Liberal Egalitarianism and the Harm Principle*  

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Abstract  
This paper analyses Rawls’s celebrated difference principle, and its lexicographic extension, in societies with a finite and an infinite number of agents. A unified framework of analysis is set up, which allows one to characterise Rawlsian egalitarian principles by means of a weaker version of a new axiom - the Harm Principle - recently proposed by [12]. This is quite surprising, because the Harm principle is meant to capture a liberal requirement of noninterference and it incorporates no obvious egalitarian content. A set of new characterisations of the maximin and of its lexicographic refinement are derived, including in the intergenerational context with an infinite number of agents.  

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1 Introduction

Almost four decades after its publication, *A Theory of Justice* ([13]) maintains a prominent role in political philosophy, economics, and social choice. Among the most influential contributions of the book is the *difference principle* contained in Rawls's second principle of justice, according to which inequalities should be allowed only insofar as they benefit the worst-off members of society ([13], p.303). Both the difference principle, formally captured by the well-known *maximin* social welfare relation, and especially its lexicographic extension, the *leximin* social welfare relation, have generated a vast literature across disciplinary borders.

Rawls’s difference principle and its extension are usually considered to have a strong egalitarian bias and are taken to represent the main alternative to libertarian and utilitarian approaches (see, e.g., [21] and the discussion in [16]). The classic characterisation of leximin, in fact, is due to Hammond ([9]) and it requires an axiom (the so-called *Hammond Equity axiom*) with a marked egalitarian content: in a welfaristic framework, Hammond Equity asserts that if \( x_i < y_i < y_j < x_j \) for two utility profiles \( x \) and \( y \), for which \( x_h = y_h \) for all agents \( h \neq i, j \), then \( y \) should be (weakly) socially preferred to \( x \). In a recent contribution, however, Mariotti and Veneziani ([12]) show that the leximin can be characterised using an axiom - the Harm Principle - that incorporates a liberal view of non-interference, without any explicit egalitarian content. This result is surprising and it raises a number of interesting issues for liberal approaches emphasising notions of individual autonomy or freedom in political philosophy and social choice, but it also sheds new light on the normative foundations of standard egalitarian principles.

This paper extends the analysis of the implications of liberal views of noninterference, as expressed in the Harm Principle, and it generalises ([12]) in a number of directions. Formally, it is shown that a weaker version of the Harm Principle, together with standard axioms in social choice, provides a unified axiomatic framework to analyse a set of social welfare relations originating from the difference principle in a welfaristic framework. Theoretically, the analysis provides a novel statement, based on liberal principles, of the ethical intuitions behind a family of normative principles stemming from Rawls’s seminal work. On the one hand, the Harm Principle is formally different from standard informational invariance axioms (see, e.g., [8]) and, unlike the latter, it has a clear normative content. On the other hand, unlike the Hammond Equity axiom, the Harm principle does not incorporate an egalitarian intuition. Therefore, quite surprisingly, the ethical foundations of two social welfare relations traditionally considered as rather egalitarian - the difference principle and its lexicographic extension - rest only on the two standard axioms of Anonymity and Pareto efficiency, and on a liberal principle incorporating a noninterfering view. No axiom with a clear
egalitarian content is necessary, and indeed our analysis provides a new meaning to the label ‘liberal egalitarianism’ usually associated with Rawls’s approach. Actually, our analysis sheds new light on the importance of the notion of justice as impartiality incorporated in the Anonymity axiom in egalitarian approaches. This is particularly clear in societies with a finite number of agents: the Harm principle and the Pareto principle are consistent with some of the least egalitarian social welfare orderings (e.g. the lexicographic dictatorships), and the Anonymity axiom plays a pivotal role in determining the egalitarian outcome. Our analysis also raises some interesting issues concerning the actual implications of liberal approaches emphasising a notion of individual autonomy, or freedom: if one endorses some standard axioms - such as Anonymity and the Pareto principle - the adoption of an arguably weak liberal view of noninterference leads straight to welfare egalitarianism. As noted by Mariotti and Veneziani ([12]), liberal noninterference implies equality, an insight that is proved to be quite robust in this paper.

To be specific, first of all, in economies with a finite number of agents, it is shown that a weaker version of the Harm Principle analysed in ([12]) is sufficient to characterise the lexicin social welfare ordering. This result is interesting because the weak Harm Principle captures liberal, noninterfering views even more clearly than the original Harm Principle. Further, based on the weak Harm Principle, a new characterisation of the maximin social welfare ordering is provided. Second, this paper analyses the maximin and the lexicin in the context of societies with an infinite number of agents. This is arguably a crucial task for egalitarians. In fact, despite Rawls’s claims to the contrary, there is no compelling reason to restrict the application of the difference principle to intra-generational justice. In the intergenerational context, a basic concern for impartiality arguably implies that principles of justice be applied to all present and future generations. The extension to the case with an infinite number of generations, however, is problematic for all the main approaches, and indeed impossibility results easily obtain, for there exists no social welfare ordering that satisfies the standard axioms of Anonymity and Strong Pareto (see [11]). A number of recent contributions have provided characterisation results for social welfare relations by dropping either completeness (see, among the others, [4], [2], [5], [7], [3]) or transitivity (see, e.g., [18]). But the definition of suitable anonymous and paretin social welfare relations is still an open question in the infinite context (for a thorough discussion, see [1]).

In this tradition, this paper provides various new characterisations of the maximin and the lexicin social welfare relations, based on the weak Harm principle in economies with an infinite number of agents. Although various formal frameworks and definitions have been proposed to analyse infinitely-lived societies, it is shown that the weak Harm Principle can be used to derive interesting results in all the main approaches.
The rest of the paper is structured as follows. Section 2 lays out the basic framework of analysis. Section 3 characterises the lexicin and the maximin social welfare orderings in economies with a finite number of agents. Section 4 provides a number of characterisation results for lexicin and maximin social welfare relations in societies with an infinite number of agents, in various different frameworks. Section 5 briefly concludes.

2 The framework

Let \( X \equiv \mathbb{R}^N \) be the set of countably infinite utility streams, where \( \mathbb{R} \) is the set of real numbers and \( N \) is the set of natural numbers. An element of \( X \) is \( 1u = (u_1, u_2, \ldots) \) and \( u_t \) is the utility level of a representative member of generation \( t \in \mathbb{N} \). For \( T \in \mathbb{N} \), \( 1u_T = (u_1, \ldots, u_T) \) denotes the T-head of \( 1u \) and \( T+1u = (u_{T+1}, u_{T+2}, \ldots) \) denotes its T-tail, so that \( 1u = (1u_T, T+1u) \); \( 1u_T \) denotes the welfare level of the worst-off generation of the T-head of \( 1u \), and \( \min(1u) = \min\{u_1, u_2, \ldots\} \) denotes the welfare level of the worst-off generation of \( 1u \) whenever it exists. \( \text{con} \epsilon \) denotes the stream of constant level of well-being equal to \( \epsilon \in \mathbb{R} \); for the sake of notational simplicity, the T-head of \( \text{con} \epsilon \) is denoted as \( 1\epsilon_T \). A permutation \( \pi \) is a bijective mapping of \( \mathbb{N} \) onto itself. A permutation \( \pi \) of \( \mathbb{N} \) is finite if there is \( T \in \mathbb{N} \) such that \( \pi(t) = t \), \( \forall t > T \), and \( \Pi \) is the set of all finite permutations of \( \mathbb{N} \). For any \( 1u \in X \) and any \( \pi \in \Pi \), let \( \pi(1u) = (u_{\pi(t)})_{t \in \mathbb{N}} \) be a permutation of \( 1u \). For any \( T \in \mathbb{N} \) and \( 1u \in X \), \( 1\tilde{u}_T \) is a permutation of \( 1u_T \) such that the components are ranked in ascending order.

For any \( 1u, 1v \in X \), we write \( 1u \succ 1v \) to mean \( u_t \succeq v_t \), \( \forall t \in \mathbb{N} \); \( 1u \succ 1v \) to mean \( 1u \succeq 1v \) and \( 1u \neq 1v \); \( 1u \not\succ 1v \) for \( (1u, 1v) \not\in \succ \); \( \succeq \) stands for “at least as good as”. The asymmetric factor \( \succ \) of \( \succeq \) is defined by \( 1u \succ 1v \) if and only if \( 1u \succeq 1v \) and \( 1v \not\succeq 1u \), and the symmetric part \( \sim \) of \( \succ \) is defined by \( 1u \sim 1v \) if and only if \( 1u \succeq 1v \) and \( 1v \succeq 1u \). They stand, respectively, for “strictly better than” and “indifferent to”. A relation \( \succeq \) on \( X \) is said to be: reflexive if, for any \( 1u \in X \), \( 1u \succeq 1u \); complete if, for any \( 1u, 1v \in X \), \( 1u \not\succeq 1v \) implies \( 1u \succ 1v \) or \( 1v \succ 1u \); transitive if, for any \( 1u, 1v, 1w \in X \), \( 1u \succeq 1v \) and \( 1v \succeq 1w \) implies \( 1u \succeq 1w \). \( \succeq \) is a quasi-ordering if it is reflexive and transitive, while \( \succ \) is an ordering if it is a complete quasi-ordering. Let \( \succ \) and \( \succ' \) be relations on \( X \); \( \succ' \) is an extension of \( \succ \) if \( \succeq \subset \succ' \) and \( \subset \succ' \).

If there are only a finite set \( \{1, \ldots, T\} = N \subset \mathbb{N} \) of agents, or generations, \( X_T \) denotes the set of utility streams of \( X \) truncated at \( T = |N| \), where \( |N| \) is the cardinality of \( N \). In order to simplify the notation, in economies with a finite number of agents we write \( u \) for \( 1u_T \). With the obvious adaptations, the notation spelled out above is carried over utility streams in \( X_T \).
3 Egalitarian Principles in Societies with a Finite Number of Agents

This section analyses liberal egalitarianism in societies with a finite number of agents. First, the characterisation of the leximin social welfare ordering (henceforth, swo) derived by Mariotti and Veneziani ([12], Theorem 1, p.126) is strengthened by weakening the main axiom incorporating a liberal view of noninterference, the Harm Principle. Then, based on the weak Harm Principle, a novel characterisation of Rawls’s difference principle, as formalised in the maximin swo, is provided.

3.1 The Leximin

According to the leximin, that society is best which lexicographically maximises the welfare of its worst-off members. Formally, the leximin relation $\succ^L \supset \succ^L \cup \sim^L$ on $X_T$ is defined as follows. The asymmetric factor $\succ^L$ of $\succ^L$ is defined by:

$$u \succ^L v \iff \bar{u}_i > \bar{v}_i \text{ or } [\exists i \in N \setminus \{1\} : \bar{u}_j = \bar{v}_j (\forall j \in N : j < i) \text{ and } \bar{u}_i > \bar{v}_i].$$

The symmetric factor $\sim^L$ of $\succ^L$ is defined by:

$$u \sim^L v \iff \bar{u}_i = \bar{v}_i, \forall i \in N.$$ 

$\succ^L$ is easily shown to be an ordering. Classic analyses of the leximin swo typically involve the following three axioms (see [9]).

**Strong Pareto Optimality**, SPO: $\forall u, v \in X_T : u \succ v \Rightarrow u \succ v$.

**Anonymity**, A: $\forall u \in X_T : \exists \pi \text{ of } N \Rightarrow u \sim \pi (u)$.

**Hammond Equity**, HE. $\forall u, v \in X_T : u_i < v_i < v_j < u_j \exists i, j \in N, u_k = v_k \forall k \in N \setminus \{i, j\} \Rightarrow v \succ u$.

The first two axioms are standard in social choice theory and need no further comment. It is important to note, instead, that HE expresses a clear concern for equality in welfare distributions, for it asserts that among any two welfare allocations which differ only in two components, society should prefer the more egalitarian one. The classic characterisation by Hammond ([9]) states that a swo is the leximin ordering if and only if it satisfies SPO, A, and HE.

In a recent contribution, Mariotti and Veneziani ([12]) drop HE and introduce a new axiom, called the Harm Principle (HP), which is meant to capture a liberal view of noninterference whenever

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1See also the related Hammond ([10]) and the generalisation by Tungodden ([20]; [21]).
individual choices have no effect on others. To be precise, starting from two welfare allocations \( u \) and \( v \) for which \( u \) is socially preferred to \( v \), consider two different allocations \( u' \) and \( v' \) such that agent \( i \) is worse off at these than at the corresponding starting allocations, the other agents are equally well off, and agent \( i \) prefers \( u' \) to \( v' \). The decrease in agent \( i \)’s welfare may be due to her negligence or her bad luck, but in any case \( \text{HP} \) states that society’s preference over \( u' \) and \( v' \) should coincide with \( i \)’s preferences. In this sense, \( \text{HP} \) requires that having already suffered a welfare loss in both allocations, agent \( i \) should not be punished in the swo by changing social preferences against her. This seems a rather intuitive way of capturing a liberal view of noninterference, and the name of the axiom is meant to echo John Stuart Mill’s famous formulation in his essay \textit{On Liberty} (see [22], and the discussion in [12]). Yet, although it has no explicit egalitarian content, quite surprisingly, Mariotti and Veneziani ([12], Theorem 1, p.126) prove that, jointly with \( \text{SPO} \) and \( \text{A} \), \( \text{HP} \) characterises the leximin swo.

In this paper, the implications of liberal, noninterfering views in social choice are explored further. As a first step, though, a weaker version of \( \text{HP} \) is presented, which can be formally stated as follows.

**Weak Harm Principle, WHP:** \( \forall u, v, u', v' \in X_T : u \succ v \text{ and } u', v' \text{ are such that, } \exists i \in N, \)

\[
\begin{align*}
    u'_i &< u_i \\
v'_i &< v_i \\
u'_j &= u_j \forall j \in N \backslash \{i\} \\
v'_j &= v_j \forall j \in N \backslash \{i\}
\end{align*}
\]

implies \( v' \not\succ u' \) whenever \( u'_i > v'_i \).

\( \text{WHP} \) weakens the axiom proposed by Mariotti and Veneziani ([12]) in that it does not require that society’s preferences over \( u' \) and \( v' \) be identical with agent \( i \)’s, but only that society should not reverse the strict preference between \( 1u \) and \( 1v \) to a strict preference for \( 1v' \) over \( 1u' \) (possibly except when \( i \) prefers otherwise). In this sense, the liberal content of \( \text{WHP} \), and the requirement that agent \( i \) should not be punished in the swo by changing social preferences against her, is even clearer, and \( \text{WHP} \) strongly emphasises the negative prescription of the Harm Principle. The surprising characterisation result provided in ([12]) can then be strengthened.

**Theorem 1** A swo \( \succeq \) on \( X_T \) is the leximin ordering if and only if it satisfies Anonymity (\( \text{A} \)), Strong Pareto Optimality (\( \text{SPO} \)), and the Weak Harm Principle (\( \text{WHP} \)).
Proof. ($\Rightarrow$) Let $\succ$ on $X_T$ be the lexicmin ordering, i.e., $\succ=\succ^{LM}$. Since WHP is weaker than HP, the proof that $\succ^{LM}$ on $X_T$ meets SPO, A, and WHP follows from the proof of necessity in ([12], Theorem 1, p.126).

($\Leftarrow$) Let $\succ$ on $X_T$ be a swo satisfying SPO, A, and WHP. We show that $\succ$ on $X_T$ is the lexicmin swo. Thus, we should prove that, $\forall u, v \in X_T,$

$$u \sim^{LM} v \Leftrightarrow u \sim v \quad (3.1)$$

and

$$u \succ^{LM} v \Leftrightarrow u \succ v \quad (3.2)$$

First, we prove the implication $\Rightarrow$ of 3.1. If $u \sim^{LM} v$, then $\bar{u} = \bar{v}$, and so $u \sim v$, by A.

Next, we prove the implication $\Rightarrow$ of 3.2. Suppose that $u \succ^{LM} v$, and so, by definition $\exists t \in \{1, ..., T\}$ such that $\bar{u}_s = \bar{v}_t \forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$. Suppose, by contradiction, that $v \succ u$. Note that since $\succ$ satisfies A, in what follows we can focus, without loss of generality, either on $u$ and $v$, or on the ranked vectors $\bar{u}$ and $\bar{v}$. Therefore, suppose $\bar{v} \succ \bar{u}$. As SPO holds it must be the case that $\bar{v}_l > \bar{u}_l$ for some $l > t$. Let

$$k = \min\{t < l \leq T | \bar{v}_l > \bar{u}_l\}.$$ 

By A, let $v_i = \bar{v}_k$ and let $u_i = \bar{u}_k$, for some $1 \leq g < k$, where $\bar{u}_k - g > \bar{v}_k - g$. Then, let two real numbers $d_1, d_2 > 0$, and consider vectors $u', v'$ and the corresponding ranked vectors $\bar{u}', \bar{v}'$ in $X$ formed from $\bar{u}$, $\bar{v}$ as follows: first, $\bar{u}_{k-g} = \bar{v}_{k-g}$ is lowered to $\bar{u}_{k-g} - d_1$ such that $\bar{u}_{k-g} - d_1 > \bar{v}_{k-g}$; next, $\bar{v}_k$ is lowered to $\bar{v}_k - d_2$ such that $\bar{u}_k > \bar{v}_k - d_2 > \bar{u}_{k-g} - d_1$; finally, all other entries of $\bar{u}$ and $\bar{v}$ are unchanged. By construction $\bar{u}'_j \geq \bar{v}'_j$ for all $j \leq k$, with at least two inequalities, $\bar{u}'_{k-g} > \bar{v}'_{k-g}$ and $\bar{u}'_k > \bar{v}'_k$, whereas WHP, combined with A, implies $\bar{v}' \succ \bar{u}'$. By SPO, $d_1, d_2 > 0$ can be chosen so that $\bar{v}' \succ \bar{u}'$, without loss of generality. Consider two cases:

a) Suppose that $\bar{v}_k > \bar{u}_k$, but $\bar{u}_l \geq \bar{v}_l$ for all $l > k$. It follows that $\bar{u}' > \bar{v}'$, and so SPO implies that $\bar{u}' \succ \bar{v}'$, a contradiction.

b) Suppose that $\bar{v}_l > \bar{u}_l$ for some $l > k$. Note that by construction $\bar{v}'_l = \bar{v}_l$ and $\bar{u}'_l = \bar{u}_l$ for all $l > k$. Then, let

$$k' = \min\{k < l \leq T | \bar{v}'_l > \bar{u}'_l\}.$$ 

where $k' > k$. The above argument can be applied to $\bar{u}'$, $\bar{v}'$ to derive vectors $\bar{u}''$, $\bar{v}''$ such that $\bar{u}''_j \geq \bar{v}''_j$ for all $j \leq k'$, whereas WHP, combined with A and SPO, implies $\bar{v}'' \succ \bar{u}''$. And so on. After a finite number of iterations $s$, two vectors $\bar{u}^s$, $\bar{v}^s$ can be derived such that, by WHP, combined with A and SPO, we have that $\bar{v}^s \succ \bar{u}^s$, but SPO implies $\bar{u}^s \succ \bar{v}^s$, yielding a contradiction.
We have proved that if \( u \succ^{LM} v \) then \( u \succ v \). Suppose now, by contradiction, that \( v \sim u \), or equivalently \( \tilde{v} \sim \tilde{u} \). Since, by our supposition, \( \tilde{u} \prec \tilde{u} \), there exists \( \varepsilon > 0 \) such that \( \tilde{v} \prec \tilde{u} - \varepsilon < \tilde{u} \).

Let \( \tilde{v}' \in X \) be a vector such that \( \tilde{u}'_t = \tilde{u} - \varepsilon \) and \( \tilde{u}'_j = \tilde{u}_j \) for all \( j \neq t \). It follows that \( \tilde{u}' \succ^{LM} \tilde{v} \) but \( \tilde{v} \succ \tilde{u}' \) by SPO and the transitivity of \( \succeq \). Hence, the above argument can be applied to \( \tilde{v} \) and \( \tilde{u}' \), yielding the desired contradiction. \( \blacksquare \)

The properties in Theorem 1 are clearly independent.

Theorem 1 has a number of interesting theoretical implications. First of all, Theorem 1 implies that HE and WHP are equivalent in the presence of A and SPO, even though they are completely independent. Actually, it can be proved that if \( N = \{1, 2\} \), then in the presence of SPO, HE implies WHP, but the converse is not true. This implies that the above characterisation is far from trivial, given that, at least in some cases, and if SPO is assumed, HE is actually stronger than WHP. Secondly, and perhaps more interestingly, Theorem 1 puts the normative foundations of leximin under a rather different light. For, unlike in standard results, the egalitarian SWO is characterised without appealing to any axioms with a clear egalitarian content. Actually, it is easily shown that SPO and WHP alone are compatible with some of the least egalitarian SWOs, namely the lexicographic dictatorships, which forcefully shows that WHP imposes no significant egalitarian restriction. As a result, Theorem 1 interestingly shows the normative strength of the Anonymity axiom in determining the egalitarian outcome, an important insight which is not obvious in standard characterisations based on HE.

The main implication of Theorem 1, however, is that it proves that the core intuition of Mariotti and Veneziani ([12], Theorem 1, p.126) concerning the implications of liberal noninterfering views is robust: a strongly egalitarian SWO can be characterised with an even weaker axiom that only incorporates a liberal view of non-interference. In the next sections, this intuition is extended further and it is shown that the counterintuitive implications of liberal noninterfering principles in terms of egalitarian orderings are quite general and robust. Analogous characterisations of a whole family of principles inspired by Rawls’s theory are obtained in societies with both finite and infinite populations, based on the weak Harm Principle.

### 3.2 The Difference Principle

The maximin relation \( \succ^M = \succ^M \cup \sim^M \) on \( X_T \) is defined as follows. The asymmetric factor \( \succ^M \) of \( \succ^M \) is defined by:

\[
\forall u, v \in X_T : u \succ^M v \iff \tilde{u}_1 > \tilde{v}_1.
\]
The symmetric factor $\sim^M$ of $\succeq^M$ is defined by:

$$u \sim^M v \iff \bar{u}_1 = \bar{v}_1.$$ 

$\succeq^M$ is easily shown to be an ordering. The maximin SWO formalises Rawls’s difference principle. As is well-known, the maximin does not satisfy SPO, and therefore the following, weaker axiom is imposed on the SWO.

**Weak Pareto Optimality**, WPO: $\forall u, v \in X_T : u \gg v \Rightarrow u \succ v$.

Second, a continuity axiom is imposed, which represents a standard interprofile condition requiring the SWO to vary continuously with variations in utility streams. This axiom is common in characterisations of the maximin SWO (see, e.g., [6]).

**Continuity**, C: $\forall u \in X_T, \{v \in X_T | v \gg u\}$ is closed and $\{v \in X_T | u \gg v\}$ is closed.

The next Theorem shows that the combination of Anonymity (A), Weak Pareto Optimality (WPO), Continuity (C), and the Weak Harm Principle (WHP) characterises the maximin SWO.

**Theorem 2** A SWO $\succeq$ on $X_T$ is the maximin ordering if and only if it satisfies Anonymity (A), Weak Pareto Optimality (WPO), Continuity (C), and the Weak Harm Principle (WHP).

**Proof.** $(\Rightarrow)$ Let $\succeq$ on $X_T$ be the maximin ordering, i.e., $\succeq = \succeq^M$. It can be easily verified that $\succeq^M$ on $X_T$ satisfies WPO, A, C, and WHP.

$(\Leftarrow)$ Let $\succeq$ on $X_T$ be a SWO satisfying A, WPO, WHP, and C. We show that $\succeq$ is the maximin SWO. We shall prove that, $\forall u, v \in X_T$,

$$u \succeq^M v \iff u \succ v \quad (3.3)$$

and

$$u \sim^M v \iff u \sim v. \quad (3.4)$$

Note that as $\succeq$ on $X_T$ satisfies A, in what follows we can focus either on $u$ and $v$, or on the ranked vectors $\bar{u}$ and $\bar{v}$, without loss of generality.

First, we show that the implication $(\Rightarrow)$ of (3.3) is satisfied. Take any $u, v \in X_T$. Suppose that $u \succeq^M v \iff \bar{u}_1 > \bar{v}_1$ and assume, by contradiction, that $v \succeq u$, or equivalently, $\bar{v} \succeq \bar{u}$. As WPO holds, $\bar{v}_j \geq \bar{u}_j$ for some $j \in N$, otherwise a contradiction immediately obtains. We proceed according to the following steps.

**Step 1.** Let

$$k = \min \{l \in N | \bar{v}_l \geq \bar{u}_l\}.$$
By A, let \( v_i = \bar{v}_k \) and let \( u_i = \bar{u}_1 \). Then, consider two real numbers \( d_1, d_2 > 0 \), and two vectors \( u', v' \) - together with the corresponding ranked vectors \( \bar{u}', \bar{v}' \in X_T \) - formed from \( \bar{u}, \bar{v} \) as follows:
\( \bar{u}_1 \) is lowered to \( \bar{u}_1 - d_1 > \bar{v}_1 \); \( \bar{v}_k \) is lowered to \( \bar{v}_k > \bar{v}_k - d_2 > \bar{u}_1 - d_1 \); and all other entries of \( \bar{u} \) and \( \bar{v} \) are unchanged. By construction \( \bar{u}'_j > \bar{v}'_j \) for all \( j \leq k \), whereas by WHP and A, we have \( \bar{v}' \succ \bar{u}' \).

**Step 2.** Let
\[
0 < \epsilon < \min\{ \bar{u}'_j - \bar{v}'_j | \forall j \leq k \}
\]
and define \( \bar{v}' = \bar{v}' + \text{con} \epsilon \). By construction, \( \bar{v}' \ll \bar{v}' \), and \( \bar{v}'_j < \bar{u}'_j \) for all \( j \leq k \). **WPO** implies \( \bar{v}' \succ \bar{v}' \). As \( \bar{v}' \succ \bar{u}' \), by step 1, the transitivity of \( \succ \) implies \( \bar{v}' \succ \bar{u}' \).

If \( \bar{u}'_j > \bar{v}'_j \) for all \( j \in N \), **WPO** implies \( \bar{u}' \succ \bar{v}' \), a contradiction. Otherwise, let \( \bar{u}'_j > \bar{v}'_j \) for some \( j > k \). Then, let
\[
k' = \min \{ l \in N | \bar{u}'_l \geq \bar{v}'_l \}
\]
where \( k' > k \).

The above steps 1-2 can be applied to \( \bar{u}', \bar{v}' \) to derive vectors \( \bar{u}''', \bar{v}''' \) such that \( \bar{u}'''_j > \bar{v}'''_j \) for all \( j \leq k' \), whereas \( \bar{v}''' \succ \bar{u}''' \). By **WPO**, a contradiction is obtained whenever \( \bar{u}'''_j > \bar{v}'''_j \) for all \( j \in N \). Otherwise, let \( \bar{v}'''_j = \bar{v}'''_l \) for some \( l > k \). And so on. After a finite number \( s \) of iterations, two vectors \( \bar{u}^s, \bar{v}^s \) can be derived such that \( \bar{v}^s \succ \bar{u}^s \), by steps 1-2, but \( \bar{u}^s \succ \bar{v}^s \), by **WPO**, a contradiction. Therefore, it must be \( \bar{u} \succ \bar{v} \) whenever \( \bar{u} \succ^M \bar{v} \). We have to rule out the possibility that \( \bar{u} \sim \bar{v} \).

We proceed by contradiction. Suppose that \( \bar{u} \sim \bar{v} \). Since \( \bar{v}_1 < \bar{u}_1 \), there exists \( \epsilon > 0 \) such that \( \bar{v}^\epsilon = \bar{v} + \text{con} \epsilon \) and \( \bar{v}^\epsilon_1 < \bar{u}_1 \) so that \( \bar{u} \succ^M \bar{v}^\epsilon \). However, by **WPO** and transitivity of \( \succ \) it follows that \( \bar{v}^\epsilon \succ \bar{u} \). Then the above reasoning can be applied to vectors \( \bar{v}^\epsilon \) and \( \bar{u} \) to obtain the desired contradiction.

Now, we show that the implication \((\Rightarrow)\) of (3.4) is met as well. Suppose that \( \bar{u} \sim^M \bar{v} \iff \bar{u}_1 = \bar{v}_1 \).

Assume, to the contrary, that \( u \not\sim v \). Without loss of generality, let \( \bar{u} \succ \bar{v} \). By A, it must be \( \bar{u} \neq \bar{v} \).

As \( \bar{u} \succ \bar{v} \), it follows from C that there exists neighborhoods \( S(\bar{u}) \) and \( S(\bar{v}) \) of \( \bar{u} \) and \( \bar{v} \) such that \( u' \succ v' \) for all \( u' \in S(\bar{u}) \) and for all \( v' \in S(\bar{v}) \). Then, there exists \( v' \in S(\bar{v}) \) such that \( v' \succ \bar{u} \) and \( \bar{u} \succ v' \sim \bar{v} \), so that \( \bar{u} \succ \bar{v} \) but \( \bar{v} \succ^M \bar{u} \). By the implication (3.3) proved above, it follows that \( \bar{v} \succ \bar{u} \), a contradiction. \( \blacksquare \)

The properties in Theorem 2 are clearly independent.\(^2\)

\(^2\)It is worth noting that a stronger characterisation result of the maximin swo can be provided by replacing the standard continuity property C with the following weaker property, where \( \epsilon \) denotes the stream of constant level of well-being equal to \( \epsilon \in \mathbb{R} \):

**Weak Continuity**, **WC**: \( \forall u, v \in X, u \succ v \Rightarrow \exists \epsilon > 0 : u \succ v + \epsilon, \exists \epsilon' > 0 : v - \epsilon' \succ^M v. \)
The theoretical relevance of the latter result can be appreciated if Theorem 2 is compared with alternative characterisations. On the one hand, unlike axioms on informational comparability often used in the literature (see, e.g., [14], [15]), the weak Harm Principle has a clear ethical foundation, but, as noted above it has no obvious egalitarian implication. In a recent contribution, Bosmans and Ooghe ([6]) characterise the maximin SWO using Anonymity (A), Weak Pareto Optimality (WPO), Continuity (C), and Hammond Equity (HE). Instead, as in the case of the lexicin ordering analysed above, Theorem 2 characterises an egalitarian SWO such as the maximin without appealing to an axiom like HE, which arguably has a marked egalitarian content, and using instead WHP, which only incorporates a liberal, noninterfering view of society.

4 Egalitarian Principles in the Infinitely-Lived Society

In this section, the axiomatic analysis of the difference principle and of its lexicographic refinement is extended to infinitely-lived economies, focusing on the role of liberal views of noninterference as formulated in the weak Harm Principle. As is well-known, the case with an infinite number of agents raises a number of issues concerning the existence and the characterisation of SWOS (see, e.g., [1]) and different definitions of the lexicin social welfare relation (henceforth, SWR) can be provided in order to compare (countably) infinite utility streams. In this section, first, the framework proposed by Asheim and Tungodden ([2]) is adopted, and an alternative characterisation of their lexicin SWRS is provided. Then, a new characterisation of an infinite-horizon ordering extension of a lexicin SWR recently proposed by Bossert, Sprumont and Suzumura ([7]) is provided. Finally, the framework of Asheim and Tungodden ([2]) is extended to analyse the maximin SWR, and a new characterisation of the difference principle is proposed in the context of infinitely-lived economies.

4.1 The Leximin SWR

Following Asheim and Tungodden, there are two different ways of formally defining the lexicin SWR. The first one is the so-called ‘weak lexicin’, or W-Leximin, and can be formalised as follows.

**Definition 1** (Definition 2, [2], p. 224) For all $u, v \in X$, $u \sim_{LM} v \iff \exists T \geq 1$ such that $\forall T \geq T': \bar{u}_T = \bar{v}_T$, and $u \succ_{LM} v \iff \exists T \geq 1$ such that $\forall T \geq T' \exists t \in \{1, ..., T\}$ $\bar{u}_t = \bar{v}_t \forall 1 \leq s < t$ and $\bar{u}_t > \bar{v}_t$.

The characterisation results below are based on a number of standard axioms. The first three axioms are similar to those used in the finite case, and need no further comment, except possibly
noting that in this context, WHP is weaker than the version in Section 2 above, since it only holds for vectors with the same tail.

**Finite Anonymity, FA:** \( \forall u \in X \) and \( \forall \pi \in \Pi, \pi(u) \sim 1u. \)

**Strong Pareto Optimality, SPO:** \( \forall 1u, 1v \in X : 1u > 1v \Rightarrow 1u > 1v. \)

**Weak Harm Principle, WHP:** \( \forall 1u, 1v \in X : \exists T \geq 1 \ 1u = (1u_{T,T+1}) > 1v, \) and \( 1u', 1v' \) are such that, \( \exists i \leq T, \)

\[
\begin{align*}
&u'_i < u_i \\
v'_i < v_i \\
u'_j = u_j \ \forall j \neq i \\
v'_j = v_j \ \forall j \neq i
\end{align*}
\]

implies \( 1v' \neq 1u' \) whenever \( u'_i > v'_i. \)

Next, following Asheim and Tungodden ([2], p. 223), an axiom is imposed, which represents a mainly technical requirement to deal with infinite dimensional vectors.

**Weak Preference Continuity, WPC:** \( \forall 1u, 1v \in X : \exists \tilde{T} \geq 1 \) such that \( (1u_{T,T+1}) > 1v \ \forall T \geq \tilde{T} \Rightarrow 1u > 1v. \)

WPC (and the same is true for the stronger SPC discussed below) provides a condition that establishes “a link to the standard finite setting of distributive justice, by transforming the comparison of any two infinite utility paths to an infinite number of comparisons of utility paths each containing a finite number of generations” ([2], p. 223). In the same vein, the next axiom states that the SWR should at least be able to compare (infinite-dimensional) vectors with the same tail. This seems an obviously desirable property which imposes a minimum requirement of completeness.

**Minimal Completeness, MC:** \( \forall 1u, 1v \in X, \exists T \geq 1 \ (1u_{T,T+1} \neq 1v \Rightarrow (1u_{T,T+1} \geq 1v \ or \ 1v \geq (1u_{T,T+1} \). \)

The next Theorem proves that the combination of Finite Anonymity (FA), Strong Pareto Optimality (SPO), Weak Harm Principle (WHP), Weak Preference Continuity (WPC), and Minimal Completeness (MC), characterises the leximin SWR.

**Theorem 3** \( \succeq \) is an extension of \( \succeq^{LM*} \) if and only if \( \succeq \) satisfies Finite Anonymity (FA), Strong Pareto Optimality (SPO), Weak Harm Principle (WHP), Weak Preference Continuity (WPC), and Minimal Completeness (MC).
Proof. (⇒) Let \( \succ^{LM} \subseteq \succ \). It is easy to see that \( \succ \) meets FA and SPO. By observing that \( \succ^{LM} \) is complete for comparisons between utility streams having the same tail it is also easy to see that \( \succ \) satisfies WPC and MC. We show that \( \succ \) meets WHP*. Take any \( u, v, u', v' \in X \) such that \( \exists T \geq 1 : (u_{T+1}, T+1) \succ (v_{T+1}, T+1) \), and \( u', v' \) are such that, \( \exists i \leq T, u'_i < u_i, v'_i < v_i, u'_j = u_j \forall j \neq i, v'_j = v_j \forall j \neq i \). We show that \( u' \succ v' \) whenever \( u'_i > v'_i \). As \( \succ^{LM} \) is complete for comparisons between utility streams having the same tail, it must be true that \( u \succ^{LM} v \). Therefore, by definition, \( \exists \tilde{T} \geq 1 \) such that \( \forall T' \geq \tilde{T}, \exists t \in \{1, \ldots, T' \} \) \( \tilde{u}_s = \tilde{v}_s \forall 1 \leq s < t \) and \( \tilde{u}_t > \tilde{v}_t \). Take any \( T' \geq \tilde{T} \). As \( T' < \infty \) it follows from Theorem 1 in [12], p. 126 that there exists \( t^* \leq t \leq T' \) such that \( \tilde{u}'_s = \tilde{v}'_s \forall 1 \leq s < t^* \) and \( \tilde{v}'_t > \tilde{u}'_t \). As it holds true for any \( T' \geq \tilde{T} \) it follows that \( u' \succ v' \) as \( \succ^{LM} \subseteq \succ \).

(⇐) Suppose that \( \succ \) satisfies FA, SPO, WHP*, WPC, and MC. We show that \( \sim^{LM} \subseteq \sim \) and \( \succ^{LM} \subseteq \succ \). Take any \( u, v \in X \).

Assume that \( u \sim^{LM} v \). By definition, \( \exists \tilde{T} \geq 1 \) such that \( \forall T \geq \tilde{T}, u_T = \tilde{u}_T, v_T = \tilde{v}_T \), and so \( T+1 u = T+1 v, \) for any \( T \geq \tilde{T} \). It follows that \( u \sim v \), by FA.

Next, suppose that \( u \succ^{LM} v \), and so, by definition, \( \exists \tilde{T} \geq 1 \) such that \( \forall T \geq \tilde{T}, \exists t \in \{1, \ldots, T \} \) such that \( \tilde{u}_s = \tilde{v}_s \forall 1 \leq s < t \) and \( \tilde{u}_t > \tilde{v}_t \). Take any such \( T \) and consider the vector \( w = (u_{T+1}, T+1 v) \).

We want to show that \( w \succ v \). By FA and transitivity, we can consider \( \tilde{w} \equiv (u_{T+1}, T+1 v) \) and \( \tilde{v} \equiv (\tilde{u}_{T+1}, T+1 v) \). Suppose that \( \tilde{v} \succ \tilde{w} \). We distinguish two cases.

Case 1. \( \tilde{v} \succ \tilde{w} \)

As SPO holds it must be the case that \( \tilde{v}_l > \tilde{w}_l \) for some \( l > t \). Let

\[ k = \min\{t < l \leq T | \tilde{v}_l > \tilde{w}_l \}. \]

Let \( v_i = \tilde{v}_k \) and let \( w_i = \tilde{w}_{k-g} \), for some \( 1 \leq g < k \), where \( \tilde{w}_{k-g} > \tilde{v}_{k-g} \). Then, let two real numbers \( d_1, d_2 > 0 \), and consider vectors \( u', v' \) formed from \( w, v \) as follows: \( \tilde{w}_{k-g} - d_1 \) is lowered to \( \tilde{w}_{k-g} - d_1 \) such that \( \tilde{w}_{k-g} - d_1 > \tilde{v}_{k-g} \), \( \tilde{v}_k \) is lowered to \( \tilde{v}_k - d_2 \) such that \( \tilde{w}_k > \tilde{v}_k - d_2 > \tilde{w}_{k-g} - d_1 \); and all other entries of \( w \) and \( v \) are unchanged. By FA, consider \( \tilde{u}' = (u_{T+1}, T+1 v) \) and \( \tilde{v}' = (v_{T+1}, T+1 v) \).

By construction \( \tilde{u}'_j \geq \tilde{v}'_j \) for all \( j \leq k \), with at least two inequalities, \( \tilde{w}'_{k-g} > \tilde{v}'_{k-g} \) and \( \tilde{w}'_k > \tilde{v}'_k \), whereas WHP*, combined with MC and FA, implies \( v' \succ w' \). Furthermore, by SPO, it is possible to choose \( d_1, d_2 > 0 \), such that \( v' \succ w' \), without loss of generality. Consider two cases:

a) Suppose that \( \tilde{v}_k > \tilde{w}_k \), but \( \tilde{w}_l \geq \tilde{v}_l \) for all \( l > k \). It follows that \( \tilde{w}'_k > \tilde{v}'_k \), and so SPO implies that \( \tilde{w}' \succ \tilde{v}' \), a contradiction.

b) Suppose that \( \tilde{v}_l > \tilde{w}_l \) for some \( l > k \). Note that by construction \( \tilde{v}'_l = \tilde{v}_l \) and \( \tilde{w}'_l = \tilde{w}_l \) for all \( l > k \). Then, let

\[ k' = \min\{k < l \leq T | \tilde{v}'_l > \tilde{w}'_l \}. \]

13
where \( k' > k \). The above argument can be applied to \( \bar{u}' \), \( \bar{v}' \) to derive vectors \( \bar{u}'' \), \( \bar{v}'' \) such that \( \bar{w}'_j \geq \bar{v}''_j \) for all \( j \leq k' \), whereas WHP*, combined with MC, FA, and SPO, implies \( \bar{v}'' \succ \bar{u}'' \).

And so on. After a finite number of iterations \( s \), two vectors \( \bar{u}^s \), \( \bar{v}^s \) can be derived such that, by WHP*, combined with MC, FA, and SPO, we have that \( \bar{v}^s \succ \bar{u}^s \), but SPO implies \( \bar{u}^s \succ \bar{v}^s \), yielding a contradiction.

**Case 2.** \( \bar{v} \sim \bar{w} \)

Since, by our supposition, \( \bar{v}_t < \bar{u}_t \equiv \bar{w}_t \), there exists \( \epsilon > 0 \) such that \( \bar{v}_t < \bar{u}_t - \epsilon \leq \bar{w}_t \). Let \( \bar{u}^s \in \bar{X} \) be a vector such that \( \bar{w}^t_1 = \bar{w}_1 - \epsilon \) and \( \bar{w}^t_j = \bar{w}_j \) for all \( j \neq t \). It follows that \( \bar{u}^s \gtrsim \bar{v} \) but \( \bar{v} \succ \bar{u}^s \) by SPO and the transitivity of \( \succ \). Hence, the argument of Case 1 above can be applied to \( \bar{v} \) and \( \bar{u}^s \), yielding the desired contradiction.

It follows from MC that \( \bar{u} \succ \bar{v} \). FA, combined with the transitivity of \( \succ \), implies that \( (1u_T, T+1v) \succ \bar{v} \). Since it holds true for any \( T \geq \bar{T} \), WPC implies \( 1u \succ \bar{v} \), as desired. ■

The properties in Theorem 3 are easily shown to be independent (see Appendix).

It is worth stressing again that in societies with an infinite number of agents, or generations, there is no obvious, and unanimously accepted, definition of the lexicin SWR. Ashim and Tungodden ([2], p. 224), for example, provide an alternative, stronger definition of the lexicin - the S-Leximin - that can be formalised as follows.

**Definition 2** (Definition 1, [2], p. 224) For all \( 1u_1v \in \bar{X} \), \( 1u \gtrsim_S \bar{v} \) \( \iff \exists \bar{T} \geq 1 \) such that \( \forall T \geq \bar{T} \): either \( \bar{u}_T = \bar{v}_T \) or \( \exists t \in \{1, ..., T\} \): \( \bar{u}_s = \bar{v}_s \forall 1 \leq s < t \) and \( \bar{u}_t > \bar{v}_t \).

The above analysis has focused on the W-Leximin, because the continuity axiom WPC is arguably more appealing than the Strong Preference Continuity property adopted by Ashim and Tungodden ([2], p. 223) to characterise the S-leximin, which seems a rather strong requirement (as forcefully argued, for example, by Basu and Mitra [5], p. 358). Strong Preference Continuity can be formalised as follows.

**Strong Preference Continuity, SPC:** \( \forall 1u_1v \in \bar{X} : \exists \bar{T} \geq 1 \) such that \( (1u_T, T+1v) \gtrsim \bar{v} \forall T \geq \bar{T} \), and \( \forall T \geq 1 \exists \bar{T} \geq \bar{T} \) such that \( (1u_T, T+1v) \gtrsim \bar{v} \) \( \forall T \geq \bar{T} \).

A result analogous to Theorem 3 can be established for the stronger definition 2 by replacing WPC with SPC. It can be easily obtained through a trivial modification of the parts of the proof of Theorem 3 that involve WPC, and by observing that the necessity of WHP* can be easily established along the same lines as in Theorem 3.
**Theorem 4** $\succeq$ is an extension of $\succeq_{S}^{LM}$ if and only if $\succeq$ satisfies Finite Anonymity ($\text{FA}$), Strong Pareto Optimality ($\text{SPO}$), Weak Harm Principle ($\text{WHP}^{*}$), Strong Preference Continuity ($\text{SPC}$), and Minimal Completeness ($\text{MC}$).

The properties in Theorem 4 are easily shown to be independent (see Appendix).

Theorems 3 and 4 identify the relevant class of lexicin swrs by postulating a continuity property on the quasi-ordering (respectively, WPC and SPC), which represents a mainly technical requirement in ranking infinite utility streams. As axioms such as SPO and FA may be considered ethically more defensible than continuity axioms, Bossert, Sprumont and Suzumura ([7]) have not postulated any continuity property on the quasi-ordering and have provided a characterisation of a subclass of the class of orderings satisfying SPO, FA, and an infinite version of HE. Formally, the relationship between the characterisation of the lexicin by Bossert et al. ([7]) and that by Asheim and Tungodden ([2]) is analogous to the relationship between the characterisation of the utilitarian swr by Basu and Mitra ([5]) and the characterisations of the more restrictive utilitarian swr induced by the overtaking criterion (see the discussion in [7], p.580). This relationship is explored below by extending the analysis of WHP* to the framework developed by Bossert et al. ([7]).

For each $T \in \mathbb{N}$, let the lexicin ordering on $X_T$ be denoted as $\succeq_{T}^{LM}$. The definition of the lexicin swr proposed by Bossert et al. ([7]) can be formulated as follows. Define a relation $\succeq_{T}^{L} \subseteq X \times X$ by letting, for all $1u, 1v \in X$,

$$1u \succeq_{T}^{L} 1v \iff 1u_T \succeq_{T}^{LM} 1v_T \text{ and } T_{+1}u \geq T_{+1}v. \tag{4.1}$$

The relation $\succeq_{T}^{L}$ can be shown to be reflexive and transitive for all $T \in \mathbb{N}$. Then the lexicin swr is $\succeq^{L} = \bigcup_{T \in \mathbb{N}} \succeq_{T}^{L}$ ([7], p. 586): it is reflexive and transitive, but not necessarily complete. Moreover, Bossert et al. ([7]) show that $\succeq^{L}$ satisfies the following property ([7], p. 586, equation (14)):

$$\forall 1u, 1v \in X : \exists T \in \mathbb{N} \text{ such that } 1u \succeq_{T}^{L} 1v \iff 1u \succeq^{L} 1v. \tag{4.2}$$

The set of ordering extensions of $\succeq^{L}$ characterised by the next theorem, based on Finite Anonymity (FA), Strong Pareto Optimality (SPO), and the weak Harm Principle (WHP*), is shown to be non-empty.

**Theorem 5** $\succeq$ is an extension of $\succeq^{L}$ if and only if $\succeq$ satisfies Finite Anonymity (FA), Strong Pareto Optimality (SPO), and Weak Harm Principle (WHP*).
Proof. (⇒) The proof that any ordering extension of \(\succeq^L\) satisfies FA and SPO is as in ([7], Theorem 2, p. 586). We only need to prove that any ordering extension \(\succeq\) of \(\succeq^L\) satisfies WHP*.

Consider any \(1u, 1v, 1u', 1v' \in X\) such that \(\exists T \geq 1\) such that \(1u = (1u_{T,T+1} v) \succ 1v\), and \(1u', 1v'\) are such that, \(\exists i \leq T\), \(u'_i < u_i\), \(v'_i < v_i\), \(u'_j = u_j \forall j \neq i\), \(v'_j = v_j \forall j \neq i\). We show that \(1u' \succ 1v'\) whenever \(u'_i > v'_i\). Since \(\succeq_T^{LM}\) is complete and \(T+1v = T+1u\) it cannot be \(1v_T \succeq_T^{LM} 1u_T\), otherwise \((1v, u) \in \succeq^L \subseteq \succeq\) which contradicts \(1u \succ 1v\). Thus, we have that \(1u_T \succeq_T^{LM} 1v_T, 1v_T \succeq_T^{LM} 1u_T\), and \(T+1v = T+1u\), so that \((1u, v) \in \succeq_T^{LM}\) by (4.1). It follows from (4.2) that \((1u, v) \in \succeq^L\). As \(1u'\) and \(1v'\) are such that, \(\exists i \leq T\), \(u'_i < u_i\), \(v'_i < v_i\), \(u'_j = u_j \forall j \neq i\), \(v'_j = v_j \forall j \neq i\), it can easily be shown, as in ([12]), that \(1u'_T \succ_T^{LM} 1v_T\) whenever \(u'_i > v'_i\). As \(T+1v' = T+1u'\) and \(1u'_T \succ_T^{LM} 1v'_T\) it follows from (4.1) that \(1u' \succ 1v'\), and therefore \(1u' \succ^L 1v'\) by (4.2). But since \(\succeq\) is an ordering extension of \(\succeq^L\) it follows that \(1u' \succ 1v'\).

(⇐) The proof is identical to ([7], Theorem 2, p. 587) using the characterisation of the T-person leximin in Theorem 1. ■

Theorem 5 leaves a larger class of orderings than that identified by Theorems 3 or 4, because in the latter results two additional axioms are employed, but it is strikingly similar to the characterisation in the finite context.

Finally, it is worth noting that the Weak Harm Principle (WHP*) can also be used to characterise the intergenerational version of the leximin SWO recently proposed by Sakai ([18]), which drops transitivity but retains completeness. In particular, if one replaces Hammond Equity with WHP*, a modified version of his characterisation results ([18], Lemma 6, p.17; and Theorem 5, p.18) can easily be proved.

4.2 The Maximin SWR

In this subsection, Rawls's difference principle is analysed in the context of economies with an infinite number of agents. First of all, the analysis focuses on the subset of utility streams that reach a minimum in a finite period. Formally, define the following subset \(Y\) of \(X\):

\[
Y = \{1u \in X | \exists T' \geq 1: 1u_T = 1u_{T'} \ \forall T \geq T'\}.
\]

The maximin SWR can be formally defined as follows.

**Definition 3** For all \(1u, 1v \in Y\), \(1u \sim_{M^*} 1v \iff \min(1u) = \min(1v)\), and \(1u \succeq_{M^*} 1v \iff \min(1u) > \min(1v)\).

Let \(\succeq_{M^*} = \sim_{M^*} \cup \succeq_{M^*}\). It is easy to show that \(\succeq_{M^*}\) is a quasi-ordering on \(X\) and that \(\succeq_{M^*}\)
is complete for any \( 1u, 1v \in Y \). In the framework proposed by Asheim and Tungodden ( [2]), Definition 3 has equivalent reformulations.

**Proposition 6** For all \( 1u, 1v \in Y \), the following statements are equivalent:

(a) either \( \min(1u) = \min(1v) \) or \( \min(1u) > \min(1v) \).
(b) \( \exists T \in \mathbb{N} : \) either \( 1u_T = 1v_T \ \forall T \geq T \) or \( 1u_T > 1v_T \ \forall T \geq T \).
(c) \( \exists T \in \mathbb{N} : \) either \( 1u_T = 1v_T \) and \( [1u_T = 1v_T = 1u_T \ \forall T \geq T] \) or \( 1u_T > 1v_T \) and \( [1u_T = 1u_T \ \forall T \geq T] \).

**Proof.** Obvious, so omitted. ■

It is worth noting that the relevant \( T \) in part (b) may be different from that in part (c).

In order to prove the main characterisation result, the following four standard axioms are imposed, which are similar to those used in the finite setting, and need no further comment, except noting that \( \text{WC} \) is a weakening of standard continuity axioms. Continuity requires that if \( 1u \) is strictly better than \( 1v \), then any vector sufficient close to \( 1u \) should be strictly better than any vector sufficient close to \( 1v \). \( \text{WC} \) only requires the existence of some vectors with the latter property.

**Finite Anonymity, FA**: \( \forall 1u \in Y \) and \( \forall \pi \in \Pi \Rightarrow \pi(1u) \sim 1u \).

**Weak Pareto Optimality, WPO**: \( \forall 1u, 1v \in Y, 1u \gg 1v \Rightarrow 1u > 1v \).

**Weak Harm Principle, WHP**: \( \forall 1u, 1v \in Y, 1u > 1v \) and \( 1u', 1v' \) are such that \( \exists i \in \mathbb{N}, \)

\[
\begin{align*}
    u'_i &< u_i, \\
    v'_i &< v_i, \\
    u'_j &= u_j \forall j \neq i \\
    v'_j &= v_j \forall j \neq i
\end{align*}
\]

implies \( 1v' \not\sim 1u' \) whenever \( u'_j > v'_j \).

**Weak Continuity, WC**: \( \forall 1u, 1v \in Y, 1u > 1v \Rightarrow \exists \epsilon > 0 : 1u > 1v + \text{con} \epsilon, \exists \epsilon' > 0 : 1u - \text{con} \epsilon' > 1v \).

In addition to the above requirements, following again Asheim and Tungodden ( [2]), a weak consistency requirement on the symmetric part of \( \gg \) is imposed.

**Weak Indifference Continuity, WIC**: \( \forall 1u, 1v \in Y, \exists T' \geq 1 \) such that \( 1u \sim (1u_{T',T+1} u) \ \forall T \geq T' \Rightarrow 1u \sim 1v \).

In analogy with \( \text{WPC} \), the latter axiom represents a mainly technical requirement that provides a link to the standard finite setting of distributive justice. \( \text{WIC} \) imposes that “an infinite utility
path should be considered indifferent to another infinite utility path if the head of the former is considered indifferent to the latter at every point in time beyond a certain initial phase” ([2], p.229). Axioms similar to WIC are common in the literature (see, e.g., [5], [1]). Finally, the next axiom requires that $\succ$ be complete at least when comparing elements of $Y$ with the same tail: this is an even weaker completeness requirement than MC, and it seems uncontroversial, for it is obviously desirable to be able to rank as many vectors as possible.

**Minimal Completeness, MC**: $\forall u, v \in Y, u \neq v \Rightarrow (u_{T+1} \succ v) \Rightarrow v \succ u \vee (u_{T+1} \succ v)$. In order to derive the main characterisation result concerning the maximin SWR, first two useful Lemmas are proved. The next result states that any SWR satisfying WC, MC*, and WPO* also satisfies monotonicity.

**Lemma 7** Let $\succ$ on $X$ be a SWR satisfying WC, MC*, and WPO*. Then, $\forall u, v \in Y : \exists T \geq 1 \succ u \succ (v_{T+1} + u) \Rightarrow u \succ (v_{T+1} + u)$.

**Proof.** See appendix. ■

Lemma 8 provides sufficient conditions to implement some type of Pareto-rankings. In particular, it proves that, for a subset of allocations in $Y$, there is a finite number $T'$ such that, starting from a given allocation, if the welfare of at least the first $T'$ generations increases uniformly, then the new allocation is strictly preferred.

**Lemma 8** Let $\succ$ on $X$ be a SWR satisfying FA*, WPO*, WC, WHP**, WIC, and MC*. Let $u, v \in Y$ be such that $u \succ^{M^*} v$, let $\hat{T} \in \mathbb{N}$ be such that $u_{T} = 1_{T}$ and $v_{T} = 1_{T}, \forall T \geq \hat{T}$, and let $\epsilon > 0$ be such that $u_{T} + \epsilon < 1_{T}$. Then, $u_{T+1} \succ (v_{T+1} + u)$.

**Proof.** See appendix. ■

Given Lemmas 7 and 8, the next Theorem proves that the combination of Finite Anonymity (FA*), Weak Pareto Optimality (WPO*), Weak Harm Principle (WHP**), Weak Continuity (WC), Weak Indifference Continuity (WIC), and Minimal Completeness (MC*), characterises the maximin SWR.

**Theorem 9** $\succ$ on $X$ is an extension of $\succ^{M^*}$ on $Y$ if and only if $\succ$ satisfies Finite Anonymity (FA*), Weak Pareto Optimality (WPO*), Weak Harm Principle (WHP**), Weak Continuity (WC), Weak Indifference Continuity (WIC), and Minimal Completeness (MC*).

**Proof.** $(\Rightarrow)$ It is easy to see that $\succ$ meets FA*, WPO*, WHP**, WC, WIC, and MC* whenever $\succ$ is an extension of $\succ^{M^*}$.
(⇐) Suppose that \( \succcurlyeq \) meets FA*, WPO*, WHP**, WC, WIC, and MC*. We show that \( \succcurlyeq^{M^*} \subseteq \succcurlyeq \), that is, \( \forall 1u,1v \in Y, \)
\[
1u \succcurlyeq^{M^*} 1v \Rightarrow 1u \succcurlyeq 1v \tag{4.3}
\]
and
\[
1u \sim^{M^*} 1v \Rightarrow 1u \sim 1v. \tag{4.4}
\]
We prove (4.3) and (4.4) by showing: 1) that (4.3) holds for vector having the same \( T \)-tail; 2) that the implication (4.4) holds \( \forall 1u,1v \in Y \); and, finally, 3) that the implication (4.3) is met \( \forall 1u,1v \in Y \) too.

**Proof of (1)**
Take any \( 1u,1v \in Y \) such that \( \min (1u) > \min (1v) \). By proposition 6 it follows that \( \exists \overline{T} \geq 1 \) such that \( 1u_{\overline{T}} > 1v_{\overline{T}} \) and \( 1u_{\overline{T}} = 1v_{\overline{T}} = 1u_{\overline{T}} \forall T \geq \overline{T} \). Take any \( T \geq \overline{T} \) and let \( 1w = (1v_{\overline{T}},T+1u) \).
Observe that \( 1w \in Y \). We show that \( 1u \succcurlyeq 1w \) since \( 1u \succcurlyeq^{M^*} 1w \), by construction. Assume, to the contrary, that \( 1u \not\succcurlyeq 1w \), so that \( 1w \not\succcurlyeq 1u \) as the premises of MC* are met. We distinguish two cases.

**Case 1:** \( 1w \succcurlyeq 1u \)
As FA* holds, let \( 1\bar{w},1\bar{u} \) be such that \( T+1\bar{w} = T+1u \) and \( 1\bar{w}_{\overline{T}},1\bar{u}_{\overline{T}} \) are such that \( \bar{w}_1 \leq \ldots \leq \bar{w}_{\overline{T}} \) and \( \bar{u}_1 \leq \ldots \leq \bar{u}_{\overline{T}} \). If \( 1\bar{w}_{\overline{T}} \ll 1\bar{u}_{\overline{T}} \), Lemma 7 implies \( 1\bar{u} \succcurlyeq 1\bar{w} \), a contradiction. Otherwise, let \( \bar{w}_i \geq \bar{u}_i \forall t \leq T \). We proceed in two steps.

**Step 1.**
Let
\[
k = \min \{ l \leq T | \bar{w}_l \geq \bar{u}_l \}.
\]
Let \( 1\bar{w} \) and \( 1\bar{u} \) be two finite permutations of \( \mathbb{N} \) such that \( T+1\bar{w} = T+1\bar{u} = T+1\bar{u} \) and, for some \( i \leq T \), \( \bar{w}_i = \bar{w}_k \) and \( \bar{u}_i = \bar{u}_i \). By FA*, \( 1\bar{w} \sim 1\bar{w} \) and \( 1\bar{u} \sim 1\bar{u} \), so that \( 1\bar{w} \succcurlyeq 1\bar{u} \). Then, let two real numbers \( d_1,d_2 > 0 \), and consider vectors \( 1u',1w' \) formed as follows: first, \( \bar{u}_i \) is lowered to \( \bar{u}_i - d_1 > \bar{u}_i \); next, \( \bar{w}_i \) is lowered to \( \bar{w}_i - d_2 > \bar{w}_i \); finally, all other entries of \( 1\bar{u} \) and \( 1\bar{w} \) are unchanged. It follows from WHP** and MC*, that \( 1w' \succcurlyeq 1u' \). Let \( 1\bar{w}' \) and \( 1\bar{u}' \) be two finite permutations of \( \mathbb{N} \) such that \( T+1\bar{w}' = T+1u' = T+1u' \) and \( 1\bar{w}'_{\overline{T}},1\bar{u}'_{\overline{T}} \) are such that \( \bar{w}'_1 \leq \ldots \leq \bar{w}'_{\overline{T}} \) and \( \bar{u}'_1 \leq \ldots \leq \bar{u}'_{\overline{T}} \). By construction, \( \bar{u}'_j > \bar{w}'_j \) for all \( j \leq k \). By FA* and transitivity, \( 1\bar{w}' \succcurlyeq 1\bar{u}' \).

**Step 2.**
Let
\[
0 < \epsilon < \min \{ \bar{u}'_j - \bar{w}'_j | \forall j \leq k \}
\]

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and define $1\tilde{w}_T' = 1\tilde{w}_T' + 1\epsilon_T$, where $1\epsilon_T$ is $T$-head of $\text{con}\epsilon$. Let $1\tilde{w}^* = (1\tilde{w}_T', T+1\tilde{w}^*)$, and observe that $1\tilde{w}^* \in Y$. Lemma 8 implies $1\tilde{w}^* \succ 1\tilde{w}'$. As $1\tilde{w}' \succ 1\tilde{u}'$, by step 1, transitivity of $\succ$ implies $1\tilde{w}^* \succ 1\tilde{u}'$.

If $\tilde{u}_j' > \tilde{w}_j'$ for all $j \leq T$, Lemma 7 implies $1\tilde{u}' \succ 1\tilde{w}'$, a contradiction. Otherwise, let $\tilde{w}_j' \geq \tilde{u}_j'$ for some $T \geq l > k$. Then, let

$$k' = \min \{l \leq T | \tilde{w}_j' \geq \tilde{u}_j'\}$$

where $k' > k$.

The above steps 1-2 can be applied to $1\tilde{u}'$, $1\tilde{w}^*$ to derive vectors $1\tilde{w}_j''$, $1\tilde{w}_j'''$ such that $\tilde{w}_j'' > \tilde{w}_j'''$ for all $j \leq k' \leq T$, whereas $1\tilde{w}'' > 1\tilde{u}'''$. By Lemma 7, a contradiction is obtained whenever $\tilde{w}_j'' > \tilde{w}_j'''$ for all $j \leq T$. Otherwise, let $\tilde{w}_j'' \geq \tilde{u}_j'$ for some $T \geq l > k'$. And so on. After a finite number $s$ of iterations, two vectors $1\tilde{w}_s$, $1\tilde{u}_s$ can be derived such that $1\tilde{w}_s \succ 1\tilde{u}_s$, but $1\tilde{u}_s \succ 1\tilde{w}_s$, by Lemma 7, yielding a contradiction.

Case 2: $1w \sim 1u$.

Since $1w_T < 1u_T$, there exists $\epsilon > 0$ such that $1w_T = 1u_T + \epsilon < 1u_T$. Then fix such $\epsilon > 0$ and let $1w^* = (1w_T + 1\epsilon_{T+1} w)$, where $1\epsilon_T$ is the $T$-head of $\text{con}\epsilon$. Thus, by construction, $1u \succ^{\text{MC}} 1w^*$.

By Lemma 8, $1w^* \succ 1w$, and thus by the transitivity of $\succ$, $1w^* \succ 1u$ and so Case 1 above obtains.

Proof of (2)

Take any $1u, 1v \in Y$ such that $\min (1u) = \min (1v)$. By proposition 6 it follows that $\exists \tilde{T} \geq 1$ such that $1u_{\tilde{T}} = 1v_{\tilde{T}}$ and $1u_T = 1v_T = 1\tilde{w}_T \forall T \geq \tilde{T}$. If $1v = \pi (1u)$ for some $\pi \in \Pi$, FA* implies $1u \sim 1v$. Otherwise, let $1v \neq \pi (1u)$ for all $\pi \in \Pi$. Take any $T \geq \tilde{T}$ and let $1w = (1v_T, T+1 w)$. We show that $1u \sim 1w$. Observe that $1u \sim^{\text{MC}} 1w$, by construction, so that $\exists \tilde{T} \geq 1$ such that $1u_{\tilde{T}} = 1w_{\tilde{T}}$ and $1u_T = 1w_T = 1w_T \forall T \geq \tilde{T}$. Assume, to the contrary, that $1u \prec 1w$, so that either $1w \succ 1u$ or $1u \succ 1w$ holds by MC*. Without loss of generality, suppose $1u \succ 1w$. As $\succ$ meets WC it follows that $\exists \epsilon > 0$ such that $1u \succ 1w + \epsilon^{\text{con}\epsilon} \equiv 1w^*$. WC implies $\exists \alpha > 0$, $1u - \text{con}\alpha \succ 1w^*$. By another application of WC, it follows that, $\exists \beta > 0$, $1u - \text{con}\alpha - \text{con}\beta \succ 1w^*$. Let us consider the feasible utility stream $(1w_T - 1\alpha_{T+1} u + T+1 \epsilon) \in Y$.

As $1w^* \succ^{\text{MC}} (1u_T - 1\alpha_{T+1} u + T+1 \epsilon)$, we have that $1w^* \succ (1u_T - 1\alpha_{T+1} u + T+1 \epsilon)$, by (1) proved above. Furthermore, $(1w_T - 1\alpha_{T+1} u + T+1 \epsilon) \succ 1u - \text{con}\alpha - \text{con}\beta$, and therefore $(1w_T - 1\alpha_{T+1} u + T+1 \epsilon) \succ 1u - \text{con}\alpha - \text{con}\beta$ by WPO*. It then follows from the transitivity of $\succ$ that $1w^* \succ 1u - \text{con}\alpha - \text{con}\beta$, a contradiction. Therefore, $1u \sim 1w \equiv (1v_T, T+1 u)$. Since $1u \sim 1w \equiv (1v_T, T+1 u)$ holds for any $T \geq \tilde{T}$, WC implies $1u \sim 1v$.

Proof of (3)
Take any \(1_u, 1_v \in Y\) such that \(\min(1_u) > \min(1_v)\). As \(1_u, 1_v \in Y\), \(\exists \tilde{T} \geq 1\) such that \(1_{u \tilde{T}} > 1_{v \tilde{T}}\) and \(1_{u \tilde{T}} = 1_{v \tilde{T}}\) and \(1_{v \tilde{T}} = 1_{v \tilde{T}}\) \(\forall \tilde{T} \geq \tilde{T}\). Take any \(T \geq \tilde{T}\) and let \(1_w = (1_{v T}, T \in \tilde{T} u)\). Observe that \(1_w \in Y\). We show that \(1_u \succ 1_v\). Observe that \(1_u \succ^M 1_w\), by construction. By (1) proved above, it follows that \(1_u \succ 1_w\). As \(1_w \sim^M 1_v\), it follows from (2) proved above that \(1_w \sim 1_v\). Transitivity of \(\succ\) implies \(1_u \succ 1_v\), as desired. ■

The properties in Theorem 9 are easily shown to be tight (see Appendix).

Theorem 9 provides an original characterisation of the maximin SWR in the context of societies with an infinite number of agents. This result is interesting per se, as compared to alternative characterisations of the maximin. For example, Lauwers ([11]) characterises the maximin SWO by an anonymous social welfare function (SWF) defined over the set of bounded infinite utility streams, by imposing a strong version of HE according to which for any two bounded infinite vectors \(1_u, 1_v\) such that \(u_i \geq u_i \geq v_j \geq u_j\) for some \(i, j \in \mathbb{N}\) and \(u_k = v_k \forall k \in \mathbb{N}\{i, j\}\), then \(1_v \succ 1_u\). The main focus of this paper is different and so the question of the characterisation of the maximin SWO by an anonymous and liberal SWF remains open. It is worth noting, however, that Theorem 9 does characterise the maximin SWR on a different set of infinite utility streams, which can be unbounded above, and to this aim neither the continuity condition, nor to the so-called “repetition approximation principle” imposed by Lauwers ([11], p.146) are necessary. Indeed, subject to the domain restriction, and except for the rather mild condition WIC, the axioms are strikingly similar to those used to characterise the maximin SWO in finite economies.\(^3\)

Perhaps more importantly, Theorem 9 provides further support to the main theoretical arguments of this paper. For it confirms that the main intuitions concerning the role of the liberal notion of noninterference embodied in the Harm Principle are robust and they do not depend on the specific definition of the maximin and lexicin SWR adopted to rank infinite utility streams (pioneered by Swensson, [19]).

5 Conclusions

This paper analyses Rawls’s celebrated difference principle, and its lexicographic refinement, in societies with a finite and an infinite number of agents. A unified framework of analysis is set up, which allows one to characterise a family of egalitarian principles by means of a weaker version of a new axiom - the Harm Principle - recently proposed in [12]. This is quite surprising, because the Harm Principle is meant to capture a liberal requirement of noninterference and it incorporates no obvious egalitarian content. A set of new characterisations of the maximin and of its lexicographic

\(^3\)This is even more evident in the light of the discussion in footnote 2 above.
refinement are derived, including in the intergenerational context with an infinite number of agents and using different definitions of the relevant social welfare relations proposed in the literature. The results presented in this paper have two main sets of implications from a theoretical viewpoint. First, they shed new light on the ethical foundations of the egalitarian approaches stemming from Rawls’s difference principle. In fact, both the leximin and the maximin are characterised by some standard axioms (such as Anonymity and the Pareto Principle) together with a liberal principle incorporating only a noninterfering view of society. No axiom with an explicitly egalitarian content is necessary in order to derive the main liberal egalitarian principles. Second, from the viewpoint of liberal approaches emphasising a notion of individual autonomy, or freedom, they have a rather counterintuitive implication. For they prove that, in a number of different contexts, liberal noninterfering principles lead straight to welfare egalitarianism.
Appendix

6 Proofs of Lemmas

Lemma 7. Let on X be a swr satisfying WPO*, WC, and MC*. Then, \( \forall u, v \in Y : \exists T \geq 1, u \succ (v_{T+1} u) \Rightarrow u \succ (v_{T+1} u) \).

Proof. Let on X be a swr satisfying WPO*, WC, and MC*. Let \( u, v \in Y \) be such that \( \exists T \geq 1, u \succ (v_{T+1} u) \). We show that \( u \succ (v_{T+1} u) \). Assume, to the contrary, that \( u \not\succ (v_{T+1} u) \). MC* implies \( (v_{T+1} u) \succ 1 \). It follows from WC that \( \exists \varepsilon > 0, (v_{T+1} u) - \varepsilon \not\succ 1 \). However, since \( u \succ (v_{T+1} u) \) we have that \( u \succ (v_{T+1} u) - \varepsilon \), so that WPO* implies \( u \succ (v_{T+1} u) - \varepsilon \), a contradiction. ■

In order to prove Lemma 8, some preliminary results are derived, which are interesting per se. They are Lemma A.1 and Lemma A.2. The former Lemma proves a property of the swr on ranking a subset of vectors with the same tail. The latter Lemma provides sufficient conditions to implement some type of Pareto-rankings. In particular, it proves that, for a subset of allocations in Y, there is a finite number T such that, starting from a given allocation, if the welfare of at least the first T generations increases uniformly, then the new allocation is strictly preferred.

Lemma A.1. Let on X be a swr satisfying FA*, WPO*, WHP**, WC, and MC*. Let \( u, v \in Y \) be such that \( u \succ 1 \). Let \( T \in \mathbb{N} \) be such that \( u_T = u_T \) and \( u_T = u_T \) \( \forall T \geq \hat{T} \). Then, \( (v_{T+1} u) \sim (v_{T+1} u) \) \( \forall T \geq \hat{T} \).

Proof. Take any \( T \geq \hat{T} \). If \( u_{T+1} = v_{T+1} \), then the result follows by FA*. Otherwise, let \( u_{T+1} \neq v_{T+1} \). Lemma 7 implies

\[
(1v_{T+1,T+2} u) \succ (1v_{T+1} u) \text{ whenever } v_{T+1} > u_{T+1},
\]

(6.1)

or

\[
(1v_{T+1} u) \succ (1v_{T+1,T+2} u) \text{ whenever } u_{T+1} > v_{T+1}.
\]

(6.2)

Let \( u_{T+1} > u_{T+1} \), and suppose that \( (1v_{T+1,T+2} u) \succ (1v_{T+1} u) \). It follows from WC that \( (1v_{T+1,T+2} u) - \alpha \not\succ (1v_{T+1} u) \) \( \forall \alpha > 0 \). If \( v_{T+1} - \alpha < u_{T+1} \), WPO* implies \( (v_{T+1} u) \succ (1v_{T+1,T+2} u) - \alpha \), a contradiction. Otherwise, let \( v_{T+1} - \alpha \geq u_{T+1} \). As \( u, v \in Y \), are such that \( u \succ 1 \) and \( T \geq \hat{T} \), then \( \exists T \leq T \) such that \( v_{t} < u_{T+1} \) and, obviously, \( v_{t} - \alpha < v_{t} \).

For the sake of notational simplicity, let \( (1v_{T+1} u) \equiv 1x \) and \( (1v_{T+1,T+2} u) - \alpha \equiv 1y \), so that \( 1y \succ 1x \). Let \( \hat{x} \) be a finite permutation of \( 1x \) such that \( \hat{x}_{T+1} = v_{t} \) and \( \hat{x}_{t} = u_{T+1} \), while
all other entries are unchanged. By $\text{FA}^*$, $\hat{x} \sim 1x$, so that $1y \succ 1\hat{x}$, by the transitivity of $\succ$. Then, let two real numbers $d_1, d_2 > 0$, and consider vectors $1y', 1\hat{x}'$ formed from $1\hat{x}$ and $1y$ as follows: first, $\hat{x}_{T+1}$ is lowered to $\hat{x}_{T+1} - d_1 > v_1 - \alpha$; next, $y_{T+1}$ is lowered to $y_{T+1} - d_2$ such that $u_{T+1} > y_{T+1} - d_2 > \hat{x}_{T+1} - d_1$; finally, all other entries of $1\hat{x}$ and $1y$ are unchanged. It follows from $\text{WHP}^*$ that $1\hat{x}' \not\succ 1y'$. Let $1\hat{x}'$ be a finite permutation of $1\hat{x}'$ such that $x'_{T+1} = u_{T+1}$ and $x'_t = \hat{x}_{T+1} - d_1$. $\text{FA}^*$ and the transitivity of $\succ$ imply $1\hat{x}' \not\succ 1y'$, but $\text{WPO}^*$ implies $1\hat{x}' \succ 1y'$, a contradiction. Therefore, $(1v_{T+1} + T + 2u) \not\succ (1v_{T+1} + T + 2u)$ whenever $v_{T+1} > u_{T+1}$. $\text{MC}^*$ implies that $(1v_{T+1} + T + 2u) \succ (1v_{T+1} + T + 2u)$, and it follows from (6.1) that $(1v_{T+1} + T + 2u) \sim (1v_{T+1} + T + 2u)$.

Let $u_{T+1} > v_{T+1}$ and suppose that $(1v_{T+1} + T + 2u) \succ (1v_{T+1} + T + 2u)$. It follows from $\text{WC}$ that $(1v_{T+1} + T + 2u) \succ (1v_{T+1} + T + 2u) + \alpha \geq 0$. If $v_{T+1} + \alpha > u_{T+1}$, $\text{WPO}^*$ implies $(1v_{T+1} + T + 2u) + \alpha \geq (1v_{T+1} + T + 2u)$, a contradiction. Otherwise, let $v_{T+1} + \alpha \leq u_{T+1}$. For the sake of notational simplicity, let $(1v_{T+1} + T + 2u) = 1x$ and $(1v_{T+1} + T + 2u) + \alpha = 1y$, so that $1x \succ 1y$. Again, note that $\exists t \leq T$ such that $v_t < u_{T+1}$. Let $1y$ be a finite permutation of $1y$ such that $\hat{y}_{T+1} = y_t$ and $\hat{y}_t = y_{T+1}$, while all other entries are unchanged. By $\text{FA}^*$, $1\hat{y} \sim 1y$, so that $1x \succ 1\hat{y}$, by the transitivity of $\succ$. Then, let two real numbers $d_1, d_2 > 0$, and consider vectors $1\hat{y}', 1x'$ formed from $1\hat{y}$ and $1x$ as follows: first, $\hat{y}_{T+1} = y_t$ is lowered to $\hat{y}_{T+1} - d_1 > x_t = v_t$; next, $x_{T+1}$ is lowered to $x_{T+1} - d_2$ such that $y_{T+1} > x_{T+1} - d_2 > \hat{y}_{T+1} - d_1$; finally, all other entries of $1\hat{y}$ and $1x$ are unchanged. It follows from $\text{WHP}^*$ that $1\hat{y}' \not\succ 1x'$. Let $1\hat{y}'$ be a finite permutation of $1\hat{y}'$ such that $y'_{T+1} = y_{T+1}$ and $y'_t = \hat{y}_{T+1} - d_1$. $\text{FA}^*$ and the transitivity of $\succ$ imply $1\hat{y}' \not\succ 1x'$, but $\text{WPO}^*$ implies $1\hat{y}' \succ 1x'$, a contradiction. Therefore, $(1v_{T+1} + T + 2u) \not\succ (1v_{T+1} + T + 2u)$ whenever $u_{T+1} > v_{T+1}$. $\text{MC}^*$ implies that $(1v_{T+1} + T + 2u) \succ (1v_{T+1} + T + 2u)$, and it follows from (6.2) that $(1v_{T+1} + T + 2u) \sim (1v_{T+1} + T + 2u)$. ■

**Lemma A.2.** Let $\succ$ on $X$ be a swr satisfying $\text{FA}^*$, $\text{WPO}^*$, $\text{WHP}^*$, $\text{WC}$, $\text{WIC}$, and $\text{MC}^*$. Let $1u, 1v \in Y$ be such that $1u \succ 1v$, let $\tilde{T} \in \mathbb{N}$ be such that $1u_{\tilde{T}} = 1v_{\tilde{T}}$ and $1u_{\tilde{T}} = 1v_{\tilde{T}} \forall T' \geq \tilde{T}$, and let $\epsilon > 0$ be such that $1u_{\tilde{T}} + \epsilon < 1v_{\tilde{T}}$. Then, $(1v_{T} + 1\epsilon T + 1u) \succ (1v_{T} + 1T + 1u) \forall T \geq T'$. Then, $(1v_{T} + 1\epsilon T + 1u) \succ (1v_{T} + 1T + 1u)$ and $\forall T \geq T'$, and so, $\forall T \geq T'$, we have that

$$(1v_{T} + 1\epsilon T + 1u) \sim (1v_{T} + 1\epsilon T + 1u) \sim (1v_{T} + 1\epsilon T + 1u) \sim (1v_{T} + 1\epsilon T + 1u)$$

By the transitivity of $\succ$, it follows that, $\forall T \in \mathbb{N} \setminus \{1, ..., T' - 1\}$,

$$(1v_{T' + 1} + 1\epsilon T' + 1u) \sim (1v_{T' + 1} + 1\epsilon T' + 1u) \sim (1v_{T' + 1} + 1\epsilon T' + 1u)$$
so that, by the transitivity of $\succ$, it follows that, $\forall T \in \mathbb{N}\setminus \{1, ..., T' - 1\}$,

\[(1v_{T', T'+1} u) \sim (1v_T + 1\epsilon_{T, T+1} u) \quad (6.3)\]

and

\[(1v_T + 1\epsilon_{T, T+1} u) \sim (1v_T, T+1 u) . \quad (6.4)\]

**WIC** and (6.3) imply $(1v_{T', T'+1} u) \sim 1v + 1\epsilon$, whereas **WIC** and (6.4) imply that $(1v_T + 1\epsilon_{T, T'+1} u) \sim 1v$. By the transitivity of $\succ$, it follows that $1v + 1\epsilon \sim 1v$, but **WPO** implies $1v + 1\epsilon \succ 1v$, a contradiction. ■

**Lemma 8.** Let $\succ$ on $X$ be a swr satisfying **FA**, **WPO**, **WHP**, **WC**, **WIC**, and **MC**. Let $1u, 1v \in Y$ be such that $1u \succ^{M^*} 1v$, let $\hat{T} \in \mathbb{N}$ be such that $1v \in \mathbb{N}$ and $1\epsilon_{T', T'+1} + 1\epsilon_{T' - 1, T} \forall T' \geq \hat{T},$

and let $\epsilon > 0$ be such that $1v_{T'} + \epsilon < 1v_T$. Then, $(1v_{T'} + 1\epsilon_{T', T'+1} u) \succ (1v_{T'} + 1u u).$

**Proof.** By Lemma A.2, $\exists T \geq T'$ such that $(1v_{T'} + 1\epsilon_{T', T'+1} u) \succ (1v_{T'} + 1\epsilon_{T', T'+1} u)$. By Lemma A.1

\[(1v_T, T+1 u) \sim (1v_{T', T'+2} u) \forall T \geq \hat{T} \quad (6.5)\]

and

\[(1v_T + 1\epsilon_{T, T+1} u) \sim (1v_{T+1} + 1\epsilon_{T+1, T+2} u) \forall T \geq \hat{T} . \quad (6.6)\]

The transitivity of $\succ$ and (6.5) imply $(1v_{T', T'+1} u) \sim (1v_{T', T'+1} u)$. And again, the transitivity of $\succ$ and (6.6) imply $(1v_{T'} + 1\epsilon_{T, T'+1} u) \sim (1v_{T} + 1\epsilon_{T', T'+1} u)$. As $(1v_{T'} + 1\epsilon_{T, T'+1} u) \succ (1v_{T'} + 1\epsilon_{T', T'+1} u)$ it follows from the transitivity of $\succ$ that $(1v_{T'} + 1\epsilon_{T, T'+1} u) \succ (1v_{T'} + 1\epsilon_{T', T'+1} u)$. ■

## 7 Independence of Axioms

The proofs of the independence of the axioms used to characterise the maximin and leximin swo are obvious and therefore they are omitted. It is worth noting, however, that some of the examples below can be easily adapted to apply to the finite context.

### 7.1 Independence of axioms used in Theorem 3

In order to complete the proof of Theorem 3, we show that the axioms are tight.

For an example violating only **FA**, define $\succ$ on $X$ in the following way: $\forall 1u, 1v \in X$,

\[1u = 1v \Rightarrow 1u \sim 1v \]

\[\exists T \in \mathbb{N} : u_t = v_t \forall t < T \quad and \quad u_T > v_T \Rightarrow 1u \succ 1v \]

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The $\succeq$ on $X$ is not an extension of the lexicin $\succeq^{LM^*}$. The $\succeq$ on $X$ satisfies all properties except FA.

For an example violating only $\textbf{SPO}$, define $\succeq$ on $X$ in the following way: $\forall u,v \in X$, $1u \sim 1v$. 

The $\succeq$ on $X$ is not an extension of the lexicin $\succeq^{LM^*}$. Clearly, the $\succeq$ on $X$ satisfies all properties except $\textbf{SPO}$.

For an example violating only $\textbf{WHP}^*$, define $\succeq$ on $X$ be the lexicin $\succeq$, i.e., $\forall u,v \in X$,

$$1u \sim^{L_X} 1v \iff \exists T \geq 1 \text{ s.t. } \forall T \geq T : 1\bar{u}_T = 1\bar{v}_T,$$

and

$$1u \succ^1_X 1v \iff \exists T \geq 1 \text{ s.t. } \forall T \geq T, \exists t \in \{1, \ldots, T\} \text{ with } \bar{u}_s = \bar{v}_s \quad (\forall t < s \leq T) \text{ and } \bar{u}_t > \bar{v}_t.$$

It follows that the $\succeq$ on $X$ is not an extension of the lexicin $\succeq$. The $\succeq$ on $X$ satisfies all properties except $\textbf{WHP}^*$.

For an example violating only $\textbf{WC}$, define $\succeq$ on $X$ in the following way: $\forall u,v \in X$,

$$1u \gg (1v) \quad \exists \pi \in \Pi \Rightarrow 1u \gg 1v.$$

The relation $\succeq$ on $X$ is the \textit{Suppes-Sen grading principle}. The $\succeq$ on $X$ satisfies all properties except $\textbf{WC}$.

For an example violating only $\textbf{WPC}$, $\forall T \in \mathbb{N}$, let the lexicin ordering on $X_T$ be denoted as $\succeq^{LM}_T$. Define $\succeq_T$ on $X$ as in (4.1). Then, let $\succeq = \bigcup_{T \in \mathbb{N}} \succeq_T$. By definition, this relation is reflexive and transitive. The $\succeq$ on $X$ satisfies all properties but $\textbf{WPC}$. [To see that $\textbf{WPC}$ is violated consider the following vectors, $1v = (3, \text{con}0)$ and $1u = (2, \text{con}1)$. Then, $(1u, 1v) \not\in \succeq$ and $((1u_{T,T+1}v), 1v) \in \succeq_T \forall T \geq 2].$

\textbf{Independence of axioms used in Theorem 4}

In order to complete the proof of Theorem 4, we show that the axioms are tight.

As Strong Preference Continuity ($\textbf{SPC}$) implies Weak Preference Continuity ($\textbf{WPC}$), the above examples show that the axioms used in Theorem 4 are tight as well.
Independence of axioms used in Theorem 9

In order to complete the proof of Theorem 9, we show that the axioms are tight.

For an example violating only FA*, define $\succ$ on $X$ in the following way: \( \forall u, v \in X, \ i) u_1 = v_1 \iff 1u \sim 1v; \ ii) u_1 > v_1 \iff 1u \succ 1v. \) The SWR $\succ$ on $X$ is not an extension of the maximin SWR $\succ^M$. The SWR $\succ$ on $X$ satisfies all properties except FA*.

For an example violating only WPO*, define $\succ$ on $X$ in the following way: \( \forall u, v \in X, 1u \sim 1v. \) The SWR $\succ$ on $X$ is not an extension of the maximin SWR $\succ^M$. Clearly, the SWR $\succ$ on $X$ satisfies all properties except WPO*.

For an example violating only WC, fix $T \in \mathbb{N}$ such that $1 < T < \infty$. For $1u \in X$, let $1\hat{u}$ denote a permutation of $1u$ such that $\hat{u}_1 \leq \hat{u}_2 \leq ... \leq \hat{u}_T$ and $\hat{u}_l \geq \hat{u}_T \forall t > T$. Let us define $\succ$ on $X$ in the following way: \( \forall 1u, 1v \in X \)

\[ 1u \succ 1v \text{ if } \exists t \leq T : \hat{u}_s = \hat{v}_s \ (\forall s < t) \ \& \ \hat{u}_l > \hat{v}_l \]

\[ 1u \sim 1v \text{ if } 1\hat{u}_T = 1\hat{v}_T. \]

The SWR $\succ$ on $X$ is not an extension of the maximin SWR $\succ^M$. Clearly, the SWR $\succ$ on $X$ satisfies all properties except WC.

For an example violating only HP*, fix $T \in \mathbb{N}$ such that $1 < T < \infty$. For $1u \in X$, let $1\hat{u}$ denote a permutation of $1u$ such that $\hat{u}_1 \leq \hat{u}_2 \leq ... \leq \hat{u}_T$ and $\hat{u}_l \geq \hat{u}_T \forall t > T$. Let us define $\succ$ on $X$ in the following way: \( \forall 1u, 1v \in X \)

\[ 1u \succ 1v \text{ if } \frac{\sum_{i=1}^{T} \hat{u}_i}{T} > \frac{\sum_{i=1}^{T} \hat{v}_i}{T} \]

and

\[ 1u \sim 1v \text{ if } \frac{\sum_{i=1}^{T} \hat{u}_i}{T} = \frac{\sum_{i=1}^{T} \hat{v}_i}{T} \]

The SWR $\succ$ on $X$ is not an extension of the maximin SWR $\succ^M$. Clearly, the SWR $\succ$ on $X$ satisfies all properties except HP*.

For an example violating only MC*, let $\sigma$ be a permutation of $\mathbb{N}$. Let $\Sigma$ be the set of all permutations of $\mathbb{N}$. Define $\succ$ on $X$ in the following way: \( \forall 1u, 1v \in X \),

\[ 1u = \pi(1v) \exists \pi \in \Pi \Rightarrow 1u \sim 1v \]

\[ 1u \succ \sigma(1v) \exists \sigma \in \Sigma \Rightarrow 1u \succ 1v \]
The SWR \( \succeq \) on \( X \) is not an extension of the maximin SWR \( \succeq^{M^*} \). Clearly, the SWR \( \succeq \) on \( X \) satisfies all properties except MC*.

For an example violating *only* WIC, define \( \succeq \) on \( Y \) in the following way: \( \forall 1u, 1v \in Y \),

\[
1u \sim_{M^*} 1v, \exists \tau \geq 1 : \tau u = \tau v \Rightarrow 1u \sim_{1} 1v \\
1u \succ_{M^*} 1v \Rightarrow 1u \succ_{1} 1v \\
1u \sim_{M^*} 1v, \exists \tau \geq 1 : \tau u = \tau v \Rightarrow 1u \not\succ_{1} 1v \text{ and } 1v \not\succ_{1} 1u.
\]

The SWR \( \succeq \) on \( Y \) is not an extension of the maximin SWR \( \succeq^{M^*} \). Clearly, the SWR \( \succeq \) on \( Y \) satisfies all properties except WIC.
References


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