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Robust consumption decisions under ambiguity for regime switching mean returns

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Robust Consumption and Portfolio Choice for Time-varying Investment Opportunities

Hening Liu

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Keywords

robustness, mean-reverting, portfolio choice

Abstract

This paper examines a continuous-time intertemporal consumption and portfolio choice problem for an investor with Du e and Epstein (1992a)'s recursive preferences who worries about model misspecification (model uncertainty) and wants to seek robust decision rules. The expected excess return of a risky asset follows a mean-reverting process. I find that whether the concern about model uncertainty decreases the total demand for equities largely depends on risk aversion and the attitude toward intertemporal substitution. When the elasticity of intertemporal substitution (EIS) is about one and risk aversion is moderate, the concern about model uncertainty increases the proportion of wealth invested in equities. The aversion to model uncertainty also increases the importance of the intertemporal hedging demand in portfolio decisions. The calibration analysis based on the detection-error probabilities shows that the quantitative effect of robustness is almost negligible for both the long sample of Campbell and Viceira (1999) and the short sample of Barberis (2000).

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Robust Consumption and Portfolio Choice for Time-varying Investment Opportunities

Hening Liu[†]

September 2009

Abstract

This paper examines a continuous-time intertemporal consumption and portfolio choice problem for an investor with Duffie and Epstein (1992a)'s recursive preferences who worries about model misspecification (model uncertainty) and wants to seek robust decision rules. The expected excess return of a risky asset follows a mean-reverting process. I find that whether the concern about model uncertainty decreases the total demand for equities largely depends on risk aversion and the attitude toward intertemporal substitution. When the elasticity of intertemporal substitution (EIS) is about one and risk aversion is moderate, the concern about model uncertainty increases the proportion of wealth invested in equities. The aversion to model uncertainty also increases the importance of the intertemporal hedging demand in portfolio decisions. The calibration analysis based on the detection-error probabilities shows that the quantitative effect of robustness is almost negligible for both the long sample of Campbell and Viceira (1999) and the short sample of Barberis (2000).

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Keywords: robustness, mean-reverting, portfolio choice

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1 Introduction

One notable puzzle in financial economics is that although the canonical models on dynamic consumption and portfolio choice (Samuelson, 1969; Merton, 1969) predict aggressive investment strategies for moderately risk-averse investors, the actual equity holdings in the real world are surprisingly low. A number of recent papers (e.g., Kim and Omberg, 1996; Brennan et al., 1997; Campbell and Viceira, 1999; Campbell et al., 2004) have analyzed dynamic portfolio choice under return predictability or mean-reverting expected returns. They find that the intertemporal hedging motives can substantially increase the demand for equities, which indeed aggravates the low equity allocation puzzle.¹ To resolve the apparent puzzle, one successful approach is to introduce model uncertainty and uncertainty aversion (e.g., Maenhout, 2004; Chen and Epstein, 2002). Taking into account model misspecification, investors are concerned with the validity of a benchmark model describing the dynamics of asset returns and are willing to consider a family of candidate models. This approach has the feature of robust decision-making, which is usually formulated into a max-min problem in the multiple priors framework. The investor's aversion to the multiplicity of beliefs over the investment opportunity set imputes pessimism into decision-making, which then translates into an uncertainty adjustment in the optimal portfolio choice. This adjustment can help turn aggressive portfolios into conservative ones and also explain the low equity demand in the real world. This approach and related results seem to make concrete the intuition that uncertainty aversion necessarily lowers the equity demand.

However, since those findings are based on the assumption of i.i.d. returns, whether the established result on the effect of model uncertainty can carry on to time-varying investment opportunities deserves further research. This paper finds that the argument that investors' aversion to model uncertainty necessarily leads to lower equity demand is neither theoretically true in general nor is empirically relevant for time-varying investment opportunities and investors with recursive preferences. To analyze the question, I consider a continuous-time intertemporal consumption and portfolio choice problem in which an investor is uncertain whether an underlying model governing return dynamics is true. The expected returns of a risky asset follow a mean-reverting Ornstein-Uhlenbeck process. To separate the effects of risk aversion and of the willingness to substitute intertemporally, I adopt Duffie and Epstein (1992a)'s continuous-time parameterization of recursive preferences. The model presented here nests the robust portfolio choice model analyzed by Maenhout (2006), where an uncertainty-averse investor with the constant relative risk aversion (CRRA) utility function

¹ The puzzle also manifests itself in a general equilibrium setting as the well-known equity premium puzzle (Mehra and Prescott (1985)).

maximizes utility over terminal wealth.

To the best of my knowledge, it is the first paper to incorporate model uncertainty into a continuous-time consumption and portfolio choice problem in which risk aversion, uncertainty aversion and the willingness to substitute intertemporally are separated among each other and investment opportunities are time varying.² I obtain an exact analytical solution when the elasticity of intertemporal substitution is equal to one. For elasticities of intertemporal substitution different from one, I use the log-linear approximation method described in Campbell et al. (2004) to solve for the optimal decision rules. I show that increasing uncertainty aversion is observationally equivalent to increasing effective risk aversion. This result extends the findings of Maenhout (2004, 2006) to the case with both recursive preferences and time-varying investment opportunities. Maenhout (2004, 2006) show that the observational equivalence holds, respectively, in a i.i.d. returns setting with recursive utility and in a mean-reverting setting with CRRA utility. Furthermore, I show that when the elasticity of intertemporal substitution is close to one, uncertainty aversion can increase the total equity demand for moderately risk-averse investors. This result stands in contrast to the findings in Maenhout (2006) where a CRRA investor maximizes utility over terminal wealth. Maenhout (2006) finds that the concern about model uncertainty always leads to lower equity demand when the coefficient of relative risk aversion is greater than one. I argue that this seeming contradiction stems from the effect of intertemporal substitution, namely, a low elasticity of intertemporal substitution implies a mitigated effect of model uncertainty on the equity demand.

In the presence of model uncertainty, the investor pursues robust decision rules that can work reasonably well not only when the underlying model is correct, but also when it is misspecified. In this paper, I use the robust control framework developed by Anderson et al. (2000) to embed model uncertainty and uncertainty aversion and adopt the homothetic robustness specification of Maenhout (2004, 2006) to solve for analytical solutions. An uncertainty-averse investor entertains a set of alternative models surrounding the benchmark model that may be misspecified; each alternative model governs a particular process of return dynamics. The investor wants to hedge against a worst-case alternative model that is difficult to distinguish from the benchmark model. The statistical criterion is a relative entropy criterion weighted by a preference parameter that measures the degree of uncertainty aversion.

² Ju and Miao (2009) and Chen, Ju and Miao (2009) use the generalized recursive smooth ambiguity preferences framework, which allows for Bayesian learning and the distinction among risk aversion, uncertainty aversion and intertemporal substitution. Schroder and Skiadas (2003) consider the generalized recursive preferences that also take into account uncertainty aversion. However, the analytical solutions for optimal consumption and portfolio policies are not available for time-varying investment opportunities in these papers.

The concern about model misspecification stems from substantial degree of uncertainty in estimating dynamics of asset returns. It has long been argued that estimating expected returns is very difficult (Merton, 1980). Further, whether asset returns are predictable has become one of the most debated question in the empirical finance literature in recent decades.³ Thus, the typical bulk of work on dynamic portfolio choice, which assumes that the model governing return dynamics is fixed and perfectly known, may encounter difficulties due to the concern that the benchmark model used to guide portfolio decisions is possibly misspecified. Introducing model uncertainty also has a well-grounded decision theoretic basis. Uncertainty is used to describe the situation where a decision-maker is uncertain over a set of probability distributions whereas risk is used to describe the situation in which a probability distribution can be precisely known. Distinguishing uncertainty from risk can provide an explanation to the well-known Ellsberg paradox.

In the calibration analysis, I empirically assess the strength of the quantitative effect of model uncertainty based on the detection error probabilities. In particular, I want to address the question of how large can the scope of robustness reach with recursive preferences and time-varying investment opportunities? According to Anderson et al. (2000), the detection error probabilities provide a useful approach to calibrate the preference parameter of robustness given a finite sample of data, which allows us to explore the quantitative effect of model uncertainty. Maenhout (2006) suggests a methodology for calculating the detection error probabilities based on Fourier inversion of the conditional characteristic functions of the Radon-Nikodym derivatives between the worst-case model and the benchmark model. In this paper, I further show that the conditional characteristic functions have analytical solutions under recursive preferences. As a result, the detection error probabilities can be explicitly characterized in closed-form (up to a Fourier inversion). I find that the scope of model uncertainty is almost negligible for the two samples considered in Campbell and Viceira (1999) and Barberis (2000). This implies that distinguishing the worst-case alternative model from the benchmark model turns out to very easy. Thus, one can argue that the attempt to explain low equity demand by introducing model misspecification as well as robust decision-making is only relevant for i.i.d. returns. My results comprise a striking difference from those of Maenhout (2004) and Chen and Epstein (2002) in which i.i.d. returns are examined. Maenhout (2004) documents a large effect of robustness on asset holdings and asset returns. Maenhout (2006) further finds that the room for

³ A number of papers argue that the empirically detected return predictability arises due to inference bias (e.g., Ang and Bekaert, 2007; Bossaerts and Hillion, 1999; Campbell and Yogo, 2006; Ferson et al., 2003; Stambaugh, 1999). Other studies suggest that the predictability may have experienced structural breaks (Goyal and Welch, 2003), and the equity premium is also subject to structural breaks (Pastor and Stambaugh, 2001). There is also a recent debate on the out-of-sample performance as a test against the predictability (e.g., Goyal and Welch, 2008; Campbell and Thompson, 2008; Cochrane, 2008).

the effect of robustness declines to a lesser extent in a time-varying investment opportunities setting than in the i.i.d. setting, though the scope is still large enough to turn highly levered portfolios into empirically plausible ones. Nevertheless, I find that this scope shrinks to a negligible level under the Duffie-Epstein's recursive preferences.

This paper is related to a large literature on the dynamic portfolio choice problem. Brennan et al. (1997) examine asset allocation decisions when returns are predictable. Kim and Omberg (1996) and Wachter (2002) derive closed-form solutions for mean-reverting risk premia in incomplete markets and complete markets, respectively. Chacko and Viceira (2005) study the effect of stochastic volatility on consumption and portfolio decisions. Liu (2007) analyzes dynamic portfolio choice when investment opportunities are governed by quadratic processes. Those papers concerning model uncertainty and uncertainty aversion include Epstein and Miao (2003), Uppal and Wang (2003), Garlappi et al. (2007) and Gollier (2007). Epstein and Schneider (2007) develop a model of learning under ambiguity and study its effect on dynamic portfolio choice. Recent papers examining the effects of Bayesian learning and model uncertainty include Miao (2001), Cagetti et al. (2002), Ju and Miao (2009) and Chen, Ju and Miao (2009).

The rest of the paper is organized as follows. Section 2 presents the dynamic optimization problem for a robust investor with recursive preferences. Section 3 gives an exact analytical solution and an approximate analytical solution when expected returns are time-varying. Section 4 describes the calculation of detection error probabilities, based on the conditional characteristic functions of the Radon-Nikodym derivatives and a Fourier inversion. Section 5 explores the quantitative effect of robustness by calibrating the model to the market data. Section 6 concludes. The appendix contains the proofs of the main results.

2 The Investor's Robust Optimization Problem

I assume that there are two assets available for investment, a riskless short-term bond paying instantaneous return r and a risky asset with the following price dynamics:

$$dS_t = S_t \mu_t dt + S_t \sigma dB_t. \quad (1)$$

The instantaneous expected return μ_t is assumed to follow a mean-reverting process:

$$d\mu_t = \kappa (\bar{\mu} - \mu_t) dt + \sigma_\mu d\tilde{B}_t. \quad (2)$$

where κ , $\bar{\mu}$ and σ_μ are all positive constants. The Brownian motions B_t and \tilde{B}_t are allowed to be correlated with correlation coefficient ρ .⁴ The process (2) can be written in terms of independent Brownian motions of the form:

$$d\mu_t = \kappa(\bar{\mu} - \mu_t) dt + \sigma_\mu \rho dB_t + \sigma_\mu \sqrt{1 - \rho^2} d\hat{B}_t$$

where \hat{B}_t and B_t are independent Brownian motions. The investor's wealth dynamics can be described by the following stochastic differential equation (SDE):

$$dW_t = [W_t(r + \alpha_t(\mu_t - r)) - C_t] dt + W_t \alpha_t \sigma_S dB_t \quad (3)$$

where α_t is the proportion of wealth W_t invested in the risky asset.

I assume that the investor has recursive preferences over consumption and adopt Duffie and Epstein's (1992a, b) continuous-time parameterization:

$$V_t = \int_t^\infty f(C_s, V_s) ds$$

where $f(C, V)$ is a normalized aggregator of current consumption and continuation utility that takes the form

$$f(C, V) = \frac{\beta}{1 - \frac{1}{\psi}} (1 - \gamma) V \left[\left(\frac{C}{((1 - \gamma) V)^{\frac{1}{1-\gamma}}} \right)^{\left(1 - \frac{1}{\psi}\right)} - 1 \right]. \quad (4)$$

Here, β is the rate of time preference, γ is the coefficient of relative risk aversion and ψ is the elasticity of intertemporal substitution. With the restriction $\psi = 1/\gamma$, the normalized aggregator (4) reduces to the standard additive power utility function, for which log utility obtains for $\gamma = \psi = 1$. When $\psi = 1$, the aggregator $f(C, V)$ takes the form

$$f(C, V) = \beta(1 - \gamma) V \left[\log(C) - \frac{1}{1 - \gamma} \log((1 - \gamma) V) \right].$$

In the absence of model uncertainty and uncertainty aversion, the investor's problem is

$$\max_{C_t, \alpha_t} E_t \left[\int_t^\infty f(C_s, V_s) ds \right]$$

subject to the state dynamics (2) and the budget constraint (3). This optimization problem has been analyzed by Campbell et al. (2004). The Hamilton-Jacobi-Bellman (HJB) equation for this problem

⁴ I would be able to proceed in the following derivations using this formulation. However, it is more convenient to work with the formulation with independent Brownian motions when model uncertainty is introduced.

is

$$0 = \sup_{\alpha_t, C_t} \left\{ f(C_t, V_t) + \mathcal{D}^{(C, \alpha)} V \right\}$$

where $\mathcal{D}^{(C, \alpha)} V$ is the infinitesimal generator applied to the value function V and is defined for an expected utility maximizer by

$$\begin{aligned} \mathcal{D}^{(C, \alpha)} V &= V_W [W_t (r + \alpha_t (\mu_t - r)) - C_t] + V_{\mu} \kappa (\bar{\mu} - \mu_t) \\ &\quad + \frac{1}{2} V_{WW} (\alpha_t \sigma_S W_t)^2 + V_{W\mu} W_t \alpha_t \rho \sigma_S \sigma_{\mu} + \frac{1}{2} V_{\mu\mu} \sigma_{\mu}^2. \end{aligned}$$

Intuitively, $\mathcal{D}^{(C, \alpha)} V$ is a measure of the instantaneous expected continuation value in the Bellman equation. As will be seen later, the adjustment to this term reflects the concern about model uncertainty. A robust investor deems the state dynamics (2) and (3) as only an approximation and possibly misspecified. The investor partly accepts the usefulness of this reference model but still doubts its validity. He therefore wants to consider a family of alternative models that are close to and hard to distinguish from the reference model.

Denote the state vector $Y_t \equiv (W_t, \mu_t)^\top$. The reference model can then be written as

$$dY_t = \Theta(Y_t) dt + \Lambda(Y_t) d\mathbf{B}_t$$

where Θ is the drift vector, Λ is the volatility matrix of the state vector Y_t , $\Sigma = \Lambda \Lambda^\top$ is the covariance matrix and \mathbf{B} is the Brownian vector $\mathbf{B}_t \equiv (B_t, \hat{B}_t)^\top$. The vector Θ and the matrix Λ have the appropriate functional forms inherited from (2) and (3). The distorted law of motion of the state vector implied by an alternative model is

$$dY_t = \Theta(Y_t) dt + \Lambda(Y_t) \left[\Lambda(Y_t)^\top u(Y_t) dt + d\mathbf{B}_t \right] \quad (5)$$

where $u(Y_t)$ is an endogenous drift adjustment vector to be determined from solving the optimization problem. Thus, model uncertainty indeed concerns the uncertainty about the drift functions of the state variables. Suppose \mathbb{P} is the subjective probability measure under the reference model and \mathbb{Q} under the alternative model. Then it follows from Girsanov's theorem that the Radon-Nikodym derivative is

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)_t = \Xi_{1,t} \quad (6)$$

where

$$\frac{d\Xi_{1,t}}{\Xi_{1,t}} = \Lambda(Y_t)^\top u(Y_t) d\mathbf{B}_t, \quad \Xi_{1,0} = 1. \quad (7)$$

The interpretation is that the investor endogenously chooses an alternative belief about the dynamics

of the state variables and accordingly the optimal consumption and portfolio rules.

The investor worries that an alternative model has adverse effect on the expected continuation payoff and wants to take into account this worst-case alternative model when making decisions. However, it is rarely possible for the investor to consider alternative models that are extremely bad and very far away from the reference model. Only those models that are reasonably close to the reference model will be considered. Thus, a penalty term should be incurred to penalize the distance of an alternative model. This penalty term is operationalized by Anderson et al. (2000) using relative entropy. Put formally, the investor seeking robustness considers the following adjustment to $\mathcal{D}^{(C,\alpha)}V$:

$$\inf_u \left\{ \mathcal{D}^{(C,\alpha)}V + u^\top \Sigma \partial V + \frac{1}{2\eta} u^\top \Sigma u \right\}.$$

The second terms reflects the adjustment to the expected continuation value when state dynamics are governed by the alternative model (5). This term captures adverse effect of model misspecification on the continuation value. The third term is the relative entropy term. Alternative models that are close to the reference model have with low entropy and low log-likelihood ratios. These models are therefore statistically difficult to distinguish from the reference model.

The preference parameter $\eta \geq 0$ measures the strength of preference for robustness, or put another way, the degree of confidence on the reference model. When $\eta = 0$, the investor has complete confidence and does not worry about model uncertainty. In this case, the problem reduces to the expected utility maximization. A higher η implies less confidence on the reference model and more desire for robustness. The investor then considers distant alternative models with large drift distortion and large entropy.

3 The Robust Intertemporal Consumption and Portfolio Decisions

I impose the set of admissible portfolio strategies with well-defined utility, namely, set \mathcal{A} given by

$$\mathcal{A} = \left\{ \alpha: W_t \geq 0 \text{ and } \mathbb{E}^\mathbb{Q} [f(C_t, V_t)] < \infty \text{ for all } t \geq 0, \right\}.$$

The HJB equation for a robust investor with recursive preferences is

$$0 = \sup_{\alpha_t, C_t} \inf_u \left\{ f(C_t, V_t) + \mathcal{D}^{(C,\alpha)}V + u^\top \Sigma \partial V + \frac{1}{2\eta} u^\top \Sigma u \right\}. \quad (8)$$

The first order conditions (FOCs) with respect to u are

$$u^* = -\eta \partial V.$$

The endogenous drift distortion depends on the state variables through the partial derivatives of the value function in an implicit way. Thus, unless I obtain a complete solution to the problem, the dependence cannot be explicitly characterized.

Substituting the FOCs back into the HJB equation gives

$$0 = \sup_{\alpha_t, C_t} \left\{ f(C_t, V_t) + \mathcal{D}^{(C, \alpha)} V - \frac{\eta}{2} \left[(W_t \alpha_t \sigma_S)^2 V_W^2 + 2W_t \alpha_t \rho \sigma_S \sigma_\mu V_W V_\mu + \sigma_\mu^2 V_\mu^2 \right] \right\}. \quad (9)$$

As noted by Anderson et al. (2000) and Maenhout (2006), robustness leads to an adjustment term in the HJB equation which reflects a concern about the quadratic variation in the partial derivatives of the value function weighted by the preference parameter η .

The first order conditions for consumption are

$$C_t = V_W^{-\psi} [(1 - \gamma) V]^{\frac{1 - \gamma \psi}{1 - \gamma}} \beta^\psi \quad \text{when } \psi \neq 1 \quad (10)$$

$$C_t = \frac{V}{V_W} (1 - \gamma) \beta \quad \text{when } \psi = 1 \quad (11)$$

The first order condition for portfolio choice is

$$\alpha_t = \frac{V_W (\mu_t - r)}{(-V_{WW} W_t + \eta V_W^2 W_t) \sigma_S^2} + \frac{V_{W\mu} \rho \sigma_\mu}{(-V_{WW} W_t + \eta V_W^2 W_t) \sigma_S} + \frac{-\eta \rho \sigma_\mu V_W V_\mu}{(-V_{WW} W_t + \eta V_W^2 W_t) \sigma_S}. \quad (12)$$

The (implicit) optimal portfolio rule has three components and is similar to the one derived in Maenhout (2006). The first component is the myopic demand, which only depends on the current risk-return trade-off. Different from the case of expected utility maximization, the standard risk aversion factor, $\frac{-V_W}{V_{WW} W_t}$, has now been adjusted to $\frac{V_W}{(-V_{WW} W_t + \eta V_W^2 W_t)}$. The second term represents the standard hedging demand as in Kim and Omberg (1996) and Campbell et al. (2004) when $\eta = 0$. This term is zero when investment opportunities are constant ($\sigma_\mu = 0$), or when the investment opportunity set cannot be used as a tool of hedging ($\rho = 0$) or when the investor has log utility ($V_{W\mu} = 0$). The third term can be interpreted as the hedging demand due to robustness. When no robustness is desired ($\eta = 0$), this term vanishes. An interesting property is that even with log utility ($V_{W\mu} = 0$), this term still persists.

To explicitly solve the model, I follow Maenhout (2004, 2006) and impose the homothetic robustness specification. In particular, I assume that the preference parameter η is state-dependent and scaled by the value function:

$$\eta(W_t, \mu_t) = \frac{\phi}{(1 - \gamma) V(W_t, \mu_t)} > 0 \quad (13)$$

where the parameter ϕ can be interpreted as the preference robustness or uncertainty aversion. The

HJB equation then becomes

$$\begin{aligned}
0 = & \sup_{\alpha_t, C_t} \left\{ f(C_t, V_t) + V_W [W_t (r + \alpha_t (\mu_t - r)) - C_t] + V_\mu \kappa (\bar{\mu} - \mu_t) \right. \\
& + \frac{1}{2} V_{WW} (\alpha_t \sigma_S W_t)^2 + V_{W\mu} W_t \alpha_t \rho \sigma_S \sigma_\mu + \frac{1}{2} V_{\mu\mu} \sigma_\mu^2 \\
& \left. - \left(\frac{\phi}{2(1-\gamma)} \frac{\alpha_t^2 \sigma_S^2 W_t^2 V_W^2}{V} + \frac{\phi}{1-\gamma} \frac{W_t \alpha_t \rho \sigma_S \sigma_\mu V_W V_\mu}{V} + \frac{\phi}{2(1-\gamma)} \frac{\sigma_\mu^2 V_\mu^2}{V} \right) \right\}.
\end{aligned}$$

For different values of the elasticity of intertemporal substitution ($\psi = 1$ or $\psi \neq 1$), the HJB equation results in different forms of partial differential equations because different normalized aggregators are used. I show these equations, respectively, for each case in Appendix A.

3.1 The analytical solution for $\psi = 1$

I present an exact analytical solution for $\psi = 1$, which is summarized in the following proposition. The solution nests the case of $\gamma = 1$, for which log utility obtains.

Proposition 1 *When $\psi = 1$ and $\gamma \neq 1$, the solution to the value function is*

$$V(W_t, \mu_t) = (I(\mu_t))^{1-\gamma} \frac{W_t^{1-\gamma}}{1-\gamma} \quad (14)$$

with

$$I(\mu_t) = \exp \left\{ A + B\mu_t + \frac{C}{2} \mu_t^2 \right\} \quad (15)$$

where A , B and C satisfy a system of recursive equations given in Appendix A and are functions of the primitive parameters of the model describing investment opportunities and preferences.

The value function for $\psi = 1$ and $\gamma = 1$ is of the form

$$V(W_t, \mu_t) = \log W + A + B\mu_t + \frac{C}{2} \mu_t^2$$

where A , B and C satisfy the system of recursive equations given in Appendix A with $\gamma = 1$.

The optimal consumption and portfolio rules are given by

$$\begin{aligned}
C_t &= \beta W_t \\
\alpha_t &= \frac{\mu_t - r}{(\gamma + \phi) \sigma_S^2} + \frac{1 - \gamma - \phi}{\gamma + \phi} \frac{\rho \sigma_\mu}{\sigma_S} (B + C\mu_t).
\end{aligned} \quad (16)$$

Proof. See Appendix A. ■

3.2 The approximate analytical solution for $\psi \neq 1$

For the more general case of $\psi \neq 1$, there is no exact analytical solution to the problem (8). But I can use the log-linear approximation method described in Campbell et al. (2004) to solve for an approximate analytical solution. The solution is summarized in the following proposition.

Proposition 2 When $\psi \neq 1$ and $\gamma \neq 1$, the solution to the value function is given by

$$V(W_t, \mu_t) = (K(\mu_t))^{-\frac{1-\gamma}{1-\psi}} \frac{W_t^{1-\gamma}}{1-\gamma} \quad (17)$$

where $K(\mu_t)$ satisfies an ordinary differential equation given in Appendix A. By applying the log-linear approximation method of Campbell et al. (2004), the analytical solution to $K(\mu_t)$ is

$$K(\mu_t) = \exp \left\{ \hat{A} + \hat{B}\mu_t + \frac{\hat{C}}{2}\mu_t^2 \right\}$$

where \hat{A} , \hat{B} and \hat{C} satisfy a system of recursive equations shown in Appendix A and are functions of the primitive parameters of the model describing investment opportunities and preferences.

The approximate analytical solution to the value function for $\psi \neq 1$ and $\gamma = 1$ is of the form

$$V(W_t, \mu_t) = \log W - \frac{1}{1-\psi} \left(\hat{A} + \hat{B}\mu_t + \frac{\hat{C}}{2}\mu_t^2 \right)$$

where \hat{A} , \hat{B} and \hat{C} satisfy the system of recursive equations given in Appendix A with $\gamma = 1$.

The optimal portfolio choice is given by

$$\alpha_t = \frac{\mu_t - r}{(\gamma + \phi)\sigma_S^2} - \frac{1}{1-\psi} \frac{1-\gamma-\phi}{\gamma+\phi} \frac{\rho\sigma_\mu}{\sigma_S} (\hat{B} + \hat{C}\mu_t) \quad (18)$$

Proof. See Appendix A. ■

I obtain the approximate solutions using a numerical recursive procedure suggested in Campbell and Viceira (1999). Appendix A describes the details. By incorporating model uncertainty and the preference for robustness, the results shown in Proposition 1 and 2 extend the work of Campbell et al. (2004) and nest their solutions by setting $\phi = 0$. The solutions shown here also extend the work of Maenhout (2004) by exploring the effects of time-varying investment opportunities and of Maenhout (2006) by disentangling risk aversion and intertemporal substitution. When $\psi = 1$, model uncertainty has no effect on the optimal consumption-wealth ratio, which only depends on the time preference parameter β . By contrast, I show through numerical examples that model uncertainty does affect the consumption-wealth ratio when $\psi \neq 1$.

In the optimal portfolio rules (16) and (18), model uncertainty affects both the myopic demand and the intertemporal hedging demand. The myopic allocation $\frac{\mu_t - r}{\sigma_S^2}$ is scaled by effective risk aversion $\gamma + \phi$. The effect on myopic demand is mirrored by the adjusted risk aversion factor $\frac{V_W}{(-V_{WW}W_t + \eta V_W^2 W_t)}$ appearing in the first term of the implicit optimal portfolio rule (12). The explicit solutions to the second term and the third term in (12) are $\frac{1-\gamma}{\gamma+\phi} \frac{\rho\sigma_\mu}{\sigma_S} (B + C\mu_t)$ and $\frac{-\phi}{\gamma+\phi} \frac{\rho\sigma_\mu}{\sigma_S} (B + C\mu_t)$ for $\psi = 1$, and $-\frac{1}{1-\psi} \frac{1-\gamma}{\gamma+\phi} \frac{\rho\sigma_\mu}{\sigma_S} (\hat{B} + \hat{C}\mu_t)$ and $\frac{1}{1-\psi} \frac{\phi}{\gamma+\phi} \frac{\rho\sigma_\mu}{\sigma_S} (\hat{B} + \hat{C}\mu_t)$ for $\psi \neq 1$. However, it is impossible to separate from the total hedging demand a hedging component in analytical form that is solely attributed

to model uncertainty because the uncertainty aversion parameter ϕ also appears in the recursive equations for $A(\hat{A})$, $B(\hat{B})$ and $C(\hat{C})$.

Maenhout (2004) derives the first term in (16) and (18) to be the optimal portfolio rule for constant investment opportunities. Thus, intertemporal substitution does not affect robust portfolio decisions. More important, this implies that a high level of uncertainty aversion always leads to a low equity demand for risk-averse investors. Maenhout (2006) assumes a mean-reverting risk premium and finds that having $\phi > 0$ lowers the optimal equity demand for a CRRA utility investor.⁵ The Duffie and Epstein (1992a)'s recursive preferences nest the CRRA utility function employed by Maenhout (2006) as a special case and allows me to distinguish the role of intertemporal substitution from risk aversion. Thus, I can examine the effects of risk aversion, uncertainty aversion and intertemporal substitution in a neatly manner.

The following proposition establishes the link between my model and the model of Campbell et al. (2004) without model uncertainty.

Proposition 3 *Given investment opportunities described by the equations (1)–(2), an uncertainty-averse investor with a homothetic preference for robustness $\eta(W, \mu) = \frac{\phi}{(1 - \gamma)V(W, \mu)}$ and Duffie and Epstein's recursive preferences is observationally equivalent to a Campbell, Chacko, Rodriguez and Viceira(2004) investor with effective risk aversion $\gamma + \phi$.*

Proof. See Appendix A. ■

The above result extends the findings of Maenhout (2004) (Proposition 4) to a time-varying investment opportunities setting. The optimal consumption and portfolio decision rules of a uncertainty-averse investor with risk aversion γ and uncertainty aversion ϕ are identical to those of an expected utility investor with risk aversion $\gamma + \phi$. An interesting property to note is that a log-utility investor also hedges for time-varying investment opportunities since his effective risk aversion accounted for uncertainty exceeds one. Moreover, the intertemporal hedging demand of an uncertainty-averse log utility investor is solely attributed to his preference for robustness. Another interesting property concerns the case of $\gamma < 1$, for which the intertemporal hedging demand may shrink to zero when the level of uncertainty aversion ϕ is nonzero. This happens when the stock allocation to hedge against risk and the allocation to hedge against model uncertainty exactly balance each other.

⁵ Through numerical calibration, Maenhout (2006) finds that increasing the desire for robustness can bring the optimal equity share below 100% without assuming a high level of risk aversion.

4 Detection error probabilities

Our task is to calibrate the price dynamics (1)–(2) to the market data and use the optimal portfolio rules derived above to find the quantitative effect of model uncertainty. A natural question is how to calibrate the preference parameter ϕ in a meaningful way? To deal with the issue, I employ the statistical model-detection tool in the form of detection error probabilities suggested in Anderson et al. (2000) to calibrate model uncertainty. According to Anderson et al. (2000), a reasonable value for uncertainty aversion ϕ should render the corresponding worst-case alternative model sufficiently close to the reference model based on a time series sample with finite length. A high detection error probability indicates a high likelihood that a robust investor will make a mistake when choosing between the worst-case alternative model and the reference model. The value of the preference parameter ϕ should be chosen in such a way that the detection error probability is high enough for the investor to face a difficult model selection problem.

In this section, I follow Maenhout (2006) and calculate the detection error probabilities based on Fourier inversion of the conditional characteristic functions of the Radon-Nikodym derivatives between the worst-case model and the reference model. I first derive the worst-case model implied by the solution to the intertemporal consumption and portfolio choice problem. The Radon-Nikodym derivatives between the two models follow naturally. Then I show that the conditional characteristic functions of the Radon-Nikodym derivatives admit closed-form solutions in explicit forms. Finally, I apply Fourier inversion to the characteristic functions and obtain detection error probabilities.

Methods for analytical calculation of the detection-error probabilities when state variables are time-varying are rare in the literature. Maenhout (2006) first use the method of the conditional characteristic functions to characterize detection error probabilities up to the solution to a system of ODEs and up to Fourier inversion. Here, I extend Maenhout's method to recursive preferences and show that the conditional characteristic functions permit analytical solutions, making calculation feasible in practice.

The solutions to the intertemporal consumption and portfolio problem (8) imply that the worst-case alternative model (model \mathbb{Q} hereafter) has the form:

$$dS_t = \left[\mu_t - \frac{\phi}{\gamma + \phi} [(\mu_t - r) + (\mathcal{B}_j + \mathcal{C}_j \mu_t) \rho \sigma_\mu \sigma_S] \right] S_t dt + \sigma_S S_t dB_t \quad (19)$$

$$d\mu_t = \left[\kappa (\bar{\mu} - \mu_t) - \frac{\phi}{\gamma + \phi} \frac{\mu_t - r}{\sigma_S} \rho \sigma_\mu - \phi \left(1 + \frac{1 - \gamma - \phi}{\gamma + \phi} \rho^2 \right) (\mathcal{B}_j + \mathcal{C}_j \mu_t) \sigma_\mu^2 \right] dt + \sigma_\mu \rho dB_t + \sigma_\mu \sqrt{1 - \rho^2} d\hat{B}_t \quad (20)$$

$$j = 0, 1$$

where $\mathcal{B}_0 = B, \mathcal{C}_0 = C$ corresponds to the case of $\psi = 1$, and $\mathcal{B}_1 = -\frac{\hat{B}}{1-\psi}, \mathcal{C}_1 = -\frac{\hat{C}}{1-\psi}$ corresponds to the case of $\psi \neq 1$. It is worth to note that unlike the i.i.d. case considered in Maenhout (2004), the implied drift distortions rely on the state variables not only explicitly but also implicitly through the intertemporal portfolio choice rule. Under recursive preferences, the risk aversion parameter γ and the elasticity of intertemporal substitution ψ are both free parameters affecting the implied drift distortions. In addition, the drift distortions include a myopic distortion and an intertemporal hedging distortion, captured respectively by the first term and the second term of the drift distortions in Equations (19) and (20). Note that the implied drift distortion is independent of the planning horizon since the horizon is infinite and the optimal decision rules are stationary. This simplification allows me to derive exact analytical solutions to the conditional characteristic functions of the Radon-Nikodym derivatives.

The column vector g , defined as $g \equiv \Lambda^\top u$ is given by

$$g(\mu_t) = -\phi \left[\begin{array}{c} \frac{1}{\gamma + \phi} \left(\frac{\mu_t - r}{\sigma_S} + (\mathcal{B}_j + \mathcal{C}_j \mu_t) \rho \sigma_\mu \right) \\ \sigma_\mu \sqrt{1 - \rho^2} (\mathcal{B}_j + \mathcal{C}_j \mu_t) \end{array} \right] \quad (21)$$

with $j = 0, 1$. It follows from (6) and (7) that the log of the Radon-Nikodym derivative of the probability measure of the worst-case model \mathbb{Q} with respect to that of the reference model \mathbb{P} is

$$\zeta_{1,t} \equiv \log \Xi_{1,t} = - \int_0^t g(\mu_s)^\top d\mathbf{B}_s - \frac{1}{2} \int_0^t g(\mu_s)^\top g(\mu_s) ds.$$

The log-likelihood ratio therefore follows a stochastic process with time-varying drift and volatility which are functions of the primitive parameters describing preferences and investment opportunities. Similarly, the log of the Radon-Nikodym derivative $\Xi_{2,t} \equiv \frac{d\mathbb{P}}{d\mathbb{Q}}$ of the probability measure of model \mathbb{P} with respect to that of model \mathbb{Q} is

$$\zeta_{2,t} \equiv \log \Xi_{2,t} = \int_0^t g(\mu_s)^\top d\mathbf{B}_s + \frac{1}{2} \int_0^t g(\mu_s)^\top g(\mu_s) ds.$$

Suppose model \mathbb{P} is true, the investor will mistakenly reject it in favor of model \mathbb{Q} based on a finite time series sample with length N when $\zeta_{1,N} > 0$. Conversely, if model \mathbb{Q} is correct, it will be rejected erroneously when $\zeta_{2,N} > 0$. Let $\{\mathcal{F}_t : t \geq 0\}$ be the filtration with respect to which expectations and probabilities are conditioned.⁶ The detection error probability $\varepsilon_N(\phi)$, which is the time-0 conditional probability of choosing a wrong model based on a finite sample of length N , is defined as

$$\varepsilon_N(\phi) \equiv \frac{1}{2} \Pr(\zeta_{1,N} > 0 \mid \mathbb{P}, \mathcal{F}_0) + \frac{1}{2} \Pr(\zeta_{2,N} > 0 \mid \mathbb{Q}, \mathcal{F}_0).$$

⁶ I maintain the usual assumption that the filtration satisfies the regular conditions.

The definition highlights the dependence of the detection error probability on the uncertainty aversion parameter ϕ . As ϕ increases, the investor desire more robustness. In this case, the worst-case alternative model looks more differently from the reference model and the model selection problem becomes easier for the investor, resulting in a lower detection error probability.

Calculating the detection error probabilities requires the (conditional) cumulative distribution functions of $\zeta_{1,t}$ and $\zeta_{2,t}$. Deriving the distribution of $\zeta_{1,t}$ and $\zeta_{2,t}$ are challenging but fortunately can be achieved by first deriving the conditional characteristic functions and then by Fourier inversion. The characteristic functions of $\zeta_{1,N}$ conditioned on model \mathbb{P} and model \mathbb{Q} are denoted by $\varphi_{\mathbb{P}}(\omega, t, N)$ and $\varphi_{\mathbb{Q}}(\omega, t, N)$ respectively, where ω is the usual transform variable. We can write $\varphi_{\mathbb{P}}(\omega, t, N)$ and $\varphi_{\mathbb{Q}}(\omega, t, N)$ as

$$\varphi_{\mathbb{P}}(\omega, t, N) \equiv \mathbb{E}^{\mathbb{P}}[\exp(i\omega\zeta_{1,N}) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\Xi_{1,N}^{i\omega} \mid \mathcal{F}_t] \quad (22)$$

$$\varphi_{\mathbb{Q}}(\omega, t, N) \equiv \mathbb{E}^{\mathbb{Q}}[\exp(i\omega\zeta_{1,N}) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[\Xi_{1,N}^{i\omega} \mid \mathcal{F}_t] \quad (23)$$

where the second equalities follow from $\zeta_{1,t} \equiv \log \Xi_{1,t}$. Applying the change of measure from \mathbb{Q} to \mathbb{P} , we can rewrite (23) as

$$\varphi_{\mathbb{Q}}(\omega, t, N) \equiv \mathbb{E}^{\mathbb{P}}[\exp(i\omega\zeta_{1,N}) \exp(\zeta_{1,N}) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\Xi_{1,N}^{i\omega+1} \mid \mathcal{F}_t]. \quad (24)$$

It can be shown by the Feynman-Kac theorem the conditional characteristic functions satisfy certain PDEs, known as Kolmogorove backward equations, together with the appropriate boundary conditions. Appendix B displays these equations and conditions. The following proposition presents the exact analytical solutions to Kolmogorove backward equations.

Proposition 4 *The analytical solution for $\varphi_{\mathbb{P}}(\omega, t, N)$ satisfying Equation (37) subject to (38) is given by*

$$\varphi_{\mathbb{P}}(\omega, t, N) = \Xi_{1,N}^{i\omega} \exp\left[D(t) + E(t)\mu_t + \frac{F(t)}{2}\mu_t^2\right] \quad (25)$$

where the functions $D(t)$, $E(t)$ and $F(t)$ satisfy a system of first-order nonlinear complex-valued coefficients ODEs with the explicit solutions given in Appendix B.

The analytical solution for $\varphi_{\mathbb{Q}}(\omega, t, N)$ satisfying Equation (39) subject to (40) is given by

$$\varphi_{\mathbb{Q}}(\omega, t, N) = \Xi_{1,N}^{i\omega+1} \exp\left[G(t) + H(t)\mu_t + \frac{J(t)}{2}\mu_t^2\right] \quad (26)$$

where the functions $G(t)$, $H(t)$ and $J(t)$ satisfy a system of first-order nonlinear complex-valued coefficients ODEs with the explicit solutions given in Appendix B.

Proof. See Appendix B. ■

Based on the result in Maenhout (2006)(Proposition 4), one can obtain the detection error proba-

bility by applying the Levy inversion formula to the conditional characteristic functions. The detection error probability at time zero based on a finite sample of length N is given by

$$\varepsilon_N(\phi) = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \left\{ \operatorname{Re} \left[\frac{\varphi_{\mathbb{Q}}(\omega, 0, N)}{i\omega} \right] - \operatorname{Re} \left[\frac{\varphi_{\mathbb{P}}(\omega, 0, N)}{i\omega} \right] \right\} d\omega \quad (27)$$

where $i \equiv \sqrt{-1}$ and $\operatorname{Re}(\cdot)$ represents the real part of the argument. For the parameter values considered in this paper, the integrand decreases rapidly with the transform variable ω . As a result, the integral in (27) can be computed using quadrature methods.

5 Calibration and Results

Parameter estimates for calibration are drawn from Barberis (2000) (the monthly data span from 1986 to 1995) and Campbell et al. (2004) (the quarterly data span from 1947 to 1995). Both papers estimate VAR models for the market returns, using dividend yield as a predictor. The continuous-time parameters are calculated from the discrete-time estimates using the methodology in Campbell et al. (2004). Table 1 displays the parameter estimates for the two samples.

[Insert Table 1 here]

In this section, I undertake two tasks. First, I analyze the role of risk aversion, uncertainty aversion (the preference for robustness) and intertemporal substitution in the intertemporal consumption and portfolio decisions.⁷ Second, I calibrate the extent of model uncertainty using the detection error probabilities.

[Insert Table 2 and Figure 1 here]

Table 2 and Figure 1 summarize the effect of model uncertainty on the optimal consumption policy. Figure 1 plots the optimal consumption-wealth ratio as a function of the state variable μ_t . A notable result is that whether the optimal consumption-wealth ratio increases or decreases with a change in the investment opportunity set is solely determined by the elasticity of intertemporal substitution, while risk aversion and uncertainty aversion has no effect on the direction of the effect.⁸ When there is an improvement in the investment opportunity set (a higher μ_t), the income effect is to increase the current consumption relative to wealth and the substitution effect tends to reduce the consumption-wealth ratio since investing in equities becomes more attractive. When the investor is

⁷ In this paper, I only show the results based on the parameter estimates of Campbell et al. (2004). The results based on the estimates of Barberis (2000) are very similar and thus not shown here.

⁸ See Campbell and Viceira (1999) and Bhamra and Uppal (2006) for related discussions.

more willing to substitute intertemporally ($\psi > 1$), the substitution effect dominates and the ratio decreases with the state variable. When the investor is less willing to substitute intertemporally ($\psi < 1$), the income effect dominates and the ratio increases with better investment opportunities. When $\psi = 1$, the two effects offset each other and the optimal consumption-wealth ratio is invariant to a change in the investment opportunity set. Risk aversion and uncertainty aversion can change the magnitude of both effects but cannot change the sign of the net effect.

Table 2 reports the exponentiated mean of the optimal log consumption-wealth ratio. The effect of model uncertainty is clear: the desire for robustness increases the average level of consumption in relation to wealth when the elasticity of intertemporal substitution ψ is greater than one, but decreases the ratio when ψ is less than one. The result can be explained by the consideration that the worst-case alternative model indeed worsens the investment opportunity set and persists in the long run. According to the findings of Campbell and Viceira (1999), the consumption-wealth ratio is determined by long-run considerations rather than the current state of the investment opportunity set. When the investor perceives that the model describing investment opportunities is misspecified, two competing effects on the average consumption-wealth ratio arise. First, he cuts consumption because a given quantity of wealth generates a lower flow of consumption under the worst-case alternative model. Second, he increases consumption as investment opportunities seem unfavorable. When $\psi > 1$, the second effect dominates and thus the investor concerned about model uncertainty increases the average consumption-wealth ratio. When $\psi < 1$, the first effect dominates and the ratio declines with the preference for robustness. Furthermore, for those who are extremely unwilling to substitute intertemporally (e.g. $\psi = 1/40$), the effect of model uncertainty is strong as compared to those with a moderately low elasticity of intertemporal substitution (e.g. $\psi = 1/2$). The investor with a low elasticity of intertemporal substitution is unwilling to rearrange consumption over time through saving. When the level of uncertainty aversion rises, he has to reduce consumption a great deal to smooth utility across states. If the investor has a high elasticity of intertemporal substitution, saving can provide a buffer stock for rearranging consumption. In this case, when the investor becomes more uncertainty averse, the incentive to save is enhanced and the consumption-wealth ratio decreases modestly.

Turning to the effect of model uncertainty on portfolio choice, Table 3 and Table 4, respectively, display the mean optimal equity allocations and the mean intertemporal hedging portfolios. Table 5 shows the the proportion of hedging demand in the mean optimal allocation. A striking result is that when the elasticity of intertemporal substitution is about 1 (e.g. $\psi = 1/0.75$ and 1), the

concern about model uncertainty increases the mean optimal equity allocation for moderate levels of risk aversion (e.g. $\gamma = 1.5$ and 2), though for other values of the preference parameters, model uncertainty reduces the mean optimal equity allocation. The result stands in contrast to the previous findings of Maenhout (2004) and Chen and Epstein (2002) that model uncertainty lowers the total equity demand, and also in contrast to the findings of Maenhout (2006) that robustness always reduces the equity demand when the coefficient of relative risk aversion is greater than one. The notable difference stems from a more general model with both time-varying investment opportunities and the distinction between risk aversion and intertemporal substitution.

[Insert Table 3 here]

To understand the effect of model uncertainty on portfolio choice, I first examine the role of the desire for robustness in the intertemporal hedging demand. As discussed above, a robust investor guards against the worst-case alternative model and has a stronger preference for smoothing future utility across states than an expected utility investor. To see this clearly, I write down the discrete-time Bellman equation analogous to (8):

$$V(W_t, \mu_t) = \max_{C_t, \alpha_t} \left\{ f(C_t, V_t) + \mathbb{E}_t^{\mathbb{Q}} [V_{t+\Delta t}] + \frac{1}{\eta(W_t, \mu_t)} \mathbb{E}_t^{\mathbb{Q}} \left[\log \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] \right\}$$

where the last term is relative entropy (expected log likelihood ratio between the worst-case model and the reference model) scaled by the preference for robustness. This term determines the “distance” of the worst-case model to the reference model. Note that a robust investor evaluates future utility $V_{t+\Delta t}$ under the endogenously selected worst-case alternative model, which reinforces his motive to smooth utility across states. Because portfolio decisions are largely driven by the desire to smooth utility across states, the preference for robustness translates into a strong desire to hold a less risky portfolio. As shocks to stock returns and revisions in expected future returns are negatively correlated ($\rho < 0$), stocks tend to have high returns when their expected future returns are low. A portfolio is deemed less risky if it can deliver wealth when investment opportunities are poor, or say when expected returns are low. Thus, the investor desiring robustness optimally increases the investment in the risky asset, leading to a higher intertemporal hedging demand than in the expected utility case when the coefficient of relative risk aversion, γ , is greater than one. When $\gamma < 1$, the effect of robustness obtains in a similar way except for the fact that the initial position in the risky asset that a non-robust investor takes in his hedging portfolio is negative. This is because the non-robust investor with a level of risk aversion less than one prefers a more risky portfolio. As noted by Campbell and Viceira (1999), when the coefficient of relative risk aversion increases further away from unity, a non-robust investor

limits his exposure to the risky asset in all states of the investment opportunity set. In this case, the desire for robustness decreases the intertemporal hedging demand. Thus, different from the findings of Bhamra and Uppal (2006) that the size of risk aversion relative to unity is solely responsible for determining the sign of the intertemporal hedging demand, the results here suggest that the sign is determined by risk aversion together with uncertainty aversion. In particular, the intertemporal hedging demand has a positive sign when the level of effective risk aversion ($\gamma + \phi$) exceeds unity and a negative sign when it is less than unity. Table 5 shows that model uncertainty greatly increases the importance of hedging demand in portfolio decisions. For example, when $\gamma = 2$, $\psi = 1/0.75$, 78% of total equity demand is due to the hedging motive for an investor with uncertainty aversion γ , as compared to 58% for a non-robust investor.

[Insert Table 4 and Table 5 here]

Having analyzed the effect of model uncertainty on the intertemporal hedging demand, one can investigate its effect on the optimal equity allocation. From the expressions for the optimal portfolio formulas (16) and (18), it is obvious that the concern about model uncertainty decreases the myopic allocation. However, as discussed above, uncertainty increases hedging demand for moderate levels of risk aversion. Whether the optimal equity demand increases or decreases with uncertainty depends on the relative strength of the two effects. When the elasticity of intertemporal substitution is near around 1 and the levels of risk aversion are not too high, the latter effect dominates over and thus the desire for robustness increases the optimal equity allocation. This can explain the results shown in Table 3.

I now explore the role of intertemporal substitution in the intertemporal hedging demand taking into account model uncertainty. As noted by Bhamra and Uppal (2006), the elasticity of intertemporal substitution affects the magnitude of the intertemporal hedging demand without changing its sign. The reason is that the parameter controlling the attitude toward intertemporal substitution, ψ , determines the desire to smooth consumption over time but not the desire to smooth future utility across states. Future utility depends on a function of the state (μ_t) multiplied by the power function of the portfolio return in that state. The function of the state is where the elasticity of intertemporal substitution appears (see the value functions (14) and (17)). An investor who is unwilling to substitute intertemporally wants to maintain a stable consumption growth rate and only has modest motive to hedge against the change in the investment opportunity set. As a result, when the desire for robustness enhances the hedging motive, its effect on the intertemporal hedging demand is stronger for investors who are more willing to substitute intertemporally than those who are less willing to do so. For

example, Table 3 shows that a robust investor with $\psi = 1/0.75$ and $\gamma = 1.5$ increases his hedging demand from 101% to 185% when the uncertainty aversion parameter ϕ increases to a level of γ , while an investor $\psi = 1/40$ and $\gamma = 1.5$ increases his hedging demand from 50% to 95%.

For the CRRA utility function, Maenhout (2006) shows that robustness always leads to a lower optimal equity allocation when the coefficient of relative risk aversion is greater than one. To reconcile the seeming contradiction, I argue that Maenhout's result is partly driven by the effect of intertemporal substitution, not entirely by the effect of robustness. The Duffie-Epstein's recursive preferences allow me to disentangle the attitude toward intertemporal substitution and risk aversion, and enable me to gain a better understanding of the effect of model uncertainty on portfolio choice. The CRRA utility function implies that the elasticity of intertemporal substitution and the coefficient of risk aversion are inversely related ($\psi = 1/\gamma$). A high level of risk aversion implies low willingness to substitute intertemporally, for which the effect of robustness on the intertemporal hedging demand is underplayed. When the effect is not strong enough, it will be dominated by the effect of model uncertainty on the myopic equity demand. Thus, the net effect is to reduce the optimal equity demand, as observed by Maenhout (2006). By disentangling risk aversion and intertemporal substitution, the effect of intertemporal substitution is controlled for and I can therefore obtain a clean result on the effect of model uncertainty.

To examine the state-dependence of portfolio decisions, Figure 2, 3 and 4 plot the optimal equity allocation and the intertemporal hedging demand as functions of the state variable for different elasticities of intertemporal substitution (Figure 2 for $\psi = 1/0.75$, Figure 3 for $\psi = 1$ and Figure 4 for $\psi = 1/4$). All graphs show that the amount of wealth allocated by the non-robust log investor to hedge against the change in the investment opportunity set is increasing in the state. However, the concern about model uncertainty does not necessarily raise the total equity demand above the myopic demand for all states. For $\psi = 1$, the optimal demand exceeds the myopic demand when the state variable is less than four-fifth of standard deviation beyond its long-run mean, while for $\psi = 1/4$ the optimal demand increases above the myopic demand when the state variable is higher than 1 standard deviation below its long-run mean. The figures also show that for a high level of risk aversion, the preference for robustness reduces the total demand for equities and the magnitude of the effect is stronger for higher values of the state variable. In other words, the slope of the optimal demand for equities declines in the presence of model uncertainty and this property also holds true for the log investor. Thus, an investor worrying model uncertainty responds less sensitively to a change in the investment opportunity set.

[Insert Figure 2, Figure 3 and Figure 4 here]

Although model uncertainty can have theoretically sizable effect on portfolio decisions (e.g. turning highly levered portfolios into empirically reasonable ones), its empirical importance cannot be rigorously assessed without a thoughtful calibration analysis. Finally, I investigate the extent of the impact of robustness based on the detection error probabilities. Anderson et al. (2000) advocate a threshold of 10% for the detection error probability $\varepsilon_N(\phi)$.

Table 5 reports $\varepsilon_N(\phi)$ for $\gamma = \{0.75, 1, 1.5, 2, 4, 6\}$ and $\psi = \{\frac{1}{0.75}, 1, \frac{1}{1.5}, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}\}$ for the short sample (monthly data from 1986 to 1995) in Barberis (2000). The values of the uncertainty aversion parameter ϕ are chosen such that $\varepsilon_N(\phi)$ stays beyond the threshold. A striking result is that the short sample allows little room for a concern about model uncertainty. To achieve meaningful detection error probabilities, the postulated uncertainty aversion parameter is of the second order as compared to the risk aversion parameter γ . For example, when $\gamma = 6$ and $\psi = 1/0.75$, the investor's concern about model misspecification can at best lead to a value of ϕ that is about a quarter of γ , which implies an effective risk aversion of $\gamma \times 1.25$. The scenario for the long sample is even worse: the concern about model misspecification can only support a level of uncertainty aversion that is about 1/10 of γ . The effect of robustness is therefore of a lower order of magnitude. These findings are in contrast to the results in Maenhout (2004) for i.i.d. returns, where a large magnitude of ϕ (larger than γ) can be allowed for, while strengthen the results in Maenhout (2006) for mean-reverting returns, where model detection becomes easier and the significance of model uncertainty shrinks. The little room for model uncertainty indicates that detecting model misspecification turns out to be very easy for a robust investor with a long horizon.

As noted by Maenhout (2006), the main reasons driving down the scope of model uncertainty when expected returns are mean-reverting include (1) the continuous observability of the state variable, (2) negative autocorrelation in returns and (3) the presence of the portfolio choice in the drift distortion⁹. The drift distortion implied by the worst-case model (19)–(20) depends on the portfolio rule. When expected returns are time-varying, a robust investor invest significantly more to the risky asset than in the i.i.d. case, due to the intertemporal hedging demand. In our model with recursive preferences and infinite horizon, the intertemporal hedging demand plays a more important role than in Maenhout (2006) with the CRRA preferences and finite horizon. This results in a larger drift distortion and more easily detectable model misspecification. Table 5 and Table 6 also show that as the elasticity of

⁹ As shown in (19)–(20), the drift distortion consists of a nonlinear function of the state variable which is continuously observable, resulting in a easier model selection problem than in the i.i.d. case. Mean-reverting returns exhibit negative autocorrelation, which facilitates the estimation of the mean of a time series data. See Maenhout (2006) for further details.

intertemporal substitution increases, detecting model misspecification turns out to be more difficult. The reason is that the investor puts less weight on the intertemporal hedging demand in his portfolio decisions as he becomes more reluctant to substitute intertemporally. As a result, the drift distortion is smaller and model selection is more obscure to the investor.

6 Conclusion

I have examined a continuous-time intertemporal consumption and portfolio choice problem in which investment opportunities are time-varying and the roles of risk aversion, uncertainty aversion and intertemporal substitution are distinguished among each other. The investor worries that his model of asset returns may be misspecified and wants robust portfolio rules that take into account model uncertainty. In contrast to the previous findings in the literature, I find that uncertainty aversion increases the total equity demand for moderately risk-averse investor when the elasticity of intertemporal substitution is about 1. The result is mainly driven by the effect of the concern about model uncertainty on the intertemporal hedging demand. Using the technique of detection error probabilities, I calibrate the quantitative effect of model uncertainty and find that the scope of model uncertainty is very small. My findings suggest that introducing model uncertainty and uncertainty aversion is insufficient to explain the low equity demand prevailing in the real world when investment opportunities are time-varying and investors have a long horizon.

In this paper, I assume that the investment opportunity set is fully observable. An interesting extension for future research is to introduce elements of learning into the model with robustness concerns. Intertemporal hedging of time-variation in perceived investment opportunity sets may generate interesting implications for dynamic portfolio choice. In addition, large scope of model uncertainty may be allowed for in a model with hidden states. Extensions along the lines of Ju and Miao (2009) and Chen, Ju and Miao (2009) are very promising.

A Proof of Proposition 1, Proposition 2 and Proposition 3

Proof of Proposition 1

Proof. Substitution of (14) into the first order conditions for consumption and portfolio choice (11) and (12) leads to the optimal consumption policy $C^* = \beta W$ and the optimal portfolio rule

$$\alpha_t = \frac{\mu_t - r}{(\gamma + \phi) \sigma_S^2} + \frac{1 - \gamma - \phi}{\gamma + \phi} \frac{\rho \sigma_\mu}{\sigma_S} (B + C \mu_t).$$

Substituting (14) and (15) into the HJB equation (9), I obtain the following equation after simplification:

$$\begin{aligned} 0 = & \beta \log \beta - \beta \left(A + B \mu_t + \frac{C}{2} \mu_t^2 \right) - \beta + r + (\mu_t - r) \alpha_t + \kappa (\bar{\mu} - \mu_t) (B + C \mu_t) + \frac{\sigma_\mu^2}{2} C \\ & + \frac{\sigma_\mu^2}{2} [1 - (\gamma + \phi)] (B + C \mu_t)^2 - \frac{1}{2} (\gamma + \phi) \sigma_S^2 \alpha_t^2 + [1 - (\gamma + \phi)] \rho \sigma_S \sigma_\mu \alpha_t (B + C \mu_t). \end{aligned}$$

Collecting terms in μ_t^2 , μ_t and constant terms, I can obtain A , B and C as the solution to the following system of recursive equations:

$$\begin{aligned} 0 = & \frac{1}{2(\gamma + \phi) \sigma_S^2} + \left(-\frac{\beta}{2} - \kappa + \frac{1 - (\gamma + \phi) \rho \sigma_\mu}{\gamma + \phi} \right) C \\ & + \frac{\sigma_\mu^2}{2} \left\{ [1 - (\gamma + \phi)] + \frac{(1 - (\gamma + \phi))^2}{\gamma + \phi} \rho^2 \right\} C^2 \end{aligned} \quad (28)$$

$$\begin{aligned} 0 = & -\frac{r}{(\gamma + \phi) \sigma_S^2} + \left(\kappa \bar{\mu} - \frac{1 - (\gamma + \phi) \rho r \sigma_\mu}{\gamma + \phi} \right) C \\ & + \left\{ -\kappa - \beta + \frac{1 - (\gamma + \phi) \rho \sigma_\mu}{\gamma + \phi} \frac{\rho \sigma_\mu}{\sigma_S} + \sigma_\mu^2 \left[1 - (\gamma + \phi) + \frac{(1 - (\gamma + \phi))^2}{\gamma + \phi} \rho^2 \right] C \right\} B \end{aligned} \quad (29)$$

$$\begin{aligned} 0 = & \beta (\log \beta - 1) + r + \frac{r^2}{2(\gamma + \phi) \sigma_S^2} - \beta A + \frac{\sigma_\mu^2}{2} C + \left(\kappa \bar{\mu} - \frac{1 - (\gamma + \phi) \rho r \sigma_\mu}{\gamma + \phi} \right) B \\ & + \frac{\sigma_\mu^2}{2} \left(1 - (\gamma + \phi) + \frac{(1 - (\gamma + \phi))^2}{\gamma + \phi} \rho^2 \right) B^2. \end{aligned} \quad (30)$$

Equation (28) is a quadratic equation which has two roots. The solution to C is the root associated with the positive discriminant of the equation. This root also ensures that $C = 0$ when $\gamma + \phi = 1$.

The solutions to A and B follow immediately from Equations (29) and (30).

Next, I show that the solution to the optimal portfolio rule when $\psi = 1$ also nests the case of logarithmic utility ($\psi = \gamma = 1$). When $\gamma \rightarrow 1$, the preference parameter η assumed in (13) converges to ϕ based on the solution to the value function when $\gamma \neq 1$:

$$\eta_{\log} = \lim_{\gamma \rightarrow 1} \frac{\phi}{W^{1-\gamma} \exp \left[(1-\gamma) \left(A + B \mu_t + \frac{C}{2} \mu_t^2 \right) \right]} = \phi.$$

Then one can show the HJB equation (9) can be solved by $V(W_t, \mu_t) = \log W + A + B\mu_t + \frac{C}{2}\mu_t^2$, where A , B and C solve the system of recursive equations (28)–(30) with $\gamma = 1$. The optimal portfolio rule is then given by

$$\alpha_t = \frac{\mu_t - r}{(1 + \phi)\sigma_S^2} - \frac{\phi}{1 + \phi} \frac{\rho\sigma_\mu}{\sigma_S} (B + C\mu_t).$$

The proof is complete. ■

Proof of Proposition 2

Proof. Substituting the value function (17) into the HJB equation (9), I obtain, after some simplification, the following ordinary differential equation

$$\begin{aligned} 0 = & -\beta^\psi K^{-1} + \beta\psi + r(1 - \psi) + \frac{1 - \psi}{2(\gamma + \phi)} \left(\frac{\mu_t - r}{\sigma_S} \right)^2 - \frac{\sigma_\mu^2}{2} \frac{K_{\mu\mu}}{K} \\ & + \frac{\sigma_\mu^2}{2} \left[1 + \frac{1 - \gamma - \phi}{1 - \psi} \left(1 + \frac{1 - \gamma - \phi}{\gamma + \phi} \rho^2 \right) \right] \left(\frac{K_\mu}{K} \right)^2 \\ & - \left[\kappa(\bar{\mu} - \mu_t) + \frac{1 - \gamma - \phi}{\gamma + \phi} \rho\sigma_\mu \left(\frac{\mu_t - r}{\sigma_S} \right) \right] \frac{K_\mu}{K}. \end{aligned} \quad (31)$$

This equation does not admit an exact analytical solution because the appearance of the term $\beta^\psi K^{-1}$. But, I can use the log-linear approximation method described in Campbell et al. (2004) to find an approximate analytical solution. Note that the term $\beta^\psi K^{-1}$ is exactly the consumption-wealth ratio, which can be approximated around the unconditional mean of the log consumption-wealth ratio as:

$$\beta^\psi K(\mu_t)^{-1} \approx k_0 + k_1(\psi \log \beta - k_t) \quad (32)$$

with $k_1 = \exp\{\mathbb{E}[c_t - w_t]\}$ and $k_0 = k_1(1 - \log k_1)$ where $c_t = \log C_t$, $w_t = \log W_t$, $k_t = \log K(\mu_t)$. Substituting the approximation (32) for the first term of (31) gives another ODE which admits an exact analytical solution. I conjecture $K(\mu_t)$ is of the exponential-quadratic form

$$K(\mu_t) = \exp \left\{ \hat{A} + \hat{B}\mu_t + \frac{\hat{C}}{2}\mu_t^2 \right\}. \quad (33)$$

Substitution of (33) into ODE (31) with the log-linear approximation (32) transform the Bellman equation into the following equation

$$\begin{aligned} 0 = & -k_0 - k_1 \left\{ \psi \log \beta - \left(\hat{A} + \hat{B} + \frac{\hat{C}}{2}\mu_t^2 \right) \right\} + \beta\psi + r(1 - \psi) - \kappa(\bar{\mu} - \mu_t) (\hat{B} + \hat{C}\mu_t) \\ & + \frac{\sigma_\mu^2}{2} \left\{ - \left[(\hat{B} + \hat{C}\mu_t)^2 + \hat{C} \right] + \left(\frac{1 - \gamma - \phi}{1 - \psi} + 1 \right) (\hat{B} + \hat{C}\mu_t)^2 \right\} + \frac{1 - \psi}{2(\gamma + \phi)} \left(\frac{\mu_t - r}{\sigma_S} \right)^2 \\ & - \frac{1 - \gamma - \phi}{\gamma + \phi} (\hat{B} + \hat{C}\mu_t) \rho\sigma_\mu \left(\frac{\mu_t - r}{\sigma_S} \right) + \frac{(1 - \gamma - \phi)^2}{2(\gamma + \phi)(1 - \psi)} (\hat{B} + \hat{C}\mu_t)^2 \rho^2 \sigma_\mu^2. \end{aligned}$$

Collecting terms in μ_t^2 , μ_t and constant terms, I can obtain \hat{A} , \hat{B} and \hat{C} as the solution to the following system of recursive equations:

$$0 = \frac{1-\psi}{2(\gamma+\phi)\sigma_S^2} + \left(\frac{k_1}{2} + \kappa - \frac{1-\gamma-\phi}{\gamma+\phi} \frac{\rho\sigma_\mu}{\sigma_S} \right) \hat{C} + \frac{\sigma_\mu^2}{2} \frac{1-\gamma-\phi}{1-\psi} \left(1 + \frac{1-\gamma-\phi}{\gamma+\phi} \rho^2 \right) \hat{C}^2 \quad (34)$$

$$0 = -\frac{1-\psi}{\gamma+\phi} \frac{r}{\sigma_S^2} - \left(\kappa\bar{\mu} + \frac{1-\gamma-\phi}{\gamma+\phi} \frac{\rho r \sigma_\mu}{\sigma_S} \right) \hat{C} + (\kappa + k_1) \hat{B} + \left\{ \frac{1-\gamma-\phi}{\gamma+\phi} \frac{\rho\sigma_\mu}{\sigma_S} + \sigma_\mu^2 \left[1 + \frac{1-\gamma-\phi}{1-\psi} \left(1 + \frac{1-\gamma-\phi}{\gamma+\phi} \rho^2 \right) \right] \right\} \hat{C} \hat{B} \quad (35)$$

$$0 = k_1 \hat{A} - k_0 - k_1 \psi \log \beta + \beta \psi + r(1-\psi) + \frac{1-\psi}{2(\gamma+\phi)} \frac{r^2}{\sigma_S^2} - \frac{\sigma_\mu^2}{2} \hat{C} + \frac{\sigma_\mu^2}{2} \frac{1-\gamma-\phi}{1-\psi} \left(1 + \frac{1-\gamma-\phi}{\gamma+\phi} \rho^2 \right) \hat{B}^2 + \left(\frac{1-\gamma-\phi}{\gamma+\phi} \frac{\rho r \sigma_\mu}{\sigma_S} - \kappa\bar{\mu} \right) \hat{B} \quad (36)$$

where the solution to \hat{C} is the root of the quadratic equation (34) that is associated with the positive root of the discriminant of the equation. The solutions to \hat{A} and \hat{B} follow immediately.

To obtain the optimal solutions to \hat{A} , \hat{B} and \hat{C} , I use a numerical procedure suggested in Campbell and Viceira (1999). I first set the exponentiated mean log consumption-wealth ratio (k_1) to β , which is the exact optimal solution for $\psi = 1$ and then find the solutions of $\{\hat{A}, \hat{B}, \hat{C}\}$ given this value of k_1 . These solutions imply a new value of k_1 , for which a new set of solutions to \hat{A} , \hat{B} and \hat{C} is computed. This recursion proceeds until the absolute value of the difference between two consecutive values of k_1 is less than a small tolerance value. Finally, given the solutions to \hat{A} , \hat{B} and \hat{C} , the optimal portfolio rule is

$$\alpha_t = \frac{\mu_t - r}{(\gamma + \phi) \sigma_S^2} - \frac{1}{1 - \psi} \frac{1 - \gamma - \phi}{\gamma + \phi} \frac{\rho \sigma_\mu}{\sigma_S} \left(\hat{B} + \hat{C} \mu_t \right).$$

The results shown above also nest the case of $\gamma = 1$, for which the preference parameter η when $\gamma = 1$, $\eta_{\gamma \rightarrow 1}$, is given by

$$\eta_{\gamma \rightarrow 1} = \lim_{\gamma \rightarrow 1} \frac{\phi}{\gamma^{1-\gamma} W^{1-\gamma} \exp \left[-\frac{1-\gamma}{1-\psi} \left(\hat{A} + \hat{B} \mu_t + \frac{\hat{C}}{2} \mu_t^2 \right) \right]} = \phi.$$

One can show that the HJB equation (9) can be solved by the value function of the form

$$V(W_t, \mu_t) = \log W - \frac{1}{1-\psi} \left(\hat{A} + \hat{B} \mu_t + \frac{\hat{C}}{2} \mu_t^2 \right)$$

together with a loglinear approximation. The optimal values of \hat{A} , \hat{B} and \hat{C} can be solved by iterating

the system of equations with $\gamma = 1$. The optimal portfolio rule is given by

$$\alpha_t = \frac{\mu_t - r}{(1 + \phi) \sigma_S^2} + \frac{1}{1 - \psi} \frac{\phi}{1 + \phi} \frac{\rho \sigma_\mu}{\sigma_S} (\hat{B} + \hat{C} \mu_t).$$

The proof is complete. ■

Proof of Proposition 3

Proof. Because the parameter ϕ appears additively to γ everywhere in Equations (28), (29), (30), (34), (35) and (36) and the optimal portfolio rules (16) and (18), it is clear that the optimal decision rules for a robust investor with risk aversion $\bar{\gamma}$ and uncertainty aversion $\bar{\phi}$ are identical to the optimal decision rules of a non-robust investor with risk aversion $\bar{\gamma} + \bar{\phi}$. ■

B Proof of Proposition 4

Proof. According to the Feynman-Kac theorem, the conditional expectation in (22) can be characterized as the solution to the following PDE

$$\begin{aligned} 0 &= \mathcal{D}\varphi_P(\omega, t, N) \\ &= \frac{1}{2} \Xi_{1,t}^2 \frac{\partial^2 \varphi_P}{\partial \Xi_{1,t}^2} g^\top g + \frac{1}{2} \frac{\partial^2 \varphi_P}{\partial \mu_t^2} \sigma_\mu^2 - \Xi_{1,t} \frac{\partial^2 \varphi_P}{\partial \Xi_{1,t} \partial \mu_t} g^\top \hat{\sigma} + \frac{\partial \varphi_P}{\partial \mu_t} \kappa (\bar{\mu} - \mu_t) + \frac{\partial \varphi_P}{\partial t} \end{aligned} \quad (37)$$

subject to the boundary condition

$$\varphi_P(\omega, N, N) = \Xi_{1,N}^{i\omega} \quad (38)$$

where $\hat{\sigma} \equiv \begin{bmatrix} \sigma_\mu \rho & \sigma_\mu \sqrt{1 - \rho^2} \end{bmatrix}^\top$ and the drift distortion vector g is given in (21).

Similarly, one can solve for the conditional expectation in (24) by solving the following PDE

$$\begin{aligned} 0 &= \mathcal{D}\varphi_Q(\omega, t, N) \\ &= \frac{1}{2} \Xi_{1,t}^2 \frac{\partial^2 \varphi_Q}{\partial \Xi_{1,t}^2} g^\top g + \frac{1}{2} \frac{\partial^2 \varphi_Q}{\partial \mu_t^2} \sigma_\mu^2 - \Xi_{1,t} \frac{\partial^2 \varphi_Q}{\partial \Xi_{1,t} \partial \mu_t} g^\top \hat{\sigma} + \frac{\partial \varphi_Q}{\partial \mu_t} \kappa (\bar{\mu} - \mu_t) + \frac{\partial \varphi_Q}{\partial t} \end{aligned} \quad (39)$$

with the boundary condition

$$\varphi_Q(\omega, N, N) = \Xi_{1,N}^{i\omega+1}. \quad (40)$$

I first solve for the analytical solution to the PDE for $\varphi_P(\omega, t, N)$. The solution to the PDE for $\varphi_Q(\omega, t, N)$ follows in a similar way.

I conjecture that the solution to $\varphi_P(\omega, t, N)$ is of the exponential-quadratic form (25) and then

verify it indeed delivers an analytical solution. Substituting (25) into PDE (37) gives

$$0 = (E(t) + F(t) \mu_t) \left[\kappa (\bar{\mu} - \mu_t) - i\omega g^\top \hat{\sigma} \right] + \frac{1}{2} i\omega (i\omega - 1) g^\top g \\ + \frac{1}{2} \left[F(t) + (E(t) + F(t) \mu_t)^2 \right] \sigma_\mu^2 + D'(t) + E'(t) \mu_t + \frac{\mu_t^2}{2} F'(t).$$

Plugging vector g and collecting terms in μ_t^2 , μ_t and the constant delivers the following system of first-order nonlinear ODEs with complex-valued coefficients:

$$0 = F'(t) + \sigma_\mu^2 F(t)^2 + 2 \left[i\omega \phi \left(\gamma_0 \frac{\rho \sigma_\mu}{\sigma_S} + \sigma_\mu^2 \gamma_1 \mathcal{C}_j \right) - \kappa \right] F(t) \\ + i\omega (i\omega - 1) \phi^2 \left[\gamma_0^2 \left(\frac{1}{\sigma_S^2} + \frac{2\rho \sigma_\mu \mathcal{C}_j}{\sigma_S} \right) + \sigma_\mu^2 \gamma_2 \mathcal{C}_j^2 \right] \quad (41)$$

$$0 = E'(t) - \left[i\omega \phi \left(\gamma_0 r \frac{\rho \sigma_\mu}{\sigma_S} - \sigma_\mu^2 \gamma_1 \mathcal{B}_j \right) - \kappa \bar{\mu} \right] F(t) + \left[i\omega \phi \left(\gamma_0 \frac{\rho \sigma_\mu}{\sigma_S} + \sigma_\mu^2 \gamma_1 \mathcal{C}_j \right) - \kappa \right] E(t) \\ + \sigma_\mu^2 E(t) F(t) + i\omega (i\omega - 1) \phi^2 \left[\gamma_0^2 \left(\frac{\rho \sigma_\mu}{\sigma_S} (\mathcal{B}_j - r \mathcal{C}_j) - \frac{r}{\sigma_S^2} \right) + \sigma_\mu^2 \gamma_2 \mathcal{B}_j \mathcal{C}_j \right] \quad (42)$$

$$0 = D'(t) + \frac{1}{2} i\omega (i\omega - 1) \phi^2 \left[\gamma_0^2 \left(\frac{r^2}{\sigma_S^2} - \frac{2\rho \sigma_\mu r \mathcal{B}_j}{\sigma_S} \right) + \sigma_\mu^2 \gamma_2 \mathcal{B}_j^2 \right] + \frac{1}{2} \sigma_\mu^2 \left[E(t)^2 + F(t) \right] \\ + i\omega \phi \left(\sigma_\mu^2 \gamma_1 \mathcal{B} - \gamma_0 r \frac{\rho \sigma_\mu}{\sigma_S} \right) E(t) + \kappa \bar{\mu} E(t) \quad (43)$$

with

$$\gamma_0 = \frac{1}{\gamma + \phi} \\ \gamma_1 = \left[1 + \frac{1 - \gamma - \phi}{\gamma + \phi} \rho^2 \right] \\ \gamma_2 = \left[1 + \frac{1 - (\gamma + \phi)^2}{(\gamma + \phi)^2} \rho^2 \right]$$

and the boundary condition

$$D(N) = E(N) = F(N) = 0$$

where $j = \{0, 1\}$ with $\{\mathcal{B}_0 = B, \mathcal{C}_0 = C\}$ for $\psi = 1$ and $\left\{ \mathcal{B}_1 = -\frac{\hat{B}}{1-\psi}, \mathcal{C}_1 = -\frac{\hat{C}}{1-\psi} \right\}$ for $\psi \neq 1$.

The above system of ODEs permits an analytical solution. Define

$$\begin{aligned}
\beta_0 &= i\omega(i\omega - 1)\phi^2 \left[\gamma_0^2 \left(\frac{1}{\sigma_S^2} + \frac{2\rho\sigma_\mu\mathcal{C}_j}{\sigma_S} \right) + \sigma_\mu^2\gamma_2\mathcal{C}_j^2 \right] \\
\beta_1 &= 2 \left[i\omega\phi \left(\gamma_0 \frac{\rho\sigma_\mu}{\sigma_S} + \sigma_\mu^2\gamma_1\mathcal{C}_j \right) - \kappa \right] \\
\beta_2 &= \sigma_\mu^2 \\
\beta_3 &= -i\omega\phi \left(\gamma_0 r \frac{\rho\sigma_\mu}{\sigma_S} - \sigma_\mu^2\gamma_1\mathcal{B}_j \right) + \kappa\bar{\mu} \\
\beta_4 &= i\omega(i\omega - 1)\phi^2 \left[\gamma_0^2 \left(\frac{\rho\sigma_\mu}{\sigma_S} (\mathcal{B}_j - r\mathcal{C}_j) - \frac{r}{\sigma_S^2} \right) + \sigma_\mu^2\gamma_2\mathcal{B}_j\mathcal{C}_j \right]
\end{aligned}$$

and

$$\lambda = \sqrt{\beta_1^2 - 4\beta_0\beta_2}.$$

The solution to $F(t)$ is given by

$$F(t) = \frac{2\beta_0(1 - e^{-\lambda(T-t)})}{2\lambda - (\beta_1 + \lambda)(1 - e^{-\lambda(T-t)})}.$$

To solve for $E(t)$, I conjecture it has the following form

$$E(t) = \frac{a_0 + a_1e^{-\lambda(T-t)/2} + a_2e^{-\lambda(T-t)}}{\lambda[2\lambda - (\beta_1 + \lambda)(1 - e^{-\lambda(T-t)})]}. \quad (44)$$

Substituting (44) back into ODE (42) and matching coefficients, I obtain

$$\begin{aligned}
E(t) &= \frac{1}{\lambda[2\lambda - (\beta_1 + \lambda)(1 - e^{-\lambda(T-t)})]} \left[(4\beta_0\beta_3 + 2\beta_4(\lambda - \beta_1)) \right. \\
&\quad \left. - 4(2\beta_0\beta_3 - \beta_1\beta_4)e^{-\lambda(T-t)/2} + (4\beta_0\beta_3 - 2\beta_4(\beta_1 + \lambda))e^{-\lambda(T-t)} \right].
\end{aligned}$$

The solution to $D(t)$ is then

$$\begin{aligned}
D(t) &= \int_t^T \left\{ -\frac{1}{2}i\omega(i\omega - 1)\phi^2 \left[\gamma_0^2 \left(\frac{r^2}{\sigma_S^2} - \frac{2\rho\sigma_\mu}{\sigma_S}r\mathcal{B}_j \right) + \sigma_\mu^2\gamma_2\mathcal{B}_j^2 \right] \right. \\
&\quad \left. - \frac{1}{2}\sigma_\mu^2 \left[E(\tau)^2 + F(\tau) \right] - i\omega\phi \left(\sigma_\mu^2\gamma_1\mathcal{B}_j - \gamma_0 r \frac{\rho\sigma_\mu}{\sigma_S} \right) E(\tau) - \kappa\bar{\mu}E(\tau) \right\} d\tau.
\end{aligned}$$

Similarly, one can obtain the analytical solution to $\varphi_Q(\omega, t, N)$ of the form

$$\varphi_Q(\omega, t, N) = \Xi_{1,t}^{i\omega+1} \exp \left[G(t) + H(t)\mu_t + \frac{J(t)}{2}\mu_t^2 \right]$$

with $J(t), H(t)$ and $G(t)$ satisfying the following system of ODEs

$$\begin{aligned}
0 &= J'(t) + \sigma_\mu^2 J(t)^2 + 2 \left[(i\omega + 1) \phi \left(\gamma_0 \frac{\rho\sigma_\mu}{\sigma_S} + \sigma_\mu^2 \gamma_1 \mathcal{C}_j \right) - \kappa \right] J(t) \\
&\quad + i\omega (i\omega + 1) \phi^2 \left[\gamma_0^2 \left(\frac{1}{\sigma_S^2} + \frac{2\rho\sigma_\mu \mathcal{C}_j}{\sigma_S} \right) + \sigma_\mu^2 \gamma_2 \mathcal{C}_j^2 \right] \\
0 &= H'(t) - \left[(i\omega + 1) \phi \left(\gamma_0 r \frac{\rho\sigma_\mu}{\sigma_S} - \sigma_\mu^2 \gamma_1 \mathcal{B}_j \right) - \kappa \bar{\mu} \right] J(t) + \left[(i\omega + 1) \phi \left(\gamma_0 \frac{\rho\sigma_\mu}{\sigma_S} + \sigma_\mu^2 \gamma_1 \mathcal{C}_j \right) - \kappa \right] H(t) \\
&\quad + \sigma_\mu^2 H(t) J(t) + i\omega (i\omega + 1) \phi^2 \left[\gamma_0^2 \left(\frac{\rho\sigma_\mu}{\sigma_S} (\mathcal{B}_j - r\mathcal{C}_j) - \frac{r}{\sigma_S^2} \right) + \sigma_\mu^2 \gamma_2 \mathcal{B}_j \mathcal{C}_j \right] \\
0 &= G'(t) + \frac{1}{2} i\omega (i\omega + 1) \phi^2 \left[\gamma_0^2 \left(\frac{r^2}{\sigma_S^2} - \frac{2\rho\sigma_\mu r \mathcal{B}_j}{\sigma_S} \right) + \sigma_\mu^2 \gamma_2 \mathcal{B}_j^2 \right] + \frac{1}{2} \sigma_\mu^2 [H(t)^2 + J(t)] \\
&\quad + (i\omega + 1) \phi \left(\sigma_\mu^2 \gamma_1 \mathcal{B}_j - \gamma_0 r \frac{\rho\sigma_\mu}{\sigma_S} \right) H(t) + \kappa \bar{\mu} H(t).
\end{aligned}$$

Define

$$\begin{aligned}
\tilde{\beta}_0 &= i\omega (i\omega + 1) \phi^2 \left[\gamma_0^2 \left(\frac{1}{\sigma_S^2} + \frac{2\rho\sigma_\mu \mathcal{C}_j}{\sigma_S} \right) + \sigma_\mu^2 \gamma_2 \mathcal{C}_j^2 \right] \\
\tilde{\beta}_1 &= 2 \left[(i\omega + 1) \phi \left(\gamma_0 \frac{\rho\sigma_\mu}{\sigma_S} + \sigma_\mu^2 \gamma_1 \mathcal{C}_j \right) - \kappa \right] \\
\tilde{\beta}_2 &= \sigma_\mu^2 \\
\tilde{\beta}_3 &= -(i\omega + 1) \phi \left(\gamma_0 r \frac{\rho\sigma_\mu}{\sigma_S} - \sigma_\mu^2 \gamma_1 \mathcal{B}_j \right) + \kappa \bar{\mu} \\
\tilde{\beta}_4 &= i\omega (i\omega + 1) \phi^2 \left[\gamma_0^2 \left(\frac{\rho\sigma_\mu}{\sigma_S} (\mathcal{B}_j - r\mathcal{C}_j) - \frac{r}{\sigma_S^2} \right) + \sigma_\mu^2 \gamma_2 \mathcal{B}_j \mathcal{C}_j \right]
\end{aligned}$$

and

$$\tilde{\lambda} = \sqrt{\tilde{\beta}_1^2 - 4\tilde{\beta}_0\tilde{\beta}_2}.$$

The analytical solutions to $J(t), H(t)$ and $G(t)$ are, respectively,

$$\begin{aligned}
J(t) &= \frac{2\tilde{\beta}_0 (1 - e^{-\tilde{\lambda}(T-t)})}{2\tilde{\lambda} - (\tilde{\beta}_1 + \tilde{\lambda}) (1 - e^{-\tilde{\lambda}(T-t)})} \\
H(t) &= \frac{1}{\tilde{\lambda} [2\tilde{\lambda} - (\tilde{\beta}_1 + \tilde{\lambda}) (1 - e^{-\tilde{\lambda}(T-t)})]} \left[(4\tilde{\beta}_0\tilde{\beta}_3 + 2\tilde{\beta}_4 (\tilde{\lambda} - \tilde{\beta}_1)) \right. \\
&\quad \left. - 4 (2\tilde{\beta}_0\tilde{\beta}_3 - \tilde{\beta}_1\tilde{\beta}_4) e^{-\tilde{\lambda}(T-t)/2} + (4\tilde{\beta}_0\tilde{\beta}_3 - 2\tilde{\beta}_4 (\tilde{\beta}_1 + \tilde{\lambda})) e^{-\tilde{\lambda}(T-t)} \right] \\
G(t) &= \int_t^T \left\{ -\frac{1}{2} i\omega (i\omega + 1) \phi^2 \left[\gamma_0^2 \left(\frac{r^2}{\sigma_S^2} - \frac{2\rho\sigma_\mu r \mathcal{B}_j}{\sigma_S} \right) + \sigma_\mu^2 \gamma_2 \mathcal{B}_j^2 \right] \right. \\
&\quad \left. - \frac{1}{2} \sigma_\mu^2 [H(\tau)^2 + J(\tau)] - (i\omega + 1) \phi \left(\sigma_\mu^2 \gamma_1 \mathcal{B}_j - \gamma_0 r \frac{\rho\sigma_\mu}{\sigma_S} \right) H(\tau) - \kappa \bar{\mu} H(\tau) \right\} d\tau.
\end{aligned}$$

The proof is complete. ■

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Table 1: This table presents two sets of parameter estimates used in the calibration analysis. The first set of estimates is drawn from Campbell et al. (2004) for the quarterly data from 1947:1–1995:4, and the second set is based on the estimates in Barberis (2000) for the monthly data from 1986:1–1995:12. Campbell et al. (2004) describes the details of recovering continuous-time estimates from discrete-time estimates.

| Parameter description | Parameter values |
|---|------------------|
| Panel A: quarterly sample (1947:1–1995:4) | |
| Rate of time preference β | 0.0153 |
| Riskless interest rate r | 0.0082 |
| Volatility of stock return σ | 0.0790 |
| Volatility of expected return σ_μ | 0.0057 |
| Mean reversion parameter κ | 0.0439 |
| Unconditional mean of expected return $\bar{\mu}$ | 0.0213 |
| Correlation between stock return and expected return ρ | -0.9626 |
| Panel B: monthly sample (1986:1–1995:12) | |
| Rate of time preference β | 0.0051 |
| Riskless interest rate r | 0.0033 |
| Volatility of stock return σ | 0.0444 |
| Volatility of expected return σ_μ | 0.0017 |
| Mean reversion parameter κ | 0.0432 |
| Unconditional mean of expected return $\bar{\mu}$ | 0.0076 |
| Correlation between stock return and expected return ρ | -0.9280 |

Table 2: This table reports the exponentiated mean optimal log consumption-wealth ratios in percentage terms, which are calculated as $\exp\{\mathbb{E}[c_t - w_t]\} = \exp\left\{\psi \log \beta - \left[\hat{A} + \hat{B}\bar{\mu} + \frac{\hat{C}}{2} \left(\bar{\mu}^2 + \frac{\sigma_\mu^2}{2\kappa}\right)\right]\right\}$ times 100. The ratios are calculated for different levels of relative risk aversion (γ), uncertainty aversion (ϕ) and elasticities of intertemporal substitution (ψ). The results are based on the parameter estimates of Campbell, Chacko, Rodriguez and Viceira (2004) for the sample period 1947:1–1995:4.

| RRA(γ), AA(ϕ) \setminus EIS (ψ) | | 1/0.75 | 1 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
|--|--------------------|--------|------|-------|------|------|------|------|------|
| $\gamma = 0.75$ | $\phi=0$ | 0.22 | 1.53 | 2.81 | 3.47 | 4.46 | 5.07 | 5.28 | 5.38 |
| | $\phi = \gamma/20$ | 0.24 | 1.53 | 2.80 | 3.45 | 4.43 | 5.03 | 5.23 | 5.33 |
| | $\phi = \gamma/10$ | 0.25 | 1.53 | 2.79 | 3.43 | 4.40 | 4.99 | 5.19 | 5.29 |
| | $\phi = \gamma/4$ | 0.29 | 1.53 | 2.77 | 3.39 | 4.32 | 4.89 | 5.07 | 5.17 |
| | $\phi = \gamma/2$ | 0.34 | 1.53 | 2.73 | 3.32 | 4.21 | 4.74 | 4.91 | 5.00 |
| | $\phi = \gamma$ | 0.42 | 1.53 | 2.66 | 3.21 | 4.03 | 4.50 | 4.66 | 4.74 |
| $\gamma = 1$ | $\phi=0$ | 0.31 | 1.53 | 2.75 | 3.36 | 4.28 | 4.83 | 5.02 | 5.11 |
| | $\phi = \gamma/20$ | 0.32 | 1.53 | 2.74 | 3.35 | 4.25 | 4.79 | 4.97 | 5.06 |
| | $\phi = \gamma/10$ | 0.33 | 1.53 | 2.73 | 3.33 | 4.22 | 4.75 | 4.93 | 5.02 |
| | $\phi = \gamma/4$ | 0.37 | 1.53 | 2.70 | 3.28 | 4.14 | 4.65 | 4.82 | 4.90 |
| | $\phi = \gamma/2$ | 0.42 | 1.53 | 2.66 | 3.21 | 4.03 | 4.50 | 4.66 | 4.74 |
| | $\phi = \gamma$ | 0.50 | 1.53 | 2.59 | 3.10 | 3.85 | 4.27 | 4.41 | 4.48 |
| $\gamma = 1.5$ | $\phi=0$ | 0.42 | 1.53 | 2.66 | 3.21 | 4.03 | 4.50 | 4.66 | 4.74 |
| | $\phi = \gamma/20$ | 0.44 | 1.53 | 2.65 | 3.20 | 4.00 | 4.46 | 4.62 | 4.69 |
| | $\phi = \gamma/10$ | 0.45 | 1.53 | 2.64 | 3.18 | 3.97 | 4.43 | 4.58 | 4.65 |
| | $\phi = \gamma/4$ | 0.49 | 1.53 | 2.61 | 3.13 | 3.89 | 4.33 | 4.47 | 4.54 |
| | $\phi = \gamma/2$ | 0.54 | 1.53 | 2.56 | 3.06 | 3.77 | 4.18 | 4.32 | 4.38 |
| | $\phi = \gamma$ | 0.62 | 1.53 | 2.48 | 2.93 | 3.58 | 3.95 | 4.07 | 4.13 |
| $\gamma = 2$ | $\phi=0$ | 0.50 | 1.53 | 2.59 | 3.10 | 3.85 | 4.27 | 4.41 | 4.48 |
| | $\phi = \gamma/20$ | 0.52 | 1.53 | 2.58 | 3.08 | 3.81 | 4.24 | 4.37 | 4.44 |
| | $\phi = \gamma/10$ | 0.53 | 1.53 | 2.56 | 3.06 | 3.78 | 4.20 | 4.33 | 4.40 |
| | $\phi = \gamma/4$ | 0.57 | 1.53 | 2.53 | 3.01 | 3.70 | 4.10 | 4.23 | 4.29 |
| | $\phi = \gamma/2$ | 0.62 | 1.53 | 2.48 | 2.93 | 3.58 | 3.95 | 4.07 | 4.13 |
| | $\phi = \gamma$ | 0.71 | 1.53 | 2.39 | 2.81 | 3.39 | 3.72 | 3.83 | 3.88 |
| $\gamma = 4$ | $\phi=0$ | 0.71 | 1.53 | 2.39 | 2.81 | 3.39 | 3.72 | 3.83 | 3.88 |
| | $\phi = \gamma/20$ | 0.72 | 1.53 | 2.38 | 2.78 | 3.36 | 3.69 | 3.79 | 3.84 |
| | $\phi = \gamma/10$ | 0.74 | 1.53 | 2.36 | 2.76 | 3.33 | 3.65 | 3.75 | 3.80 |
| | $\phi = \gamma/4$ | 0.78 | 1.53 | 2.32 | 2.70 | 3.24 | 3.54 | 3.64 | 3.69 |
| | $\phi = \gamma/2$ | 0.84 | 1.53 | 2.26 | 2.61 | 3.11 | 3.39 | 3.48 | 3.53 |
| | $\phi = \gamma$ | 0.93 | 1.53 | 2.16 | 2.46 | 2.90 | 3.15 | 3.22 | 3.26 |
| $\gamma = 10$ | $\phi=0$ | 1.01 | 1.53 | 2.08 | 2.34 | 2.73 | 2.95 | 3.02 | 3.06 |
| | $\phi = \gamma/20$ | 1.02 | 1.53 | 2.06 | 2.32 | 2.69 | 2.91 | 2.97 | 3.01 |
| | $\phi = \gamma/10$ | 1.04 | 1.53 | 2.04 | 2.29 | 2.66 | 2.86 | 2.93 | 2.96 |
| | $\phi = \gamma/4$ | 1.08 | 1.53 | 2.00 | 2.23 | 2.56 | 2.75 | 2.81 | 2.84 |
| | $\phi = \gamma/2$ | 1.15 | 1.53 | 1.93 | 2.13 | 2.42 | 2.59 | 2.64 | 2.67 |
| | $\phi = \gamma$ | 1.24 | 1.53 | 1.83 | 1.98 | 2.20 | 2.33 | 2.37 | 2.39 |
| $\gamma = 20$ | $\phi=0$ | 1.24 | 1.53 | 1.83 | 1.98 | 2.20 | 2.33 | 2.37 | 2.39 |
| | $\phi = \gamma/20$ | 1.26 | 1.53 | 1.81 | 1.95 | 2.16 | 2.29 | 2.33 | 2.35 |
| | $\phi = \gamma/10$ | 1.27 | 1.53 | 1.80 | 1.93 | 2.13 | 2.25 | 2.29 | 2.31 |
| | $\phi = \gamma/4$ | 1.31 | 1.53 | 1.76 | 1.87 | 2.04 | 2.14 | 2.17 | 2.19 |
| | $\phi = \gamma/2$ | 1.36 | 1.53 | 1.70 | 1.79 | 1.91 | 1.99 | 2.02 | 2.03 |
| | $\phi = \gamma$ | 1.44 | 1.53 | 1.62 | 1.67 | 1.74 | 1.78 | 1.79 | 1.80 |
| $\gamma = 40$ | $\phi=0$ | 1.44 | 1.53 | 1.62 | 1.67 | 1.74 | 1.78 | 1.79 | 1.80 |
| | $\phi = \gamma/20$ | 1.45 | 1.53 | 1.61 | 1.65 | 1.71 | 1.75 | 1.76 | 1.76 |
| | $\phi = \gamma/10$ | 1.46 | 1.53 | 1.60 | 1.63 | 1.68 | 1.71 | 1.73 | 1.73 |
| | $\phi = \gamma/4$ | 1.49 | 1.53 | 1.57 | 1.59 | 1.62 | 1.63 | 1.64 | 1.64 |
| | $\phi = \gamma/2$ | 1.53 | 1.53 | 1.53 | 1.53 | 1.53 | 1.53 | 1.53 | 1.53 |
| | $\phi = \gamma$ | 1.58 | 1.53 | 1.48 | 1.45 | 1.41 | 1.39 | 1.38 | 1.38 |

Table 3: This table reports the mean optimal percentage allocation to equities, for different levels of relative risk aversion (γ), uncertainty aversion (ϕ) and elasticities of intertemporal substitution (ψ). The mean optimal equity share is calculated as $\alpha_t = \frac{\mu_t - r}{(\gamma + \phi)\sigma_S^2} + \frac{1 - \gamma - \phi}{\gamma + \phi} \frac{\rho\sigma_\mu}{\sigma_S} (\mathcal{B}_j + \mathcal{C}_j\mu_t)$, $j = \{0, 1\}$ with $\{\mathcal{B}_0 = B, \mathcal{C}_0 = C\}$ for $\psi = 1$ and $\{\mathcal{B}_1 = -\frac{\hat{B}}{1-\psi}, \mathcal{C}_1 = -\frac{\hat{C}}{1-\psi}\}$ for $\psi \neq 1$ where μ_t is at its long-run mean. The results are based on the parameter estimates of Campbell, Chacko, Rodriguez and Viceira (2004) for the sample period 1947:1–1995:4.

| RRA(γ), AA(ϕ) \setminus EIS (ψ) | | 1/0.75 | 1 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
|--|--------------------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\gamma = 0.75$ | $\phi=0$ | 174.88 | 197.34 | 211.00 | 216.19 | 222.65 | 225.94 | 226.95 | 227.45 |
| | $\phi = \gamma/20$ | 181.63 | 199.69 | 210.83 | 215.07 | 220.33 | 223.00 | 223.83 | 224.23 |
| | $\phi = \gamma/10$ | 187.74 | 201.86 | 210.67 | 214.02 | 218.18 | 220.29 | 220.94 | 221.26 |
| | $\phi = \gamma/4$ | 203.02 | 207.37 | 210.18 | 211.25 | 212.58 | 213.25 | 213.45 | 213.55 |
| | $\phi = \gamma/2$ | 221.11 | 214.09 | 209.34 | 207.51 | 205.27 | 204.14 | 203.79 | 203.62 |
| | $\phi = \gamma$ | 241.56 | 221.83 | 207.47 | 201.92 | 195.08 | 191.66 | 190.62 | 190.11 |
| $\gamma = 1$ | $\phi=0$ | 209.90 | 209.90 | 209.90 | 209.90 | 209.90 | 209.90 | 209.90 | 209.90 |
| | $\phi = \gamma/20$ | 214.75 | 211.71 | 209.68 | 208.90 | 207.95 | 207.46 | 207.31 | 207.24 |
| | $\phi = \gamma/10$ | 219.11 | 213.34 | 209.45 | 207.96 | 206.13 | 205.20 | 204.92 | 204.78 |
| | $\phi = \gamma/4$ | 229.71 | 217.35 | 208.74 | 205.44 | 201.36 | 199.32 | 198.70 | 198.39 |
| | $\phi = \gamma/2$ | 241.56 | 221.83 | 207.47 | 201.92 | 195.08 | 191.66 | 190.62 | 190.11 |
| | $\phi = \gamma$ | 252.95 | 225.74 | 204.64 | 196.34 | 186.11 | 181.00 | 179.44 | 178.69 |
| $\gamma = 1.5$ | $\phi=0$ | 241.56 | 221.83 | 207.47 | 201.92 | 195.08 | 191.66 | 190.62 | 190.11 |
| | $\phi = \gamma/20$ | 244.11 | 222.77 | 207.07 | 200.98 | 193.49 | 189.74 | 188.60 | 188.04 |
| | $\phi = \gamma/10$ | 246.30 | 223.56 | 206.66 | 200.09 | 192.00 | 187.96 | 186.72 | 186.12 |
| | $\phi = \gamma/4$ | 251.14 | 225.21 | 205.38 | 197.61 | 188.04 | 183.26 | 181.80 | 181.10 |
| | $\phi = \gamma/2$ | 255.18 | 226.17 | 203.12 | 193.97 | 182.68 | 177.04 | 175.32 | 174.48 |
| | $\phi = \gamma$ | 255.10 | 224.20 | 198.26 | 187.72 | 174.61 | 168.03 | 166.03 | 165.05 |
| $\gamma = 2$ | $\phi=0$ | 252.95 | 225.74 | 204.64 | 196.34 | 186.11 | 181.00 | 179.44 | 178.69 |
| | $\phi = \gamma/20$ | 254.04 | 226.01 | 204.04 | 195.37 | 184.67 | 179.34 | 177.71 | 176.92 |
| | $\phi = \gamma/10$ | 254.86 | 226.14 | 203.43 | 194.43 | 183.32 | 177.78 | 176.09 | 175.27 |
| | $\phi = \gamma/4$ | 256.04 | 225.95 | 201.53 | 191.77 | 179.68 | 173.64 | 171.80 | 170.91 |
| | $\phi = \gamma/2$ | 255.10 | 224.20 | 198.26 | 187.72 | 174.61 | 168.03 | 166.03 | 165.05 |
| | $\phi = \gamma$ | 247.79 | 217.85 | 191.51 | 180.51 | 166.62 | 159.61 | 157.46 | 156.41 |
| $\gamma = 4$ | $\phi=0$ | 247.79 | 217.85 | 191.51 | 180.51 | 166.62 | 159.61 | 157.46 | 156.41 |
| | $\phi = \gamma/20$ | 245.90 | 216.34 | 190.15 | 179.16 | 165.25 | 158.20 | 156.04 | 154.99 |
| | $\phi = \gamma/10$ | 243.93 | 214.79 | 188.80 | 177.83 | 163.92 | 156.86 | 154.69 | 153.64 |
| | $\phi = \gamma/4$ | 237.74 | 209.97 | 184.77 | 174.00 | 160.21 | 153.17 | 151.00 | 149.95 |
| | $\phi = \gamma/2$ | 227.07 | 201.73 | 178.22 | 167.97 | 154.67 | 147.81 | 145.68 | 144.65 |
| | $\phi = \gamma$ | 206.64 | 185.84 | 165.99 | 157.06 | 145.17 | 138.91 | 136.95 | 136.00 |
| $\gamma = 10$ | $\phi=0$ | 188.61 | 171.56 | 155.00 | 147.37 | 137.01 | 131.44 | 129.69 | 128.83 |
| | $\phi = \gamma/20$ | 184.50 | 168.26 | 152.44 | 145.12 | 135.12 | 129.73 | 128.02 | 127.19 |
| | $\phi = \gamma/10$ | 180.54 | 165.07 | 149.96 | 142.93 | 133.29 | 128.06 | 126.41 | 125.61 |
| | $\phi = \gamma/4$ | 169.51 | 156.11 | 142.92 | 136.71 | 128.09 | 123.35 | 121.85 | 121.11 |
| | $\phi = \gamma/2$ | 153.65 | 143.00 | 132.43 | 127.38 | 120.24 | 116.26 | 114.98 | 114.36 |
| | $\phi = \gamma$ | 129.10 | 122.20 | 115.29 | 111.91 | 107.03 | 104.23 | 103.32 | 102.87 |
| $\gamma = 20$ | $\phi=0$ | 129.10 | 122.20 | 115.29 | 111.91 | 107.03 | 104.23 | 103.32 | 102.87 |
| | $\phi = \gamma/20$ | 125.07 | 118.73 | 112.36 | 109.24 | 104.71 | 102.10 | 101.25 | 100.83 |
| | $\phi = \gamma/10$ | 121.28 | 115.44 | 109.56 | 106.68 | 102.48 | 100.05 | 99.26 | 98.87 |
| | $\phi = \gamma/4$ | 111.14 | 106.56 | 101.94 | 99.66 | 96.31 | 94.35 | 93.71 | 93.39 |
| | $\phi = \gamma/2$ | 97.50 | 94.42 | 91.31 | 89.76 | 87.46 | 86.10 | 85.65 | 85.43 |
| | $\phi = \gamma$ | 78.21 | 76.85 | 75.48 | 74.79 | 73.75 | 73.13 | 72.92 | 72.82 |
| $\gamma = 40$ | $\phi=0$ | 78.21 | 76.85 | 75.48 | 74.79 | 73.75 | 73.13 | 72.92 | 72.82 |
| | $\phi = \gamma/20$ | 75.22 | 74.09 | 72.94 | 72.36 | 71.49 | 70.97 | 70.80 | 70.71 |
| | $\phi = \gamma/10$ | 72.45 | 71.52 | 70.57 | 70.09 | 69.37 | 68.94 | 68.79 | 68.72 |
| | $\phi = \gamma/4$ | 65.25 | 64.77 | 64.29 | 64.04 | 63.67 | 63.45 | 63.38 | 63.34 |
| | $\phi = \gamma/2$ | 55.96 | 55.96 | 55.97 | 55.97 | 55.98 | 55.98 | 55.98 | 55.98 |
| | $\phi = \gamma$ | 43.54 | 43.99 | 44.45 | 44.68 | 45.03 | 45.25 | 45.32 | 45.35 |

Table 4: This table reports the mean proportion of wealth allocated to equities due to the intertemporal hedging demand for different levels of relative risk aversion (γ), uncertainty aversion (ϕ) and elasticities of intertemporal substitution (ψ). The mean optimal hedging allocation to equities is calculated as $\alpha_{hedge} = \frac{1-\gamma-\phi}{\gamma+\phi} \frac{\rho\sigma_\mu}{\sigma_S} (\mathcal{B}_j + \mathcal{C}_j\mu_t)$, $j = \{0, 1\}$ with $\{\mathcal{B}_0 = B, \mathcal{C}_0 = C\}$ for $\psi = 1$ and $\{\mathcal{B}_1 = -\frac{\hat{B}}{1-\psi}, \mathcal{C}_1 = -\frac{\hat{C}}{1-\psi}\}$ for $\psi \neq 1$ where μ_t is at its long-run mean. The results are based on the parameter estimates of Campbell, Chacko, Rodriguez and Viceira (2004) for the sample period 1947:1–1995:4.

| RRA(γ), AA(ϕ) \setminus EIS (ψ) | | 1/0.75 | 1 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
|--|--------------------|---------|--------|--------|--------|--------|--------|--------|--------|
| $\gamma = 0.75$ | $\phi=0$ | -104.99 | -82.53 | -68.87 | -63.68 | -57.22 | -53.93 | -52.92 | -52.42 |
| | $\phi = \gamma/20$ | -84.92 | -66.85 | -55.71 | -51.47 | -46.21 | -43.54 | -42.71 | -42.31 |
| | $\phi = \gamma/10$ | -66.68 | -52.57 | -43.76 | -40.41 | -36.25 | -34.13 | -33.48 | -33.17 |
| | $\phi = \gamma/4$ | -20.88 | -16.53 | -13.72 | -12.65 | -11.32 | -10.65 | -10.44 | -10.34 |
| | $\phi = \gamma/2$ | 34.53 | 27.51 | 22.76 | 20.93 | 18.69 | 17.56 | 17.21 | 17.04 |
| | $\phi = \gamma$ | 101.63 | 81.89 | 67.54 | 61.98 | 55.15 | 51.73 | 50.68 | 50.17 |
| $\gamma = 1$ | $\phi=0$ | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | $\phi = \gamma/20$ | 14.85 | 11.80 | 9.77 | 9.00 | 8.04 | 7.56 | 7.41 | 7.34 |
| | $\phi = \gamma/10$ | 28.29 | 22.52 | 18.63 | 17.14 | 15.31 | 14.38 | 14.10 | 13.96 |
| | $\phi = \gamma/4$ | 61.79 | 49.43 | 40.82 | 37.51 | 33.44 | 31.40 | 30.77 | 30.47 |
| | $\phi = \gamma/2$ | 101.63 | 81.89 | 67.54 | 61.98 | 55.15 | 51.73 | 50.68 | 50.17 |
| | $\phi = \gamma$ | 148.00 | 120.79 | 99.69 | 91.39 | 81.16 | 76.05 | 74.49 | 73.74 |
| $\gamma = 1.5$ | $\phi=0$ | 101.63 | 81.89 | 67.54 | 61.98 | 55.15 | 51.73 | 50.68 | 50.17 |
| | $\phi = \gamma/20$ | 110.84 | 89.50 | 73.80 | 67.71 | 60.22 | 56.47 | 55.32 | 54.77 |
| | $\phi = \gamma/10$ | 119.08 | 96.35 | 79.45 | 72.87 | 64.78 | 60.74 | 59.51 | 58.91 |
| | $\phi = \gamma/4$ | 139.19 | 113.27 | 93.44 | 85.66 | 76.09 | 71.32 | 69.86 | 69.15 |
| | $\phi = \gamma/2$ | 161.89 | 132.88 | 109.83 | 100.68 | 89.39 | 83.75 | 82.03 | 81.19 |
| | $\phi = \gamma$ | 185.13 | 154.23 | 128.29 | 117.76 | 104.64 | 98.07 | 96.06 | 95.09 |
| $\gamma = 2$ | $\phi=0$ | 148.00 | 120.79 | 99.69 | 91.39 | 81.16 | 76.05 | 74.49 | 73.74 |
| | $\phi = \gamma/20$ | 154.08 | 126.05 | 104.09 | 95.41 | 84.72 | 79.38 | 77.75 | 76.96 |
| | $\phi = \gamma/10$ | 159.45 | 130.73 | 108.02 | 99.02 | 87.91 | 82.37 | 80.68 | 79.86 |
| | $\phi = \gamma/4$ | 172.07 | 141.99 | 117.57 | 107.81 | 95.72 | 89.68 | 87.84 | 86.94 |
| | $\phi = \gamma/2$ | 185.13 | 154.23 | 128.29 | 117.76 | 104.64 | 98.07 | 96.06 | 95.09 |
| | $\phi = \gamma$ | 195.32 | 165.37 | 139.03 | 128.03 | 114.15 | 107.13 | 104.98 | 103.94 |
| $\gamma = 4$ | $\phi=0$ | 195.32 | 165.37 | 139.03 | 128.03 | 114.15 | 107.13 | 104.98 | 103.94 |
| | $\phi = \gamma/20$ | 195.92 | 166.37 | 140.17 | 129.18 | 115.27 | 108.22 | 106.06 | 105.02 |
| | $\phi = \gamma/10$ | 196.23 | 167.09 | 141.09 | 130.13 | 116.22 | 109.16 | 106.99 | 105.94 |
| | $\phi = \gamma/4$ | 195.76 | 167.99 | 142.79 | 132.02 | 118.23 | 111.19 | 109.02 | 107.97 |
| | $\phi = \gamma/2$ | 192.08 | 166.74 | 143.24 | 132.99 | 119.69 | 112.82 | 110.70 | 109.67 |
| | $\phi = \gamma$ | 180.40 | 159.60 | 139.75 | 130.82 | 118.94 | 112.67 | 110.71 | 109.76 |
| $\gamma = 10$ | $\phi=0$ | 167.62 | 150.57 | 134.01 | 126.38 | 116.02 | 110.45 | 108.70 | 107.84 |
| | $\phi = \gamma/20$ | 164.51 | 148.27 | 132.45 | 125.13 | 115.13 | 109.73 | 108.03 | 107.20 |
| | $\phi = \gamma/10$ | 161.46 | 145.99 | 130.88 | 123.85 | 114.21 | 108.98 | 107.33 | 106.52 |
| | $\phi = \gamma/4$ | 152.72 | 139.31 | 126.12 | 119.91 | 111.29 | 106.56 | 105.06 | 104.32 |
| | $\phi = \gamma/2$ | 139.65 | 129.01 | 118.44 | 113.39 | 106.25 | 102.27 | 100.99 | 100.36 |
| | $\phi = \gamma$ | 118.60 | 111.71 | 104.79 | 101.41 | 96.53 | 93.73 | 92.82 | 92.38 |
| $\gamma = 20$ | $\phi=0$ | 118.60 | 111.71 | 104.79 | 101.41 | 96.53 | 93.73 | 92.82 | 92.38 |
| | $\phi = \gamma/20$ | 115.08 | 108.73 | 102.36 | 99.24 | 94.71 | 92.11 | 91.26 | 90.84 |
| | $\phi = \gamma/10$ | 111.74 | 105.90 | 100.02 | 97.14 | 92.94 | 90.51 | 89.72 | 89.33 |
| | $\phi = \gamma/4$ | 102.75 | 98.17 | 93.55 | 91.27 | 87.91 | 85.96 | 85.32 | 85.00 |
| | $\phi = \gamma/2$ | 90.50 | 87.43 | 84.31 | 82.76 | 80.46 | 79.10 | 78.65 | 78.43 |
| | $\phi = \gamma$ | 72.96 | 71.60 | 70.23 | 69.54 | 68.50 | 67.88 | 67.68 | 67.57 |
| $\gamma = 40$ | $\phi=0$ | 72.96 | 71.60 | 70.23 | 69.54 | 68.50 | 67.88 | 67.68 | 67.57 |
| | $\phi = \gamma/20$ | 70.22 | 69.09 | 67.94 | 67.37 | 66.50 | 65.98 | 65.80 | 65.71 |
| | $\phi = \gamma/10$ | 67.68 | 66.75 | 65.80 | 65.32 | 64.60 | 64.17 | 64.02 | 63.95 |
| | $\phi = \gamma/4$ | 61.05 | 60.57 | 60.09 | 59.84 | 59.47 | 59.25 | 59.18 | 59.14 |
| | $\phi = \gamma/2$ | 52.46 | 52.47 | 52.47 | 52.47 | 52.48 | 52.48 | 52.48 | 52.48 |
| | $\phi = \gamma$ | 40.92 | 41.37 | 41.82 | 42.05 | 42.41 | 42.62 | 42.69 | 42.73 |

Table 5: This table reports the fraction of the mean optimal allocation due to the intertemporal hedging demand. The ratio is calculated as $\frac{\alpha_{hedge}}{\alpha}$, where $\alpha_t = \frac{\mu_t - r}{(\gamma + \phi)\sigma_S^2} + \frac{1 - \gamma - \phi}{\gamma + \phi} \frac{\rho\sigma_\mu}{\sigma_S} (\mathcal{B}_j + \mathcal{C}_j\mu_t)$, and $\alpha_{hedge} = \frac{1 - \gamma - \phi}{\gamma + \phi} \frac{\rho\sigma_\mu}{\sigma_S} (\mathcal{B}_j + \mathcal{C}_j\mu_t)$, with $\{\mathcal{B}_0 = B, \mathcal{C}_0 = C\}$ for $\psi = 1$ and $\left\{\mathcal{B}_1 = -\frac{\hat{B}}{1 - \psi}, \mathcal{C}_1 = -\frac{\hat{C}}{1 - \psi}\right\}$ for $\psi \neq 1$, and μ_t is at its long-run mean. The results are based on the parameter estimates of Campbell, Chacko, Rodriguez and Viceira (2004) for the sample period 1947:1–1995:4.

| RRA(γ), AA(ϕ) \setminus EIS (ψ) | | 1/0.75 | 1 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
|--|--------------------|---------|---------|---------|---------|---------|---------|---------|---------|
| $\gamma = 0.75$ | $\phi=0$ | -60.04% | -41.82% | -32.64% | -29.45% | -25.7% | -23.87% | -23.32% | -23.05% |
| | $\phi = \gamma/20$ | -46.75% | -33.48% | -26.42% | -23.93% | -20.97% | -19.52% | -19.08% | -18.87% |
| | $\phi = \gamma/10$ | -35.52% | -26.04% | -20.77% | -18.88% | -16.61% | -15.5% | -15.15% | -14.99% |
| | $\phi = \gamma/4$ | -10.28% | -7.97% | -6.53% | -5.99% | -5.32% | -4.99% | -4.89% | -4.84% |
| | $\phi = \gamma/2$ | 15.62% | 12.85% | 10.87% | 10.09% | 9.1% | 8.6% | 8.45% | 8.37% |
| | $\phi = \gamma$ | 42.07% | 36.92% | 32.55% | 30.7% | 28.27% | 26.99% | 26.59% | 26.39% |
| $\gamma = 1$ | $\phi=0$ | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% |
| | $\phi = \gamma/20$ | 6.91% | 5.57% | 4.66% | 4.31% | 3.87% | 3.64% | 3.57% | 3.54% |
| | $\phi = \gamma/10$ | 12.91% | 10.55% | 8.89% | 8.24% | 7.43% | 7.01% | 6.88% | 6.82% |
| | $\phi = \gamma/4$ | 26.9% | 22.74% | 19.56% | 18.26% | 16.61% | 15.75% | 15.49% | 15.36% |
| | $\phi = \gamma/2$ | 42.07% | 36.92% | 32.55% | 30.7% | 28.27% | 26.99% | 26.59% | 26.39% |
| | $\phi = \gamma$ | 58.51% | 53.51% | 48.72% | 46.55% | 43.61% | 42.02% | 41.51% | 41.27% |
| $\gamma = 1.5$ | $\phi=0$ | 42.07% | 36.92% | 32.55% | 30.7% | 28.27% | 26.99% | 26.59% | 26.39% |
| | $\phi = \gamma/20$ | 45.4% | 40.18% | 35.64% | 33.69% | 31.12% | 29.76% | 29.33% | 29.13% |
| | $\phi = \gamma/10$ | 48.35% | 43.1% | 38.44% | 36.42% | 33.74% | 32.32% | 31.87% | 31.65% |
| | $\phi = \gamma/4$ | 55.42% | 50.29% | 45.49% | 43.35% | 40.47% | 38.91% | 38.42% | 38.18% |
| | $\phi = \gamma/2$ | 63.44% | 58.75% | 54.07% | 51.9% | 48.93% | 47.3% | 46.79% | 46.53% |
| | $\phi = \gamma$ | 72.57% | 68.79% | 64.71% | 62.73% | 59.93% | 58.36% | 57.86% | 57.61% |
| $\gamma = 2$ | $\phi=0$ | 58.51% | 53.51% | 48.72% | 46.55% | 43.61% | 42.02% | 41.51% | 41.27% |
| | $\phi = \gamma/20$ | 60.65% | 55.77% | 51.01% | 48.84% | 45.88% | 44.26% | 43.75% | 43.5% |
| | $\phi = \gamma/10$ | 62.56% | 57.81% | 53.1% | 50.93% | 47.96% | 46.33% | 45.82% | 45.56% |
| | $\phi = \gamma/4$ | 67.21% | 62.84% | 58.34% | 56.22% | 53.27% | 51.65% | 51.13% | 50.87% |
| | $\phi = \gamma/2$ | 72.57% | 68.79% | 64.71% | 62.73% | 59.93% | 58.36% | 57.86% | 57.61% |
| | $\phi = \gamma$ | 78.82% | 75.91% | 72.6% | 70.93% | 68.51% | 67.12% | 66.67% | 66.45% |
| $\gamma = 4$ | $\phi=0$ | 78.82% | 75.91% | 72.6% | 70.93% | 68.51% | 67.12% | 66.67% | 66.45% |
| | $\phi = \gamma/20$ | 79.68% | 76.9% | 73.72% | 72.1% | 69.76% | 68.41% | 67.97% | 67.76% |
| | $\phi = \gamma/10$ | 80.44% | 77.79% | 74.73% | 73.17% | 70.9% | 69.59% | 69.16% | 68.95% |
| | $\phi = \gamma/4$ | 82.34% | 80.01% | 77.28% | 75.87% | 73.8% | 72.59% | 72.2% | 72% |
| | $\phi = \gamma/2$ | 84.59% | 82.66% | 80.37% | 79.17% | 77.38% | 76.33% | 75.99% | 75.82% |
| | $\phi = \gamma$ | 87.3% | 85.88% | 84.19% | 83.29% | 81.93% | 81.11% | 80.84% | 80.71% |
| $\gamma = 10$ | $\phi=0$ | 88.87% | 87.76% | 86.46% | 85.76% | 84.68% | 84.03% | 83.81% | 83.71% |
| | $\phi = \gamma/20$ | 89.16% | 88.12% | 86.89% | 86.22% | 85.21% | 84.59% | 84.39% | 84.28% |
| | $\phi = \gamma/10$ | 89.43% | 88.44% | 87.28% | 86.65% | 85.68% | 85.1% | 84.91% | 84.81% |
| | $\phi = \gamma/4$ | 90.09% | 89.24% | 88.25% | 87.72% | 86.89% | 86.39% | 86.22% | 86.14% |
| | $\phi = \gamma/2$ | 90.89% | 90.21% | 89.43% | 89.01% | 88.36% | 87.96% | 87.83% | 87.76% |
| | $\phi = \gamma$ | 91.87% | 91.41% | 90.9% | 90.62% | 90.19% | 89.93% | 89.84% | 89.8% |
| $\gamma = 20$ | $\phi=0$ | 91.87% | 91.41% | 90.9% | 90.62% | 90.19% | 89.93% | 89.84% | 89.8% |
| | $\phi = \gamma/20$ | 92.01% | 91.58% | 91.1% | 90.85% | 90.45% | 90.21% | 90.13% | 90.09% |
| | $\phi = \gamma/10$ | 92.13% | 91.73% | 91.29% | 91.06% | 90.69% | 90.46% | 90.39% | 90.35% |
| | $\phi = \gamma/4$ | 92.45% | 92.12% | 91.76% | 91.58% | 91.28% | 91.1% | 91.04% | 91.01% |
| | $\phi = \gamma/2$ | 92.82% | 92.59% | 92.34% | 92.21% | 92% | 91.87% | 91.83% | 91.81% |
| | $\phi = \gamma$ | 93.29% | 93.17% | 93.05% | 92.98% | 92.88% | 92.82% | 92.8% | 92.79% |
| $\gamma = 40$ | $\phi=0$ | 93.29% | 93.17% | 93.05% | 92.98% | 92.88% | 92.82% | 92.8% | 92.79% |
| | $\phi = \gamma/20$ | 93.36% | 93.25% | 93.15% | 93.09% | 93.01% | 92.96% | 92.94% | 92.93% |
| | $\phi = \gamma/10$ | 93.42% | 93.33% | 93.24% | 93.19% | 93.12% | 93.08% | 93.07% | 93.06% |
| | $\phi = \gamma/4$ | 93.57% | 93.52% | 93.47% | 93.44% | 93.41% | 93.38% | 93.38% | 93.37% |
| | $\phi = \gamma/2$ | 93.75% | 93.75% | 93.75% | 93.75% | 93.75% | 93.75% | 93.75% | 93.75% |
| | $\phi = \gamma$ | 93.97% | 94.04% | 94.1% | 94.13% | 94.17% | 94.2% | 94.21% | 94.21% |

Table 6: This table reports the mean optimal allocation to equities for different levels of relative risk aversion (γ), uncertainty aversion (ϕ) and elasticities of intertemporal substitution (ψ). The number in parenthesis is the detection error probability $\varepsilon_N(\phi) \equiv \frac{1}{2} \Pr(\zeta_{1,N} > 0 \mid P, \mathcal{F}_0) + \frac{1}{2} \Pr(\zeta_{1,N} < 0 \mid Q, \mathcal{F}_0)$ for different values of γ , ϕ and ψ , where $\zeta_{1,t} = -\int_0^t g(\mu_s)^\top d\mathbf{B}_s - \frac{1}{2} \int_0^t g(\mu_s)^\top g(\mu_s) ds$. The results are based on the parameter estimates of Barberis (2000) for the sample period 1986–1995.

| RRA, UA \ EIS | $\psi = 1/0.75$ | $\psi = 1$ | $\psi = 1/1.5$ | $\psi = 1/2$ | $\psi = 1/4$ | $\psi = 1/6$ |
|--------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\gamma = 0.75$ | | | | | | |
| $\phi = 0$ | 229.78 | 234.41 | 238.39 | 240.19 | 242.67 | 243.44 |
| $\phi = \gamma/12$ | 226.75 (0.4389) | 229.94 (0.4394) | 232.69 (0.4396) | 233.92 (0.4396) | 235.62 (0.4395) | 236.15 (0.4395) |
| $\phi = \gamma/8$ | 225.26 (0.4046) | 227.82 (0.4054) | 230.02 (0.4058) | 231.00 (0.4059) | 232.36 (0.4059) | 232.79 (0.4059) |
| $\phi = \gamma/4$ | 220.92 (0.3035) | 221.84 (0.3058) | 222.62 (0.3071) | 222.97 (0.3075) | 223.45 (0.3079) | 223.60 (0.3080) |
| $\gamma = 1$ | | | | | | |
| $\phi = 0$ | 218.12 | 218.12 | 218.12 | 218.12 | 218.12 | 218.12 |
| $\phi = \gamma/12$ | 214.50 (0.4306) | 213.47 (0.4316) | 212.59 (0.4322) | 212.19 (0.4324) | 211.65 (0.4326) | 211.48 (0.4326) |
| $\phi = \gamma/8$ | 212.73 (0.3919) | 211.25 (0.3935) | 209.99 (0.3945) | 209.42 (0.3949) | 208.65 (0.3953) | 208.40 (0.3954) |
| $\phi = \gamma/4$ | 207.61 (0.2786) | 205.00 (0.2823) | 202.77 (0.2849) | 201.78 (0.2858) | 200.41 (0.2870) | 199.99 (0.2873) |
| $\gamma = 1.5$ | | | | | | |
| $\phi = 0$ | 198.08 | 193.92 | 190.39 | 188.82 | 186.66 | 185.99 |
| $\phi = \gamma/12$ | 193.63 (0.4178) | 188.96 (0.4195) | 184.99 (0.4207) | 183.23 (0.4213) | 180.81 (0.4219) | 180.06 (0.4221) |
| $\phi = \gamma/8$ | 191.49 (0.3724) | 186.61 (0.3751) | 182.46 (0.3771) | 180.61 (0.3779) | 178.08 (0.3789) | 177.30 (0.3793) |
| $\phi = \gamma/4$ | 185.33 (0.2407) | 179.96 (0.2465) | 175.40 (0.2509) | 173.37 (0.2528) | 170.59 (0.2551) | 169.73 (0.2559) |
| $\gamma = 2$ | | | | | | |
| $\phi = 0$ | 181.43 | 175.85 | 171.09 | 168.98 | 166.08 | 165.18 |
| $\phi = \gamma/12$ | 176.49 (0.4080) | 170.70 (0.4102) | 165.77 (0.4120) | 163.58 (0.4127) | 160.57 (0.4137) | 159.64 (0.4139) |
| $\phi = \gamma/8$ | 174.12 (0.3576) | 168.26 (0.3611) | 163.27 (0.3638) | 161.05 (0.3650) | 158.01 (0.3665) | 157.06 (0.3669) |
| $\phi = \gamma/4$ | 167.38 (0.2123) | 161.41 (0.2196) | 156.32 (0.2253) | 154.05 (0.2277) | 150.94 (0.2309) | 149.97 (0.2319) |
| $\gamma = 4$ | | | | | | |
| $\phi = 0$ | 135.79 | 130.55 | 126.01 | 123.96 | 121.14 | 120.25 |
| $\phi = \gamma/12$ | 130.32 (0.3834) | 125.33 (0.3866) | 121.00 (0.3893) | 119.05 (0.3904) | 116.34 (0.3920) | 115.49 (0.3925) |
| $\phi = \gamma/8$ | 127.74 (0.3210) | 122.89 (0.3257) | 118.66 (0.3297) | 116.75 (0.3315) | 114.11 (0.3339) | 113.28 (0.3346) |
| $\phi = \gamma/4$ | 120.59 (0.1433) | 116.12 (0.1525) | 112.22 (0.1602) | 110.44 (0.1636) | 107.98 (0.1683) | 107.20 (0.1698) |
| $\gamma = 6$ | | | | | | |
| $\phi = 0$ | 108.43 | 104.68 | 101.37 | 99.85 | 97.73 | 97.06 |
| $\phi = \gamma/12$ | 103.23 (0.3698) | 99.79 (0.3730) | 96.74 (0.3757) | 95.34 (0.3769) | 93.38 (0.3786) | 92.76 (0.3791) |
| $\phi = \gamma/8$ | 100.80 (0.3008) | 97.52 (0.3055) | 94.60 (0.3095) | 93.25 (0.3114) | 91.37 (0.3139) | 90.77 (0.3147) |
| $\phi = \gamma/4$ | 94.17 (0.1062) | 91.30 (0.1148) | 88.72 (0.1223) | 87.53 (0.1257) | 85.86 (0.1304) | 85.33 (0.1319) |

Table 7: This table reports the mean optimal allocation to equities for different levels of relative risk aversion (γ), uncertainty aversion (ϕ) and elasticities of intertemporal substitution (ψ). The number in parenthesis is the detection error probability $\varepsilon_N(\phi) \equiv \frac{1}{2} \Pr(\zeta_{1,N} > 0 \mid P, \mathcal{F}_0) + \frac{1}{2} \Pr(\zeta_{1,N} < 0 \mid Q, \mathcal{F}_0)$ for different values of γ , ϕ and ψ , where $\zeta_{1,t} = -\int_0^t g(\mu_s)^\top d\mathbf{B}_s - \frac{1}{2} \int_0^t g(\mu_s)^\top g(\mu_s) ds$. The results are based on the parameter estimates of Campbell, Chacko, Rodriguez and Viceira (2004) for the sample period 1947:1–1995:4.

| RRA, UA \ EIS | $\psi = 1/0.75$ | $\psi = 1$ | $\psi = 1/1.5$ | $\psi = 1/2$ | $\psi = 1/4$ | $\psi = 1/6$ |
|-----------------|------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\gamma = 0.75$ | $\phi = 0$ 174.88 | 197.34 | 211.00 | 216.19 | 222.65 | 224.52 |
| | $\phi = \gamma/50$ 177.66 (0.4630) | 198.31 (0.4723) | 210.93 (0.4736) | 215.73 (0.4735) | 221.70 (0.4729) | 223.43 (0.4726) |
| | $\phi = \gamma/30$ 179.45 (0.4338) | 198.93 (0.4496) | 210.89 (0.4519) | 215.43 (0.4517) | 221.08 (0.4507) | 222.72 (0.4503) |
| | $\phi = \gamma/15$ 183.73 (0.3568) | 200.44 (0.3883) | 210.78 (0.3933) | 214.71 (0.3931) | 219.60 (0.3914) | 221.01 (0.3906) |
| $\gamma = 1$ | $\phi = 0$ 209.90 | 209.90 | 209.90 | 209.90 | 209.90 | 209.90 |
| | $\phi = \gamma/50$ 211.91 (0.4573) | 210.65 (0.4660) | 209.81 (0.4682) | 209.49 (0.4684) | 209.10 (0.4682) | 208.99 (0.4681) |
| | $\phi = \gamma/30$ 213.20 (0.4234) | 211.13 (0.4383) | 209.75 (0.4421) | 209.23 (0.4425) | 208.58 (0.4423) | 208.39 (0.4421) |
| | $\phi = \gamma/15$ 216.26 (0.3335) | 212.27 (0.3635) | 209.60 (0.3719) | 208.58 (0.3730) | 207.32 (0.3730) | 206.96 (0.3728) |
| $\gamma = 1.5$ | $\phi = 0$ 241.56 | 221.83 | 207.47 | 201.92 | 195.08 | 193.13 |
| | $\phi = \gamma/50$ 242.63 (0.4466) | 222.22 (0.4548) | 207.32 (0.4584) | 201.54 (0.4592) | 194.43 (0.4599) | 192.40 (0.4601) |
| | $\phi = \gamma/30$ 243.30 (0.4040) | 222.47 (0.4180) | 207.21 (0.4244) | 201.29 (0.4260) | 194.01 (0.4274) | 191.92 (0.4277) |
| | $\phi = \gamma/15$ 244.88 (0.2904) | 223.05 (0.3192) | 206.94 (0.3331) | 200.68 (0.3368) | 192.98 (0.3403) | 190.78 (0.3410) |
| $\gamma = 2$ | $\phi = 0$ 252.95 | 225.74 | 204.64 | 196.34 | 186.11 | 183.19 |
| | $\phi = \gamma/50$ 253.42 (0.4365) | 225.86 (0.4446) | 204.40 (0.4494) | 195.95 (0.4509) | 185.52 (0.4524) | 182.55 (0.4528) |
| | $\phi = \gamma/30$ 253.71 (0.3858) | 225.93 (0.3999) | 204.24 (0.4084) | 195.69 (0.4111) | 185.14 (0.4139) | 182.13 (0.4146) |
| | $\phi = \gamma/15$ 254.34 (0.2502) | 226.06 (0.2796) | 203.84 (0.2982) | 195.05 (0.3043) | 184.22 (0.3108) | 181.12 (0.3125) |
| $\gamma = 4$ | $\phi = 0$ 247.79 | 217.85 | 191.51 | 180.51 | 166.62 | 162.62 |
| | $\phi = \gamma/50$ 247.05 (0.4010) | 217.25 (0.4108) | 190.96 (0.4191) | 179.96 (0.4225) | 166.07 (0.4266) | 162.05 (0.4278) |
| | $\phi = \gamma/30$ 246.54 (0.3221) | 216.85 (0.3394) | 190.60 (0.3542) | 179.60 (0.3602) | 165.70 (0.3676) | 161.68 (0.3697) |
| | $\phi = \gamma/15$ 245.25 (0.1119) | 215.83 (0.1488) | 189.70 (0.1809) | 178.71 (0.1940) | 164.80 (0.2104) | 160.77 (0.2151) |
| $\gamma = 6$ | $\phi = 0$ 227.07 | 201.73 | 178.22 | 167.97 | 154.67 | 150.76 |
| | $\phi = \gamma/50$ 225.79 (0.3721) | 200.74 (0.3837) | 177.46 (0.3943) | 167.28 (0.3989) | 154.05 (0.4048) | 150.16 (0.4065) |
| | $\phi = \gamma/30$ 224.94 (0.2709) | 200.09 (0.2913) | 176.95 (0.3100) | 166.82 (0.3182) | 153.64 (0.3287) | 149.76 (0.3318) |
| | $\phi = \gamma/15$ 222.83 (0.0024) | 198.45 (0.0457) | 175.68 (0.0860) | 165.68 (0.1036) | 152.63 (0.1265) | 148.78 (0.1332) |

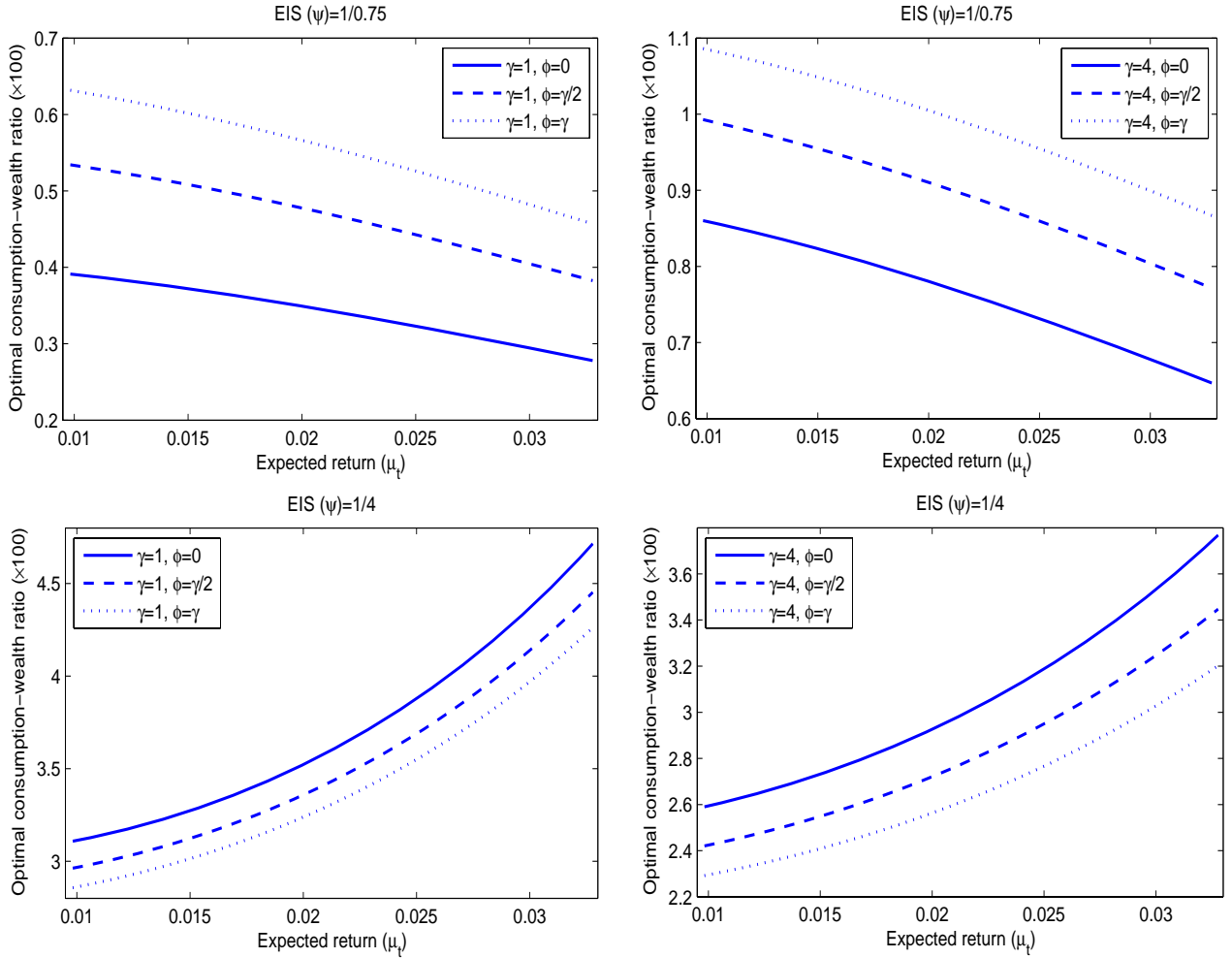


Figure 1: This figure plots the optimal consumption-wealth ratio against μ_t over the interval $[\bar{\mu} - 2\sigma_\mu, \bar{\mu} + 2\sigma_\mu]$ for different levels of risk aversion γ and uncertainty aversion ϕ . The top panels are for $\psi = 1/0.75$ and the bottom panels are for $\psi = 1/4$. The results are based on the parameter estimates of Campbell et al. (2004).

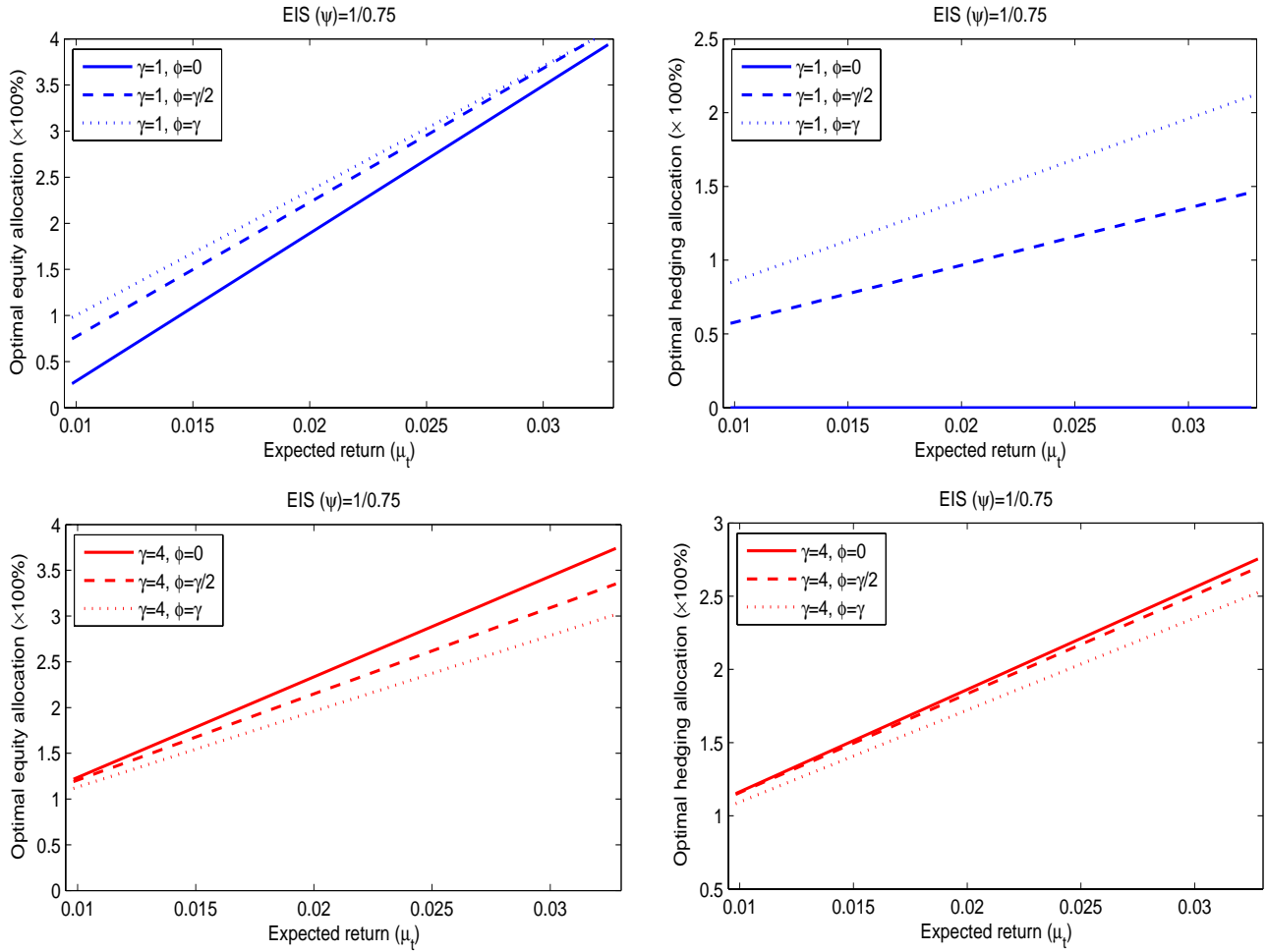


Figure 2: This figure plots the optimal proportion of wealth allocated to equities and the proportion due to hedging demand against μ_t over the interval $[\bar{\mu} - 2\sigma_\mu, \bar{\mu} + 2\sigma_\mu]$ for different levels of risk aversion γ and uncertainty aversion ϕ . The leftmost panels show the optimal equity allocation and the rightmost panels show the optimal hedging allocation. The elasticity of intertemporal substitution ψ is equal to 1/0.75. The results are based on the parameter estimates of Campbell et al. (2004).

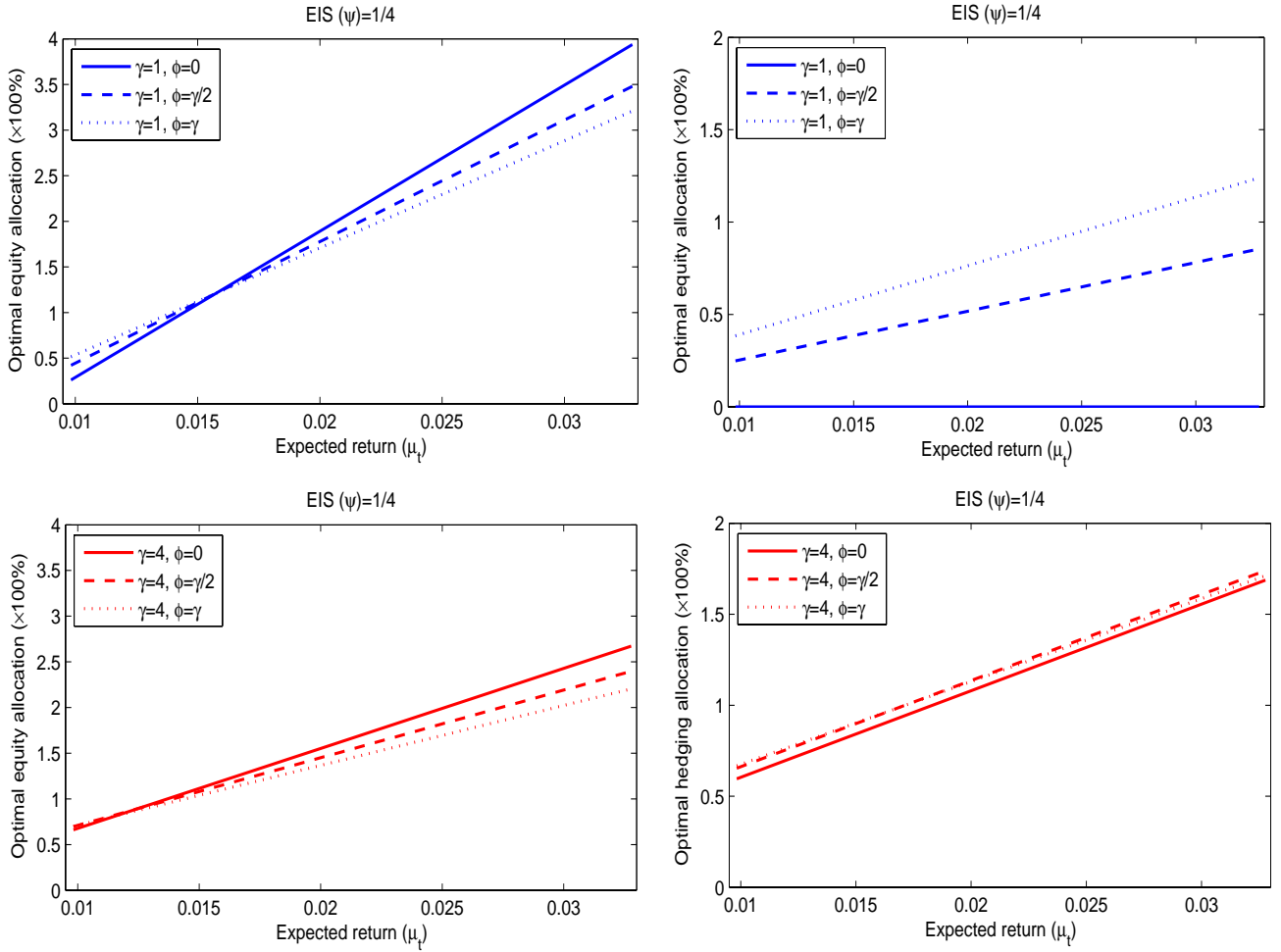


Figure 3: This figure plots the optimal proportion of wealth allocated to equities and the proportion due to hedging demand against μ_t over the interval $[\bar{\mu} - 2\sigma_\mu, \bar{\mu} + 2\sigma_\mu]$ for different levels of risk aversion γ and uncertainty aversion ϕ . The leftmost panels show the optimal equity allocation and the rightmost panels show the optimal hedging allocation. The elasticity of intertemporal substitution ψ is equal to 1/4. The results are based on the parameter estimates of Campbell et al. (2004).

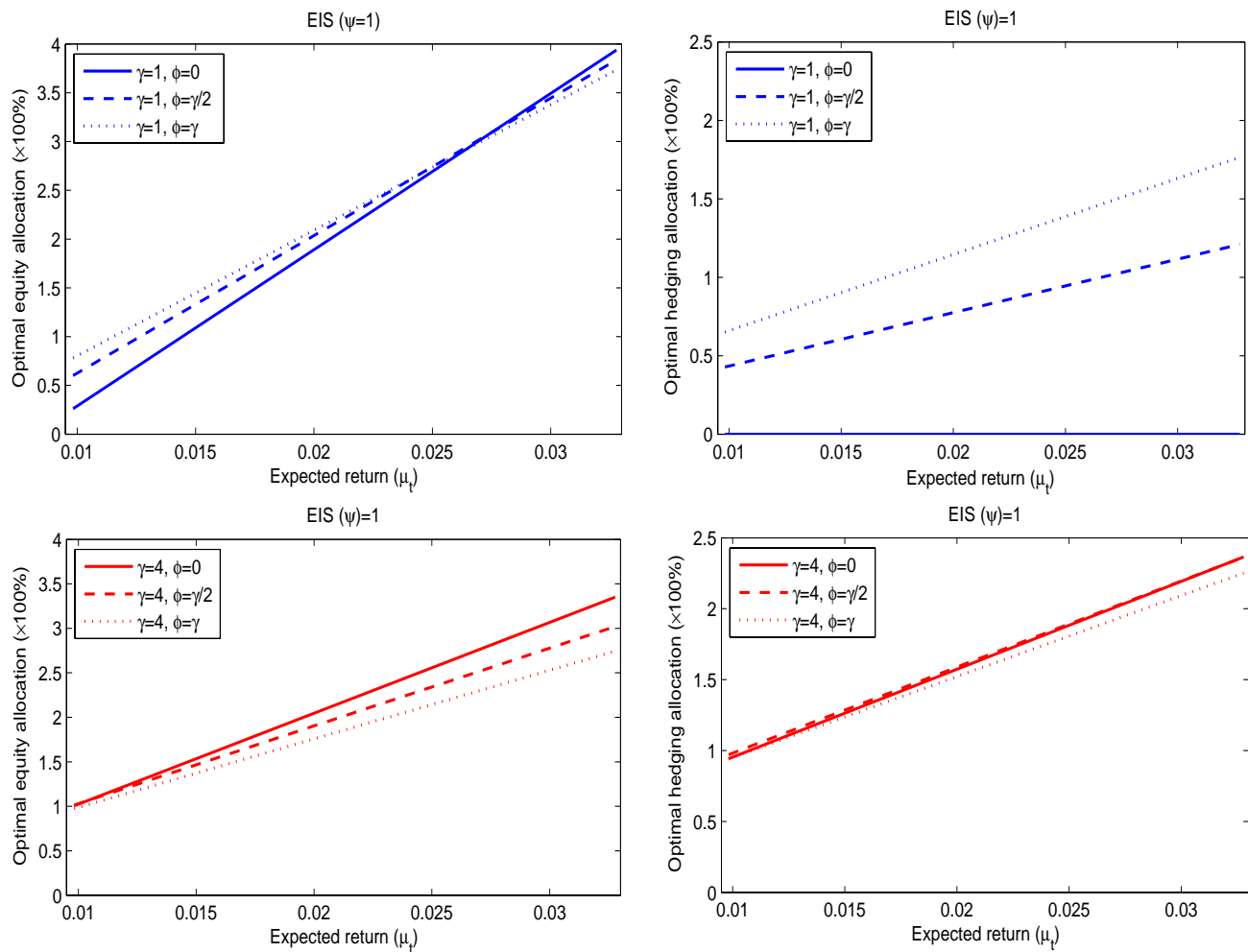


Figure 4: This figure plots the optimal proportion of wealth allocated to equities and the proportion due to hedging demand against μ_t over the interval $[\bar{\mu} - 2\sigma_\mu, \bar{\mu} + 2\sigma_\mu]$ for different levels of risk aversion γ and uncertainty aversion ϕ . The leftmost panels show the optimal equity allocation and the rightmost panels show the optimal hedging allocation. The elasticity of intertemporal substitution ψ is equal to 1. The results are based on the parameter estimates of Campbell et al. (2004).