathbf{R}_+^L$  and a consumption bundle of public goods is denoted by  $g = (g^1, \dots, g^K) \in \mathbf{R}_+^K$ . The private and public goods consumption set is  $\mathbf{R}_+^{L+K}$ , for each consumer  $i$ . The production technology for public goods is described by an aggregate production set  $Y \subset \mathbb{R}^L \times \mathbb{R}_+^K$ . A typical production plan will be written  $(y, g)$ , where  $y \in \mathbb{R}^L$  denotes *inputs of private goods* and  $g \in \mathbb{R}_+^K$  denotes *outputs of public goods*. Each consumer  $i$  has strictly positive endowments of private goods, denoted by  $w_i \in \mathbb{R}_{++}^L$ , and has no endowments of public goods. The preferences of each consumer  $i$  may be represented by a utility function  $u_i(x_i, g_i, G_{-i})$ , where  $x_i$  is consumer’s  $i$  private goods consumption,  $g_i$  is consumer’s  $i$  public goods consumption, and  $G_{-i} = \sum_{j \neq i} g_j$  is the total provision of public goods

minus consumer's  $i$  provision. The utility function  $u_i$  satisfies the following properties:

**[A.1] Monotonicity:** The utility function  $u_i(\cdot, \cdot, \cdot)$  is increasing. Moreover, given any  $G_{-i} \in \mathbb{R}_+^K$ , the function  $u_i(\cdot, \cdot, G_{-i})$  is strictly increasing on  $\mathbb{R}_+^L \times \mathbb{R}_{++}^K$ .

**[A.2] Continuity:** The utility function  $u_i(\cdot, \cdot, \cdot)$  is continuous.

**[A.3] Convexity:** The utility function  $u_i(\cdot, \cdot, \cdot)$  is quasi-concave.

**[A.4] Warm glow indispensability:** For every  $(x_i, g_i, G_{-i}) \in \mathbb{R}_+^{L+K}$ , if  $g_i \notin \mathbb{R}_{++}^K$  then  $u_i(x_i, g_i, G_{-i}) = \inf u_i(\cdot, \cdot, \cdot)$ .

For simplicity, we consider a constant returns to scale technology. Thus, we assume that  $Y$  is a closed convex cone with vertex the origin; satisfying the usual conditions of irreversibility, no free production, and free disposal. In addition, we assume the possibility of producing public goods, that is,  $Y \cap (\mathbb{R}^L \times \mathbb{R}_{++}^K) \neq \emptyset$ , and that public goods are inessential in production, that is, if  $(y, g^1, \dots, g^K) \in Y$ , then  $(y, \sup\{g^1, 0\}, \dots, \sup\{g^K, 0\}) \in Y$ .

### 3 Edgeworth allocations

Let  $S$  be a non-empty subset of  $N$ . An allocation  $((x_i, g_i), i \in S)$  is  $S$ -feasible if

$$\left( \sum_{i \in S} (x_i - w_i), \sum_{i \in S} g_i \right) \in Y.$$

For simplicity of notations, an  $N$ -feasible allocation will be simply called a feasible allocation.

A coalition  $S \subset N$  can *improve upon* an allocation  $((\bar{x}_i, \bar{g}_i), i \in N)$  if there exists an  $S$ -feasible allocation  $((x_i, g_i), i \in S)$ , such that

$$u_i(x_i, g_i, \sum_{j \in S \setminus \{i\}} g_j) > u_i(\bar{x}_i, \bar{g}_i, \bar{G}_{-i}), \text{ for each consumer } i \in S.$$

That is to say, coalition  $S$  could do better for its members by breaking away from the grand coalition and proposing another allocation that is feasible with its own available resources. An allocation  $((\bar{x}_i, \bar{g}_i), i \in N)$  is in the *core* if it is feasible and cannot be improved upon by any coalition  $S \subset N$ .

For each positive integer  $r$ , we define the  $r^{\text{th}}$  *replica economy*, denoted by  $\mathcal{E}_r$ , as the economy with a set of consumers

$$N_r = \{(i, q) \mid i = 1, \dots, N \text{ and } q = 1, \dots, r\}.$$

Consumer  $(i, q)$  is called the  $q^{\text{th}}$  *consumer of type  $i$* . It will be the case that all consumers of type  $i$  are identical in terms of their consumption sets, endowments and preferences to consumer  $i$ . Let  $S$  be a non-empty subset of  $N_r$ . An allocation  $((x_{(i,q)}, g_{(i,q)}), (i, q) \in S)$  is  $S$ -feasible in the economy  $\mathcal{E}_r$  if

$$\left(\sum_{(i,q) \in S} (x_{(i,q)} - w_{(i,q)}), \sum_{(i,q) \in S} g_{(i,q)}\right) \in Y.$$

For each integer  $r$ , we define the  $r^{\text{th}}$  replica of allocation  $((x_i, g_i), i \in N)$ , denoted by  $((x_{(i,q)}, g_{(i,q)}), (i, q) \in N_r)$ , as follows:

$$x_{(i,q)} = x_i \text{ and } g_{(i,q)} = g_i, \text{ for each } (i, q) \in N_r$$

That is, in the  $r^{\text{th}}$  replica economy  $\mathcal{E}_r$ , each of the  $r^{\text{th}}$  replica consumers of type  $i$  has the same private goods and public goods consumption as consumer  $i$ .

An allocation  $((\bar{x}_i, \bar{g}_i), i \in N)$  is an *Edgeworth allocation*, if for each positive integer  $r$ , the  $r^{\text{th}}$  replica of  $((\bar{x}_i, \bar{g}_i), i \in N)$  is in the core of the  $r^{\text{th}}$  replica economy  $\mathcal{E}_r$ .

### 3.1 Non-vacuity of the set of Edgeworth allocations

Brenheim and Rangel (2005) and Andreoni (2006) argues that asymptotically consumers charitable giving is more due to the act of giving itself rather than concerns about the aggregate provision of public goods. Our main economic assumption attempts to formalize the above intuition.

**[WGD] Warm glow dominance**<sup>1</sup> For every consumer  $i \in N$ , there exists a bundle public goods  $G_{-i}^* \in \mathbb{R}_{++}^K$ , such that for all  $(x_i, g_i, G_{-i}) \in \mathbb{R}_+^{L+2K}$ , it holds that

$$u_i(x_i, g_i, G_{-i}^*) \geq u_i(x_i, g_i, G_{-i}).$$

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<sup>1</sup>We borrowed this term from Andreoni (2006).

The [WGD] assumption ensures that beyond a public goods bundle<sup>2</sup> for other consumers' provision, each consumer receives utility only from his public goods provision and his private goods consumption. It is worth noticing that the [WGD] assumption does not imply the asymptotic satiation assumption in public goods of Conley (1994) since the utility functions could still be unbounded due to an ever-increasing own provision of public goods.

Our theorem below states that, for a sufficiently large replica, an Edgeworth allocation exists.

**Theorem 1.** Assume [A.1]-[A.4] and [WGD]. Then, there exists a positive integer  $r^*$  such that the set of Edgeworth allocation for the  $r^*$  replica economy  $\mathcal{E}_{r^*}$  is non-empty.

**Proof.** We first construct an auxiliary private goods production economy  $\hat{\mathcal{E}}$  with  $N$  consumers. Each consumer  $i$  is described by a consumption set  $\mathbf{R}_+^{L+K}$ , an endowment  $(w_i, 0) \in \mathbf{R}_+^{L+K}$ , and a utility function  $\hat{u}_i$  defined as follows

$$\hat{u}_i(x_i, g_i) = u_i(x_i, g_i, G_{-i}^*), \quad \text{for all } (x_i, g_i) \in \mathbf{R}_+^{L+K}.$$

The production technology of the auxiliary economy  $\hat{\mathcal{E}}$  is characterized by the production set  $Y$ . Hence, it is obvious that the set of feasible allocations of the auxiliary economy  $\hat{\mathcal{E}}$  coincides with the set of feasible allocations of the economy  $\mathcal{E}$ .

From standard results for the non-vacuity of the set of Edgeworth allocations for private goods production economies (for example, see Florenzano (1990, 2003)) it follows that there exists an Edgeworth allocation  $((\bar{x}_i, \bar{g}_i), i \in N)$  for the economy  $\hat{\mathcal{E}}$ . Then, by warm glow indispensability and possibility of producing public goods assumptions, it follows that  $\bar{g}_i \in \mathbf{R}_{++}^K$ . Therefore, we can choose an  $r^*$  such that  $(r^* - 1)\bar{g}_i \geq G_{-i}^*$ , for every consumer  $i$ . Hence, it is easy to check that for the allocation  $((\bar{x}_{(i,q)}, \bar{g}_{(i,q)}), (i, q) \in N_{r^*})$ , the  $r^*$  replica of  $((\bar{x}_i, \bar{g}_i), i \in N)$ , it holds that  $u_i(\bar{x}_{(i,q)}, \bar{g}_{(i,q)}, \bar{G}_{-(i,q)}) = \hat{u}_i(\bar{x}_i, \bar{g}_i)$ .

We claim that  $((\bar{x}_{(i,q)}, \bar{g}_{(i,q)}), (i, q) \in N_{r^*})$  is in an Edgeworth allocation of the economy  $\mathcal{E}_{r^*}$ . Suppose not. Then, there exists a positive integer  $n$ , and a coalition  $S \subset N_{nr^*}$  that can improve upon the  $n^{\text{th}}$  replica of  $((\bar{x}_{(i,q)}, \bar{g}_{(i,q)}), (i, q) \in N_{r^*})$ . Thus, there exists an  $S$ -feasible allocation

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<sup>2</sup>This public goods bundle may not be feasible in the economy.



$((x_{(i,q)}, g_{(i,q)}), (i, q) \in S)$ , such that

$$u_i(x_{(i,q)}, g_{(i,q)}, \sum_{j \in S \setminus \{(i,q)\}} g_j) > u_i(\bar{x}_{(i,q)}, \bar{g}_{(i,q)}, \bar{G}_{-(i,q)}), \text{ for all } (i, q) \in S.$$

By the definition of the utility functions  $\hat{u}_i$ , it holds that

$$\hat{u}_i(x_{(i,q)}, g_{(i,q)}) \geq u_i(x_{(i,q)}, g_{(i,q)}, \sum_{j \in S \setminus \{(i,q)\}} g_j), \text{ for all } (i, q) \in S.$$

Hence,

$$\hat{u}_i(x_{(i,q)}, g_{(i,q)}) > \hat{u}_i(\bar{x}_i, \bar{g}_i), \text{ for all } (i, q) \in S.$$

This contradicts the fact that  $((\bar{x}_i, \bar{g}_i), i \in N)$  is an Edgeworth allocation for the auxiliary economy  $\tilde{\mathcal{E}}$ . Hence,  $((\bar{x}_{i,q}, \bar{g}_{i,q}), (i, q) \in N_{r^*})$  is an Edgeworth allocation for the economy  $\mathcal{E}_{r^*}$ .  $\square$

## 4 Warm glow equilibrium

In a recent paper, Allouch (2008) introduces the warm glow equilibrium concept as a competitive equilibrium for a warm glow economy.<sup>3</sup> In a warm glow equilibrium each consumer faces a common price for the private goods, a personalized price for his own public goods provision, and another personalized price for other consumers's public goods provision. These personalized prices arise due to the externalities brought about by each consumer public goods provision.

**Definition:** A warm glow equilibrium is  $((\bar{x}_i, \bar{g}_i, \pi_i, \pi_{-i})_{i \in N}, p, p^g)$ , where  $((\bar{x}_i, \bar{g}_i), i \in N)$  is a feasible allocation,  $p \in \mathbb{R}_+^L$  is a price system for private goods,  $p^g \in \mathbb{R}_+^K$  a price system for public goods,  $\pi_i \in \mathbb{R}_+^K$  is the personalized price of consumer's  $i$  own public goods provision, and  $\pi_{-i} \in \mathbb{R}_+^K$ , is consumer's  $i$  personalized price for other consumers' public goods provision.

(i). For all  $(y, g) \in Y$ ,

$$(p, p^g) \cdot (y, g) \leq (p, p^g) \cdot \left( \sum_i (\bar{x}_i - w_i), \sum_i \bar{g}_i \right) = 0;$$

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<sup>3</sup>It is worth noticing that the warm glow equilibrium coincides with the Lindahl equilibrium if we consider the standard public good model, where preferences of consumer  $i$  are represented by the utility function  $u_i(x_i, g_i + G_{-i})$  instead of  $u_i(x_i, g_i, G_{-i})$ .

(ii). for each consumer  $i \in N$ ,

$$p \cdot x_i + \pi_i \cdot g_i + \pi_{-i} \cdot G_{-i} = p \cdot w_i,$$

and if

$$u_i(x_i, g_i, G_{-i}) > u_i(\bar{x}_i, \bar{g}_i, \bar{G}_{-i})$$

then

$$p \cdot x_i + \pi_i \cdot g_i + \pi_{-i} \cdot G_{-i} > p \cdot w_i;$$

(iii). for each consumer  $i \in N$ ,

$$\pi_i + \sum_{j \neq i} \pi_{-j} = p^g.$$

Condition (i) is the profit maximization, Condition (ii) is the utility maximization and Condition (iii) ensures the personalized prices sum up to the public goods prices.

We now show that warm glow equilibrium is in the core.

**Theorem 2.** If  $((\bar{x}_i, \bar{g}_i, \pi_i, \pi_{-i})_{i \in N}, p, p^g)$  is a warm glow equilibrium, then  $((\bar{x}_i, \bar{g}_i), i \in N)$  is in the core.

**Proof.** First, it follows from the definition of warm glow equilibrium that  $((\bar{x}_i, \bar{g}_i), i \in N)$  is feasible. Now, suppose that there is a coalition  $S \subset N$  and an  $S$ -feasible allocation  $((x_i, g_i), i \in S)$  such that

$$u_i(x_i, g_i, \sum_{j \in S \setminus \{i\}} g_j) > u^i(\bar{x}_i, \bar{g}_i, \bar{G}_{-i}), \quad \text{for each } i \in S$$

From (ii) in the definition of warm glow equilibrium, for each consumer  $i$ , it holds that

$$p \cdot x_i + \pi_i \cdot g_i + \pi_{-i} \cdot \sum_{j \in S \setminus \{i\}} g_j > p \cdot w_i.$$

Summing up the above inequalities over  $i \in S$ , it holds that

$$\sum_{i \in S} p \cdot x_i + \sum_{i \in S} (\pi_i + \sum_{j \in S \setminus \{i\}} \pi_{-j}) \cdot g_i > \sum_{i \in S} p \cdot w_i. \quad (1)$$

Now, we extend the allocation  $((x_i, g_i), i \in S)$  in the following way. For each  $i \notin S$ , we set  $(x_i, g_i) = (w_i, 0)$ . Since  $((x_i, g_i), i \in S)$  is an  $S$ -feasible allocation it follows that  $((x_i, g_i), i \in N)$  is a feasible allocation.

Then, (1) implies that

$$\sum_{i \in N} p \cdot x_i + \sum_{i \in N} (\pi_i + \sum_{j \in N \setminus \{i\}} \pi_{-j}) \cdot g_i > \sum_{i \in N} p \cdot w_i.$$

Since  $p^g = \pi_i + \sum_{j \in N \setminus \{i\}} \pi_{-i}$ , it follows that

$$p \cdot \sum_{i \in N} (x_i - w_i) + p^g \cdot \sum_{i \in N} g_i > 0,$$

which contradicts (i) in the definition of equilibrium.  $\square$

Now, we state our main existence theorem, where we decentralize an Edgeworth allocation as a warm glow equilibrium.

**Theorem 3.** Assume [A.1]-[A.4] and [WGD]. Then, there exists a positive integer  $r^*$ , such that the  $r^*$  replica economy  $\mathcal{E}_{r^*}$  has a warm glow equilibrium

**Proof.** From Theorem 1, there exists a positive integer  $r^*$ , such that the set of Edgeworth allocations is non-empty in the economy  $\mathcal{E}_{r^*}$ . For simplicity of notations, we can assume that  $r^* = 1$  and let  $((\bar{x}_i, \bar{g}_i), i \in N)$  be an Edgeworth allocation for  $\mathcal{E}_1$ .

In the following, we expand the economy in the spirit of Foley (1970). However, our decentralization argument differs from Foley's since we expand the economy in a specific way in order to take account of warm glow effects in preferences. We also consider individual consumers' preferred sets rather than an aggregate preferred set for the economy. For each consumer  $i$ , we define the set  $\Gamma_i \subset \mathbb{R}^{L+2NK}$ ; where  $N$  is the set of consumers,  $L$  is the number of private goods and  $K$  is the number of public goods, as follows:

$$\Gamma_i = \{(x_i - w_i, 0, \dots, g_i, G_{-i}, \dots, 0) \mid u_i(x_i, g_i, G_{-i}) > u_i(\bar{x}_i, \bar{g}_i, \bar{G}_{-i})\}$$

The set  $\Gamma_i$  is consumer  $i$ 's expanded preferred set, listing his net trade in private goods, his public goods provision and other consumers' public goods provision. Since [A.1],  $\Gamma_i$  is non empty for each consumer  $i$ . In addition, it is

easy to check that the convexity of the preferences implies that  $\Gamma_i$  is convex for each consumer  $i$ . Let  $\Gamma$  denote the convex hull of the union of the sets  $\Gamma_i$ ,  $i = 1, \dots, N$ .

Now, we define the set:

$$\tilde{Y} = \{(y, g_1, G_{-1}, \dots, g_N, G_{-N}) \mid \text{for each } i, G_{-i} = \sum_{j \neq i} g_j \text{ and } (y, \sum_{j \in N} g_j) \in Y\}.$$

The set  $\tilde{Y}$  is a convex cone with vertex the origin since  $Y$  is a convex cone with vertex the origin. We claim that

$$\Gamma \cap \tilde{Y} = \emptyset.$$

To see this, assume, on the contrary, that  $\Gamma \cap \tilde{Y} \neq \emptyset$ . Then, there exists  $(x_i - w_i, 0, \dots, g_i, G_{-i}, \dots, 0) \in \Gamma_i$  and  $\lambda \in \mathbb{R}_+^N$  such that  $\sum_i \lambda_i = 1$ , and

$$\sum \lambda_i (x_i - w_i, 0, \dots, g_i, G_{-i}, \dots, 0) \in \tilde{Y}.$$

Let  $S = \{i \mid \lambda_i > 0\}$ . It is obvious that  $S \neq \emptyset$  since  $\sum_i \lambda_i = 1$ . For each  $i \in S$  and each positive integer  $n$ , let  $n_i$  be the smallest integer which is greater than or equal to  $n\lambda_i$ . For each  $i \in S$ , define

$$(x_i^n, g_i^n, G_{-i}^n) = \frac{n\lambda_i}{n_i} (x_i, g_i, G_{-i}) + (1 - \frac{n\lambda_i}{n_i})(w_i, 0, 0). \quad (2)$$

From continuity of preferences, for all  $n$  sufficiently large, it holds that

$$u_i(x_i^n, g_i^n, G_{-i}^n) > u_i(\bar{x}_i, \bar{g}_i, \bar{G}_{-i}), \quad \text{for each } i \in S.$$

It follows from (2) that

$$\sum_{i \in S} \left(\frac{n_i}{n}\right) \frac{n\lambda_i}{n_i} (x_i - w_i, 0, \dots, g_i, G_{-i}, \dots, 0) \in \tilde{Y}.$$

Then,

$$\sum_{i \in S} \frac{n_i}{n} (x_i^n - w_i, 0, \dots, g_i^n, G_{-i}^n, \dots, 0) \in \tilde{Y}.$$

Since  $\tilde{Y}$  is a cone with vertex zero, it holds that

$$\sum_{i \in S} n_i (x_i^n - w_i, 0, \dots, g_i^n, G_{-i}^n, \dots, 0) \in \tilde{Y}.$$

Thus, we have constructed a blocking coalition, which is a contradiction to the assumption that  $((\bar{x}_i, \bar{g}_i), i \in N)$  is an Edgeworth allocation. Therefore,  $\Gamma \cap \tilde{Y} = \emptyset$ .

From Minkowski's separating hyperplane theorem, there exists a hyperplane with normal  $(p, \pi_1, \pi_{-1}, \dots, \pi_N, \pi_{-N}) \neq 0$ , and a scalar  $r$  such that

(i). for all  $(y, g_1, G_{-1}, \dots, g_N, G_{-N}) \in \Gamma$

$$(p, \pi_1, \pi_{-1}, \dots, \pi_N, \pi_{-N}) \cdot (y, g_1, G_{-1}, \dots, g_N, G_{-N}) \geq r.$$

(ii). for all  $(y, g_1, G_{-1}, \dots, g_N, G_{-N}) \in \tilde{Y}$

$$(p, \pi_1, \pi_{-1}, \dots, \pi_N, \pi_{-N}) \cdot (y, g_1, G_{-1}, \dots, g_N, G_{-N}) \leq r,$$

Since  $\tilde{Y}$  is a closed convex cone with vertex zero, we can choose  $r = 0$ . It follows from (i) in the separation theorem that for any consumer  $i$ , and any consumption bundle  $(x_i, g_i, G_{-i})$  such that  $u_i(x_i, g_i, G_{-i}) > u_i(\bar{x}_i, \bar{g}_i, \bar{G}_{-i})$ , it holds that

$$p \cdot (x_i - w_i) + \pi_i \cdot g_i + \pi_{-i} \cdot G_{-i} \geq 0. \quad (3)$$

Thus, by continuity and monotonicity of preferences with respect to private goods, we obtain

$$p \cdot (\bar{x}_i - w_i) + \pi_i \cdot \bar{g}_i + \pi_{-i} \cdot \bar{G}_{-i} \geq 0$$

Summing up over the consumers and rearranging terms, we get

$$(p, \pi_1, \pi_{-1}, \dots, \pi_N, \pi_{-N}) \cdot \left( \sum (\bar{x}_i - w_i), \bar{g}_1, \bar{G}_{-1}, \dots, \bar{g}_N, \bar{G}_{-N} \right) \geq 0. \quad (4)$$

By feasibility of  $((\bar{x}_i, \bar{g}_i), i \in N)$  and (ii) in the separation theorem, it holds that

$$(p, \pi_1, \pi_{-1}, \dots, \pi_N, \pi_{-N}) \cdot \left( \sum (\bar{x}_i - w_i), \bar{g}_1, \bar{G}_{-1}, \dots, \bar{g}_N, \bar{G}_{-N} \right) \leq 0. \quad (5)$$

Hence, it follows from (4) and (5) that

$$(p, \pi_1, \pi_{-1}, \dots, \pi_N, \pi_{-N}) \cdot (\sum (\bar{x}_i - w_i), \bar{g}_1, \bar{G}_{-1}, \dots, \bar{g}_N, \bar{G}_{-N}) = 0. \quad (6)$$

And, therefore

$$p \cdot (\bar{x}_i - w_i) + \pi_i \cdot \bar{g}_i + \pi_{-i} \cdot \bar{G}_{-i} = 0. \quad (7)$$

We claim that, for any two consumers  $j_1$  and  $j_2$ , it holds that

$$\pi_{j_1} + \sum_{i \neq j_1} \pi_{-i} = \pi_{j_2} + \sum_{i \neq j_2} \pi_{-i}.$$

Suppose this was not the case, then, without loss of generality, one could assume that for some public good, say the  $k^{th}$ , it holds that

$$\pi_{j_1}^k + \sum_{i \neq j_1} \pi_{-i}^k > \pi_{j_2}^k + \sum_{i \neq j_2} \pi_{-i}^k.$$

Let  $\delta_k$  be a vector in  $\mathbb{R}_+^K$  consisting of one unit of the  $k^{th}$  public good and nothing else. For a small enough  $\varepsilon > 0$ , let us consider the following public goods bundle,  $\bar{G}^\varepsilon = (\bar{g}_i^\varepsilon, \dots, \bar{g}_N^\varepsilon)$ , defined as follows

$$\bar{G}^\varepsilon = \begin{cases} \bar{g}_{j_1}^\varepsilon = \bar{g}_{j_1} - \varepsilon \delta_k, \\ \bar{g}_{j_2}^\varepsilon = \bar{g}_{j_2} + \varepsilon \delta_k, \\ \bar{g}_i^\varepsilon = \bar{g}_i, \end{cases} \quad \text{if } i \in N \setminus \{j_1, j_2\}.$$

It is obvious that

$$(\sum (\bar{x}_i - e_i), \bar{g}_1^\varepsilon, \bar{G}_{-1}^\varepsilon, \dots, \bar{g}_N^\varepsilon, \bar{G}_{-N}^\varepsilon) \in \tilde{Y}.$$

Moreover, since (6) it follows that

$$(p, \pi_1, \pi_{-1}, \dots, \pi_N, \pi_{-N}) \cdot (\sum (\bar{x}_i - e_i), \bar{g}_1^\varepsilon, \bar{G}_{-1}^\varepsilon, \dots, \bar{g}_N^\varepsilon, \bar{G}_{-N}^\varepsilon) > 0,$$

but, this contradicts property (i) of the separation theorem. Thus, we set up

$$p^g = \pi_i + \sum_{j \neq i} \pi_{-j}, \quad \text{for all } i \in N.$$

In view of this, property (ii) of the separation theorem and (6) imply that for all  $(y, g) \in Y$ ,

$$(p, p^g) \cdot (y, g) \leq (p, p^g) \cdot \left( \sum_i (\bar{x}_i - w_i), \sum_i \bar{g}_i \right) = 0.$$

This proves (i) and (iii) in the definition of warm glow equilibrium.

From [A.1] and the separation theorem it follows that  $p \in \mathbb{R}_+^L \setminus \{0\}$  and for each consumer  $i$ ,  $(\pi_i, \pi_{-i}) \in \mathbb{R}_+^K \setminus \{0\} \times \mathbb{R}_+^K$ . Now, we show that for any consumer  $i$  and any consumption bundle  $(x_i, g_i, G_{-i})$ , such that  $u_i(x_i, g_i, G_{-i}) > u_i(\bar{x}_i, \bar{g}_i, \bar{G}_{-i})$ , it holds that

$$p \cdot x_i + \pi_i \cdot g_i + \pi_{-i} \cdot G_{-i} > p \cdot w_i.$$

Assume that this was not the case. By indispensability of warm glow provision, it follows that  $g_i \in \mathbb{R}_{++}^K$ . Then, there exists  $g'_i \in \mathbb{R}_+^K$ , such that  $g'_i \ll g_i$ . Therefore, by quasi-concavity and continuity, along the line joining  $(x_i, g'_i, G_{-i})$  and  $(x_i, g_i, G_{-i})$ , there is a point in the consumption set of  $i$  which is strictly preferred to  $(\bar{x}_i, \bar{g}_i, \bar{G}_{-i})$  and costs strictly less than  $p \cdot w_i$ . This contradicts (3). This and (7) prove (ii) in the definition of warm glow equilibrium.  $\square$

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