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Swap market model: Theory and empirical evidence

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Swap Market Model: Theory and Empirical Evidence

Bing Gan, Eric Guan and Ser-Huang Poon

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Abstract

This paper tests the co-terminal swap market model (SMM) pricing and hedging performance on Bermudan swaptions. To our knowledge, the drift for SMM is derived explicitly for the first time here, and the procedures for calibration and simulation using a collection of forward swap rates are also shown in detail. The Longstaff-Schwartz least square method is used to approximate the early exercise decision in Bermudan swaption. By introducing individual parameters for volatility of each co-terminal forward swap rate, the model can match the market quoted European swaption price perfectly. It is noted that, for the SMM, one particular volatility formula may not be enough to capture the term structure of different markets. Hedging performance of the model is tested on Euro and USD European co-terminal swaption using a set of swaps as hedge instrument. Principle component analysis (PCA) is adopted to capture the trend of the forward rates' movement and absolute mean of the PCA factors is used for the bumping of forward rate curve. Hedge ratio is calculated based on the delta ratios with respect to PCA factors. P&L of the hedge portfolio generated by co-terminal SMM is examined on Euro and USD market. The result shows that more factors may be needed in order to improve the hedge performance of the model. Nevertheless, the SMM is very similar to LMM in terms of implementation and model performance, and the SMM is more convenient when dealing with exotic interest rate derivatives where swap rate is the underlying.

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Swap Market Model: Theory and Empirical Evidence

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Abstract

This paper tests the co-terminal swap market model (SMM) pricing and hedging performance on Bermudan swaptions. To our knowledge, the drift for SMM is derived explicitly for the first time here, and the procedures for calibration and simulation using a collection of forward swap rates are also shown in detail. The Longstaff-Schwartz least square method is used to approximate the early exercise decision in Bermudan swaption. By introducing individual parameters for volatility of each co-terminal forward swap rate, the model can match the market quoted European swaption price perfectly. It is noted that, for the SMM, one particular volatility formula may not be enough to capture the term structure of different markets. Hedging performance of the model is tested on Euro and USD European co-terminal swaption using a set of swaps as hedge instrument. Principle component analysis (PCA) is adopted to capture the trend of the forward rates' movement and absolute mean of the PCA factors is used for the bumping of forward rate curve. Hedge ratio is calculated based on the delta ratios with respect to PCA factors. P&L of the hedge portfolio generated by co-terminal SMM is examined on Euro and USD market. The result shows that more factors may be needed in order to improve the hedge performance of the model. Nevertheless, the SMM is very similar to LMM in terms of implementation and model performance, and the SMM is more convenient when dealing with exotic interest rate derivatives where swap rate is the underlying.

Swap Market Model: Theory and Empirical Evidence

1 Introduction

Ho and Lee (1986) first propose the arbitrage free approach for interest rate term structure modelling. This class of models take initial term structure of interest rate as input which have made great advances along with the development of modeling techniques. The Asset Liability Management (ALM) department in global financial institutions face the problem like: Is it feasible for a global bank to use a single model for its interest rate businesses around the world stretching from the US, Europe? Would there be any inconsistency and risk of arbitrage if the bank adopts different interest rate models for different interest rate markets. These are profound questions, and this paper aims to address a simple question by looking at the effectiveness of the swap market model (SMM) for the purpose of Asset Liability Management. This paper concentrates on the implementation of SMM.

Both the SMM and LMM fall in the category of market model where market observable interest rates such as LIBOR rates and swap rates are modeled under the arbitrage free dynamics, and give prices of European caplet/swaption compatible with the standard Black formula.

Under the market model dynamics, there are basically two market observable forward rates being modeled, the LIBOR rates and the swap rate. Brace, Gatarek and Musiela (1997) firstly introduced the arbitrage free process for forward LIBOR rates which lead to the LIBOR Market Model (LMM, also known as BGM model) under the HJM framework. Miltersen, Sandmann and Sondermann (1997) derive a unified interest rates term structure which gives closed form solution for caplet under the assumption of log-normally distributed interest rates. Different methods of parameterization and calibration of LMM are examined in Brigo, Mercurio and Mrini (2003). The term structure of swap rates is firstly developed in Jamshidian (1997) which is known as Swap Market Model (SMM). Jamshidian (1997) also proposed the concept of co-terminal market model at the earliest. Base on the

research of the above papers, Galluccio, Huang, Ly and Scaillet (2007) use graph theory to classify the “admissible” market models into three subclasses named co-initial, co-sliding and co-terminal. Among other things, they show that the LMM is the only admissible model for swaps of a co-sliding type.

There are comparatively fewer papers on the implementation of SMM at the time of writing while extensive research have been done on the theories and implementation of the LMM have already been published. This is partly because the libor rate is often taken as the basic instrument in pricing and hedging interest rate derivatives and partly due to the misperception that swap rate is more difficult to model because of the overlapping interest rate periods. However, it cannot be denied that SMM is natural for pricing many exotic interest rate derivatives since it is designed to capture the term structure of volatilities of swaps. This misconception is noticed and partly corrected in Galluccio et al (2007) by giving numerical results on the pricing and calibration of co-terminal SMM to European swaptions. Their co-terminal SMM set is built under different measures and pricing test is done on European swaptions. However, the drift term of SMM has not been discussed in their paper. Jamshidian (1997) examined the drift of SMM and give examples on Bermudan swaption but the implementation is restricted to two exercise dates due to the limitation in simulation method at that time.

In this paper, the drift of the ctSMM under the terminal measure is explicitly derived. This is a crucial for pricing Bermudan swaptions. The model is then implemented by Monte Carlos simulation using least square (Longstaff and Schwartz) method to price Bermudan swaption.

The rest of this paper is organised as follows. Section 2 introduces fundamental definitions and equations that will be used in later implementation. The tenor structure is defined and the co-terminal SMM comes next by assigning arbitrage free dynamics to a collection of forward swap rates of the same terminal date. It is assumed that the forward swap rates are log-normally distributed and satisfy the arbitrage free condition under their respective measures. Drift are derived in Sec-

tion 3 with further details given in the Appendix. Cross-variation method proposed by Joshi and Liesch (2006) is used in the derivation. Section 4 gives details of the Monte Carlo simulation under the terminal measure and procedure of the calibration to co-terminal European swaptions. A linear-exponential formula is used to replicate term structure of implied volatilities of a collection of one-year swaptions with different maturity length. Section 5 derives the discounting factor needed for pricing Bermudan swaption, and the Longstaff-Schwartz least square approximation for early exercise decision. Section 6 investigates the idea of bumping the rate curve using results from principle component analysis on forward rates. Details of deriving hedge ratios with respect to principle component factors are explained. Hedge ratios with respect to hedge instruments can be calculated by minimizing the delta mismatch error. The equation for calculating profit and loss of hedge portfolio is also given in this section. Section 7 provides details of data and procedure of testing the co-terminal SMM on Euro and USD market. The results of numerical test are reported and discussed as well. Section 8 concludes.

2 Theory of Co-terminal Swap Market Model

2.1 Basic Theories and Definitions

Assume that there is a set of reset/exercise dates $0 := T_0 < T_1 < T_2 < \dots < T_M$, where T_0 is the contract starting date and T_M is the maturity of underlying derivative. This is the tenor structure under which the interest rate model is constructed in this paper. Also define $\tau_{n,m} = T_m - T_n$, ($0 \leq n < m < M$), as the time interval between two reset dates. For simplicity, we assume that all time intervals between two adjacent reset dates are the same, so $\tau_{n-1,n}$ ($n = 1, \dots, M$) can be reduced to τ . The time t value of a zero-coupon bond that pays one unit currency at maturity T_n ($n = 0, \dots, M$) is denoted by $B_{T_n}(t)$. For any n , $B_{T_n}(T_n) = 1$. In the following section, $B_n(t)$ is used as short version of $B_{T_n}(t)$.

The spot LIBOR rate at time T_n prevailing for period T_n to T_m ($0 \leq n < m < M$)

is denoted by $L(T_n, T_m)$ and defined as:

$$L(T_n, T_m) = \frac{1}{T_m - T_n} \left(\frac{B_n(T_n) - B_m(T_n)}{B_m(T_n)} \right) = \frac{1 - B_m(T_n)}{\tau_{n,m} B_m(T_n)} \quad (1)$$

The forward LIBOR rate at time t for period prevailing for period T_n to T_m is denoted by $F(t, T_n, T_m)$ where $t \leq T_n < T_m < T_{M+1}$. It is defined by:

$$F(t, T_n, T_m) = \frac{1}{T_m - T_n} \left(\frac{B_n(t) - B_m(t)}{B_m(t)} \right) = \frac{1}{\tau_{n,m}} \left(\frac{B_n(t)}{B_m(t)} - 1 \right) \quad (2)$$

A plain vanilla interest rate swap is a contract which exchanges fixed-rate interest payments at swap rate K for floating interest rate payments based on LIBOR on each reset date during period T_n to T_m ($0 < T_n < T_m$). That is, the contract starts from T_n while the first cash flow will happen at time T_{n+1} and the last cash flow will take place at time T_m . Also note that the related floating rate used to calculate cash flow of time T_k is known at time T_{k-1} ($k = 1, \dots, M$). A swap can also be viewed as a collection of Forward Rate Agreements (FRAs). Then the value of a payer swap at time t ($0 \leq t < T_n < T_m$) can be calculated by the following formula.

$$PayerSwap_{n,m}(t) = \sum_{i=n+1}^m \tau B_i(t) \left[\tilde{L}(T_{i-1}, T_i) - K \right]$$

Here, $\tilde{L}(T_{i-1}, T_i)$ represents the unknown spot LIBOR rate at future time T_{i-1} . Recall that a payer swap is a contract where the buyer will pay fixed rate and receive floating rates, whereas a receiver swap involves paying floating rates and receiving fixed rate on the buyer part. In a risk neutral world by arbitrage free arguments, the above formula can also be written in terms of forward libor rate from (2) as follows:

$$\begin{aligned} PayerSwap_{n,m}(t) &= \sum_{i=n+1}^m \tau B_i(t) [F(t, T_{i-1}, T_i) - K] \\ &= \sum_{i=n+1}^m \tau B_i(t) \left[\frac{1}{\tau} \left(\frac{B_{i-1}(t) - B_i(t)}{B_i(t)} \right) - K \right] \\ &= B_n(t) - B_m(t) - \sum_{i=n+1}^m \tau K B_i(t). \end{aligned} \quad (3)$$

The forward swap rate $S_{n,m}(t)$ is defined as the swap rate K that makes the swap contract value at starting date T_n equal to zero. Hence, by setting $PayerSwap_{n,m}(t)$

in equation (3) to be zero, we can get the forward swap rate $S_{n,m}(t)$:

$$S_{n,m}(t) = \frac{B_n(t) - B_m(t)}{\sum_{i=n+1}^m \tau B_i(t)} \quad (4)$$

From equation (4), we can also derive the spot swap rate at T_n by setting $t = T_n$ and then $B_n(T_n) = 1$ leads to the equation as below.

$$S_{n,m}(T_n) = \frac{1 - B_m(T_n)}{\sum_{i=n+1}^m \tau B_i(T_n)} \quad (5)$$

Furthermore, from equation (4), $B_n(t) - B_m(t) = S_{n,m}(t) \sum_{i=n+1}^m \tau B_i(t)$, we can write $PayerSwap_{n,m}(t)$ in equation (3) as:

$$PayerSwap_{n,m}(t) = [S_{n,m}(t) - K] \sum_{i=n+1}^m \tau B_i(t) \quad (3')$$

Let $(\Omega, \mathcal{F}, \mathcal{Q}, F_t)$, $t \in [0, T_M]$, be a filtered probability space and W_t be a Wiener process (in case of one dimensional model). Let $C_{n,m}(t) = \sum_{i=n+1}^m \tau B_i(t)$ represent the value of the annuity from T_{n+1} to T_m . Jamshidian (1997) proves that $C_{n,m}(t)$ induces a probability measure $Q^{n,m}$ such that $\frac{B_i(t)}{C_{n,m}(t)}$ for $(n+1 \leq i \leq m)$ are $Q^{n,m}$ martingales. It follows that with the given tenor structure, each forward swap rate $S_{n,m}(t)$ is a martingale under the associated forward swap measure $Q^{n,m}$ and numeraire $C_{n,m}(t)$. It should be noted that $C_{n,m}(t)$ is uniquely determined by the two reset dates T_n and T_m . Furthermore, we assume that $S_{n,m}(t)$ follows a geometric Brownian motion, which leads to the general SDE:

$$dS_{n,m}(t) = S_{n,m}(t) \sigma_{n,m}(t) dW_t^{n,m} \quad \text{for } t \in [0, T_M] \quad (6)$$

Here $W_t^{n,m}$ is a Wiener process under measure $Q^{n,m}$ and $\sigma_{n,m}(t)$ is deterministic. By Ito's formula, we can obtain the solution of the SDE as:

$$S_{n,m}(T_n) = S_{n,m}(t) e^{-\frac{1}{2} \int_t^{T_n} \sigma_{n,m}^2(s) ds + \int_t^{T_n} \sigma_{n,m}(s) dW_s^{n,m}} \quad (7)$$

Now let's recall a European swaption that gives the holder the right to enter into a swap contract on option maturity date which involves cash flows taking place

over period T_{n+1} to T_m ($0 < T_n < T_m$). From equation (3'), the value of a payer swaption at option maturity date T_n is given by:

$$PayerSw_{n,m}(T_n) = [S_{n,m}(T_n) - K]^+ \sum_{i=n+1}^m \tau B_i(T_n) \quad (8)$$

It can be viewed as a call option on forward swap rate $S_{n,m}(t)$. Under measure $Q^{n,m}$ with associated numeraire $C_{n,m}(t) = \sum_{i=n+1}^m \tau B_i(t)$, the value of the payer swaption at time t can be obtained by taking the risk neutral expectation of $[S_{n,m}(t) - K]^+$, then multiply it with the numeraire under the martingale measure. Here we get the classic Black formula for European payer swaption used by practitioners.

$$\begin{aligned} PayerSw_{n,m}(t) &= E^{n,m} \left\{ [S_{n,m}(T_n) - K]^+ \sum_{i=n+1}^m \frac{\tau B_i(T_n)}{C_{n,m}(T_n)} C_{n,m}(T_n) \right\} B_n(t) \\ &= C_{n,m}(t) E^{n,m} \left\{ [S_{n,m}(T_n) - K]^+ \sum_{i=n+1}^m \frac{\tau B_i(T_n)}{C_{n,m}(T_n)} \right\} \\ &= C_{n,m}(t) E^{n,m} \{ [S_{n,m}(T_n) - K]^+ \} \\ &= C_{n,m}(t) [S_{n,m}(t) N(d_1) - K N(d_2)] \end{aligned} \quad (9)$$

where

$$\begin{aligned} d_1 &= \frac{\ln \left(\frac{S_{n,m}(t)}{K} \right) + \frac{1}{2} v_{n,m}^2 (T_n - t)}{v_{n,m} \sqrt{T_n - t}}, \\ d_2 &= d_1 - v_{n,m} \sqrt{T_n - t}, \end{aligned}$$

$N(\cdot)$ is the cumulative distribution function of a standard normal distribution. The Black implied volatility can be expressed as

$$v_{n,m}^2 = \frac{1}{T_n - t} \int_t^{T_n} \sigma_{n,m}^2(s) ds.$$

2.2 Co-terminal Swap Market Model

Given the tenor structure, a co-terminal Swap Market Model refers to a model which assigns the arbitrage free dynamic to a set of forward swap rates that have different swap starting date T_n ($n = 1, \dots, M-1$) but conclude on the same maturity date

T_M . One important feature about the co-terminal swaption is that it is internally consistent with a Bermudan swaption that gives the holder the right to enter into a swap at each reset date during period T_1 to T_{M-1} with T_M being the terminal maturity of the underlying swap. When considering at each tenor date whether or not to exercise the option to enter into a swap contract, the holder need to consider the forward swap rate dynamics from that tenor date till final maturity, which is actually driven by the volatility prevailing at that time. The advantage of co-terminal SMM over other market models in pricing Bermudan swaptions has already been noted and discussed in Jamshidian (1997) and Galluccio et al (2007).

Although most research concentrates on pricing swaption under LMM framework, it has been widely accepted that LMM is the natural model for cap/caplet whereas SMM is the natural model for European swaptions. Galluccio et al (2007) point out that it involves strong assumptions and complicated algorithm to force LMM to re-produce the swaption dynamics. As to co-terminal SMM, the calibration to the market quoted implied volatilities (swaption prices) can be done directly and efficiently since the modelling objectives of the SMM are actually swap rates. Here, our objective is to price and hedge a Bermudan swaption (or a portfolio of European swaptions), the SMM is an ideal candidate for this job.

In Galluccio et al (2007) the co-terminal SMM is defined by introducing a collection of mutually equivalent probability measures and a family of Brownian motions such that for any the forward swap rate satisfies the SDE (6) for all . They also test the calibration and pricing of co-terminal SMM to European swaptions under different measure. Since our goal here is to price and hedge a Bermudan swaption, where early exercise is permitted, it is necessary to fix one measure and derive the drift of different forward swap rates under this measure.

3 Drift of Co-terminal SMM under the Terminal Measure

If we take $C_{M-1,M}(t) = \tau_M B_M(t)$ as the numeraire and use the risk neutral measure $Q^{M-1,M}$ for all co-terminal forward swap rates, then only the last (one period) swap rate $S_{M-1,M}(t)$ is a martingale. As mentioned before, the co-terminal forward swap rates satisfy the following general SDE under the terminal measure:

$$dS_{n,m}(t) = S_{n,m}(t) \sigma_{n,m}(t) dW_t^{M-1,M} + \text{drift} \quad \text{for } n = 1, \dots, M-1 \quad (10)$$

where $W_t^{M-1,M}$ is the Wiener process under the terminal measure $Q^{M-1,M}$.

Joshi and Liesch (2006) recommend using the cross-variation in assessing the impact of numeraire changing on a drift. Let X_t and Y_t be two continuous semimartingales in . From Ito Formula:

$$dX_t Y_t = X_t dY_t + Y_t dX_t + \langle X_t, Y_t \rangle dt$$

where $\langle X_t, Y_t \rangle$ denotes the quadratic co-variation process of X_t and Y_t . Given the SDEs:

$$\begin{aligned} dX_t &= \mu_X(t) dt + \sigma_X(t) dW_t^X \\ dY_t &= \mu_Y(t) dt + \sigma_Y(t) dW_t^Y \end{aligned}$$

then, we have $\langle X_t, Y_t \rangle = \rho_{X,Y} \sigma_X(t) \sigma_Y(t)$ where $\rho_{X,Y}$ is the correlation between W_t^X and W_t^Y . If we substitute X_t with $S_{n,M}(t)$ and Y_t with $S_{m,M}(t)$ with $0 < n < m < M$, the co-variation between two forward swap rates defined in SDE (10) is then $\rho_{n,m} \sigma_{n,M}(t) \sigma_{m,M}(t)$.

In order to derive the dynamic of the ratio of two SDEs, Joshi and Liesch (2006, p.4) first assume A , B and C are three tradable assets. If we take C as numeraire, then under the associated measure, A/C and B/C are martingales while A/B is an Ito process. By Ito formula we have:

$$d\frac{A}{C} = d\left(\frac{A}{B} \frac{B}{C}\right) = \frac{A}{B} d\frac{B}{C} + \frac{B}{C} d\frac{A}{B} + \left\langle \frac{A}{B}, \frac{B}{C} \right\rangle dt \quad (11)$$

from which we obtain

$$d\frac{A}{B} = -\frac{C}{B} \left\langle \frac{A}{B}, \frac{B}{C} \right\rangle dt + \frac{C}{B} d\frac{A}{C} - \frac{C}{B} \frac{A}{B} d\frac{B}{C} \quad (11)$$

Since the last two terms on the RHS are martingales and driftless, the drift of A/B is given by:

$$E \left[d\frac{A}{B} \right] = -\frac{C}{B} \left\langle \frac{A}{B}, \frac{B}{C} \right\rangle dt$$

Now apply the formula to the co-terminal SMM defined under the given tenor structure in this paper. Let $A = B_n(t) - B_m(t)$, $B = C_{n,M}(t) = \sum_{i=n+1}^M \tau B_i(t)$ and $C = C_{M-1,M}(t) = \tau_M B_M(t)$. By equations (4) and (11), the drift of a forward swap rate $S_{n,M}(t)$, $n = 1, \dots, M-1$ under the terminal measure $Q^{M-1,M}$ is given by:

$$\begin{aligned} E^{M-1,M} [dS_{n,M}(t)] &= \mu_{n,M} = -\frac{C_{M-1,M}(t)}{C_{n,M}(t)} \left\langle \frac{B_n(t) - B_m(t)}{C_{n,M}(t)}, \frac{C_{n,M}(t)}{C_{M-1,M}(t)} \right\rangle \\ &= -\frac{C_{M-1,M}(t)}{C_{n,M}(t)} \left\langle S_{n,M}(t), \frac{C_{n,M}(t)}{C_{M-1,M}(t)} \right\rangle \end{aligned} \quad (12)$$

Note that when $n = M-1$, $\mu_{M-1} = 0$ because $\langle S_{n,M}(t), 1 \rangle = 0$.

In Joshi and Liesch (2006), they simplify the covariance term by moving the forward swap rate and volatility out (See Joshi and Liesch (2006), page 6, equation 5.3):

$$\left\langle SR_j, \frac{A_j}{P_n} \right\rangle = SR_j \sum_{k=1}^F a_{jk} \left\langle W_k, \frac{A_j}{P_n} \right\rangle$$

where W_k is independent and a_{jk} is the volatility. We don't move $SR_j \sum_{k=1}^F a_{jk}$ out of the sharp bracket.

In Joshi and Liesch (2006), the $S_{n,M}(t)$ term inside the square brackets is simplified into independent Wiener process W_k ,¹ which explains why their drift term

¹In their paper, they simplify the covariance term by moving the forward swap rate and volatility out (See Joshi and Liesch (2006), page 6, equation 5.3):

$$\left\langle SR_j, \frac{A_j}{P_n} \right\rangle = SR_j \sum_{k=1}^F a_{jk} \left\langle W_k, \frac{A_j}{P_n} \right\rangle$$

where W_k is independent and a_{jk} is the volatility. The equation in this paper doesn't move $SR_j \sum_{k=1}^F a_{jk}$ out.

does not contain the correlation terms $\rho_{n,m}$. Although for one factor model, the correlation terms $\rho_{n,m} = 1$ can be ignored, it cannot be omitted when the dimension of W_k is two or more. Expanding the cross-variation term in equation (12) (as shown in the appendix), the drift becomes a complicated function of numeraires and cross-variation of two forward swap rates. By induction, the general form of drift under terminal measure can also be written as:

$$\mu_{n,M} = \sum_{i=n}^{M-2} \left(\tau_{i+1} \frac{C_{i+1,M}}{C_{M-1,M}} \rho_{n,i+1} S_{n,M} \sigma_{n,M} S_{i+1,M} \sigma_{i+1,M} \prod_{j=n+1}^i (1 + \tau_j S_{j,M}) \right) \quad (12')$$

Details of the derivation are given in appendix. However, the above equation is not an explicit function of $S_{n,M}(t)$ for $n = 1, \dots, M-1$, because the $C_{n,M}$ term is derived from a set of $S_{i,M}(t)$ for $n \leq i \leq M-1$. Detail on the derivation of $C_{n,M}$ is provided in Section 4.1.1. This drift term is complicated in appearance but not time-consuming in computation.

4 Pricing and Calibration of European Swaptions

In this section, the co-terminal SMM under the terminal measure is used to price European swaption. The price of the swaption can be calculated by Black formula given the market quoted Black implied volatility. The co-terminal SMM is naturally fit for a collection of co-terminal European swaptions in a sense that it can capture the term structure of volatilities of forward swap rates prevailing at the given tenor period. As mentioned in Galluccio et al (2007), co-terminal SMM can be easily and effectively calibrated to co-terminal European swaptions. As LMM is not a natural model for swaptions, complicated equations of libor rates have been used to replicate the movement of swap rate. In practice, this may lead to overfitting during calibration. While Galluccio et al (2007) calibrated their model under different forward swap measures to caplets, the co-terminal SMM built here is under (one) terminal measure with time dependent drift terms because we need to accommodate early exercise decision.

4.1 Pricing of Co-terminal European Swaptions

To simulate the forward swap rates within the tenor structure, we need the time T_0 forward swap rate as the original input of the model. The forward rate can be easily derived from a set of zero coupon bonds by equation (4) while the latter can be found from many interest rate derivatives on the market. For example, given a set of time T_0 interest rate swaps, we can calculate the bond price by equation (5). The latter is used in the numerical test of this paper.

4.1.1 Simulation of Forward Swap Rate

Given tenor structure $0 := T_0 < T_1 < T_2 < \dots < T_M$ and taking $C_{M-1,M}(t) = \tau_M B_M(t)$ as the numeraire, under terminal measure $Q^{M-1,M}$, the distribution of the forward swap rate follows the SDE as below:

$$dS_{n,M}(t) = \mu_{n,M}(t) dt + S_{n,M}(t) \sigma_{n,M}(t) dW_t^{M-1,M} \quad \text{for } n = 1, \dots, M-1$$

By Ito formula we can get the solution as:

$$\begin{aligned} S_{n,M}(T_n) &= S_{n,M}(t) e^{\int_t^{T_n} [\mu_{n,M}(s) - \frac{1}{2} \sigma_{n,M}^2(s)] ds + \int_t^{T_n} \sigma_{n,M}(s) dW_s^{M-1,M}} \\ &= S_{n,M}(t) e^{[\mu_{n,M}(t) - \frac{1}{2} \sigma_{n,M}^2(t)] \tau + \sigma_{n,M}(t) \sqrt{\tau} z_t} \end{aligned} \quad (13)$$

where the volatility is periodically deterministic and the drift is derived by equation (12)'. Here z_t for $T_0 \leq t \leq T_n$ are independent random numbers drawn from standard normal distribution $N(0, 1)$.

In each simulation path, only $S_{M-1,M}(T_n)$ is evolved without the drift term. Other co-terminal forward swap rates are driven by drift determined by other forward swap rates of the previous step as well as the volatility terms. That means, each step we move forward, new numeraires need to be recalculated from the evolved and new set of swap rates and new drifts will be re-derived for next step's simulation. This makes the drifts become time dependent. Also note that with each time step forward, we will throw away one forward swap rate that becomes spot swap rate at the next reset date.

In order to compute time dependent bond prices and numeraire at each time step, we need to solve equations defining relative swap rates. At time T_{M-1} , we have

$$S_{M-1,M}(T_{M-1}) = \frac{B_{M-1}(T_{M-1}) - B_M(T_{M-1})}{C_{M-1,M}(T_{M-1})} = \frac{1 - B_M(T_{M-1})}{\tau B_M(T_{M-1})}.$$

It is easy to deduce the bond price using

$$B_M(T_{M-1}) = \frac{1}{1 - \tau S_{M-1,M}(T_{M-1})}.$$

At time T_{M-2} , we have

$$\begin{aligned} S_{M-2,M}(T_{M-2}) &= \frac{B_{M-2}(T_{M-2}) - B_M(T_{M-2})}{C_{M-2,M}(T_{M-2})} = \frac{1 - B_M(T_{M-2})}{\tau [B_{M-1}(T_{M-2}) + B_M(T_{M-2})]}, \\ S_{M-1,M}(T_{M-2}) &= \frac{B_{M-1}(T_{M-2}) - B_M(T_{M-2})}{\tau B_M(T_{M-2})} \end{aligned}$$

With two equations and two unknowns, we can get time T_{M-2} bond prices $B_{M-1}(T_{M-2})$ and $B_M(T_{M-2})$ as well as numeraire $C_{M-2,M}(T_{M-2})$.

Similarly, at each subsequent reset date T_n for $n = M-3, \dots, 1$, we can deduce a collection of forward bond prices and the relative numeraire given the related set of simulated forward swap rates although it requires more labour to solve multi-variable linear equation. In this paper, NAG C function (f07aec) is called to solve the real system of linear equations of different zero coupon bonds. This NAG routine solves a real system of linear equations by forward and backward substitution and the computation has been proved to be quite efficient.

It is worth noting that we can derive bond prices at any reset date only if we have complete set of relevant forward swap rates. Therefore it is not necessary to work backward from time T_{M-1} . The algorithm is actually forward deduction in practice. First, we should use time T_0 swap rates and bond prices derived from time T_0 market quoted interest rate derivatives (may be called initial term structure) to calculate the drift prevailing from time T_0 to T_1 with equation (12)'. Then we can simulate time T_1 forward swap rates based on equation (13). These newly simulated forward swap rates can be decomposed as functions of different zero coupon bonds

Table 1: Simulated Forward Swap Rates

$T_0=0$	T_1	T_2		T_{M-2}	T_{M-1}	T_M
$S_{1,M}(T_0)$	$S_{1,M}(T_1)^*$					
$S_{2,M}(T_0)$	$S_{2,M}(T_1)$	$S_{2,M}(T_2)^*$				
...			
$S_{M-2,M}(T_0)$	$S_{M-2,M}(T_1)$	$S_{M-2,M}(T_2)$...	$S_{M-2,M}(T_{M-2})^*$		
$S_{M-1,M}(T_0)$	$S_{M-1,M}(T_1)$	$S_{M-1,M}(T_2)$...	$S_{M-1,M}(T_{M-2})$	$S_{M-1,M}(T_{M-1})^*$	

and by solving these functions with method described above, we can derive time T_1 bond prices and numeraire. The next step is to calculate drifts for time T_2 again by equation (12)'. Repeat these steps from time T_2 till time T_M . After one simulation, we can get a set of co-terminal forward swap rates as below. The forward swap rates marked with “*” are finished at that time step.

The co-terminal SMM is built using Monte Carlo simulation with 10,000 paths due to the limit on the matrix size that the library regression function can handle.

4.1.2 Calculation of Discounting Factor

In order to discount the cash flow in the case where the option is exercised before maturity, we need to deduce the discounting bond values at the maturity of the European swaption. For simulation, the discounting factor is path dependent. Under the terminal measure, time T_n cash flow should be discounted back to time T_0 by the discounting factor

$$D_{0,n}(T_n) = \frac{B_M(T_0)}{B_M(T_n)} \quad \text{for } n = 2, \dots, M$$

where $B_M(T_0)$ is already known at time T_0 and $B_M(T_n)$ can be derived by method described in Section 4.1.1.

4.1.3 Calculation of Exercise Value at Maturity

Given the simulated forward swap rate and relative numeraire, it is trivial to calculate the exercise value at swaption maturity date T_n where “ n ” represents dif-

ferent type of European swaption. For a payer swaption $PayerSw_{n,m}(t)$ for $n = 1, \dots, M - 1$, if we exercise the option at maturity, the exercise value (discounted cash flow) is $[S_{n,M}(T_n) - K]^+ C_{n,M}(T_n)$ as equation (8) indicates.

4.1.4 Swaption Value

With N paths simulation and corresponding discounted exercise values, the swaption value at time T_0 can be calculated by summing them up and dividing by simulation number N according to law of large numbers. For a European payer swaption under the terminal measure, after simulating N possible paths for $S_{n,M}(T_n)$, the swaption value is then calculated as

$$PayerSw_{n,m}(t) = \frac{1}{N} \sum_{j=1}^N \left[S_{n,M}^j(T_n) - K \right]^+ C_{n,M}^j(T_n) D_{0,M}^j(T_n)$$

4.2 Calibration of Co-terminal SMM

Brigo, Mercurio and Morini(2003), Rebonato(2003), and other literatures attribute a lot to the parameterisation and calibration of *LMM*. Based on their research, Galluccio et al (2007) test calibration of co-terminal *SMM* using similar methodology as the one advocated by Rebonato (2003) in the case of *LMM*. By calibrating to swaption and caplet ATM volatilities, they finally get different set of parameters $\{\phi_j, a_j, b_j, c_j, d_j\}$ where $j = 1, \dots, M - 1$ for $M - 1$ number of co-terminal swaptions under different forward measure. Since the model built in this paper is under one measure and the volatility of different co-terminal swap rate will affect the drift terms in a complicated way, it is unrealistic to use different parameter set for different co-terminal swaption. Therefore, a similar parameter formula is followed here, but four common parameters will be allocated to volatilities of all co-terminal swaptions.

In order to capture the term structure of swaption volatilities which are periodically deterministic, the linear-exponential formulation proposed in Brigo, Mercurio and Morini (2003) is adopted here as swaption volatility function. Further more

the swaption vol is formulated as

$$\begin{aligned}\sigma_{n,m}(t) &= \phi_{n,M}\psi(T_n - t, a, b, c, d) \\ \psi(T_n - t, a, b, c, d) &= [a(T_n - t) + b]e^{-c(T_n - t)} + d\end{aligned}\quad (14)$$

The term ψ can preserve the well-known humped shape of market quoted Black implied volatility.

The method adopted in this paper is to calibrate parameter a , b , c , and d to the implied volatilities of a set of co-terminal swaptions with different maturities but associated with same length of swaps. The parametric form is as below

$$v(t, M) = \psi(t, a, b, c, d) = [at + b]e^{-ct} + d \text{ for } T_0 < t < T_M$$

where $v(t, M)$ is the market implied volatility, and t denotes the maturity of swaption. Because of its simple log-linear character, calibration algorithm using this parametric formula is quite fast and robust.

5 Pricing of Bermudan Swaption

After the calibration to co-terminal swaptions, we obtain a set of parameters $\{\phi_{n,M}, a, b, c, d\}$ that can be used to re-produce the term structure of volatilities of different forward swap rates during the given tenor period. The problem now lies on how to decide the optimal exercise boundary and get the value of the Bermudan swaption. As mentioned before, under the given tenor structure $0 := T_0 < T_1 < T_2 < \dots < T_M$, the Bermudan swaption is exercisable from time T_1 to time T_{M-1} with T_M being the terminal maturity of the underlying swap.

5.1 Deriving the Discount Factor $D_{n,n+1}(T_n)$

The procedure of simulation is exactly the same as that in the pricing of European swaptions. But it should be noted that, here, we need one step forward discounting factor to discount cash flows between two reset dates instead of the one used to discount cash flow from maturity directly back to the value date since we want to

calculate and compare the immediate exercise value and holding value of the option. Under the terminal measure, the discounting factor is given by

$$D_{n,n+1}(T_n) = \frac{B_M(T_n)}{B_M(T_{n+1})} \text{ for } n = 1, \dots, M-1$$

Note that $B_M(T_n)$ and $B_M(T_{n+1})$ are derived at time T_n and T_{n+1} respectively by method described before. At the swaption maturity date T_{M-1} , $D_{M-1,M}(T_{M-1})$ is exactly $B_M(T_{M-1})$. For each simulation path, we need to derive the discounting factor for any two adjacent exercise dates.

5.2 Least Square Approach

After generating N paths co-terminal forward swap rates on every reset date that corresponds to an exercise point of the Bermudan swaption, we can use the least squares method to determine the expected continuation value of the option. Least squares approach, firstly proposed in Longstaff and Schwartz (2001, P.114) and also called Longstaff and Schwartz method, is a method used to estimate the conditional expectation of the payoff of holding the option from cross-sectional information in the simulation. The key step of this method is to run a regression of future realised option payoffs as a function of the underlying assets values.

At each exercise point T_n for $n = 1, \dots, M-1$, holder of the swaption needs to know the exercise value and continuous value in order to decide whether to exercise the option or not. That is, at one particular exercise date, if the immediate exercising payoff $[S_{n,M}(T_n) - K]^+ C_{n,M}(T_n)$ is larger than the expected payoff from holding the swaption $E_{T_n} \{ [S_{n,M}(T_n) - K]^+ C_{n,M}(T_n) \}$, one will exercise the option and get the cash flow at that date. Just as other ways to price Bermudan option, we work backward to decide the holding value of the swaption.

In view of simplicity and computation efficiency, I use quadratic regression formula

$$y^j = F(x^j) = a_0 + a_1 x^j + a_2 (x^j)^2 \quad (1)$$

and do regression through NAG library routine. The procedure of pricing Bermudan

(payer) swaption using least square method is stated as follows.

- (i) At the last exercise/reset date , we need to record the cash flows of N different paths , which is exactly the exercise values at maturity.
- (ii) Moving one step backward, we should also calculate the time T_{M-2} exercise values by equation (8) for all paths. Since regression will be done only on in-the-money (ITM) paths, it is necessary to pick out those paths where the time T_{M-2} exercise value is larger than zero. Assume that the number of in-the-money items is N^* , then for N^* number of selected paths, we can make x^j for $j = 1, \dots, N^*$ equals to the corresponding simulated forward swap rates $S_{M-2,M}^j(T_{M-2})$. Meanwhile, let y^j be the continuation value which is actually the discounted time T_{M-1} cash flow given by

$$\left[S_{M-1,M}^j(T_{M-1}) - K \right]^+ C_{M-1,M}^j(T_{M-1}) D_{M-2,M}^j(T_{M-2}) \text{ for } j = 1, \dots, N^*$$

- (iii) For all ITM paths, we can now perform the regression x^j of on y^j for $j = 1, \dots, N^*$ and get value of parameter a_0, a_1, a_2 . With these parameters, the time T_{M-2} continuation values of each ITM path can be recalculated by the same quadratic formula (1). In the numerical test, NAG C routine (g02dac) is employed to perform a general multiple linear regression using least squares estimation.
- (iv) For all N paths, revalue the new cash flow at time T_{M-2} of each path. Let it be the time T_{M-2} exercise value if the later is larger than the recalculated time T_{M-2} continuation value. Otherwise, we should discount time T_{M-1} cash flow back to time T_{M-2} and make it the time T_{M-2} cash flow.
- (v) Keep moving backward step by step until time T_1 . On each exercise date, repeat procedure (i) to (iv). Then we can get the exercise point and discounted cash flows of every simulation path.

- (vi) The Bermudan swaption value can be obtained if we sum up time T_1 cash flows of N paths discounted to time T_0 by $D_{0,1}^j(T_0)$ for $j = 1, \dots, N$, and divide it by number of simulation N .

One may have noticed that vectors with at least two dimensions are required here to record the relative information for different path at every reset date. This may cause memory problem when the simulation number become very large. Besides, the vector size that the regression function can handle will also set constraint on the number of simulation path.

6 PCA and Hedging of Bermudan Swaption

Here, we assume that a 10×1 Bermudan swaption at time T_0 and want to hedge it with some interest rate derivatives. The hedged portfolio will be unwound on next exercise date T_1 , which is one year later. By analysing the P&L (profit and loss) of the hedged portfolio on the unwind date, we can check the hedge performance of the interest rate models. A good model should, on average, produce a hedged portfolio P&L close to zero. First of all, we need to decide the hedge instruments and calculate corresponding hedge ratios. Since the underlying assets of swaptions are swaps, it seems logical to hedge the Bermudan swaption using a set of swaps. One could also use a set of zero coupon bond which is the basic elements of swap. However, swap has zero value at inception, whereas zero coupon bond as a hedge instrument will produce a lot of extra cash flows which may not be desirable. Hence, in this paper, swaps are chosen to hedge the 10×1 Bermudan swaption.

6.1 PCA on Change of Forward Rates

When calculating the hedge ratio, it would be inexpedient to take into account the change of only one single forward swap rate and compare it with the change of the swaption value because the forward swap rates are related to each other and they don't moving independently. In other words, if the yield curve at a particular time

is changed, all forward swap rates will be affected. To overcome this problem, we derive the hedge ratios using Principle Component Analysis (PCA) on the historical change of forward rates. Since the swap rate is a function of forward rates, making PCA analysis directly on the forward rates should be more efficient. Yallup (2006) shows that immunization scheme based on forward curve, compared with other types of yield curves, produced the best results.

To avoid complications due to time series dynamics of interest rate factors measured at different time frequencies, it is important to conduct the PCA analysis (for hedging purpose) on forward rates with data frequency the same as the unwind interval of the hedged portfolio. Hence, PCA was carried out using forward rates measured at annual interval and derived from zero yields, forward LIBOR and forward swap rates under the given tenor structure. Let L_t^j denote the j -th forward libor rate at time t and

$$\Delta L_t^j = L_{t+\tau}^j - L_t^j, \quad j = 1, \dots, J$$

where τ is the data frequency, which in this case is one year. Since we use 11 forward rates as input, $j = 1, \dots, 11$.

The outcome of the PCA analysis is a set of PCA scores, a_k^j , from the following multivariate relationship

$$\Delta L_t^j = \alpha^j + a_1^j P_{1,t} + a_2^j P_{2,t} + \dots + a_m^j P_{m,t} + \varepsilon_t$$

$P_{k,t}$ for $k = 1, \dots, m$ are m number of PCA factors. By theoretical construct $m \leq J$. By changing these PCA factors, we know in which direction and by how much will each of the forward rates move wrt to changes in the PCA factor(s).

In practice, three components were reported to drive the interest rate curves. The first component corresponds to parallel shifts, where a shock has an uniform impact on the whole curve. The second factor affects the slope of the interest rate curve causing the short and the long rates to move in opposite direction. The third factor affects the curvature, where short and long rates move in the same direction whereas intermediate maturity rates move in the opposite direction. Hence, the

following relationship is used in the hedging exercise (omitting the j superscript for the time being)

$$\begin{aligned} L_{t+\tau} - L_t &= \alpha + a_1 P_{1,t} + a_2 P_{2,t} + a_3 P_{3,t} + \varepsilon_t \\ \widehat{L}_{t+\tau} &= L_t + \alpha + \alpha + a_1 P_{1,t} + a_2 P_{2,t} + a_3 P_{3,t}. \end{aligned}$$

If we increase each of the three PCA components by a ΔP_k amount, this will increase the forecast $\widehat{L}_{t+\tau}$ by $\Delta \widehat{L}_{t+\tau}$:

$$\Delta \widehat{L}_{t+\tau} = \sum_{k=1}^3 a_k \Delta P_{k,t}.$$

We introduce an upward and a downward shift, ΔP_k^\pm , of each of the three PCA factors. The amount of factor move could have been based on (i) the mean absolute change in the history (ii) standard deviation of changes in the history or (iii) ARIMA forecast of using the entire PCA history. Since, we do not have a very long time series of annual forward rates, method (iii) is ruled out. In this paper, we use the mean absolute change of PCA factor to produce ΔP_k^\pm . Once the change in the PCA factor is determined, the corresponding change in the forward rate can be derived from equation (DeltaL) above. The same process of bumping the forward rate is repeated for each forward rate on the forward curve.

6.2 Calculation of Hedge Ratio

After bumping the forward rate curve by PCA factors, we can recalculate new values of Bermudan swaption and swaps at the time of hedging. Here the bumping behaviour includes adding ΔP_k to the PCA factor (bumping up) and subtracting ΔP_k from the PCA factor (bumping down). The delta ratios of these derivatives with respect to PCA factors can be derived by equations as below:

$$\begin{aligned} \Delta_k^{BerSwn} &= \frac{\partial BerSwn(T_0)}{\partial P_k} = \frac{[BerSwn(T_0)]_k^+ - [BerSwn(T_0)]_k^-}{P_k^+ - P_k^-} \\ \Delta_k^{Swap_{0,m}} &= \frac{\partial Swap_{0,m}(T_0)}{\partial P_k} = \frac{[Swap_{0,m}(T_0)]_k^+ - [Swap_{0,m}(T_0)]_k^-}{P_k^+ - P_k^-} \end{aligned} \quad (15)$$

Here the value of Bermudan swaption is given by the co-terminal SMM while values of hedging swaps are calculated by equation (3)'. It should be noted that the strike rate of Bermudan ATM swaption is the relevant time T_0 market observable forward swap rate no matter how the forward curve is bumped. Similarly, the strike rate of the swap is exactly the relevant time T_0 spot swap rate. Therefore, without bumping the curve, the value of swap is zero at T_0 under no arbitrage condition.

Here, we construct our hedging portfolio with 1-year, 5-year and 11-year swaps with hedge ratios, x_t , y_t and z_t . These hedge ratios are derived by minimizing the amount of delta mismatch (w.r.t. the first three PCA factors). So we want to optimize the following

$$\min_{x_t, y_t, z_t} \sum_{k=1}^3 w_k \varepsilon_{t,k}^2 \quad (2)$$

where

$$\varepsilon_{t,k} = \Delta_k^{BerSwan} - x_t \Delta_k^{Swap0,1} - y_t \Delta_k^{Swap0,5} - z_t \Delta_k^{Swap0,11}$$

is the amount of delta mismatch with respect to PCA factor k , and w_k is the weight associated with each PCA factor. It is indisputable that the first factor will be given the largest weight considering the much larger proportion it contributes to explaining the forward rate curve movements. Simplex algorithm from NAG C library is used to find the optimal solution of x_t , y_t and z_t so that equation(2) is satisfied.

$$\begin{aligned} F(\varepsilon) &= \sum_{k=1}^m w_k \varepsilon_k^2 \\ \varepsilon_k^2 &= \Delta_k^{BerSwan} + \sum x_n \Delta_k^{Swap_{n,M}} \text{ for } 1 < n < M \end{aligned} \quad (16)$$

6.3 Profit & Loss of Hedge Portfolio

On the unwind date T_1 , hedge performance of the model can be checked by calculating the profit and loss of the hedged portfolio. After interval τ , the maturity of the Bermudan swaption becomes one period shorter but the strike rate will remain unchanged. It is necessary to recalibrate the co-terminal SMM to the market implied volatilities at time T_1 under new tenor structure $0 := T_1 < T_2 < \dots < T_M$, so

that we can use the T_1 -calibrated model to price the Bermudan swaption (now 10 years to maturity). Meanwhile, the length of hedge instruments will also become one period shorter. Their prices are calculated using the yield curves prevailing at time T_1 and the same strike rate as time T_0 . P&L at time T_1 can be calculated as follows

$$P\&L(T_1) = \frac{BerSw_n(T_0)}{B_1(T_0)} - BerSw_n(T_1) - \sum x_n Swap_{n-1, M-1}(T_1) \quad (17)$$

7 Data and Empirical Study

In this paper, the numerical test is carried out on both Euro and USD markets. The tenor structure is set to be $0 := T_0 < T_1 < T_2 < \dots < T_{11}$ with interval τ equal to one year. Then there are ten reset/exercise dates between now (T_0) and final swap maturity date (T_{11}). A set of zero curves (zero yields) with length ranging from one year to eleven years are used in the test. In calibration, the zero curves of different calibrating dates are used as inputs of the model. When carrying out the PCA on change of forward rates, monthly zero yields are used to derive the forward LIBOR rates of each month. Model calibration will be conducted at the end of each month from February 2005 to September 2006. So there are nineteen calibration dates and corresponding parameter sets. The matrix of market implied volatility of ATM swaptions includes 1-year forward 10-year swaption to 10-year forward 1-year swaption, which are also observed on the above nineteen dates.

7.1 Data

The data comprises annual forward rates and implied volatility of at-the-money European swaptions from the Euro and USD interest rate markets. All data for this study is downloaded from Datastream. We have monthly data (last business day of each month) from 28 February 2005 till 30 September 2007 for Euro and USD payer swaptions. We also downloaded the corresponding zero-yield curve to derive the forward Libor rates spanning from 1 year to 10 year.

Table 2: Example of Implied Volatilities of ATM swaptions on 2006-4-28 (Euro Market)

	1	2	3	4	5	6	7	8	9	10
1	0.167	0.166	0.165	0.162	0.158	0.160	0.145	0.171	0.137	0.134
2	0.165	0.163	0.162	0.158	0.153	0.147	0.142	0.138	0.135	0.132
3	0.164	0.160	0.158	0.153	0.148	0.142	0.137	0.133	0.130	0.127
4	0.161	0.157	0.152	0.149	0.144	0.138	0.133	0.130	0.127	0.124
5	0.157	0.152	0.148	0.145	0.140	0.135	0.131	0.128	0.125	0.122
6	0.153	0.149	0.145	0.141	0.137	0.132	0.129	0.125	0.122	0.120
7	0.149	0.145	0.142	0.138	0.134	0.130	0.127	0.124	0.121	0.118
8	0.146	0.142	0.139	0.136	0.132	0.129	0.125	0.122	0.120	0.117
9	0.144	0.140	0.136	0.134	0.131	0.127	0.124	0.121	0.119	0.117
10	0.141	0.137	0.134	0.132	0.129	0.126	0.123	0.121	0.118	0.116

7.2 The Interpolation for the Swaption Matrix

As mentioned before, the co-terminal SMM will be calibrated to the market price that requires the Black implied volatilities as inputs. Table (2) gives an example of the implied volatilities matrix of swaptions.

In the table, the first row indicates the maturity of the swaption, while the first column is the index of the length of swaps involved with the swaption. We can see that several volatilities from column of 6-year forward, 8-year forward and 9-year forward swap are written in bold letters because these data are unavailable on the market and have been interpolated using the adjacent market data on the same row. The following log-linear (or power) functional parametric formula for interpolation is recommended in Brigo and Morini (2005, P.24)

$$Y = \beta_1 X^{-\beta_2}$$

where X is the maturity of the swaption and Y is the volatility of swaption. Since the calibration target is a set of co-terminal European payer swaptions (the diagonal cells), we care only about the upper triangular part of the matrix.

It is noticed that in the Euro market, Black implied volatilities of 6-year forward 1-year swaption and 8-year forward 1-year swaption are fixed at abnormally

higher level in comparison with other market quotations from Jan 2006 to Jul 2006. Therefore they are not selected in interpolation and calibration during that period. From experience in the numerical test, the pricing results will be affected by the way we interpolate the swaption's implied volatilities to a large extent, especially in some abnormal market conditions.

7.3 Calibration of Co-terminal SMM

As mentioned in Section IV, zero curves and market implied volatilities are used as input of the model in calibration. Given a collection of zero curves, we can conveniently calculate the price of zero coupon bond by formula:

$$B_n(T_0) = \frac{1}{(1 + y_n)^n} \quad \text{for } n = 1, \dots, 11$$

where y_n is the zero yield for n year(s). Then the strike rate and the time T_0 forward swap rate can be derived by equation (4).

The first step is to calibrate parameter a , b , c , and d to implied volatility of t year forward and $(M - t)$ -year forward swaption. Here t denotes the swaption's maturity time and varies from 1 year to 10 years. In other words, we want to find a set of $\{a, b, c, d\}$ values that can replicate the term structure of implied volatilities of the co-terminal swaption as shown below

$$v_{t,M} = \psi(t, a, b, c, d) = [at + b] e^{-ct} + d \text{ for } t = 1, \dots, 10 \quad (18)$$

Details of the procedure have been discussed in Section 4. NAG C function (e04unc) is employed here to do the calibration by minimizing an arbitrary smooth sum of squares function subject to constraints using a sequential quadratic programming (SQP) method.

According to experience in calibrating, sometimes the library routine may give combination of parameters whose values have no economic meaning. For example, it can fit the function ideally under given tolerance of error where parameter $a = -100$ and $c = 200$. But these values have no realistic meaning. To avoid such result and

Table 3: Initial settings for parameters a, b, c, d.

	a	b	c	d
Initial estimate	0.04	0.06	0.5	0.1
Upper bound	1	1	1	1
Lower bound	0	-1	0	0

to improve the computational efficiency, we need to set initial values and boundaries for these parameters. Actually, these initial settings play an important role on the performance of the library routine. Taking into account both the parameter range given by FINCAD for their LMM calibration functions and distinctiveness of the SMM, the initial constraints of the parameter set are chosen for both markets as shown in Table 3.

For the purpose of hedging test where we need to recalculate the value of 9*1 Bermudan swaption (maturity is 9 years), it is necessary to do the calibration on the unwind date again so that the SMM can match the term structure of the set of European payer swaption with common maturity date T_{10} . Here, parameter set $\{a, b, c, d\}$ is calibrated to volatility of a collection of European payer swaptions with common maturity date T_{10} .

Table 5 and Figure 2 present the parameter values from calibration. The parameters are quite stable through time.

7.4 Bermudan Pricing Results

We compare the price calculated from one-factor SMM with those from one-factor LMM. The results are summarized in Table 6 and Table 7 and Figure 3. The first part, we are pricing a 11-year Bermudan swaption whereas in the second part we are pricing the same set of Bermudan swaptions one year later when the option maturity has reduced to 10 year. As we could see from the table, given one factor LMM and SMM model, the Bermudan swaption prices are very close to each other.

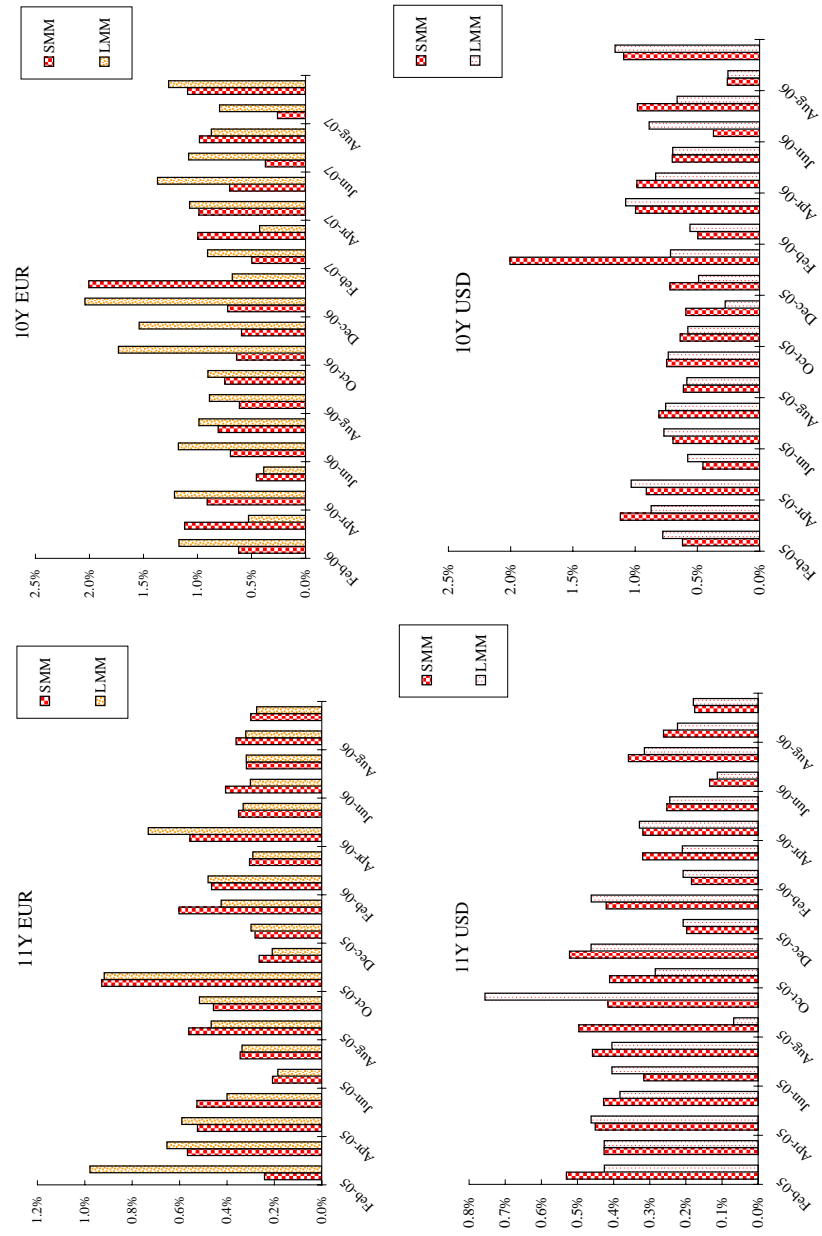
Table 4: Root mean square error to European co-terminal swaptions (11Y and 10Y) in EUR and USD markets from February

2005 to September 2007

Date	EUR		11Y		USD		11Y		Date	EUR		10Y		USD		10Y	
	SMM	LMM	SMM	LMM	SMM	LMM	SMM	LMM		SMM	LMM	SMM	LMM	SMM	LMM	SMM	LMM
2005-2-28	0.2%	1.0%	0.5%	0.4%	0.4%				2006-2-28	0.6%	1.2%	0.6%	0.8%	0.6%	0.8%		
2005-3-31	0.6%	0.7%	0.4%	0.4%					2006-3-31	1.1%	0.5%	1.1%	0.9%	1.1%	0.9%		
2005-4-29	0.5%	0.6%	0.5%	0.5%					2006-4-28	0.9%	1.2%	0.9%	1.0%	0.9%	1.0%		
2005-5-31	0.5%	0.4%	0.4%	0.4%					2006-5-31	0.5%	0.4%	0.5%	0.6%	0.5%	0.6%		
2005-6-30	0.2%	0.2%	0.3%	0.4%					2006-6-30	0.7%	1.2%	0.7%	0.8%	0.7%	0.8%		
2005-7-29	0.3%	0.3%	0.5%	0.4%					2006-7-31	0.8%	1.0%	0.8%	0.8%	0.8%	0.8%		
2005-8-31	0.6%	0.5%	0.5%	0.1%					2006-8-31	0.6%	0.9%	0.6%	0.6%	0.6%	0.6%		
2005-9-30	0.5%	0.5%	0.4%	0.8%					2006-9-29	0.7%	0.9%	0.7%	0.7%	0.7%	0.7%		
2005-10-31	0.9%	0.9%	0.4%	0.3%					2006-10-31	0.6%	1.7%	0.6%	0.6%	0.6%	0.6%		
2005-11-30	0.3%	0.2%	0.5%	0.5%					2006-11-30	0.6%	1.5%	0.6%	0.3%	0.6%	0.3%		
2005-12-30	0.3%	0.3%	0.2%	0.2%					2006-12-29	0.7%	2.0%	0.7%	0.5%	0.7%	0.5%		
2006-1-31	0.6%	0.4%	0.4%	0.5%					2007-1-30	2.0%	0.7%	2.0%	0.7%	2.0%	0.7%		
2006-2-28	0.5%	0.5%	0.2%	0.2%					2007-2-28	0.5%	0.9%	0.5%	0.6%	0.5%	0.6%		
2006-3-31	0.3%	0.3%	0.3%	0.2%					2007-3-31	1.0%	0.4%	1.0%	1.1%	1.0%	1.1%		
2006-4-28	0.6%	0.7%	0.3%	0.3%					2007-4-30	1.0%	1.1%	1.0%	0.8%	1.0%	0.8%		
2006-5-31	0.4%	0.3%	0.3%	0.2%					2007-5-31	0.7%	1.4%	0.7%	0.7%	0.7%	0.7%		
2006-6-30	0.4%	0.3%	0.1%	0.1%					2007-6-30	0.4%	1.1%	0.4%	0.9%	0.4%	0.9%		
2006-7-31	0.3%	0.3%	0.4%	0.3%					2007-7-29	1.0%	0.9%	1.0%	0.7%	1.0%	0.7%		
2006-8-31	0.4%	0.3%	0.3%	0.2%					2007-8-31	0.3%	0.8%	0.3%	0.3%	0.3%	0.3%		
2006-9-29	0.3%	0.3%	0.2%	0.2%					2007-9-28	1.1%	1.3%	1.1%	1.2%	1.1%	1.2%		
Total	8.6%	9.0%	7.1%	6.6%					Total	15.8%	21.0%	15.8%	14.3%	15.8%	14.3%		

Note: SMM stands for Swap Market Model and LMM stands for Libor Market Model. '11Y' denotes calibration results for 11Y co-terminal ATM European swaptions from February 2005 to September 2006; '10Y' denotes calibration results for 10Y co-terminal European swaptions from Feb 2006 to Sep 2007. 'EUR' denotes Euro market and 'USD' denotes US-dollar market.

Figure 1: Root mean square error of prices by date from calibration to 11Y co-terminal European swaptions over the period of February 2005 to September 2006



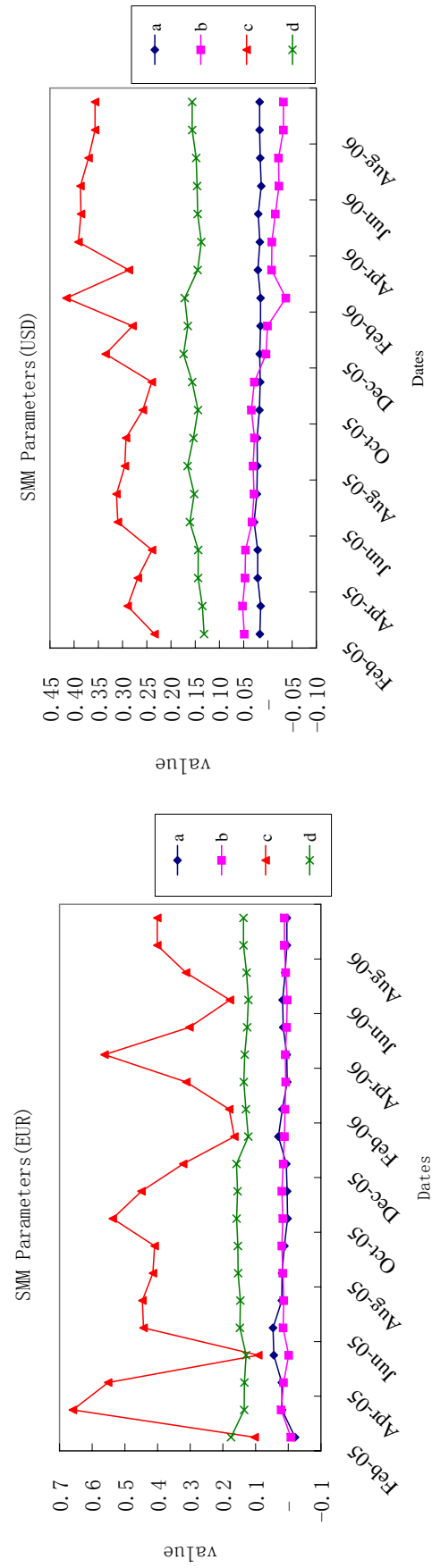
Note: SMM stands for Swap Market Model and LMM stands for Libor market Model.

Table 5: Model calibrated parameter values for EUR and USD market from February 2005 to September 2006

Date	SMM (EUR)				Date	SMM (USD)			
	a	b	c	d		a	b	c	d
2005-2-28	-0.0208	-0.0082	0.1031	0.1753	2005-2-28	0.0166	0.0488	0.2342	0.1317
2005-3-31	0.0188	0.0215	0.6595	0.1351	2005-3-31	0.0149	0.0524	0.2897	0.1354
2005-4-29	0.0197	0.0143	0.5514	0.1343	2005-4-29	0.0211	0.0470	0.2687	0.1440
2005-5-31	0.0441	-0.0016	0.0919	0.1288	2005-5-31	0.0211	0.0461	0.2394	0.1438
2005-6-30	0.0465	0.0158	0.4431	0.1480	2005-6-30	0.0284	0.0326	0.3098	0.1610
2005-7-29	0.0196	0.0136	0.4466	0.1466	2005-7-29	0.0228	0.0286	0.3125	0.1521
2005-8-31	0.0190	0.0164	0.4144	0.1533	2005-8-31	0.0219	0.0306	0.2954	0.1659
2005-9-30	0.0117	0.0196	0.4089	0.1542	2005-9-30	0.0223	0.0277	0.2931	0.1537
2005-10-31	0.0019	0.0164	0.5374	0.1580	2005-10-31	0.0174	0.0340	0.2581	0.1442
2005-11-30	0.0032	0.0197	0.4492	0.1552	2005-11-30	0.0158	0.0278	0.2398	0.1563
2005-12-30	0.0060	0.0153	0.3217	0.1585	2005-12-30	0.0169	0.0039	0.3350	0.1740
2006-1-31	0.0300	0.0114	0.1652	0.1221	2006-1-31	0.0151	0.0007	0.2796	0.1654
2006-2-28	0.0181	0.0096	0.1807	0.1296	2006-2-28	0.0152	-0.0372	0.4159	0.1716
2006-3-31	0.0030	0.0075	0.3124	0.1360	2006-3-31	0.0206	-0.0077	0.2870	0.1450
2006-4-28	0.0045	0.0089	0.5624	0.1332	2006-4-28	0.0166	-0.0082	0.3910	0.1374
2006-5-31	0.0152	0.0049	0.3026	0.1259	2006-5-31	0.0199	-0.0154	0.3859	0.1449
2006-6-30	0.0171	0.0030	0.1787	0.1218	2006-6-30	0.0137	-0.0230	0.3872	0.1463
2006-7-31	0.0094	0.0075	0.3131	0.1274	2006-7-31	0.0160	-0.0220	0.3703	0.1479
2006-8-31	0.0041	0.0118	0.4010	0.1371	2006-8-31	0.0171	-0.0322	0.3567	0.1564
2006-9-30	0.0041	0.0118	0.4010	0.1371	2006-9-30	0.0171	-0.0322	0.3567	0.1564

Note: SMM stands for Swap Market Model .These are the calibrated parameters for 11Y co-terminal ATM European swaptions from February 2005 to September 2006; 'EUR' denotes Euro market and 'USD' denotes US-dollar market.

Figure 2: Model calibrated parameter values for EUR and USD market from February 2005 to September 2006



Note: SMM stands for Swap Market Mode. a,b,c,d in SMM are the parameters of the volatility functional form as suggested by Rebonato 1999. We calibrated to co-terminal 11Y European swaptions from February 2005 to September 2006. All calibration are performed with both EUR and USD market.

7.5 PCA on Change of Forward LIBOR Rates

A set of monthly zero yields ranging from Feb 1999 to Jul 2007 is used here to derive the relative zero coupon bond prices. Then a set of forward LIBOR rates $L(T_0, T_n, T_{n+1})$ for $n = 0, \dots, 10$ for each month can be calculated by equation (2). The PCA was carried out on the time series of the yearly change of these forward LIBOR rates. Here, τ is equal to 12 months. From PCA results, it is confirmed that three factors can describe 93% of the pattern of $\Delta L^j(T_0, T_n, T_{n+1})$ on the Euro market with factor one (P_1) accounting for 69.2%, factor two (P_2) accounting for 16.2% and factor three (P_3) accounting for 7.6% and the corresponding data from USD market are 94.2%, 79.6%, 8.7% and 5.9%.

Table 8 and 9 and Figure 4 and 5 display the loadings of the first three factors. We have used absolute of mean of scores for computing the shocks. We could have used mean of scores or standard deviation of scores or some other statistical criteria. Litterman and Scheikman (1991) used one standard deviation to compute the shocks. The values show that for our data set the standard deviations are quite large and can estimate rather unrealistic shocks, thereby we used mean absolute of scores for estimating the forward rate bumps.

Since the standard deviations for three factors are all larger than the absolute means, the later is chosen as ΔP_k for $k = 1, 2, 3$ to bump the LIBOR curve. The ARIMA prediction approach is not used here and is left for future research because the time series used for PCA has covered those hedging dates. After getting the bumped forward LIBOR curves by methods discussed in section VI, we can transfer it into new set of forward swap rates through the medium of zero coupon bonds by equation (2) and (4).

Table 6: 11Y and 10Y Bermudan swaption prices in EUR market from February 2005 to September 2007.

DATE(11Y)	SMM	LMM	AVERAGE
2005-2-28	0.0463	0.0459623	0.0462
2005-3-31	0.0479	0.0479900	0.0479
2005-4-29	0.0481	0.0484811	0.0483
2005-5-31	0.0499	0.0500876	0.0500
2005-6-30	0.0534	0.0549168	0.0541
2005-7-29	0.0484	0.0478462	0.0481
2005-8-31	0.0483	0.0474547	0.0479
2005-9-30	0.0458	0.0464061	0.0461
2005-10-31	0.0450	0.0452753	0.0451
2005-11-30	0.0444	0.0447728	0.0446
2005-12-30	0.0415	0.0414789	0.0415
2006-1-31	0.0395	0.0393072	0.0394
2006-2-28	0.0379	0.0376263	0.0378
2006-3-31	0.0375	0.0373956	0.0375
2006-4-28	0.0386	0.0389507	0.0388
2006-5-31	0.0397	0.0403900	0.0401
2006-6-30	0.0384	0.0387323	0.0386
2006-7-31	0.0382	0.0384757	0.0384
2006-8-31	0.0384	0.0385994	0.0385
2006-9-29	0.0379	0.0381707	0.0380

DATE(10Y)	SMM	LMM	Strike
2006-2-28	0.0273	0.0271	0.0402
2006-3-31	0.0365	0.0364	0.0395
2006-4-28	0.0519	0.0521	0.0377
2006-5-31	0.0591	0.0592	0.0363
2006-6-30	0.0722	0.0723	0.0344
2006-7-31	0.0585	0.0586	0.0356
2006-8-31	0.0575	0.0576	0.0339
2006-9-29	0.0519	0.0522	0.0344
2006-10-31	0.0399	0.0397	0.0369
2006-11-30	0.0371	0.0369	0.0371
2006-12-29	0.0540	0.0537	0.0357
2007-1-31	0.0504	0.0502	0.0377
2007-2-28	0.0423	0.0422	0.0378
2007-3-30	0.0328	0.0329	0.0409
2007-4-30	0.0293	0.0290	0.0431
2007-5-31	0.0370	0.0369	0.0433
2007-6-29	0.0464	0.0466	0.0442
2007-7-31	0.0473	0.0474	0.0428
2007-8-31	0.0502	0.0504	0.0408
2007-9-28	0.0573	0.0576	0.0401

Note: SMM stands for Swap Market Model and LMM stands for Libor Market Model. 11-y Bermudan swaption are priced as at-the-money from February 2005 to September 2006; 10 year Bermudan swaptions are priced with the corresponding 11-y ATM strikes one year ago. For example, on February 2006, the strike rate for 10-y Bermudan swaption is 0.0402 which is the ATM strike rate for 11-y Bermudan swaption on February 2005. 10-y Bermudan swaption are priced from February 2006 to September 2007.

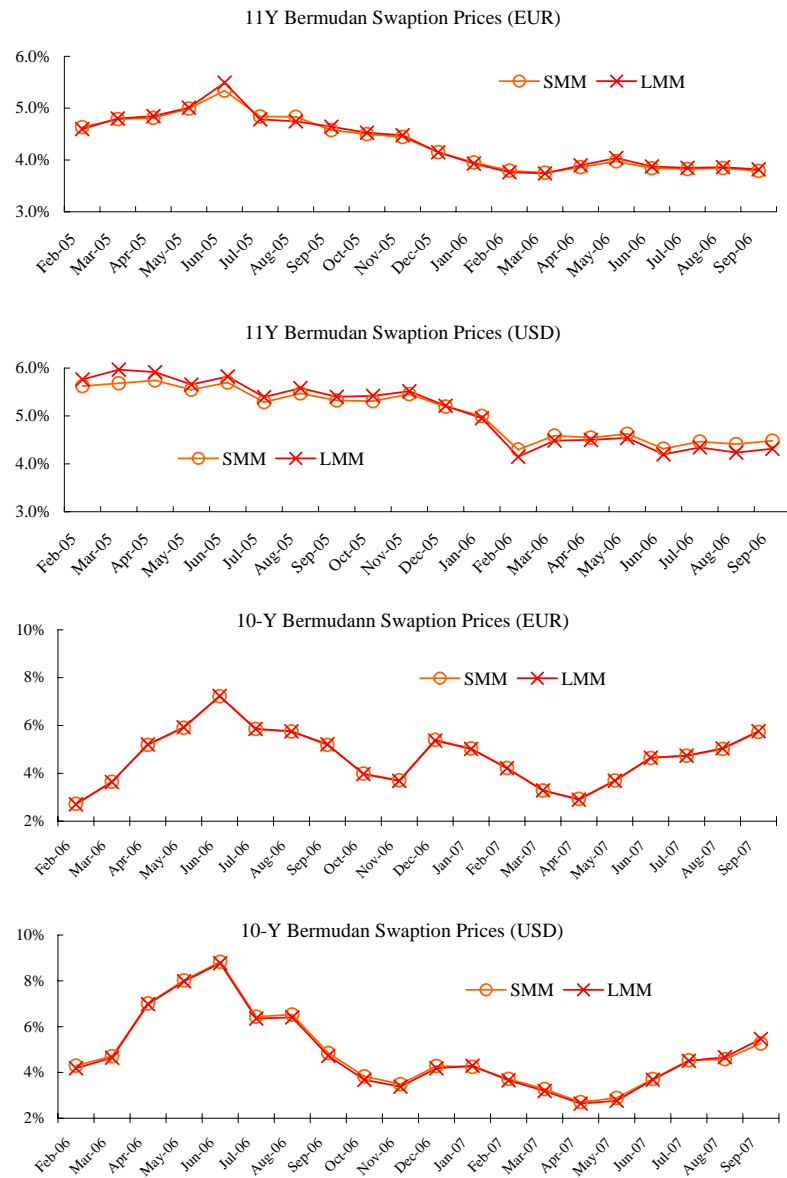
Table 7: 11Y and 10Y Bermudan swaption prices in USD market from February 2005 to September 2007.

DATE(11Y)	SMM	LMM	AVERAGE
2005-2-28	0.0562	0.0576	0.0569
2005-3-31	0.0568	0.0597	0.0582
2005-4-29	0.0574	0.0592	0.0583
2005-5-31	0.0555	0.0566	0.0560
2005-6-30	0.0570	0.0582	0.0576
2005-7-29	0.0529	0.0539	0.0534
2005-8-31	0.0547	0.0558	0.0553
2005-9-30	0.0532	0.0540	0.0536
2005-10-31	0.0531	0.0542	0.0536
2005-11-30	0.0545	0.0552	0.0549
2005-12-30	0.0520	0.0521	0.0520
2006-1-31	0.0500	0.0495	0.0498
2006-2-28	0.0430	0.0415	0.0422
2006-3-31	0.0459	0.0448	0.0454
2006-4-28	0.0455	0.0450	0.0452
2006-5-31	0.0463	0.0454	0.0458
2006-6-30	0.0432	0.0420	0.0426
2006-7-31	0.0446	0.0434	0.0440
2006-8-31	0.0441	0.0424	0.0432
2006-9-29	0.0448	0.0431	0.0440

DATE(10Y)	SMM	LMM	Strike
2006-2-28	0.0430	0.0418	0.0496
2006-3-31	0.0471	0.0464	0.0522
2006-4-28	0.0701	0.0698	0.0482
2006-5-31	0.0802	0.0798	0.0464
2006-6-30	0.0883	0.0877	0.0454
2006-7-31	0.0643	0.0635	0.0486
2006-8-31	0.0653	0.0640	0.0456
2006-9-29	0.0485	0.0473	0.0492
2006-10-31	0.0383	0.0368	0.0520
2006-11-30	0.0349	0.0338	0.0517
2006-12-29	0.0428	0.0418	0.0503
2007-1-31	0.0424	0.0429	0.0512
2007-2-28	0.0372	0.0366	0.0514
2007-3-30	0.0327	0.0319	0.0548
2007-4-30	0.0270	0.0264	0.0573
2007-5-31	0.0289	0.0276	0.0577
2007-6-29	0.0371	0.0368	0.0585
2007-7-31	0.0453	0.0451	0.0566
2007-8-31	0.0458	0.0467	0.0538
2007-9-28	0.0526	0.0547	0.0525

Note:SMM stands for Swap Market Model and LMM stands for Libor Market Model. 11-y Bermudan swaption are priced as at-the-money from February 2005 to September 2006; 10 year Bermudan swaptions are priced with the corresponding 11-y ATM strikes one year ago. For example, on February 2006, the strike rate for 10-y Bermudan swaption is 0.0402 which is the ATM strike rate for 11-y Bermudan swaption on February 2005. 10-y Bermudan swaption are priced from February 2006 to September 2007.

Figure 3: 11Y and 10Y Bermudan swaption prices in EUR and USD markets from February 2005 to September 2007.



Note: SMM stands for Swap Market Model and LMM stands for Libor Market Model.

Figure 4: Factor loadings for the first three principle components of forward rates term structure in the EUR and USD markets

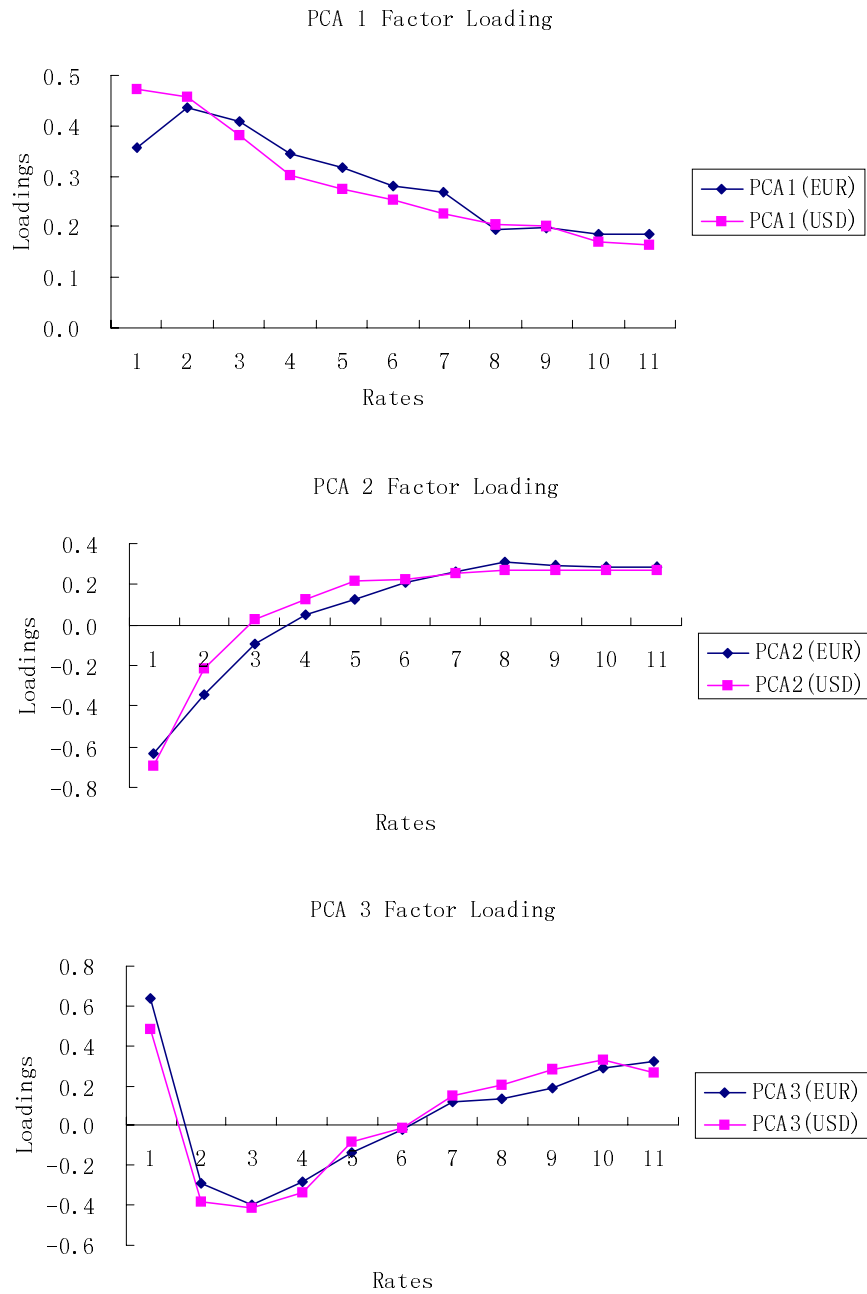


Figure 5: PCA factor loadings (mean of the absolute change) for the first three principle components in EUR and USD markets.

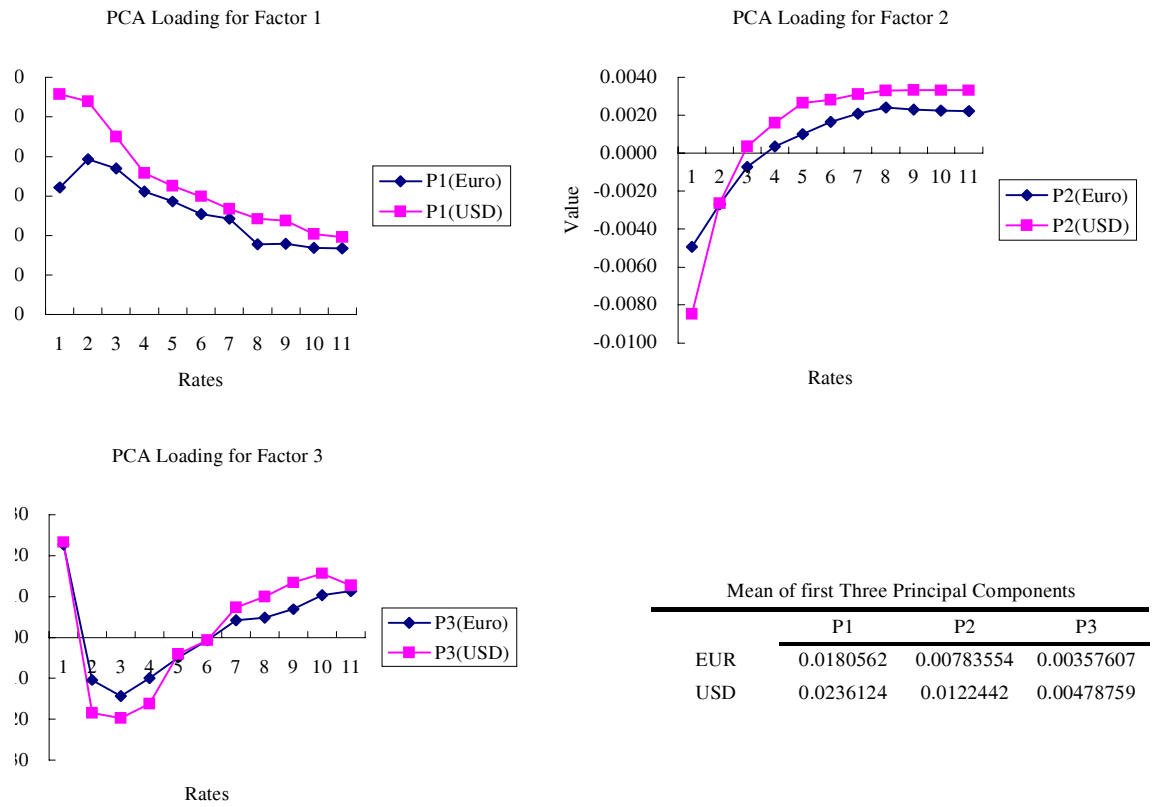


Table 8: PCA factor loadings of Libor rates from January 2000 to September 2007.

Rates	EUR			USD		
	PCA1(EUR)	PCA2(EUR)	PCA3(EUR)	PCA1(USD)	PCA2(USD)	PCA3(USD)
1	0.36	-0.63	0.64	0.47	-0.69	0.49
2	0.43	-0.34	-0.29	0.46	-0.22	-0.39
3	0.41	-0.09	-0.40	0.38	0.03	-0.41
4	0.35	0.05	-0.28	0.30	0.13	-0.34
5	0.32	0.13	-0.14	0.28	0.22	-0.09
6	0.28	0.21	-0.02	0.25	0.23	-0.01
7	0.27	0.26	0.12	0.23	0.25	0.15
8	0.20	0.31	0.14	0.20	0.27	0.21
9	0.20	0.29	0.19	0.20	0.27	0.28
10	0.19	0.29	0.29	0.17	0.27	0.33
11	0.18	0.28	0.32	0.17	0.27	0.26

Table 9: Explanatory power of the first three principle components.

EUR	Percentage variance	Cumulative variance	USD	Percentage variance	Cumulative variance
PCA 1	82.37%	82.37%	PCA 1	74.22%	74.22%
PCA 2	13.87%	96.24%	PCA 2	20.59%	94.81%
PCA 3	2.81%	99.05%	PCA 3	2.96%	97.77%

7.6 Hedge Performance

Here, seventeen dates (annual interval) are selected for test of hedge performance. They are the last business day of each month from February, 2005 to September, 2006. Then the corresponding unwound dates are the last business day of each month from February 2006 to September 2007, which result in seventeen sets of P&L.

Given the tenor structure, ten swaps with length ranging from 2 years to 11 years are used to hedge a 10×1 Bermudan swaption. Six sets of values of the 10×1 Bermudan swaption and ten different swaps are calculated using new forward rate curves bumped up and bumped down by amount of with . Then these values are taken into equation (15) to derive the delta ratios with respect to three PCA factor.

On the unwind date (one year later), the value of 9×1 Bermudan swaption is given by the recalibrated co-terminal SMM. Meanwhile, the length of ten hedging swaps all becomes one year shorter with length ranging from 1 years to 10 years, and their prices are calculated using zero curve of the unwind date by equation (3)'. The P&L given by equation (17) for seven unwind days are shown in Table 10 and

Figure 6.

From Figure 6, it is noted that the P&L given by the model for USD market is a bit higher than those for the Euro market, which may indicate that this co-terminal SMM with volatility parameterised by that particular formula performs better on Euro market than on USD market.

8 Conclusions

The study shows that simulation of co-terminal forward swap rates evolved with drift terms under terminal measure is fast in computation although complex in theory. In simulation, the drift term is determined by volatilities, correlations and values of a set of co-terminal forward swap rates evolved from the previous step, although the correlation term can be ignored if we are doing calibration of one factor co-terminal SMM only to swaptions. In pricing of Bermudan swaption, vector of two dimensions may be required for the implementation of least squares method.

It is noticed that model calibration plays an important role in pricing performance of the model. The initial values and constraints set for parameters will affect the efficiency and result of the calibration. It is quite time consuming to calibrate several parameters simultaneously to market quoted price while calibrating to initial term structure of market implied volatilities is quite efficient and robust. Furthermore, a particular volatility formula may not be applicable to different market.

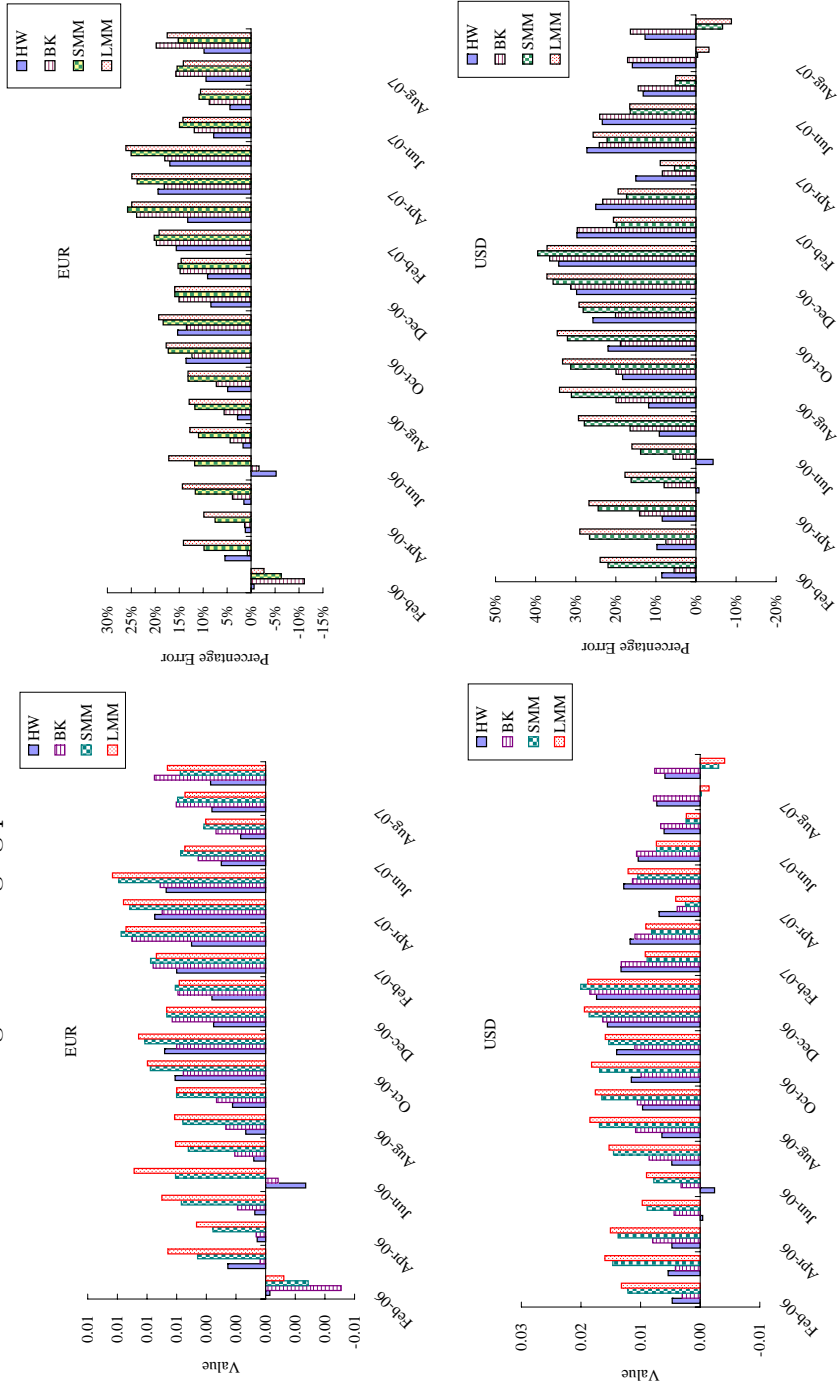
Although the pricing part is satisfactory, hedge performance of one factor model is not good enough. More factors may be needed in order to improve the hedge results of the model. Because of the limitation of time calibration of co-terminal SMM with multiple factors to caps and swaptions jointly will be left for future research. Also, performance of the model for Brazil market will be tested when the data are available.

Table 10: Hedging P&L

EUR	SMM	LMM	USD	SMM	LMM
2006-02-28	-0.0029	-0.0012	2006-02-28	0.0121	0.0132
2006-03-31	0.0046	0.0066	2006-03-31	0.0147	0.0160
2006-04-28	0.0036	0.0047	2006-04-28	0.0138	0.0151
2006-05-31	0.0057	0.0070	2006-05-31	0.0089	0.0097
2006-06-30	0.0061	0.0089	2006-06-30	0.0078	0.0090
2006-07-31	0.0052	0.0061	2006-07-31	0.0145	0.0153
2006-08-31	0.0056	0.0061	2006-08-31	0.0169	0.0185
2006-09-29	0.0060	0.0060	2006-09-29	0.0165	0.0176
2006-10-31	0.0078	0.0080	2006-10-31	0.0169	0.0182
2006-11-30	0.0081	0.0086	2006-11-30	0.0153	0.0159
2006-12-29	0.0067	0.0067	2006-12-29	0.0186	0.0194
2007-01-31	0.0061	0.0058	2007-01-31	0.0200	0.0188
2007-02-28	0.0078	0.0074	2007-02-28	0.0089	0.0092
2007-03-30	0.0097	0.0094	2007-03-30	0.0081	0.0091
2007-04-30	0.0092	0.0096	2007-04-30	0.0025	0.0041
2007-05-31	0.0099	0.0103	2007-05-31	0.0105	0.0121
2007-06-29	0.0057	0.0055	2007-06-29	0.0073	0.0074
2007-07-31	0.0042	0.0040	2007-07-31	0.0024	0.0023
2007-08-31	0.0059	0.0054	2007-08-31	-0.0002	-0.0015
2007-09-28	0.0058	0.0066	2007-09-28	-0.0031	-0.0041
RMSS	0.0066	0.0070	RMSS	0.0553	0.0587

Note:SMM stands for Swap Market Model and LMM stands for Libor Market Model. All numbers are in real value. RMSS is the root mean sum of square of all P&L in each period.

Figure 6: Hedging profit and loss in EUR and USD markets



Note: HW stands for Hull-White model, BK stands for Black-Karasinski model, SMM stands for Swap Market Model and LMM stands for Libor Market Model. All the values are in real monetary value and the percentage error is the hedging P&L in percentage of the Bermudan swaption prices.

Appendix: Derivation of Drift

Three rules in computation of cross variation products will be obeyed, which are mentioned in Joshi and Liesch (2006, P4):

- (i) For any constant c , $\langle X, c \rangle = 0$.
- (ii) For any $a_j \in \mathcal{R}$, $\left\langle X, \sum_j a_j Y_j \right\rangle = \sum_j a_j \langle X, Y_j \rangle$.
- (iii) $\langle X, YZ \rangle = \langle X, Y \rangle Z + \langle X, Z \rangle Y$

Note that $\frac{B_M(t)}{C_{M-1,M}(t)} = \frac{1}{\tau_M}$ is a constant.

When $n = M - 1$

$$\left\langle S_{M-1,M}, \frac{C_{n,M-1}}{C_{M-1,M}} \right\rangle = \langle S_{M-1,M}, 1 \rangle = 0$$

and $\mu_{M-1} = 0$.

When $n = M - 2$

$$\begin{aligned} \left\langle S_{M-2,M}, \frac{C_{M-2,M}}{C_{M-1,M}} \right\rangle &= \left\langle S_{M-2,M}, \frac{C_{M-1,M}}{C_{M-1,M}} \right\rangle + \left\langle S_{M-2,M}, \frac{\tau_{M-1} B_{M-1}}{C_{M-1,M}} \right\rangle \\ &= 0 + \tau_{M-1} \left\langle S_{M-2,M}, S_{M-1,M} + \frac{B_M}{C_{M-1,M}} \right\rangle \\ &= \tau_{M-1} \langle S_{M-2,M}, S_{M-1,M} \rangle \\ &= \tau_{M-1} \rho_{M-2,M-1} S_{M-2,M} \sigma_{M-2,M} S_{M-1,M} \sigma_{M-1,M} \end{aligned}$$

$$\begin{aligned} \mu_{M-2} &= -\frac{C_{M-1,M}}{C_{M-2,M}} \left\langle S_{M-2,M}, \frac{C_{M-2,M}}{C_{M-1,M}} \right\rangle \\ &= -\frac{C_{M-1,M}}{C_{M-2,M}} \tau_{M-1} \langle S_{M-2,M}, S_{M-1,M} \rangle \\ &= -\frac{C_{M-1,M}}{C_{M-2,M}} \tau_{M-1} \rho_{M-2,M-1} S_{M-2,M} \sigma_{M-2,M} S_{M-1,M} \sigma_{M-1,M} \end{aligned}$$

When $n = M - 3$

$$\begin{aligned}
& \left\langle S_{M-3,M}, \frac{C_{M-3,M}}{C_{M-1,M}} \right\rangle \\
&= \left\langle S_{M-3,M}, \frac{C_{M-2,M}}{C_{M-1,M}} \right\rangle + \left\langle S_{M-3,M}, \frac{\tau_{M-2} B_{M-2}}{C_{M-1,M}} \right\rangle \\
&= \left\langle S_{M-3,M}, \frac{C_{M-2,M}}{C_{M-1,M}} \right\rangle + \tau_{M-2} \left\langle S_{M-3,M}, \frac{S_{M-2,M} C_{M-2,M}}{C_{M-1,M}} + \frac{B_M}{C_{M-1,M}} \right\rangle \\
&= \left\langle S_{M-3,M}, \frac{C_{M-2,M}}{C_{M-1,M}} \right\rangle + \tau_{M-2} \left\langle S_{M-3,M}, \frac{C_{M-2,M}}{C_{M-1,M}} \right\rangle S_{M-2,M} \\
&\quad + \tau_{M-2} \langle S_{M-3,M}, S_{M-2,M} \rangle \frac{C_{M-2,M}}{C_{M-1,M}} \\
&= (1 + \tau_{M-2} S_{M-2,M}) \left\langle S_{M-3,M}, \frac{C_{M-2,M}}{C_{M-1,M}} \right\rangle + \tau_{M-2} \frac{C_{M-2,M}}{C_{M-1,M}} \langle S_{M-3,M}, S_{M-2,M} \rangle \\
&= (1 + \tau_{M-2} S_{M-2,M}) \tau_{M-1} \langle S_{M-3,M}, S_{M-1,M} \rangle + \tau_{M-2} \frac{C_{M-2,M}}{C_{M-1,M}} \langle S_{M-3,M}, S_{M-2,M} \rangle
\end{aligned}$$

Finally, taking $S_{M-3,M}$ back into the procedure where we get $\left\langle S_{M-2,M}, \frac{C_{M-2,M}}{C_{M-1,M}} \right\rangle$.

$$\begin{aligned}
\mu_{M-3} &= -\frac{C_{M-1,M}}{C_{M-3,M}} \left\langle S_{M-3,M}, \frac{C_{M-3,M}}{C_{M-1,M}} \right\rangle \\
&= -\frac{C_{M-1,M}}{C_{M-3,M}} (1 + \tau_{M-2} S_{M-2,M}) \tau_{M-1} \rho_{M-3,M-1} S_{M-3,M} \sigma_{M-3,M} S_{M-1,M} \sigma_{M-1,M} \\
&\quad - \frac{C_{M-2,M}}{C_{M-3,M}} \tau_{M-2} \rho_{M-3,M-2} S_{M-3,M} \sigma_{M-3,M} S_{M-2,M} \sigma_{M-2,M}
\end{aligned}$$

When $n = M - 4$

$$\begin{aligned}
& \left\langle S_{M-4,M}, \frac{C_{M-4,M}}{C_{M-1,M}} \right\rangle \\
&= \left\langle S_{M-4,M}, \frac{C_{M-3,M}}{C_{M-1,M}} \right\rangle + \left\langle S_{M-4,M}, \frac{\tau_{M-3} B_{M-3}}{C_{M-1,M}} \right\rangle \\
&= \left\langle S_{M-4,M}, \frac{C_{M-3,M}}{C_{M-1,M}} \right\rangle + \tau_{M-3} \left\langle S_{M-4,M}, \frac{S_{M-3,M} C_{M-3,M}}{C_{M-1,M}} + \frac{B_M}{C_{M-1,M}} \right\rangle \\
&= \left\langle S_{M-4,M}, \frac{C_{M-3,M}}{C_{M-1,M}} \right\rangle + \tau_{M-3} \left\langle S_{M-4,M}, \frac{C_{M-3,M}}{C_{M-1,M}} \right\rangle S_{M-3,M} \\
&\quad + \tau_{M-3} \frac{C_{M-3,M}}{C_{M-1,M}} \langle S_{M-4,M}, S_{M-3,M} \rangle \\
&= (1 + \tau_{M-3} S_{M-3,M}) [(1 + \tau_{M-2} S_{M-2,M}) \tau_{M-1} \langle S_{M-4,M}, S_{M-1,M} \rangle \\
&\quad + \tau_{M-2} \frac{C_{M-2,M}}{C_{M-1,M}} \langle S_{M-4,M}, S_{M-2,M} \rangle] + \tau_{M-3} \frac{C_{M-3,M}}{C_{M-1,M}} \langle S_{M-4,M}, S_{M-3,M} \rangle
\end{aligned}$$

Again, the last step can be deduced by taking $S_{M-4,M}$ back into the procedure where we get $\left\langle S_{M-3,M}, \frac{C_{M-3,M}}{C_{M-1,M}} \right\rangle$. Follow this procedure and by induction, the general form of drift under terminal measure can be written as

$$\mu_n = \sum_{i=n}^{M-2} \left[\tau_{i+1} \frac{C_{i+1,M}}{C_{M-1,M}} \rho_{n,i+1} S_{n,M} \sigma_{n,M} S_{i+1,M} \sigma_{i+1,M} \prod_{j=n+1}^i (1 + \tau_j S_{j,M}) \right]$$

Also note that the above equation is not an explicit function of $S_{n,M}$, because the $C_{n,M}$ term is calculated from a set of $S_{n,M}$.

Appendix: Libor Market Model Calibration

There have been many empirical studies and theoretical discussion about how best to calibrate the LMM model (see Pedersen (1998), Schoenmakers (2002), Brigo and Mercurio (2003, 2004), Rebonato (1999, 2000, 2002)). Given that there is a huge number of interest rate instruments and interest rate derivatives, it is impossible to achieve a perfect fit to all market prices. Not only that it will be technically challenging, it is also costly in terms of time. In general, one should calibrate the interest rate model to the most relevant subset of market data. Since, our primary concern is the pricing and hedging of Bermudan swaption, it seems appropriate to calibrate the model to all co-terminal European swaptions of the same tenor structure as the Bermudan swaption.

The past literature on calibration includes ways of imposing smoothing criteria; parameterize the volatility and correlation function, choice of minimizing pricing error or volatility mismatch. Here we adopt the volatility function proposed in Rebonato (1999) and calibrate it directly to European swaption (Black) implied volatility. This method has been used extensively in the literature and was proved to possess very good properties. Since the one-factor LMM has no correlation parameter, the volatility has the following specification

$$\sigma_i(t) = \phi(i)[(a + b(T_i - t))\exp(-c(T_i - t)) + d]. \quad (3)$$

With appropriate parameter values, this volatility function will produce a hump share (wrt time) as observed among the real market data.

The one-factor calibration results are summarized in Table (5). The calibration is performed on the last business date of each month from February 2005 to September 2007. From the table, we could see that the parameters from calibration is quite stable. We also plot the percentage difference between market data and the calibrated result in Figure (1).

The Sequential Quadratic Programming (SQP) algorithm from NAG C library was used for calibration. This routine is based on the algorithm suggested in Gill et

al (1986) . Due to the complexity of our target function, we approximate all partial derivatives using finite differences. This however does not appear to preclude the convergence of the optimization routine. Calibration settings are summarized in Table (3).

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