

Guan, Zhenke; Gan, Bing; Khan, Aisha; Poon, Ser-huang

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Choice of interest rate term structure models for assets and liability management

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Choice of Interest Rate Term Structure Models for Assets and Liability Management

Eric Guan, Bing Gan, Aisha Khan and Ser-Huang Poon

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Author(s) and affiliation

Professor Ser-Huang Poon
Professor of Finance
Manchester Business School
University of Manchester
Crawford House
Oxford Road
Manchester M13 9PL
United Kingdom
Tel: +44 161 275 0431 (direct) 0429/0436 (secretary)
Fax: +44 161 275 4023
Email: ser-huang.poon@mbs.ac.uk
<http://www.personal.mbs.ac.uk/ser-huang-poon/>
<http://ssrn.com/author=44637/>

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Keywords

Asset-liability management, Hull-White, Black-Karasinski, Libor Market Model, Swap Market Model, Bermudan swaptions

Abstract

This paper compares the pricing and hedging performance of the LMM model against two spot-rate models, namely Hull-White and Black-Karasinski, and the more recent Swap Market Model from an Asset-Liability-Management (ALM) perspective. In contrast to previous studies in the literature, our emphasis here is on ALM and we use hedging performance on Bermudan swaptions to proxy risk management outcome of long-term mortgage loans. Our tests involve calibrating the four interest rate models to European swaption prices for EURO and USD over the period February 2005 to September 2007. The calibrated models are then used to price and hedge a constant 11-year Bermudan swaption portfolio using a series of interest rate swaps and a 1-year holding-revision period. Our empirical results show that, the calibrated parameters of all four models are stable and their pricing errors are small and comparable. No single model dominates in the pricing exercise. The hedging performance of all four models is similar for the Euro market. For the USD market, the short rate models perform marginally better than SMM and LMM. The HW model is marginally better than BK model in terms of model parameter stability and smaller pricing and hedging errors.

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Choice of Interest Rate Term Structure Models for Assets and Liability Management

Zhenke Guan, Bing Gan, Aisha Khan and Ser-Huang Poon *

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*All authors are from Manchester Business School; Zhenke Guan is a doctoral candidate, Bing Gan and Aisha Khan are MSc Quantitative Finance and Financial Engineering students. Please send all correspondence to Professor Ser-Huang Poon (ser-huang.poon@mbs.ac.uk), Manchester Business School, Crawford House, University of Manchester, Oxford Road, Manchester M13 9PL, UK. Tel: +44 161 275 0431, Fax: +44 161 27 54023. We are grateful to Rijkman Groenink and Geerten Battjes from ABN AMRO for proposing this project. We would like to thank Peter van der Wal, Vincent van Bergen and Jan Remmerswaal from ABN AMRO for many useful suggestions and data, Michael Croucher for NAG software support, and Dick Stapleton and Marti Subrahmanyam for many helpful advice.

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Abstract

This paper compares the pricing and hedging performance of the LMM model against two spot-rate models, namely Hull-White and Black-Karasinski, and the more recent Swap Market Model from an Asset-Liability-Management (ALM) perspective. In contrast to previous studies in the literature, our emphasis here is on ALM and we use hedging performance on Bermudan swaptions to proxy risk management outcome of long-term mortgage loans. Our tests involve calibrating the four interest rate models to European swaption prices for EURO and USD over the period February 2005 to September 2007. The calibrated models are then used to price and hedge a constant 11-year Bermudan swaption portfolio using a series of interest rate swaps and a 1-year holding-revision period. Our empirical results show that, the calibrated parameters of all four models are stable and their pricing errors are small and comparable. No single model dominates in the pricing exercise. The hedging performance of all four models is similar for the Euro market. For the USD market, the short rate models perform marginally better than SMM and LMM. The HW model is marginally better than BK model in terms of model parameter stability and smaller pricing and hedging errors.

Choice of Interest Rate Term Structure Model for Assets and Liability Management

1 Introduction

This chapter studies the choice of interest rate term structure models for the purpose of asset-liability management (ALM) in a global bank. In ALM, stochastic interest rate models are used to price embedded options in banking book products. An important example of such an embedded option is the prepayment option on fixed rate mortgages. A prepayment option closely resembles a Bermudan swaption, apart from the fact that the customer's exercise strategy is more complex in the case of mortgages. The maturity of the swap underlying the Bermudan has been chosen at 10 years because it is the most common choice of period for fixing mortgage rates in some countries, for example, the Netherlands. Prepayment options are hedged to stabilize value changes.¹ In this paper, we compare the performance of four 1-factor models, viz., Hull-White (HW), Black-Karasinski (BK), Swap Market Model (SMM) and Libor Market Model (LMM), in hedging an 11-year Bermudan swaption (i.e., 1-year option on 10-year swap) on an annual basis from February 2006 to September 2007 in both Euro and USD markets. Unlike the case of the short-term pricing problem, the one-factor model is often preferred for low frequency ALM purpose because of its simplicity. For long horizon hedging, the multi-factor model could produce more noise as it requires more parameter input. For the same reason, we do not consider the “smile” effect in this ALM study. Pricing performance measures a model's capability of capturing the current term structure and market prices of interest rate sensitive instruments. Pricing performance can always be improved, in an almost sure sense, by adding more explanatory variables and complexities to the dynamics. However, pricing performance alone cannot reflect

¹In a trading environment, value changes and P&L are the same things. However, in ALM, a different accounting policy is used (accrual accounting) so that value changes do not immediately run through the P&L.

the model's ability to capture the true term structure dynamics as it evolves through time. To assess the appropriateness of model dynamics, one has to study the model's out-of-sample forecasting and hedging performance.

The last few decades have seen the development of a great variety of interest rate models for estimating prices and risk sensitivities of interest rate derivatives. These models can be broadly divided into short rate, forward rate, and market models. The class of short-rate models, among others, includes Vasicek (1977) [52], Hull and White (1990) [26], and Black and Karasinski (1991) [6]. A generalized framework for arbitrage-free forward-rate modeling originates from the work of Heath, Jarrow and Morton (HJM, 1992) [24]. Market models are a class of models within the HJM framework that model the evolution of rates that are directly observable in the market. Brace, Gatarek and Musiela (1997) [5] and Miltersen, Sandmann and Sondermann (1997) [45] and Jamshidian (1997) [34] are among the first to introduce the arbitrage free process for forward Libor rates which lead to the Libor Market Model under the HJM framework. The SMM was first proposed by Jamshidian (1997) [34] who also introduces the concept of the co-terminal swap market model.

All these models have their own strengths and weaknesses. Short-rate models are tractable, easy to understand and implement, but do not provide complete freedom in choosing the volatility structure. The HJM framework is popular due to its flexibility in terms of the number of factors that can be used and it permits different volatility structures for forward rates of different maturities. The key problem associated with the HJM is that instantaneous forward rates are not directly observable in the market and hence models under this framework are difficult to calibrate. Market models, such as LMM and SMM, overcome these limitations but are more complex and computationally expensive when compared with the short rate models. The question is whether one should use simple models like Gaussian HW model or the lognormal BK model, or a more complicated model like SMM and LMM. In this paper, our focus is on the difference between the four interest rate models for longer term asset liability management in two interest rate regimes (i.e., Euro and USD). The choice of HW and BK is simple: at the

time of writing, they are the most important and popular short rate models used by the industry; the choice of Swap Market Model and Libor Market Model is also simple because they are new or more recent and are widely used among practitioners.

This study extends the current interest rate empirical literature by comparing a wide range of models, from spot and forward to swap rate models; by focusing on the hedging performance on the more complicated Bermudan swaptions (instead of caplets, floorlets or European swaptions); by taking a broader ALM perspective and by comparing model performance across currencies. At the time of writing, hedging of Bermudan swaptions has only been done by two papers with real financial data: Longstaff, Schwartz and Santa-Clara (2001a) [43] and Andersen and Andreason (2002) [2]. Pietersz and Pelser (2001) [46] present a new way to calculate vega for hedging Bermudan swaption but they do not perform any empirical tests. In addition, none of them compare the short rate models with market models and from an ALM perspective.

Specifically, the objective here is to select an interest rate term structure model that produces the best hedging performance of a portfolio of 11-year Bermudan swaption and hedging instruments over a 1-year horizon. The hedging profit and loss is calculated as the difference between the values of the hedged portfolio on initialization date and on the date 1 year later when it was unwound. This exercise is repeated every month over the data sample period spanning from February 2005 to September 2006. The data are from Datastream and ABN AMRO bank.

Despite the vastly different implementation and calibration procedures, and the different dynamics of the key rates, the pricing and hedging results of the four term structure models tested are not markedly different. For the Euro market, the performance difference is hardly noticeable. In the USD market, the short rate models are marginally better than the two market models. Since parameter estimates are more stable for the HW model and the fact that Gaussian (in contrast to lognormal) facilitates the use of analytical results, we would therefore conclude by recommending the HW model, but stress the fact that the differences among the four models

are not great.

This remaining of the paper is organized as follows. Section 2 discusses the previous studies and section 3 briefly describes the models we use and the research design. Data is described in Section 4. Section 5 deals with calibration procedures of all models and calibration instruments, as well as calibration algorithm. In Section 6, Bermudan swaption prices from the four models are compared. Hedging performance and P&L are reported in Section 7 and, finally, we conclude in Section 8.

2 Previous Studies

Significant effort has been placed on developing pricing models for interest rate claims. However, the empirical evaluation of these models, especially in the swaption market, has lagged behind the theoretical advances made in this area. Much of the literature on multi-factor term structure models has focused on explaining bond yield and swap rates, as outlined in Dai and Singleton (2003) [15]. Therefore, despite the importance of caps and swaptions, there is still wide divergence of opinion on how to best value these claims. It is widely believed that since the term structure of interest rates is driven by multiple factors, interest rate claims should be valued using multi-factor models.

Amin and Morton (1994) [4] implement and test six HJM interest rate models using an implied volatility technique with Eurodollar futures and Eurodollar futures options data. They find that two-parameter models tend to fit prices better, but their parameter estimates are less stable and they earn less from the perceived mispricings. Although the one-parameter models fit slightly less well, their implied parameter values are more stable over time and they are able to earn significantly larger and more consistent abnormal profits from the mispricings they detect. Using caplet data, where maturities ranged from 1 year to 10 years, Ritchken and Chuang (1999) [49] show that generalized Vasicek model captures the hump in the volatility of forward rates, leads to significant improvements on pricing. The interest rate claims are priced in the Heath-Jarrow-Morton paradigm and the structure of

volatilities is captured without using time varying parameters. As a result, the volatility structure is stationary. Gupta and Subrahmanyam (2005) [21] examine many one- and two-factor models (HW, HJM and LMM) for pricing and hedging interest rate caps and floors. Unlike Amin and Morton, they conclude that a one-factor lognormal forward rate model outperforms other competing one-factor models in pricing accuracy, with two factor models improving pricing performance only marginally. However, for hedging, they find a significant advantage in moving from one to two-factor models. The caplets (four maturities: 2-, 3-, 4-, and 5-year) in their paper are hedged using Eurodollar futures contracts up to a maturity of 10 years, in increments of 3 months. The hedging rebalancing interval is 5-day and 20-day. Buhler, Uhrig, Walter and Weber (1999) [11] test different one- and two-factor models (four forward rate models and three spot rate models) in the German fixed-income warrants market. They found that the one-factor forward rate model with linear proportional volatility and the two spot-rate models with two factors outperform the other models. Unlike Gupta and Subrahmanyam (2005) [21], they find no advantage in moving beyond a one-factor model. Using 3 years of interest rate caps price data accross strikes, Jarrow, Li, and Zhao (2007) [37] show that even a three-factor model with stochastic volatility and jumps cannot completely capture the smile/skew patterns observed in this market.

In contrast to the cap/floor market, few empirical studies have been conducted on swaptions. Longstaff, Santa-Clara, Schwartz (LSS, 2001a) [43] use a string model to test the relative valuation of caps and swaptions using at-the-money cap and swaptions data, and find support for using model with at least four factors for pricing swaptions. They use 34 swaptions data where the final maturity dates of the underlying swap are less than or equal to 10 years and the interest rate cap data consists of weekly midmarket implied volatility for 2-, 3-, 4-, 5-, 7- and 10 years. Their criterion for evaluating models is based on the sum of squared percentage pricing errors. In other words, their criterion is based on pricing accuracy, not on hedging precision. Peterson, Stapleton and Subrahmanyam (2003) [48] develop an extension of the lognormal model of Black and Karasinski (1991) [6] to multiple factors

and provide evidence that the addition of a third factor is helpful in pricing swaptions.

Not all studies, however, indicate that multiple factors are necessary for improving pricing performance for swaptions. For example, Driessen, Klaassen, and Melenberg (2003) (hereafter DKM) [16] investigate the performance of several Gaussian models, where volatility structures are deterministic functions of their maturities. They show that the out-of-sample pricing performance of swaption pricing models does not necessarily improve as the number of factors increases. Indeed, one of their one-factor models prices swaptions no worse than their multi-factor models and to the same degree of accuracy as LSS's multi-factor model. Jagannathan, Kaplin and Sun (2003) [35] investigate the pricing of swaptions using multifactor Cox, Ingersoll and Ross models. Their preliminary conclusions suggest that increasing the number of factors does not necessarily improve pricing performance. Indeed, adding factors makes the pricing of short term contracts worse. Fan, Gupta and Ritchken (2007) [18] compare the pricing performance of several single and multi-factor models with different volatility structures and identify those models that eliminate most of the pricing biases in the swaption market. In this regard, their paper is closely related to DKM and LSS. They find that for pricing swaptions, the benefit of increasing the number of factors beyond one is minor. Their results also show that incorporating level dependence in the volatility structure is extremely important for away-from-the-money caps, and that proportional dependent structures are better than both square root and level independent structures. For at-the-money swaptions, the level dependence issue is minor.

Very few studies have compared the abilities of different models for hedging swaptions. LSS briefly consider hedging, in the context of their four-factor model, relative to the Black model. The most related studies on swaptions hedging are by DKM (2003) [16] and Fan, Gupta and Ritchken (2003) [17]. DKM uses the caps with maturities ranging from 1 to 10 years. For the swaptions, the option maturities range from one month to five years while the swaps maturities range from 5 to 10 years. The hedging interval is two weeks, i.e., the value of the hedge portfolio is calculated two weeks after

the hedge portfolio is constructed. DKM use the HJM model to demonstrate that if the number of hedge instruments (zero coupon bonds with different maturities) is equal to the number of factors, multi-factor models outperform one-factor models in hedging caps and swaptions. However, they claim that by using a large set of hedge instruments, their one-factor models perform as well as multi-factor models. This last finding is the opposite of what Gupta and Subrahmanyam (2005) [21] find in the cap market. In the context of Unspanned Stochastic Volatility (USV), Fan, Gupta, and Ritchken (2003) [17] show that even swaption straddles can be well hedged using Libor bonds alone if at least a three factor model is used. HJM is used as the basic model in the hedging test. The volatilities of U.S. dollar swaptions with expiration dates of six months and one, two, three, four, and five years, with underlying swap maturities of two, three, four, and five years. Different Libor discount bonds are used as the (factor) hedging instruments. The hedge position is maintained unchanged for one week, and the hedged and unhedged residuals are obtained and stored. They then repeat this analysis for holding periods of two, three, and four weeks. Fan, Gupta and Ritchken (2007) [18] find that multifactor models are essential for reducing the risk in hedged positions. They also demonstrate that allowing additional hedging instruments in a one and two factor model does not improve the results. Their main conclusion is that while accurate swaption prices can be obtained from a one-factor model, one- and even two-factor models cannot hedge swaptions well, and the benefits of multifactor models are significant. The data for the study consists of USD swaption and cap prices. The swaptions data set comprises volatilities of swaptions of maturities 6 months, 1-, 2-, 3-, 4-, and 5-years, with the underlying swap maturities of 1-, 2-, 3-, 4, and 5-years each. The cap prices are for a ten-month period (March 1 - December 31, 1998), across four different strikes (6.5%, 7%, 7.5%, and 8%) and four maturities (2-, 3-, 4-, and 5-year).

Some research has also been done regarding the importance of factors for pricing Bermudan swaptions. Longstaff, Santa-Clara and Schwartz (2001b) [44] show that exercise strategies based on one-factor models understate the true option value for Bermudans. They contend that the current market

practice of using one-factor models leads to suboptimal exercise policies and a significant loss of value for the holders of these contracts. However, Andersen and Andreasen (2001) [1] conclude that the standard market practice of recalibrating one-factor models does not necessarily understate the price of Bermudan swaptions. The authors do not investigate any hedging issues however. Pietersz and Pelsser (2005) [47] compare single factor Markov-function and multifactor market models for hedging Bermudan swaptions. They find that on most trade days the Bermudan swaption prices estimated from these two models are similar and co-move together. Their results also show that delta and delta-vega hedging performances of both models are comparable. The delta hedge in their study is set up in terms of discount bonds, one discount bond for each tenor associated with the deal. In the case of 10Y Bermudan with annual coupon, there are 11 such discount bonds. The hedging is carried out daily. For the swaptions, the option maturities range from one month to five years while the swaps maturities range from 1 to 30 years.

3 Description of Term Structure Models Implementation

In this study, we compare the pricing and hedging performance of four of the most important term structure models, viz. HW, BK, SMM and LMM. In this section, we briefly describe the four models and their characteristics.

3.1 The Hull-White and BK Model

In the Hull-White model, also referred to as the extended-Vasicek model, the instantaneous short-rate process evolves under the risk-neutral measure as follows:

$$dr_t = [\theta_t - a_t r_t]dt + \sigma_t dz, \quad (1)$$

where θ_t , a_t and σ_t are deterministic functions of time. The function θ_t is chosen so that the model fits the initial term structure of interest rates. The other two time-varying parameters, a_t and σ_t , enable the model to be

fitted to the initial volatility of all zero coupon rates and to the volatility of short rate at all future times.² Hull and White (1994a [28], 1994b [29]) note, however, that while a_t and σ_t allow the model to be fitted to the volatility structure at time zero, the resulting volatility term structure could be non-stationary in the sense that the future volatility structure implied by the model can be quite different from the volatility structure today. On the other hand, when these two parameters are kept constant, the volatility structure stays stationary but the model's consistency with market prices of e.g. caps or swaptions can suffer considerably. Thus there is a trade-off between tighter fit and model stationary.

In the HW model the distribution of short rate is Gaussian. Gaussian distribution leads to a theoretical possibility of short rate going below zero. A model that addresses this negative interest rate issue is the Black and Karasinski (1991) [6] model. In this model, the risk neutral process for logarithm of the instantaneous spot rate, $\ln r_t$, is

$$d \ln r_t = [\theta_t - a_t \ln r_t] dt + \sigma_t dz, \quad (2)$$

where r_0 (at $t = 0$) is a positive constant, θ_t , a_t and σ_t are deterministic functions of time as before.

Both HW and BK model are implemented by constructing a recombining trinomial lattice for the short-term interest rate following Hull and White (1996 [30], 2001 [32]).³

3.2 The Libor Market Model

The Libor Market Model (and the Swap Market Model described in the next section) have been popular among practitioners because it is consistent with market standard of using Black formula to price caplets (European swaptions).

Let $L_i(t)$, $i = 1, 2, \dots, N$, as the forward Libor rates, and δ_j be the interest rate reset interval between T_j and T_{j+1} . The dynamics of forward Libor rate

²The initial volatility of all rates depends on $\sigma(0)$ and $a(t)$. The volatility of short rate at future times is determined by $\sigma(t)$ (Hull and White 1996, p.9) [30].

³See Khan, Guan and Poon (2007) [39] for further details.

under the T_{n+1} terminal measure is

$$\frac{dL_i(t)}{L_i(t)} = - \sum_{j=i+1}^N \frac{\delta_j L_j(t) \sigma_{ij}(t)}{1 + \delta_j L_j(t)} dt + \sigma_i(t) \cdot dW^{Q_{n+1}^T}(t) \quad (3)$$

for $0 \leq t \leq \min(T_i, T_{n+1})$, $i = 1, \dots, n, \dots, N$, where $\sigma_{ij}(t)$ is the covariance between $L_i(t)$ and $L_j(t)$, and $\sigma_i(t)$ is the volatility of $L_i(t)$. Note that if $B_{n+1}(t)$ is the price of a zero coupon bond maturing at T_{n+1} , then $L_n(t)$ is driftless under the T_{n+1} terminal measure with $B_{n+1}(t)$ as the numeraire. This special case is called the forward measure and it leads to the Black formula for caplet and floorlet.

Unlike caplet valuation, to implement LMM for pricing swaptions, it is necessary to simulate all forward rates under one measure. This means we have to calculate the drift term for all $L_n(t)$ under the terminal measure in (3).

Following Glasserman and Zhao (1999) [22], the Euler discretization scheme of the LMM dynamics under the T_{N+1} terminal measure in (3) is, for a time step h ,

$$L_i((k+1)h) = L_i(kh) \exp \left[\left(\mu_i(kh) - \frac{1}{2} \sigma_i(kh)^2 \right) h + \sigma_i(kh) \sqrt{h} \epsilon_{j+1} \right],$$

where

$$\mu_i(kh) = - \sum_{j=i+1}^N \frac{\delta L_j(kh) \sigma_{ij}(kh)}{1 + \delta L_j(kh)}. \quad (4)$$

One difficulty in implementing LMM is that the drift is state-dependent. One therefore needs ways to approximate the evolution. The industry standard used to be Euler stepping using many steps, but Glasserman and Zhao (1999) [22] note that forward prices of bonds are not arbitrage free under the Euler discretization scheme. Hence, predictor-corrector scheme is used here to make the drift approximation more accurate (see Hunter, Joshi and Jackel (2001) [23] for more details). For the Euler stepping, we have the following approximation

$$\int_S^T \mu_i(kh) = \int_S^T - \sum_{j=i+1}^N \frac{\delta L_j(t)}{1 + \delta L_j(t)} \rho_{ij} \sigma_i \sigma_j dt \approx - \sum_{j=i+1}^N \frac{\delta L_j(S)}{1 + \delta L_j(S)} \int_S^T \rho_{ij} \sigma_i \sigma_j dt \quad (5)$$

For the predictor-corrector method, we first estimate the Libor rates using Euler stepping in (5) and use the resulting forward rates to adjust the drift. The adjusted drift is used to re-compute Libor rate $L_i(T)$ using the same set of random numbers. Thus the following equation is used instead of equation (5)

$$\int_S^T \mu_i(kh) = -\frac{1}{2} \sum_{j=i+1}^N \left(\frac{\delta L_j(T)}{1 + \delta L_j(T)} + \frac{\delta L_j(S)}{1 + \delta L_j(S)} \right) \int_S^T \rho_{ij} \sigma_i \sigma_j dt \quad (6)$$

3.3 The Swap Market Model

The modeling framework of the SMM is similar to that of the LMM, but the SMM has not been as intensively studied as the LMM. Here, we use a particular version of co-terminal swap market model. Given the tenor structure, a co-terminal Swap Market Model refers to a model which assigns the arbitrage free dynamic to a set of forward swap rates that have different swap starting date T_n ($n = 1, \dots, M-1$) but all conclude on the same maturity date T_M . One important feature about the co-terminal swaption is that it is internally consistent with a Bermudan swaption that gives the holder the right to enter into a swap at each reset date during period T_1 to T_{M-1} with T_M being the terminal maturity of the underlying swap. When considering, at each tenor date, whether or not to exercise the option to enter into a swap contract, the holder needs to consider the forward swap rate dynamics from that tenor date until final maturity.

The advantage of co-terminal SMM over other market models in pricing Bermudan swaptions has already been noted and discussed in Jamshidian (1997) [34] and Galluccio, Huang, Ly and Scaillet (2007) [20]. Furthermore, Galluccio, Huang, Ly and Scaillet (2007) [20] use graph theory to classify the “admissible” market models into co-initial, co-sliding and co-terminal subgroups. They show that the LMM is the only admissible model for swaps of a co-sliding type.

In Galluccio et al (2007) [20], the co-terminal SMM is defined by introducing a collection of mutually equivalent probability measures and a family of Brownian motions such that all forward swap rates satisfy the

chosen SDE. Let $C_{n,M}(t) = \sum_{i=n+1}^M \delta B_i(t)$ represent the value of the annuity from T_{n+1} to T_M . $B_i(t)$ is the i th bond price at time t and δ is the interest rate reset interval (we assume all intervals are the same for simplicity). The co-terminal forward swap rates satisfy the following general SDE under the terminal measure $Q^{M-1,M}$ (with $C_{M-1,M}(t)$ as the numeraire):

$$dS_{n,M}(t) = S_{n,M}(t) \sigma_{n,M}(t) dW_t^{M-1,M} + \text{drift for } n = 1, \dots, M-1 \quad (7)$$

We follow Joshi and Liesch (2007) [36], who recommend using the cross-variation in assessing the impact of changing numeraire on the drift in (7). The drift of a forward swap rate $S_{n,M}(t)$, $n = 1, \dots, M-1$ under the terminal measure $Q^{M-1,M}$ is given by:

$$\begin{aligned} E^{M-1,M}[dS_{n,M}(t)] &= \mu_{n,M} = -\frac{C_{M-1,M}(t)}{C_{n,M}(t)} \left\langle \frac{B_n(t) - B_m(t)}{C_{n,M}(t)}, \frac{C_{n,M}(t)}{C_{M-1,M}(t)} \right\rangle \\ &= -\frac{C_{M-1,M}(t)}{C_{n,M}(t)} \left\langle S_{n,M}(t), \frac{C_{n,M}(t)}{C_{M-1,M}(t)} \right\rangle \end{aligned} \quad (8)$$

Note that when $n = M-1$, $\mu_{M-1,M} = 0$ because $\langle S_{n,M}(t), 1 \rangle = 0$. Here $\langle \cdot, \cdot \rangle$ means the quadratic variation.

In Joshi and Liesch (2007) [36], the $S_{n,M}(t)$ term inside the square brackets is simplified into F independent Wiener process $\sum_{k=1}^F W_k$. For one factor model, all correlation terms $\rho_{n,m} = 1$ and hence can be ignored. The correlation term has to be explicitly included when the dimension of W_k is two or more, however. Expanding the cross-variation term in equation (8), the drift becomes a complicated function of numeraire and cross-variation of two forward swap rates. By induction, the general form of drift under the terminal measure can also be written as:

$$\mu_{n,M} = \sum_{i=n}^{M-2} \left(\tau_{i+1} \frac{C_{i+1,M}}{C_{M-1,M}} \rho_{n,i+1} S_{n,M} \sigma_{n,M} S_{i+1,M} \sigma_{i+1,M} \prod_{j=n+1}^i (1 + \tau_j S_{j,M}) \right) \quad (9)$$

Details of the derivation are given in the working paper by Gan, Guan and Poon (2007) [19]. The above equation is not an explicit function of $S_{n,M}(t)$ for $n = 1, \dots, M-1$, because the $C_{n,M}$ term is derived from a set of $S_{i,M}(t)$

for $n \leq i \leq M-1$. This drift term is complicated in appearance but is quite straightforward to compute.

The SMM is implemented using Monte Carlo simulations using the predictor-corrector drift approximation for more accuracy. This is done in a similar way to the LMM case in equation (5). First, using Euler stepping, we approximate

$$\int_S^T \mu_{n,M}(t) dt \approx \int_S^T \sum_{i=n}^{M-2} (\mu_{n,M}^*(S) \rho_{n,i+1}(t) \sigma_{n,M}(t) \sigma_{i+1,M}(t)) dt \quad (10)$$

where $\mu_{n,M}^*(S) = \tau_{i+1} \frac{C_{i+1,M}(S)}{C_{M-1,M}(S)} S_{n,M}(S) S_{i+1,M}(S) \prod_{j=n+1}^i (1 + \tau_j S_{j,M}(S))$.

Then we re-compute the state-dependent part at the end of each step and re-evolve the swap rates using average drift using the same set of random numbers.

$$\int_S^T \mu_{n,M}(t) dt \approx \frac{1}{2} \int_S^T \sum_{i=n}^{M-2} (\mu_{n,M}^*(S) + \mu_{n,M}^*(T) \rho_{n,i+1}(t) \sigma_{n,M}(t) \sigma_{i+1,M}(t)) dt \quad (11)$$

4 Research Design and Data

4.1 Choosing Calibration Instruments

A common financial practice is to calibrate the interest rate model using the instruments that are as closely related as possible to the instrument being valued and hedged, rather than attempting to fit the models to all available market data. This approach is appropriate here since the problem we have at hand is narrowly focused. In this study, the objective is to price and hedge 10×1 Bermudan. For this 10×1 Bermudan swaption, the most relevant calibrating instruments are the 1×10 , 2×9 , $3 \times 8, \dots$, 10×1 co-terminal European swaptions. (An $n \times m$ swaption is an n -year European option to enter into a swap lasting for m years after option maturity.) These European swaptions are the closest hedging instruments for the Bermudan. When fully calibrated, the pricing error of these individual instruments are minimized and should therefore be the closest to achieving

no-arbitrage condition. Hence, the 10 ATM co-terminal European swaptions are used to calibrate the four interest rate models for pricing the 10×1 Bermudan swaption.⁴ In the subsequent unwinding period, when the models are used for pricing the 9×1 Bermudan swaption we use the corresponding nine ATM co-terminal European swaptions, i.e., $1 \times 9, 2 \times 8, \dots, 9 \times 1$, to calibrate the term structure models.

4.2 Model Calibration

All four TS models are no-arbitrage interest rate models in the sense that they could exactly match the current term structure of interest rates. Both HW and BK have mean-reverting drift. Practitioners and econometricians often use historical data for inferring the rate of mean reversion (e.g., Candelon and Gil-Alana (2006) [13]).

Bermudan swaptions are typically hedged with co-terminal at-the-money European swaptions. Thus, the co-terminal swaption implied volatilities are used as the common calibration instrument for all the four models.

Given the calibrating instruments, the immediate question that follows is the specification of the parameters. Since we are dealing with only 1-factor models, we do not need to consider the functional form of the correlation coefficient. The only specification needed is that for volatilities σ_t for the short rate models. For the short rate model, there should be eleven volatility parameters, $\sigma_0, \sigma_1, \dots, \sigma_{10}$. We use the European swaption prices to calibrate $\sigma_0, \sigma_3, \sigma_{11}$ and assume the other volatility parameters are linear interpolations of these three estimates. The volatility functions for LMM and SMM are more complicated and described, in full detail, in Section 5.2 and 5.3.

Given that the European swaption prices are quoted in Black implied volatility, the only model that is consistent with these Black implied volatility is the SMM. So for the SMM, we can fit the Black implied volatility directly using the pre-specified volatilities specification and obtain parameter values without calculating the European swaption prices.

Since swap rates are linear combinations of forward rates (with stochastic

⁴Pietersz and Pelsser (2005) [47] follow the same approach.

weights), it follows that swap rate dynamics are also fully determined once the volatilities and correlations of the forward rates have been specified. In practice, it is extremely important to ascertain the implications for the dynamics of the swap rates, given a particular choice of dynamics for the forward rates and vice versa. Jackel and Rebonato (2003b) [38] produce an approximate closed form solution for the pricing of European swaptions under the LMM framework based on log-normally evolving forward rates.⁵ Using the total variance of the lognormal swap rate in conjunction with the Black formula, one can price European swaptions with a remarkable degree of accuracy. Let $\sigma^{n \times M}(t)$ denote the volatility at time t of a swap rate $SR^{n \times M}$ expiring n years from today and maturing M years thereafter. Let B_{i+1} denote the zero coupon bond maturing at the payment time of the i th forward rate L_i . The swap rate can be approximated by a linear function of all relevant forward rates as follows

$$SR^{n \times M}(t) = \sum_{i=n}^M w_i L_i(t)$$

with the weights w_i given by

$$w_i = \frac{\delta_\alpha \Pi_{j=n+1}^i \frac{1}{1 + \delta_j L_j(t)}}{\sum_{k=n+1}^M \delta_k \Pi_{j=n+1}^k \frac{1}{1 + \delta_j L_j(t)}}$$

With Ito formula, then

$$[\sigma^{n \times M}(t)]^2 = \frac{\sum_{j,k=1}^M w_j(t) w_k(t) L_j(t) L_k(t) \rho_{jk}(t) \sigma_j(t) \sigma_k(t)}{\left[\sum_{i=1}^M w_i(t) L_i(t) \right]^2} \quad (12)$$

As usual, $\sigma_j(t)$ is the time- t instantaneous volatility of the forward rate L_j , and ρ_{jk} is the instantaneous correlation between forward rate L_j and L_k . By “freezing” the drift, equation (12) can be approximated by

$$[\sigma^{n \times M}(t)]^2 = \frac{\sum_{j,k=1}^M w_j(0) w_k(0) L_j(0) L_k(0) \rho_{jk}(t) \sigma_j(t) \sigma_k(t)}{\left[\sum_{i=1}^M w_i(0) L_i(0) \right]^2} \quad (13)$$

⁵Hull and White (2000) [31] derive a similar swaption approximation method under a different volatility structure.

This is the swap rate volatility to use in the Black formula to price the European swaptions. The empirical result in Jackel and Rebonato (2003b) [38] shows that equation (13) is very accurate.

4.2.1 Goodness-of-fit Measure for the Calibration

The four interest rate models are calibrated by minimizing the sum of squared percentage pricing errors between the model and the market prices of the co-terminal European swaptions, i.e. the goodness-of-fit measure is

$$\min \sum_{i=1}^n \left(\frac{P_{i,n,\text{model}}}{P_{i,n,\text{market}}} - 1 \right)^2$$

where $P_{i,n,\text{market}}$ is the market price and $P_{i,n,\text{model}}$ is the model generated price of the $i \times (n - i)$ European swaptions, with $n = 11$ when the models are calibrated to price 10×1 year Bermudan swaptions, and $n = 10$ when the models are calibrated to price 9×1 year Bermudan swaptions. Instead of minimizing the sum of squared percentage price errors, we could have minimized the sum of squared errors in prices. However, such a minimization strategy would place more weights on the expensive instruments. Minimization of squared percentage pricing error is typically used as a goodness-of-fit measure for similar calibrations in the literature and by practitioners.

4.3 Testing the Hedging Performance

Let t denote a particular month in the sample period from February 2005 to September 2007. The procedure for calibrating, hedging and unwinding a 10×1 Bermudan swaption are as follows:

- (i) At month t , an interest rate model is calibrated to 10 ATM co-terminal European swaptions underlying the 10×1 ATM Bermudan swaption by minimizing the root mean square of the percentage pricing errors.
- (ii) The calibrated model from (i) is then used to price the 10×1 ATM Bermudan swaption at t , and to calculate the hedge ratios using, as hedge instruments, 1-year, 5-year and 11-year swaps. At the time of

initiation, t , the values of all three swaps (1-year, 5-year and 11-year) are zero.

- (iii) A delta hedged portfolio is formed by minimizing the amount of delta mismatch (i.e., the delta of the portfolio of the Bermudan swaption and the swaps has to be as close to zero as possible).
- (iv) At time $t + 1$ (i.e. one year later), the interest rate model is calibrated to 9 co-terminal ATM European swaptions ($9 \times 1, 8 \times 2, \dots$) underlying the Bermudan swaption in (ii) which is now 9×1 .
- (v) The calibrated model from (iv) is used to price the 9×1 Bermudan swaption using the strike rate determined in (ii). The time $t + 1$ yield curve is used to price the three swaps in (ii), which are now 0-year, 4-years and 10-year to maturity.
- (vi) The profit and loss is calculated for the delta hedged portfolio formed at t and unwound at $t + 1$ with any net cash flow at t suitably compounded.
- (vii) Steps (i) to (vi) are repeated every month for $t+1, t+2, \dots$ till T where T is the last month of the sample period for the hedging exercise.
- (viii) The whole process from (i) to (vii) is performed four times since we have four competing term structure interest rate models.

4.4 Data

To perform the valuation and hedging analysis described above, the following data sets are collected from Datastream:

- (a) Monthly prices (quoted in Black implied volatility) of ATM European swaptions in Euro and USD. Two sets of implied volatilities were collected: from February 2005 to September 2006, prices of co-terminal ATM European swaptions underlying the 10×1 Bermudan swaption, and from February 2006 to September 2007, prices of co-terminal ATM European swaptions underlying the 9×1 Bermudan swaption. These

prices are quoted in Black implied volatility. The implied volatility matrix downloaded has a number of missing entries especially in the earlier part of the sample period. The missing entries were filled in using log-linear interpolation following Brigo and Morini (2005, p 9, 24 and 25) [9].

- (b) Annual yields, $R_{0,t}$, for maturities up to 11 years are downloaded from Datastream. For short rate models, these annual yields are converted into continuously compounded yields using $r_{0,t} = \ln(1 + R_{0,t})$, which in turns are converted into discrete forward and swap rates as input in the LMM and SMM, respectively.
- (c) Monthly data of the annual yield curve for the period January 1999 to July 2007, i.e. total 103 observations are downloaded for Euro and USD. This data was transformed into discrete forward rates for use in the principal component analysis.
- (iv) Monthly data of 1-month yield for the period January 2000 to July 2007 was downloaded for estimating the “mean-reversion” parameter of the short rate model. In the implementation, a time step (Δt) of 0.1 year is used for constructing the trinomial tree, which means that the rates on nodes of the tree are continuously compounded Δt -period rates. All the required yields are linearly interpolated from these annual yields. At the very short end, the 0.1 year rate is approximated by the 1-month rate.

All monthly data are collected for the last business day of the month in the sample period.

5 Model Calibration

Before we price Bermudan swaptions, we need to find the parameters that give the correct European swaptions to avoid any arbitrage opportunities. This section discusses the calibration method for each model.

5.1 HW&BK Model Calibration

To calibrate the model in (1) or (2), we need to fix θ_t , σ_t and a_t . Given the models are recalibrated every month, it does not make sense to allow the long term mean reversion rate to change every month. Here, we have decided to fix the mean reversion rate $a_t = a$ and estimate a from historical time series of 1-month yield following the approach of Candelon and Gil-Alana (2006) [13]. The discretised version of the HW model in (1) with $a_t = a$ is as follows:

$$\begin{aligned} r_{t+1} - r_t &= [\theta_t - ar_t] + \varepsilon_{t+1}, \\ r_{t+1} &= \theta_t + (1 - a)r_t + \varepsilon_{t+1}, \end{aligned} \tag{14}$$

where ε_{t+1} is a draw from a normal distribution. Equation (14) represents an AR(1) process. An ordinary least square (OLS) estimate of coefficient $(1 - a)$ in equation (14) $\hat{\beta}$ would be⁶

$$\begin{aligned} 1 - a &= \hat{\beta} = \frac{\rho\sigma(r_{t+1})\sigma(r_t)}{\sigma^2(r_t)} = \rho \\ a &= 1 - \rho \end{aligned}$$

where ρ is the correlation coefficient between r_{t+1} and r_t . For the BK model we perform this regression using time series of $\ln(r)$, where as before r , is 1-month interest rate (as the time step in the trinomial tree is 0.1).

There can be different ways to parameterize σ_t . It could be piecewise linear, piecewise constant or σ_t could follow some parametric functional form. In this study, we chose to fix σ_t at three fixed points: σ_0 , σ_3 and σ_{11} for the 10x1 Bermudan swaption. These three volatility parameters are estimated through the calibration process.⁷ The volatilities for the time periods in between these points are linearly interpolated. For the 9×1

⁶For the regression $y_i = \alpha + \beta x_i + \varepsilon_i$, the OLS estimate of β is

$$\hat{\beta}_{xy} = \frac{\rho_{xy}\sigma_x\sigma_y}{\sigma_{x_y}^2}$$

⁷We are very grateful to Peter Van de Wal for making this suggestion, which made the calibration exercise a lot more manageable.

Bermudan swaptions, the fixed point used are σ_0 , σ_2 and σ_{10} . All the volatility parameters are bounded between 0.0001 and 1.

5.2 Libor Market Model Calibration

We observed in the previous section that, in LMM, the joint evolution of forward Libor rates under the chosen pricing measure is fully determined by their instantaneous volatilities and correlations. Among the different parameterizations of the volatility term structure proposed in the literature, we adopt the one suggested by Rebonato (1999) [50] which is tested extensively in the literature. The parametric form in Rebonato (1999) [50] is

$$\sigma_i(t) = \phi(i)[(a + b(T_i - t))e^{-c(T_i - t)} + d] \quad (15)$$

where $\phi(i)$ is specified for each i and a , b , c , d are parameters which are bounded in the range reported in Table 1.⁸

Table 1: The parameters range

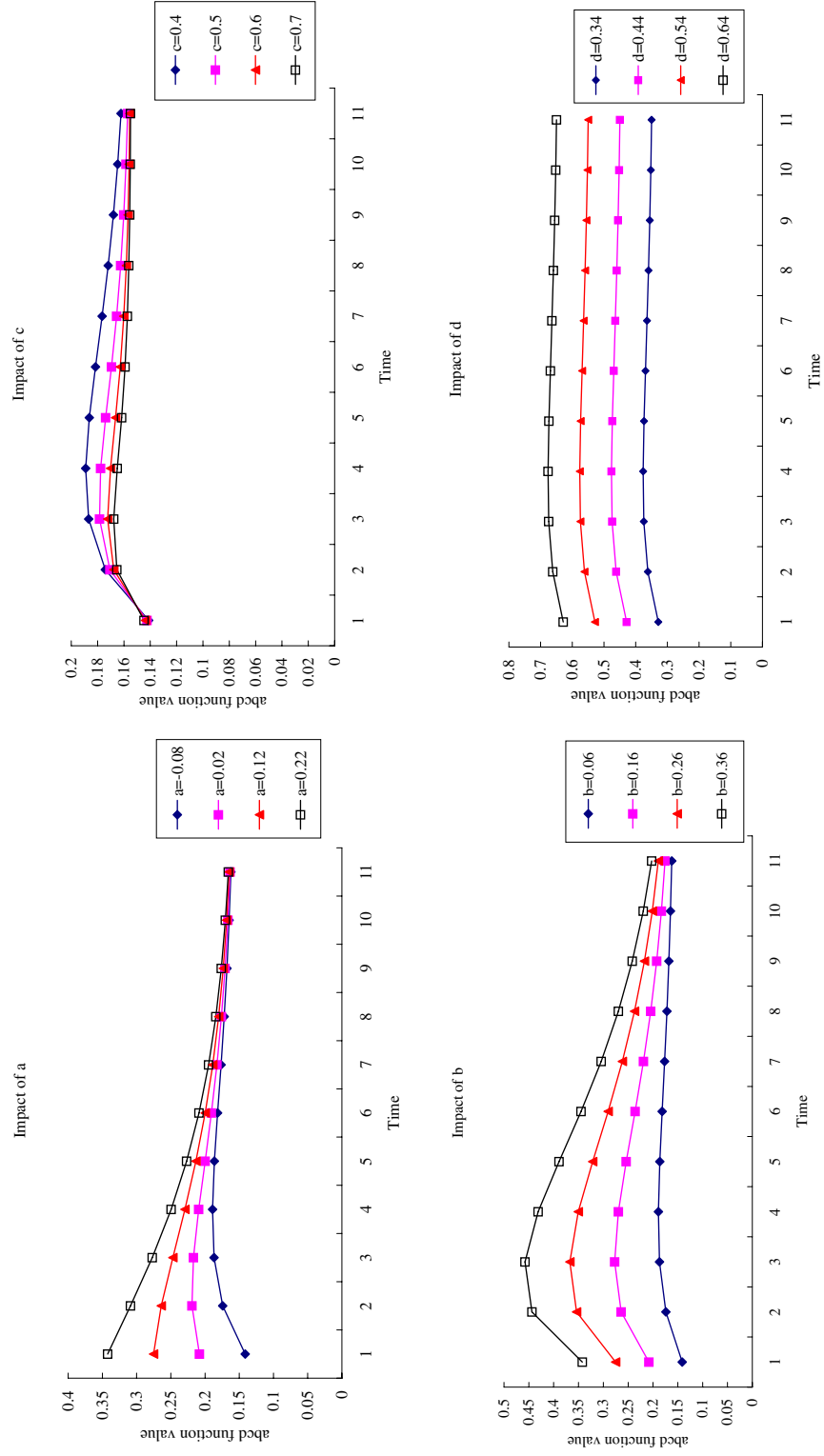
Parameter	Range
a	-1 to 1
b	-1 to 1
c	0 to 5
d	0 to 0.5

Assume the base case scenario is $a = -0.08$, $b = 0.06$, $c = 0.40$ and $d = 0.15$. Now we plot the graphs of the impact of each parameters on the shape of the volatility curve in Figure 1. Parameter a influence the short end of the volatility; parameter b has more impact on the humped shape of the curve; the volatility function appears to be not sensitive to the value of parameter c , and parameter d determine the volatility level.

In contrast to, e.g., a piecewise-constant, this parametric specification of the instantaneous volatility preserves the qualitative shape of the implied

⁸See <http://www.fincad.com/support/developerFunc/mathref/LIBORMarketModel.htm>.

Figure 1: Example of impact of each parameter on the volatility shape with initial parameter values as $a=-0.08$, $b=0.06$, $c=0.40$ and $d=0.15$.



volatility observed in the market, i.e., a hump at around year two. In a full calibration, $\phi(i)$ is different for each European swaption. Here, we set $\phi(i) = 1$ so that the calibration procedure and the degree of freedom are comparable to those of the short rate models.⁹¹⁰

5.3 Swap Market Model Calibration

Brigo, Mercurio and Morini (2003) [8], Rebonato (2002) [51], among others, contribute to the parameterisation and calibration of the LMM. Based on their research, Galluccio et al (2007) [20] test calibration of co-terminal SMM using similar functional form as the one advocated by Rebonato (2002) [51] in (15). By calibrating to swaption and caplet ATM volatilities, Galluccio et. al (2007) get different set of parameters $\{\phi_j, a_j, b_j, c_j, d_j\}$ where $j = 1, \dots, M - 1$ for $M - 1$ number of co-terminal swaptions under different forward measure. Since the model we implement here is under one measure, and the volatility of different co-terminal swap rates affect the drift terms in a complicated way, it is unrealistic to use different parameter set for different co-terminal swaption. Therefore, we adopt only one volatility form and the four parameters are common to volatilities of all co-terminal swaptions. We assume the volatility for the swap rate $S_{n,M}(t)$ follows

$$\sigma_{n,M}(t) = \psi(\tau_n, a, b, c, d) = [a + b\tau_n]e^{-c\tau_n} + d, \text{ for } T_0 < \tau_n < T_M \quad (16)$$

where $\tau_n = T_n - t$. The parameters a , b , c , and d are calibrated directly to the implied volatilities of the set of co-terminal swaptions with different maturities but associated with same length of swaps. Because of its simple log-linear character, the calibration algorithm for this parametric formula is quite fast and robust. Again, the fitted volatility function can preserve the well-known humped shape of market quoted Black implied volatility.

⁹NAG C Sequential Quadratic Programming algorithm is used to find the parameters of the caplet volatility so that the root mean square error of the differences between the approximated swaption volatility and market data is minimized.

¹⁰The correlation between Libor rates is set to 1 in the one-factor LMM. The assumption is comparable with the properties of the short rate models.

5.4 Calibration Results

In comparison, the short rate models calibration is relatively computationally costly as they do not have an analytical (approximation) formula for European swaptions. For the LMM, the calibration is fast with the approximation formula which links swap rate volatility to caplet volatility. The SMM model is straightforward and faster since it calibrates directly on swaption volatilities.

Calibration results are summarized in Table 2. The calibration of 11-year European co-terminal swaptions is performed on the last business date of each month from February 2005 to September 2006. The calibration of 10-year European co-terminal swaptions, on the other hand, is performed on the last business date of each month from February 2006 to September 2007. All the root mean square percentage pricing errors are reported in Table 2. Figure 2 presents the same information by date and Figure 3 reproduces the results and arrange them according to contract maturity.

It is clear that the pricing error of the two short rate models are similar, whereas the pricing errors of the two market models are similar. Except for 11-year USD market, the short rate model errors are smaller than those of the market models. Although the short rate models are, supposedly, less flexible, it is implemented with a smaller time step, i.e., 0.1 year. The market models are calibrated by fitting the volatility functions directly to the swaption Black implied volatilities. Such a pricing error result was quite unexpected and is useful information for practice.

The other key issue in model calibration is the stability of the parameters calibrated. The calibrated parameter values are reported in Table 3 and 4 (Table 3 for Euro market and Table 4 for USD market) and presented in Figure 4. All parameters, except the parameter c of LMM and SMM, are reasonably stable through time. The volatility is insensitive to parameter c .

The stability of the model parameter values will have a direct impact on the hedging result. It appears that the HW model values are more stable than BK, and the SMM model values are more stable than LMM.

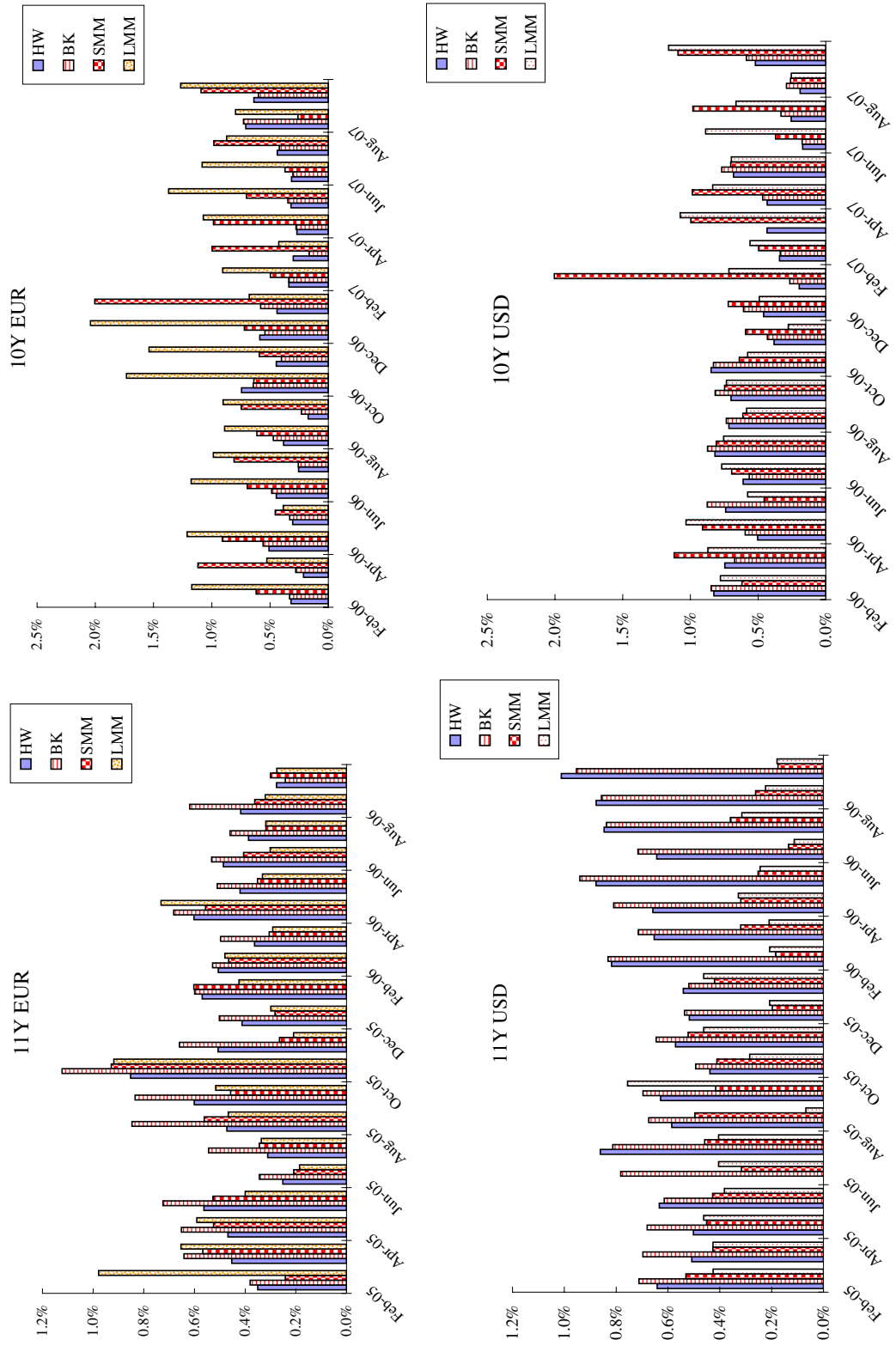
Table 2: Root mean square error to European co-terminal swaptions (11Y and 10Y) in EUR and USD markets from February 2005 to September 2007

EUR 11Y					USD 11Y				
Date	HW	BK	SMM	LMM	Date	HW	BK	SMM	LMM
2005-2-28	0.4%	0.4%	0.2%	1.0%	2005-2-28	0.6%	0.7%	0.5%	0.4%
2005-3-31	0.5%	0.6%	0.6%	0.7%	2005-3-31	0.5%	0.7%	0.4%	0.4%
2005-4-29	0.5%	0.7%	0.5%	0.6%	2005-4-29	0.5%	0.7%	0.5%	0.5%
2005-5-31	0.6%	0.7%	0.5%	0.4%	2005-5-31	0.6%	0.6%	0.4%	0.4%
2005-6-30	0.3%	0.3%	0.2%	0.2%	2005-6-30	0.0%	0.8%	0.3%	0.4%
2005-7-29	0.3%	0.5%	0.3%	0.3%	2005-7-29	0.9%	0.8%	0.5%	0.4%
2005-8-31	0.5%	0.8%	0.6%	0.5%	2005-8-31	0.6%	0.7%	0.5%	0.1%
2005-9-30	0.6%	0.8%	0.5%	0.5%	2005-9-30	0.6%	0.7%	0.4%	0.8%
2005-10-31	0.9%	1.1%	0.9%	0.9%	2005-10-31	0.4%	0.5%	0.4%	0.3%
2005-11-30	0.5%	0.7%	0.3%	0.2%	2005-11-30	0.6%	0.6%	0.5%	0.5%
2005-12-30	0.4%	0.5%	0.3%	0.3%	2005-12-30	0.5%	0.5%	0.2%	0.2%
2006-1-31	0.6%	0.6%	0.6%	0.4%	2006-1-31	0.5%	0.5%	0.4%	0.5%
2006-2-28	0.5%	0.5%	0.5%	0.5%	2006-2-28	0.8%	0.8%	0.2%	0.2%
2006-3-31	0.4%	0.5%	0.3%	0.3%	2006-3-31	0.7%	0.7%	0.3%	0.2%
2006-4-28	0.6%	0.7%	0.6%	0.7%	2006-4-28	0.7%	0.8%	0.3%	0.3%
2006-5-31	0.4%	0.5%	0.4%	0.3%	2006-5-31	0.9%	0.9%	0.3%	0.2%
2006-6-30	0.5%	0.5%	0.4%	0.3%	2006-6-30	0.6%	0.7%	0.1%	0.1%
2006-7-31	0.4%	0.5%	0.3%	0.3%	2006-7-31	0.8%	0.8%	0.4%	0.3%
2006-8-31	0.4%	0.6%	0.4%	0.3%	2006-8-31	0.9%	0.9%	0.3%	0.2%
2006-9-29	0.3%	0.2%	0.3%	0.3%	2006-9-29	1.0%	1.0%	0.2%	0.2%
Total	9.3%	11.9%	8.6%	9.0%	Total	12.8%	14.5%	7.1%	6.6%

EUR 10Y					USD 10Y				
Date	HW	BK	SMM	LMM	Date	HW	BK	SMM	LMM
2006-2-28	0.3%	0.3%	0.6%	1.2%	2006-2-28	0.8%	0.8%	0.6%	0.8%
2006-3-31	0.2%	0.3%	1.1%	0.5%	2006-3-31	0.7%	0.7%	1.1%	0.9%
2006-4-28	0.5%	0.6%	0.9%	1.2%	2006-4-28	0.5%	0.6%	0.9%	1.0%
2006-5-31	0.3%	0.3%	0.5%	0.4%	2006-5-31	0.7%	0.9%	0.5%	0.6%
2006-6-30	0.4%	0.5%	0.7%	1.2%	2006-6-30	0.6%	0.6%	0.7%	0.8%
2006-7-31	0.3%	0.3%	0.8%	1.0%	2006-7-31	0.8%	0.9%	0.8%	0.8%
2006-8-31	0.4%	0.5%	0.6%	0.9%	2006-8-31	0.7%	0.7%	0.6%	0.6%
2006-9-29	0.2%	0.2%	0.7%	0.9%	2006-9-29	0.7%	0.8%	0.7%	0.7%
2006-10-31	0.7%	0.6%	0.6%	1.7%	2006-10-31	0.8%	0.8%	0.6%	0.6%
2006-11-30	0.4%	0.4%	0.6%	1.5%	2006-11-30	0.4%	0.4%	0.6%	0.3%
2006-12-29	0.6%	0.5%	0.7%	2.0%	2006-12-29	0.5%	0.6%	0.7%	0.5%
2007-1-30	0.4%	0.6%	2.0%	0.7%	2007-1-30	0.2%	0.3%	2.0%	0.7%
2007-2-28	0.3%	0.3%	0.5%	0.9%	2007-2-28	0.3%	0.3%	0.5%	0.6%
2007-3-31	0.3%	0.2%	1.0%	0.4%	2007-3-31	0.4%	0.0%	1.0%	1.1%
2007-4-30	0.3%	0.3%	1.0%	1.1%	2007-4-30	0.4%	0.5%	1.0%	0.8%
2007-5-31	0.3%	0.3%	0.7%	1.4%	2007-5-31	0.7%	0.8%	0.7%	0.7%
2007-6-30	0.3%	0.3%	0.4%	1.1%	2007-6-30	0.2%	0.2%	0.4%	0.9%
2007-7-29	0.4%	0.4%	1.0%	0.9%	2007-7-29	0.3%	0.3%	1.0%	0.7%
2007-8-31	0.7%	0.7%	0.3%	0.8%	2007-8-31	0.2%	0.3%	0.3%	0.3%
2007-9-28	0.6%	0.6%	1.1%	1.3%	2007-9-28	0.5%	0.6%	1.1%	1.2%
Total	8.2%	8.3%	15.8%	21.0%	Total	10.6%	11.1%	15.8%	14.3%

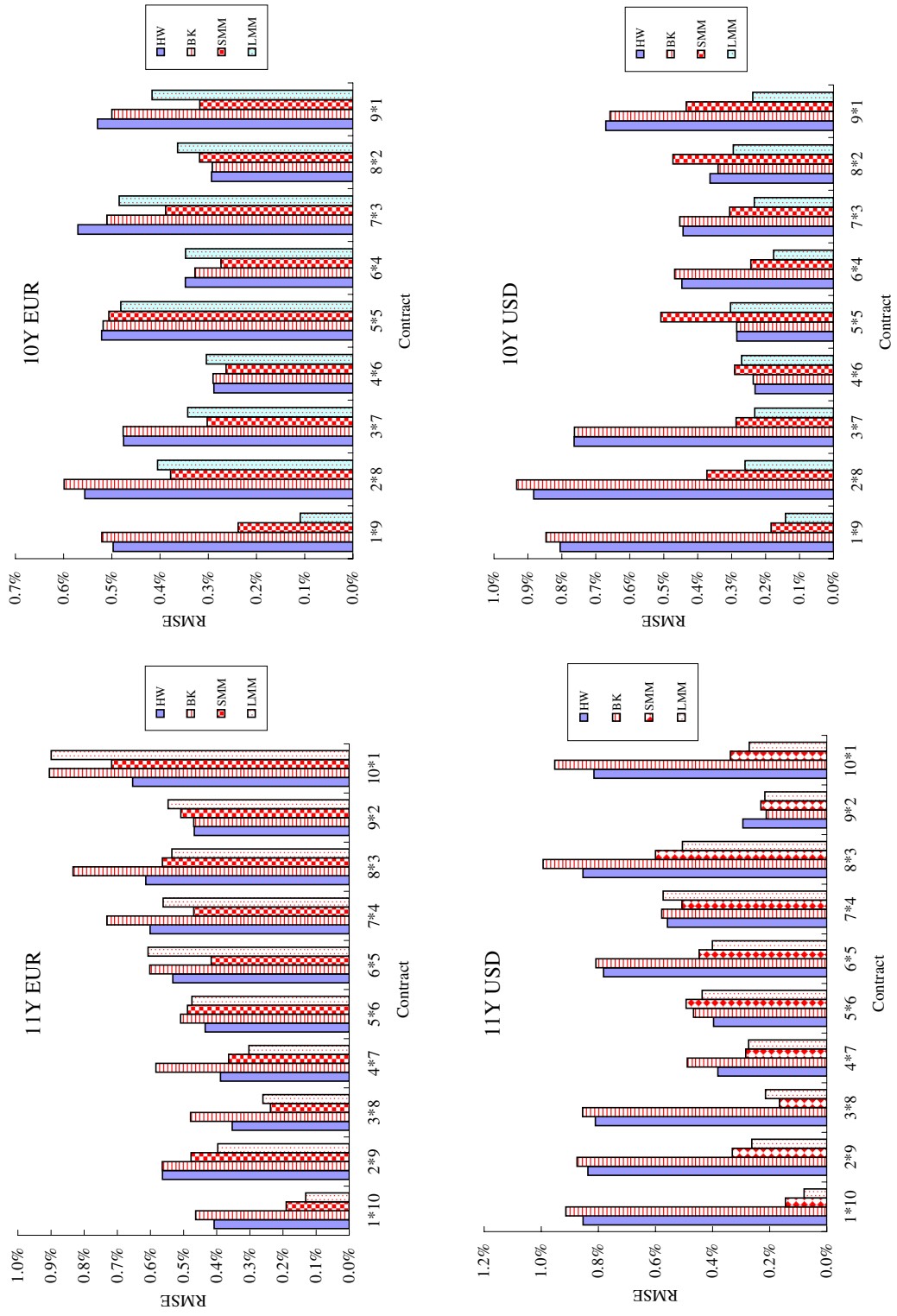
Note: HW stands for Hull-White model, BK stands for Black-Karasinski model, SMM stands for Swap Market Model and LMM stands for Libor Market Model. '11Y' denotes calibration results for 11Y co-terminal ATM European swaptions from February 2005 to September 2006; '10Y' denotes calibration results for 10Y co-terminal European swaptions from February 2006 to September 2007. 'EUR' denotes Euro market and 'USD' denotes US-dollar market.

Figure 2: Root mean square error of prices by date from calibration to 11Y co-terminal European swaptions over the period of February 2005 to September 2006



Note: HW stands for Hull-White model; BK stands for Black-Karasinski model; SMM stands for Swap Market Model and LMM stands for Libor market Model.

Figure 3: Root mean square error of prices by contract in EUR and USD markets from February 2005 to September 2006



Note: HW stands for Hull-White model; BK stands for Black-Karasinski model; SMM stands for Swap Market Model and LMM stands for Libor market Model. $x*y$ means x -year option on y -year swap. Calibration is made simultaneously to 11-year $(x+y)$ co-terminal European swaption.

Table 3: Model calibrated parameter values for EUR market from February 2005 to September 2006

HW (EUR)					SMM (EUR)				
Date	a	σ_0	σ_3	σ_{11}	Date	a	b	c	d
2005-2-28	0.0100	0.0062	0.0062	0.0072	2005-2-28	-0.0208	-0.0082	0.1031	0.1753
2005-3-31	0.0100	0.0065	0.0061	0.0067	2005-3-31	0.0188	0.0215	0.6595	0.1351
2005-4-29	0.0100	0.0061	0.0060	0.0064	2005-4-29	0.0197	0.0143	0.5514	0.1343
2005-5-31	0.0100	0.0064	0.0064	0.0059	2005-5-31	0.0441	-0.0016	0.0919	0.1288
2005-6-30	0.0100	0.0069	0.0063	0.0064	2005-6-30	0.0465	0.0158	0.4431	0.1480
2005-7-29	0.0100	0.0063	0.0064	0.0066	2005-7-29	0.0196	0.0136	0.4466	0.1466
2005-8-31	0.0100	0.0063	0.0066	0.0065	2005-8-31	0.0190	0.0164	0.4144	0.1533
2005-9-30	0.0100	0.0063	0.0066	0.0064	2005-9-30	0.0117	0.0196	0.4089	0.1542
2005-10-31	0.0100	0.0065	0.0067	0.0071	2005-10-31	0.0019	0.0164	0.5374	0.1580
2005-11-30	0.0100	0.0066	0.0068	0.0065	2005-11-30	0.0032	0.0197	0.4492	0.1552
2005-12-30	0.0100	0.0064	0.0068	0.0064	2005-12-30	0.0060	0.0153	0.3217	0.1585
2006-1-31	0.0100	0.0061	0.0067	0.0058	2006-1-31	0.0300	0.0114	0.1652	0.1221
2006-2-28	0.0100	0.0059	0.0065	0.0059	2006-2-28	0.0181	0.0096	0.1807	0.1296
2006-3-31	0.0100	0.0061	0.0063	0.0060	2006-3-31	0.0030	0.0075	0.3124	0.1360
2006-4-28	0.0100	0.0063	0.0061	0.0065	2006-4-28	0.0045	0.0089	0.5624	0.1332
2006-5-31	0.0100	0.0063	0.0061	0.0062	2006-5-31	0.0152	0.0049	0.3026	0.1259
2006-6-30	0.0100	0.0063	0.0062	0.0062	2006-6-30	0.0171	0.0030	0.1787	0.1218
2006-7-31	0.0100	0.0062	0.0062	0.0060	2006-7-31	0.0094	0.0075	0.3131	0.1274
2006-8-31	0.0100	0.0063	0.0063	0.0060	2006-8-31	0.0041	0.0118	0.4010	0.1371
2006-9-29	0.0100	0.0064	0.0063	0.0060	2006-9-30	0.0041	0.0118	0.4010	0.1371

BK (EUR)					LMM (EUR)				
Date	a	σ_0	σ_3	σ_{11}	Date	a	b	c	d
2005-2-28	0.0087	0.1642	0.1328	0.1521	2005-2-28	-0.05583	0.20987	1.11543	0.13725
2005-3-31	0.0087	0.1740	0.1312	0.1405	2005-3-31	-0.01693	0.16650	0.82451	0.13515
2005-4-29	0.0087	0.1716	0.1337	0.1351	2005-4-29	0.02031	0.09971	0.69715	0.13380
2005-5-31	0.0087	0.1860	0.1465	0.1229	2005-5-31	0.11632	0.00516	0.25772	0.12712
2005-6-30	0.0087	0.2123	0.1488	0.1412	2005-6-30	0.15546	0.08563	0.56811	0.14679
2005-7-29	0.0087	0.1867	0.1530	0.1401	2005-7-29	-0.00889	0.08907	0.59491	0.14592
2005-8-31	0.0087	0.1966	0.1654	0.1406	2005-8-31	-0.05369	0.10831	0.58707	0.15424
2005-9-30	0.0087	0.1933	0.1719	0.1348	2005-9-30	-0.09584	0.10510	0.51949	0.15207
2005-10-31	0.0087	0.1856	0.1657	0.1562	2005-10-31	-0.10225	0.10169	0.66646	0.15752
2005-11-30	0.0087	0.1864	0.1708	0.1397	2005-11-30	-0.11746	0.10204	0.55246	0.15365
2005-12-30	0.0087	0.1882	0.1850	0.1425	2005-12-30	-0.08355	0.06340	0.40016	0.15472
2006-1-31	0.0087	0.1677	0.1705	0.1257	2006-1-31	-0.00311	0.03619	0.21188	0.10617
2006-2-28	0.0087	0.1627	0.1659	0.1310	2006-2-28	-0.02168	0.03187	0.23717	0.12016
2006-3-31	0.0087	0.1539	0.1503	0.1300	2006-3-31	-0.04290	0.03204	0.40280	0.13422
2006-4-28	0.0087	0.1523	0.1366	0.1354	2006-4-28	-0.02980	0.05099	0.67779	0.13294
2006-5-31	0.0087	0.1523	0.1350	0.1213	2006-5-31	0.01428	0.02403	0.41302	0.12467
2006-6-30	0.0087	0.1484	0.1373	0.1231	2006-6-30	0.02557	0.00861	0.21430	0.11649
2006-7-31	0.0087	0.1505	0.1406	0.1205	2006-7-31	-0.01818	0.03045	0.39833	0.12580
2006-8-31	0.0087	0.1597	0.1500	0.1266	2006-8-31	-0.05753	0.05362	0.48683	0.13540
2006-9-29	0.0087	0.1638	0.1552	0.1283	2006-9-29	0.01479	0.02075	0.24451	0.12278

Note: HW stands for Hull-White model, BK stands for Black-Karasinski model, SMM stands for Swap Market Model and LMM stands for Libor Market Model. '11Y' denotes calibration results for 11Y co-terminal ATM European swaptions from February 2005 to September 2006; '10Y' denotes calibration results for 10Y co-terminal European swaptions from Feb 2006 to Sep 2007. 'EUR' denotes Euro market and 'USD' denotes US-dollar market.

Table 4: Model calibrated parameter values for USD market from February 2005 to September 2006.

HW (USD)					SMM (USD)				
Date	a	σ_0	σ_3	σ_{11}	Date	a	b	c	d
2005-2-28	0.0060	0.0092	0.0089	0.0062	2005-2-28	0.0166	0.0488	0.2342	0.1317
2005-3-31	0.0060	0.0102	0.0087	0.0062	2005-3-31	0.0149	0.0524	0.2897	0.1354
2005-4-29	0.0060	0.0098	0.0094	0.0064	2005-4-29	0.0211	0.0470	0.2687	0.1440
2005-5-31	0.0060	0.0094	0.0094	0.0064	2005-5-31	0.0211	0.0461	0.2394	0.1438
2005-6-30	0.0060	0.0097	0.0096	0.0065	2005-6-30	0.0284	0.0326	0.3098	0.1610
2005-7-29	0.0060	0.0096	0.0090	0.0066	2005-7-29	0.0228	0.0286	0.3125	0.1521
2005-8-31	0.0060	0.0097	0.0094	0.0070	2005-8-31	0.0219	0.0306	0.2954	0.1659
2005-9-30	0.0060	0.0097	0.0094	0.0069	2005-9-30	0.0223	0.0277	0.2931	0.1537
2005-10-31	0.0060	0.0099	0.0095	0.0067	2005-10-31	0.0174	0.0340	0.2581	0.1442
2005-11-30	0.0060	0.0101	0.0101	0.0076	2005-11-30	0.0158	0.0278	0.2398	0.1563
2005-12-30	0.0060	0.0097	0.0098	0.0083	2005-12-30	0.0169	0.0039	0.3350	0.1740
2006-1-31	0.0060	0.0092	0.0099	0.0079	2006-1-31	0.0151	0.0007	0.2796	0.1654
2006-2-28	0.0060	0.0081	0.0097	0.0079	2006-2-28	0.0152	-0.0372	0.4159	0.1716
2006-3-31	0.0060	0.0083	0.0097	0.0066	2006-3-31	0.0206	-0.0077	0.2870	0.1450
2006-4-28	0.0060	0.0081	0.0088	0.0069	2006-4-28	0.0166	-0.0082	0.3910	0.1374
2006-5-31	0.0060	0.0084	0.0095	0.0071	2006-5-31	0.0199	-0.0154	0.3859	0.1449
2006-6-30	0.0060	0.0080	0.0093	0.0077	2006-6-30	0.0137	-0.0230	0.3872	0.1463
2006-7-31	0.0060	0.0080	0.0094	0.0076	2006-7-31	0.0160	-0.0220	0.3703	0.1479
2006-8-31	0.0060	0.0077	0.0096	0.0077	2006-8-31	0.0171	-0.0322	0.3567	0.1564
2006-9-29	0.0060	0.0078	0.0096	0.0077	2006-9-30	0.0171	-0.0322	0.3567	0.1564

BK (USD)					LMM (USD)				
Date	a	σ_0	σ_3	σ_{11}	Date	a	b	c	d
2005-2-28	0.0060	0.1985	0.1737	0.1004	2005-2-28	0.0255	0.0606	0.3062	0.1223
2005-3-31	0.0060	0.2028	0.1576	0.1058	2005-3-31	0.0674	0.0612	0.3763	0.1303
2005-4-29	0.0060	0.2133	0.1854	0.1044	2005-4-29	0.0253	0.0697	0.3149	0.1300
2005-5-31	0.0060	0.2123	0.1957	0.1076	2005-5-31	-0.0080	0.0756	0.3045	0.1310
2005-6-30	0.0060	0.2239	0.2044	0.1138	2005-6-30	0.0082	0.0757	0.2864	0.1300
2005-7-29	0.0060	0.2045	0.1858	0.1144	2005-7-29	-0.0484	0.0885	0.3866	0.1454
2005-8-31	0.0060	0.2214	0.2037	0.1316	2005-8-31	-0.0370	0.0814	0.3598	0.1570
2005-9-30	0.0060	0.2053	0.1923	0.1170	2005-9-30	-0.0454	0.0802	0.3516	0.1441
2005-10-31	0.0060	0.1969	0.1819	0.1150	2005-10-31	-0.0001	0.0577	0.3029	0.1314
2005-11-30	0.0060	0.2034	0.1958	0.1371	2005-11-30	-0.0213	0.0536	0.3024	0.1478
2005-12-30	0.0060	0.2013	0.1995	0.1552	2005-12-30	-0.0858	0.0648	0.4062	0.1701
2006-1-31	0.0060	0.1872	0.1959	0.1502	2006-1-31	-0.0801	0.0512	0.3185	0.1563
2006-2-28	0.0060	0.1628	0.1940	0.1586	2006-2-28	-0.2150	0.0609	0.4785	0.1700
2006-3-31	0.0060	0.1613	0.1827	0.1182	2006-3-31	-0.1376	0.0707	0.3364	0.1361
2006-4-28	0.0060	0.1513	0.1548	0.1175	2006-4-28	-0.1236	0.0674	0.4668	0.1352
2006-5-31	0.0060	0.1559	0.1677	0.1202	2006-5-31	-0.1638	0.0786	0.4562	0.1420
2006-6-30	0.0060	0.1455	0.1645	0.1316	2006-6-30	-0.1527	0.0527	0.4492	0.1443
2006-7-31	0.0060	0.1501	0.1696	0.1297	2006-7-31	-0.1650	0.0617	0.4344	0.1453
2006-8-31	0.0060	0.1513	0.1824	0.1378	2006-8-31	-0.2088	0.0661	0.4190	0.1533
2006-9-29	0.0060	0.1574	0.1869	0.1411	2006-9-29	-0.2322	0.0762	0.4611	0.1591

Note: HW stands for Hull-White model; BK stands for Black-Karasinski model; SMM stands for Swap Market Model and LMM stands for Libor Market Model. a in HW and BK models is the mean reversion rate, and is assumed to be constant through out the sample period. a, b, c, d in SMM and LMM are the parameters of the volatility functional form as suggested by Rebonato 1999. The parameter values are obtained by calibrating 11-Y co-terminal European swaptions.

Figure 4: Model calibrated parameter values for EUR market from February 2005 to September 2006.



Note: HW stands for Hull-White model; BK stands for Black-Karasinski model. a in HW and BK model is the mean reversion rate; SMM stands for Swap Market Model and LMM stands for Libor Market Model. a, b, c, d in SMM and LMM are the parameters of the volatility functional form as suggested by Rebonato 1999. We calibrated to co-terminal 11Y European swaptions from February 2005 to September 2006. All calibration are performed with both EUR and USD market.

6 Pricing Bermudan Swaption

Let us briefly turn to previous work on the factor dependence of Bermudan swaption. To the best of our knowledge, Longstaff, Schwartz and Santa-Clara (LSS hereafter, 2001a [43]) are the first to investigate this subject. They study the costs of applying single-factor exercise strategies to American swaptions when the term structure is actually driven by multiple factors. However, LSS does not directly consider the effects of applying within a high-dimensional model a simple exercise strategy parameterized exclusively in intrinsic swap value. Comparing this strategy with a more complicated exercise rule that is allowed to depend on more information about the state of the forward curve, they find that the latter leads to an increase in model-generated option prices, typically in the order of 1 – 3%. From this result, LSS conclude that Bermudan swaptions holders following one-factor exercise strategies suffer significant losses. In contrast, Andersen and Andreasen (2001) [1] argue that the LSS paper does not provide a conclusive analysis of the consequences of market practices. The approach taken in LSS does not consider the re-calibration procedures employed in practice, and also offer no evidence that their simple exercise strategy within a multi-factor model will be representative of any or all of the possible exercise strategies that can be implied from a best-fit one factor model. Andersen and Andreasen (2001) [1] use mean-reverting Gaussian model and lognormal Libor Market Model for pricing Bermudan swaption. They find that for both models, Bermudan swaption prices change only moderately when the number of factors in the underlying interest rate model is increased from one to two.

We use recombining trinomial tree for pricing Bermudan swaptions with short rate models. Early exercise decision is straightforward in trinomial tree. For Swap Market Model and Libor Market Model, we perform Monte Carlo simulation (with 10,000 simulation paths) using Longstaff-Schwartz least square method to approximate the early exercise decision. As there is no market quote for Bermudan swaptions (which are mostly traded on OTC), we compare the price calculated from HW, BK, SMM and LMM. As all the four models have been widely used by practitioners, it is reasonable

to expect that interest rate derivative values produced should not be too drastically different (otherwise, they cannot co-exist). So it is possible that, when calibrated to the same instruments, all four models might give similar prices for the same Bermudan swaption. We price both 11-year and 10-year Bermudan swaptions. The 11-year Bermudan swaptions are priced at-the-money while 10-year Bermudan swaptions are priced using the strike rates set when they were 11-year ATM in the previous year. The pricing results are summarized in Table 5 (for Euro) and Table 6 (for USD).

As we can see from Tables 5 and 6 and Figure 5, the prices given by four models are quite close on most dates. The 11-year Bermudans are all ATM. The 10-year Bermudan swaptions, however, would be in-the-money or out-the-money depending on how the interest rates had moved. It is very encouraging to see that, again, prices produced by all four models are similar and co-move together.

7 Hedging Bermudan Swaption

Changes in the term structure can significantly affect the value of any interest rate sensitive instruments. Therefore, protecting fixed income securities from unfavorable term structure movements, or hedging, is one of the most demanding tasks for any financial institution and for the ALM group in particular. Inefficient hedging strategies can cost these institutions dearly. In order to protect a position from adverse future interest rate changes, first one needs to generate realistic scenarios and assess the impact of these risk scenarios. Thus, two important issues to be addressed by any interest rate model risk are: (i) how to perturb the term structure to imitate possible term structure movements; and (ii) how to immunize the portfolio against these movements. In the following subsections, we explain the methodologies, applied in this study, for

- (i) estimating the perturbations by which the input term structure should be ‘bumped’ to simulate the possible future changes;
- (ii) selecting hedge instruments;

Table 5: 11Y and 10Y Bermudan swaption prices in EUR market from February 2005 to September 2007.

DATE(11Y)	HW	BK	SMM	LMM
2005-2-28	0.0465	0.0441	0.0463	0.0460
2005-3-31	0.0467	0.0441	0.0479	0.0480
2005-4-29	0.0476	0.0447	0.0481	0.0485
2005-5-31	0.0491	0.0460	0.0499	0.0501
2005-6-30	0.0508	0.0474	0.0534	0.0549
2005-7-29	0.0485	0.0455	0.0484	0.0478
2005-8-31	0.0486	0.0455	0.0483	0.0475
2005-9-30	0.0464	0.0438	0.0458	0.0464
2005-10-31	0.0459	0.0437	0.0450	0.0453
2005-11-30	0.0452	0.0432	0.0444	0.0448
2005-12-30	0.0429	0.0414	0.0415	0.0415
2006-1-31	0.0411	0.0398	0.0395	0.0393
2006-2-28	0.0396	0.0384	0.0379	0.0376
2006-3-31	0.0386	0.0376	0.0375	0.0374
2006-4-28	0.0387	0.0377	0.0386	0.0390
2006-5-31	0.0396	0.0384	0.0397	0.0404
2006-6-30	0.0386	0.0376	0.0384	0.0387
2006-7-31	0.0387	0.0376	0.0382	0.0385
2006-8-31	0.0389	0.0379	0.0384	0.0386
2006-9-29	0.0381	0.0373	0.0379	0.0382

DATE(10Y)	HW	BK	SMM	LMM	Strike
2006-2-28	0.0271	0.0272	0.0273	0.0271	0.0402
2006-3-31	0.0379	0.0367	0.0365	0.0364	0.0395
2006-4-28	0.0534	0.0510	0.0519	0.0521	0.0377
2006-5-31	0.0610	0.0582	0.0591	0.0592	0.0363
2006-6-30	0.0744	0.0720	0.0722	0.0723	0.0344
2006-7-31	0.0610	0.0584	0.0585	0.0586	0.0356
2006-8-31	0.0600	0.0573	0.0575	0.0576	0.0339
2006-9-29	0.0545	0.0522	0.0519	0.0522	0.0344
2006-10-31	0.0417	0.0404	0.0399	0.0397	0.0369
2006-11-30	0.0390	0.0378	0.0371	0.0369	0.0371
2006-12-29	0.0563	0.0543	0.0540	0.0537	0.0357
2007-1-31	0.0526	0.0508	0.0504	0.0502	0.0377
2007-2-28	0.0443	0.0427	0.0423	0.0422	0.0378
2007-3-30	0.0379	0.0332	0.0328	0.0329	0.0409
2007-4-30	0.0308	0.0301	0.0293	0.0290	0.0431
2007-5-31	0.0391	0.0380	0.0370	0.0369	0.0433
2007-6-29	0.0476	0.0461	0.0464	0.0466	0.0442
2007-7-31	0.0487	0.0471	0.0473	0.0474	0.0428
2007-8-31	0.0510	0.0492	0.0502	0.0504	0.0408
2007-9-28	0.0578	0.0557	0.0573	0.0576	0.0401

Note: HW stands for Hull-White model, BK stands for Black-Karasinski model, SMM stands for Swap Market Model and LMM stands for Libor Market Model. 11-y Bermudan swaption are priced as at-the-money from February 2005 to September 2006; 10 year Bermudan swaptions are priced with the corresponding 11-y ATM strikes one year ago. For example, on February 2006, the strike rate for 10-y Bermudan swaption is 0.0402 which is the ATM strike rate for 11-y Bermudan swaption on February 2005. 10-y Bermudan swaption are priced from February 2006 to September 2007.

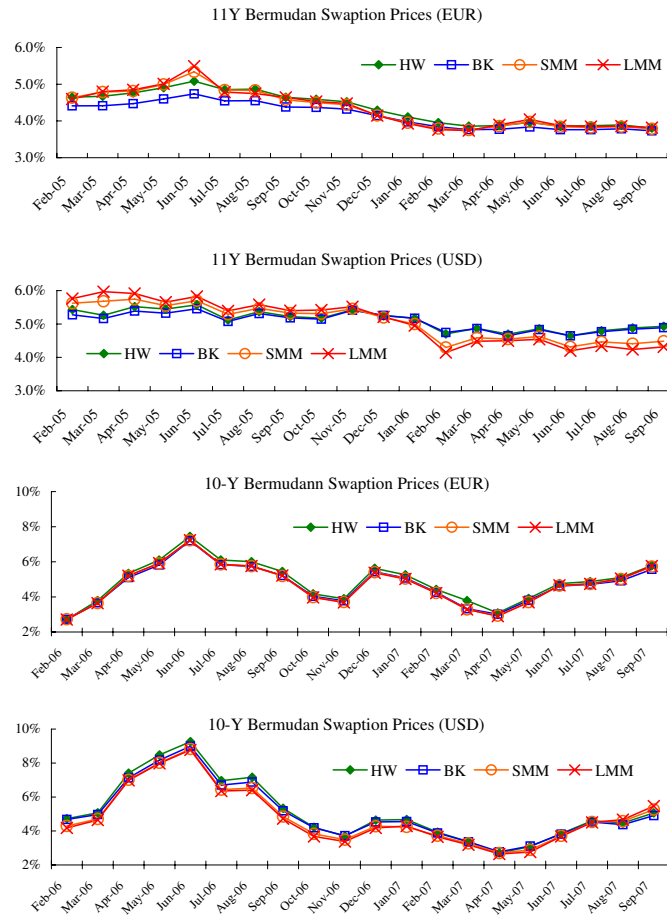
Table 6: 11Y and 10Y Bermudan swaption prices in USD market from February 2005 to September 2007.

DATE(11Y)	HW	BK	SMM	LMM
2005-2-28	0.0543	0.0528	0.0562	0.0576
2005-3-31	0.0526	0.0516	0.0568	0.0597
2005-4-29	0.0552	0.0539	0.0574	0.0592
2005-5-31	0.0545	0.0532	0.0555	0.0566
2005-6-30	0.0557	0.0546	0.0570	0.0582
2005-7-29	0.0512	0.0508	0.0529	0.0539
2005-8-31	0.0537	0.0531	0.0547	0.0558
2005-9-30	0.0522	0.0518	0.0532	0.0540
2005-10-31	0.0517	0.0515	0.0531	0.0542
2005-11-30	0.0543	0.0541	0.0545	0.0552
2005-12-30	0.0525	0.0526	0.0520	0.0521
2006-1-31	0.0517	0.0517	0.0500	0.0495
2006-2-28	0.0470	0.0475	0.0430	0.0415
2006-3-31	0.0487	0.0486	0.0459	0.0448
2006-4-28	0.0470	0.0465	0.0455	0.0450
2006-5-31	0.0485	0.0483	0.0463	0.0454
2006-6-30	0.0465	0.0464	0.0432	0.0420
2006-7-31	0.0480	0.0477	0.0446	0.0434
2006-8-31	0.0488	0.0484	0.0441	0.0424
2006-9-29	0.0493	0.0489	0.0448	0.0431

DATE(10Y)	HW	BK	SMM	LMM	Strike
2006-2-28	0.0472	0.0468	0.0430	0.0418	0.0496
2006-3-31	0.0507	0.0497	0.0471	0.0464	0.0522
2006-4-28	0.0741	0.0713	0.0701	0.0698	0.0482
2006-5-31	0.0848	0.0817	0.0802	0.0798	0.0464
2006-6-30	0.0926	0.0899	0.0883	0.0877	0.0454
2006-7-31	0.0696	0.0670	0.0643	0.0635	0.0486
2006-8-31	0.0716	0.0688	0.0653	0.0640	0.0456
2006-9-29	0.0534	0.0519	0.0485	0.0473	0.0492
2006-10-31	0.0422	0.0419	0.0383	0.0368	0.0520
2006-11-30	0.0369	0.0372	0.0349	0.0338	0.0517
2006-12-29	0.0464	0.0453	0.0428	0.0418	0.0503
2007-1-31	0.0467	0.0455	0.0424	0.0429	0.0512
2007-2-28	0.0393	0.0388	0.0372	0.0366	0.0514
2007-3-30	0.0337	0.0336	0.0327	0.0319	0.0548
2007-4-30	0.0271	0.0277	0.0270	0.0264	0.0573
2007-5-31	0.0308	0.0311	0.0289	0.0276	0.0577
2007-6-29	0.0385	0.0381	0.0371	0.0368	0.0585
2007-7-31	0.0460	0.0453	0.0453	0.0451	0.0566
2007-8-31	0.0447	0.0437	0.0458	0.0467	0.0538
2007-9-28	0.0506	0.0491	0.0526	0.0547	0.0525

Note: HW stands for Hull-White model, BK stands for Black-Karasinski model, SMM stands for Swap Market Model and LMM stands for Libor Market Model. 11-y Bermudan swaption are priced as at-the-money from February 2005 to September 2006; 10 year Bermudan swaptions are priced with the corresponding 11-y ATM strikes one year ago. For example, on February 2006, the strike rate for 10-y Bermudan swaption is 0.0402 which is the ATM strike rate for 11-y Bermudan swaption on February 2005. 10-y Bermudan swaption are priced from February 2006 to September 2007.

Figure 5: 11Y and 10Y Bermudan swaption prices in EUR and USD markets from February 2005 to September 2007.



Note: HW stands for Hull-White model, BK stands for Black-Karasinski model, SMM stands for Swap Market Model and LMM stands for Libor Market Model. The data for this figure is from Table 5.6 and 5.7.

- (iii) delta-hedging the underlying Bermudan swaption using the selected instruments;
- (iv) calculating the profits and losses (P&L) of the hedged portfolio.

In each subsection below, we use examples from the literature to justify our model choice where appropriate.

7.1 Perturbing the Term Structure

Over the years, researchers and practitioners have been using duration analysis for interest rate risk management; i.e. they shift the entire yield curve upward and downward in a parallel manner and then estimate how the value of their interest rate portfolio is affected as a result of these parallel shifts. Parallel shifts are unambiguously the most important kind of yield curve shift but alone cannot completely explain the variations of yield curve movements observed in the markets. Three most commonly observed term structure shifts are: *Parallel Shift*, where the entire curve goes up or down by same amount; *Tilt*, also known as slope shift, in which short yields fall and long yields rise (or vice versa); and *Curvature shift*, in which short and long yields rise while mid-range yields fall (or vice versa). These three types of term structure movements together can explain almost all the variations present in any term-structure and thus one should not completely rely on duration and convexity measures for estimating the risk sensitivity of a fixed income position. There are numerous examples in the literature to support this argument. Here, we mention a few studies that lead to this conclusion. Litterman and Scheikman (1991) [41] perform principal components analysis (PCA) and found that, on average, the first three factors can explain 98.4% of the variations in Treasury bond returns. They suggested that “by considering the effects of each of these three factors on a portfolio, one can achieve a better hedged position than by holding only a zero-duration portfolio”.¹¹ Knez, Litterman, and Scheikman (1994) [40] investigate the common factors in money markets and find that, on average, the first three

¹¹Litterman and Scheikman (1991, page 54) [41].

factors explain 86% of the total variations in most money market returns whereas four factors can explain, on average, 90% of the variations. Chen and Fu (2002) [12] perform a PCA on yield curve and find that the first four factors capture over 99.99% of the yield curve variations. They too claim that hedging against these factors would lead to a more stable portfolio and thus superior hedging performance.

Based on the findings of these studies, we perform PCA on historical data to estimate historical term structure shifts. There are not many examples in interest rate derivative literature of estimating price sensitivities with regard to multiple factors (like 3 principal components here) in the context of a one-factor term structure model. Generally, the risk sensitivities are often calculated by perturbing only the model intrinsic factors, i.e. for the one-factor model, only one-factor is perturbed and so on.

In this study, PCA has been performed on annual changes of forward Libor rates. Annual changes have been used because each hedge is maintained for one year. The reason for using forward rates rather than the spot yield curve for doing PCA is due to findings in Yallup (2006) [53]. Using Gilt coupon bond and Gilt strips, he uses various spline method to model the term structure as a polynomial. He finds that the smoothed forward rate splines have the required combination of both low out-of-sample errors and low forward rate curvature. The smoothed discount rate spline does not have the required out-of-sample fit and forward rate curvature properties. Here, scores of the first three principal components of the changes of forward curves are used to estimate the shifts by which we bumped the forward rate curves.

Tables 7, 8 and Figure 6 present our PCA results, which appear to be in line with the findings in the literature, e.g., Knez, Litterman, and Scheikman (1994) [40].

Next, we briefly describe how PCA was performed on the forward rates in order to estimate the term structure shifts/bumps in the hedging study:

- (i) Monthly observations of 11 forward Libor rates ($L_{0,0,1}, L_{0,1,2}, L_{0,2,3}, \dots, L_{0,10,11}$) for the period January 1999 to December 2006 have been used for PCA.

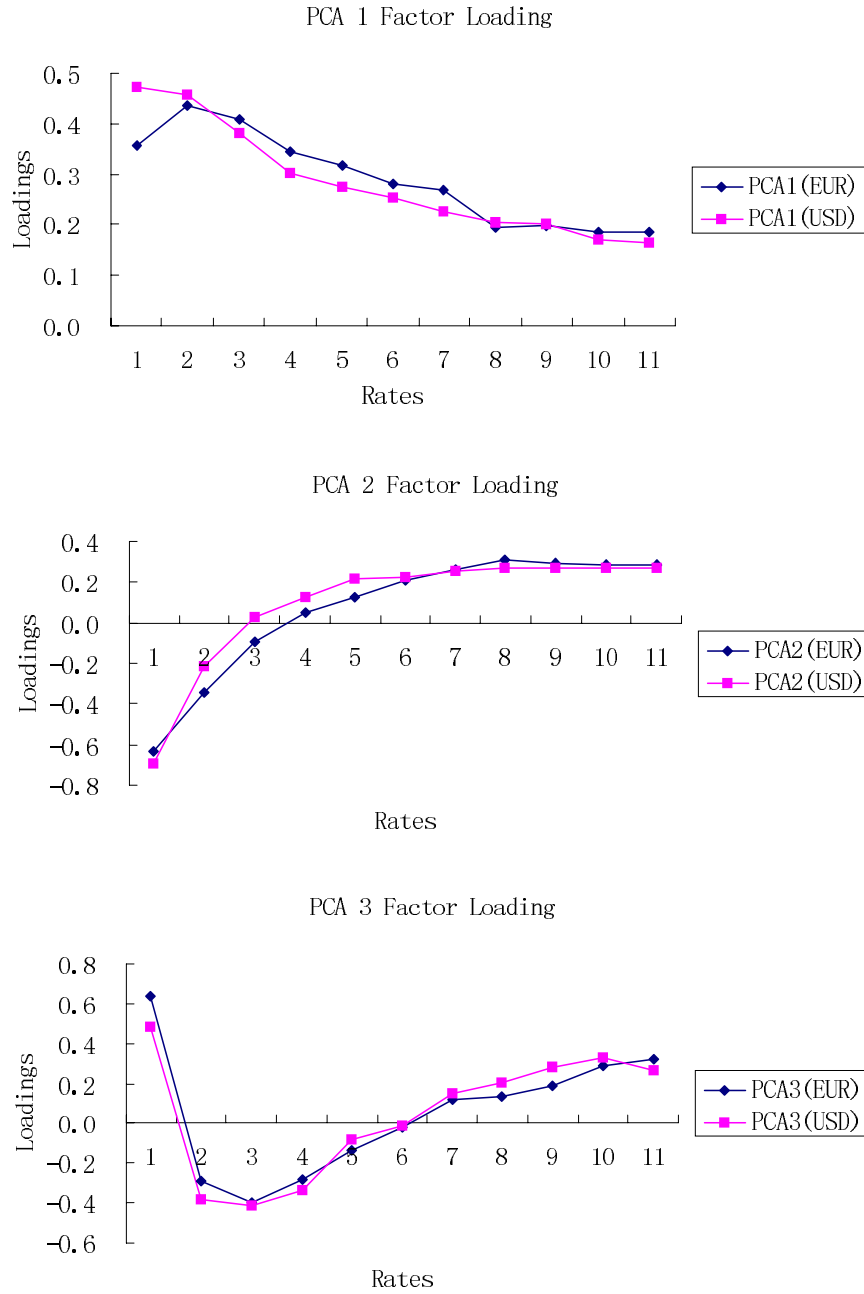
Table 7: PCA factor loadings of Libor rates from January 2000 to September 2007.

Rates	PCA1(EUR)	EUR		PCA1(USD)	USD	
		PCA2(EUR)	PCA3(EUR)		PCA2(USD)	PCA3(USD)
1	0.36	-0.63	0.64	0.47	-0.69	0.49
2	0.43	-0.34	-0.29	0.46	-0.22	-0.39
3	0.41	-0.09	-0.40	0.38	0.03	-0.41
4	0.35	0.05	-0.28	0.30	0.13	-0.34
5	0.32	0.13	-0.14	0.28	0.22	-0.09
6	0.28	0.21	-0.02	0.25	0.23	-0.01
7	0.27	0.26	0.12	0.23	0.25	0.15
8	0.20	0.31	0.14	0.20	0.27	0.21
9	0.20	0.29	0.19	0.20	0.27	0.28
10	0.19	0.29	0.29	0.17	0.27	0.33
11	0.18	0.28	0.32	0.17	0.27	0.26

Table 8: Explainary power of the first three principle components.

EUR			USD		
	Percentage variance	Cumulative variance		Percentage variance	Cumulative variance
PCA 1	82.37%	82.37%	PCA 1	74.22%	74.22%
PCA 2	13.87%	96.24%	PCA 2	20.59%	94.81%
PCA 3	2.81%	99.05%	PCA 3	2.96%	97.77%

Figure 6: Factor loadings for the first three principle components of forward rates term structure in the EUR and USD markets



- (ii) We calculate annual changes for each of the 11 forward rates as follows. Suppose we have at time t monthly observations of the 11 forward rates, $L_{t,t+\tau_j,t+\tau_j+1}$, where $\tau_j = 0, 1, 2, \dots, 10$. The annual change of each of the forward Libor rates can be calculated as

$$\Delta L_{t,\cdot,\cdot} = L_{t,\cdot,\cdot} - L_{t-12,\cdot,\cdot}$$

- (iii) Principal component analysis was performed on the eleven time series of $\Delta L_{t,\cdot,\cdot}$ to produce principal factor loadings and the factor scores. The other output from the PCA are the proportion of variations explained by each principal component.
- (iv) From the PCA results (to be discussed later), we chose to concentrate on the first three factors. Each of these factors were perturbed in order to calculate the delta of the Bermudan swaptions (and the hedging instruments) w.r.t the factor. For each forward rate, $L_{t,t+\tau_j,t+\tau_j+1}$ and each factor k ,

$$\begin{aligned} \pm \Delta L_{t,\tau_j,\tau_j+1} &= \pm \beta_{\tau_j,k} \Delta P_k \\ L_{t,\tau_j,\tau_j+1}^\pm(k) &= L_{t,\tau_j,\tau_j+1} \pm \beta_{\tau_j,k} \Delta P_k \end{aligned}$$

for $\tau_j = 0, 1, 2, \dots, 10$ and P_k is the vector of scores for the k th factor, and $k = 1, 2, 3$.

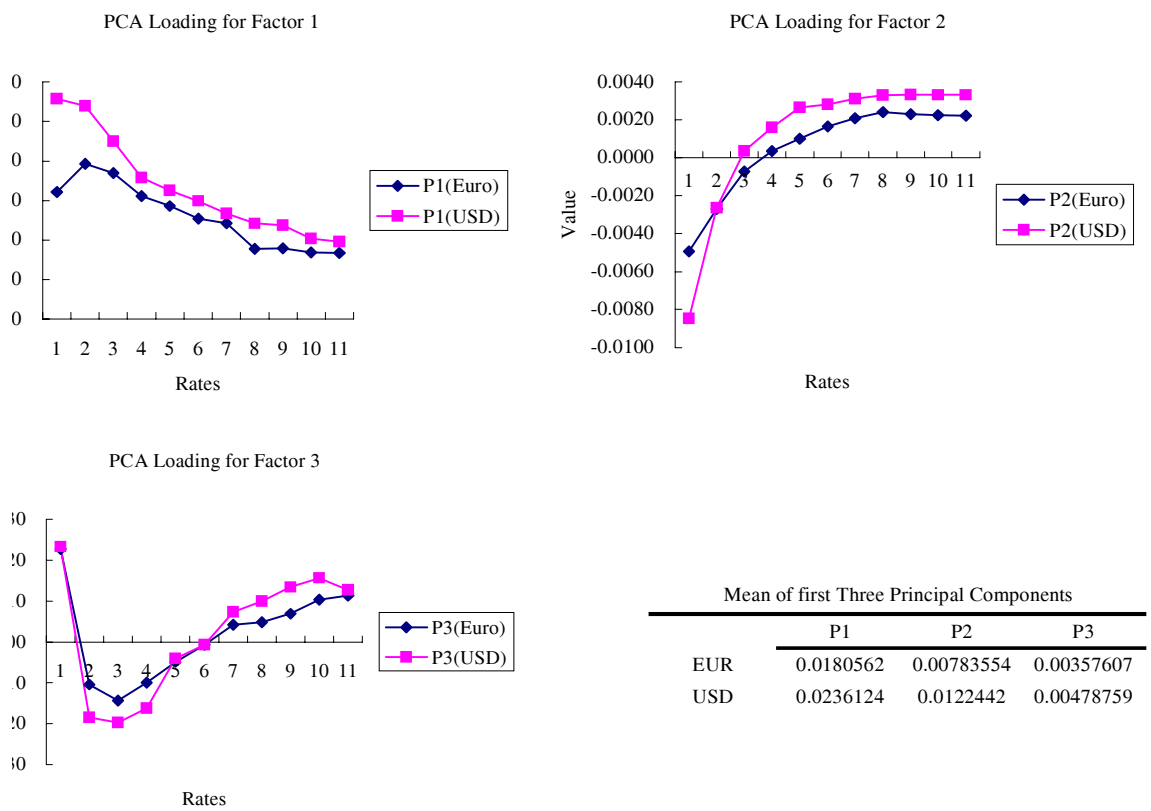
There are many ways of setting ΔP_k ; here, we have decided to use the mean of absolute change calculated from the entire sample period. So ΔP_k is the same for all t . Figure 7 presents the mean of absolute change for the first three principle factors.

- (v) After performing the regressions in step (iv), the 11 forward rates are bumped by shocks corresponding to the first three factors as:

$$L_{t,\tau_j,\tau_j+1}^\pm = L_{0,\tau_j,\tau_j+1} \pm \beta_{\tau_j,k} \Delta P_k, \quad \tau_j = 0, \dots, 10, \quad k = 1, 2, 3.$$

This gives us 6 new sets of term structure, along with the original input term structure for each of the days on which we will hedge the Bermudan swaption.

Figure 7: PCA factor loadings (mean of the absolute change) for the first three principle components in EUR and USD markets.



7.2 Choosing the Hedge Instruments

In the literature there are two main approaches to hedging: factor hedging and bucket hedging. For *factor hedging* in a K -factor model, typically K different instruments (together with the money market account) are used as hedging instruments. The choice of hedge instruments is independent of the derivative to be hedged (i.e., the same K hedging instruments can be used for hedging any derivative in a K -factor model), and depend only on the number of factors in the model. For *bucket hedging* the choice of hedge instruments depend on the instrument to be hedged and not on the number of factors in the model. In this hedging strategy, the number of hedge instruments is equal to the number of total payoffs provided by the instrument. The hedge instruments are chosen so their maturities correspond to different payment or decision dates of the underlying derivative.

When using any other criteria for selecting the hedge instruments, the number of hedge instruments will lie between the numbers of hedge instruments associated with these two hedging strategies. Before discussing the instruments that are used in this study to delta-hedge the Bermudan swaption, we first briefly review a few previous studies in the literature. Driessen, Klaassen and Melenberg (2003) [16] used (delta-) hedging of caps and swaptions as criteria for comparing the hedge performance of HJM class models and the Libor Market Model. They used zero coupon bonds as hedge instruments. For each model, they considered factor and bucket hedging strategies. DKM show that when bucket strategies are used for hedging, the performance of the one-factor models improves significantly. Fan, Gupta and Ritchken (2007) [18] compare the swaption delta neutral hedging performance of single factor and multi-factor factor term structure models using discount bonds as hedge instruments to delta-hedge swaptions. They conclude that multi-factor models outperform single factor models. Next in light of the DKM results, they repeated their experiments using additional hedging instruments. They find that for both the one-factor and the two-factor model adding more instruments did not result in better hedging results. The general implications of these examples are:

- (i) Effectiveness of delta neutral hedges is often used to evaluate the hedging performance of term structure models.
- (ii) Using multiple instruments can improve the hedge performance of one factor models. Practitioners also prefer this practice (called bucket hedging).

Therefore, in this study, we decide to use three swaps of different maturities 1-, 5- and 11-year, as hedge instruments. Maturity of 11-year swap coincides with the maturity of the co-terminal Bermudan swaption to be hedged and the length of the other two swaps is used to hedge the short and medium term interest rate risk. We could have used discount bonds to hedge this Bermudan swaption but the use of swap is cheaper and is more in line with the general practice in banks.¹²

7.3 Constructing Delta Hedged Portfolio

Delta hedging is the process of keeping the delta of the hedged portfolio as close as possible to zero. Since delta measures the exposure of a derivative to the changes in the value of the underlying, the overall value of a portfolio remains unchanged for small changes in the underlying factor. A delta hedged portfolio is established by buying or selling an amount of the hedge instrument(s) that corresponds to the delta of the exposure.

For interest rate derivatives, if the entire initial term structure is perturbed by same amount, say ε (parallel shift), then the risk sensitivity of a fixed income security w.r.t. this perturbation can be estimated as

$$\frac{V(\varepsilon) - V}{\varepsilon}$$

where V is the value of the derivative calculated using initial term structure and $V(\varepsilon)$ is the value of the derivative after the initial term structure is perturbed by ε . If we first increase the entire initial term structure by ε and then next decrease it by ε , then the risk sensitivity can be estimated as

$$\frac{V(\varepsilon^+) - V(\varepsilon^-)}{2\varepsilon} \tag{17}$$

¹²Pietersz and Pelsser (2005) [47] used discount bonds to hedge Bermudan swaption.

where $V(\varepsilon^+)$ is the value of the derivative calculated after initial term structure has been shifted up by ε and $V(\varepsilon^-)$ is the value of the derivative after the initial term structure has been shifted down by ε . In our case we have bumped the initial forward rate curve by three factors. Also for each factor, we have bumped the forward rate curve both up and down. Using the idea presented in equation (17), we estimate the sensitivity (delta) of the Bermudan swaption and the three swaps with respect to the three PCA factors as follows

$$\begin{aligned}\Delta_k^{Berm} &= \frac{\partial BSwn}{\partial P_k} = \frac{BSwn_k^+ - BSwn_k^-}{2\Delta P_k} \\ \Delta_k^{S_1} &= \frac{\partial S_1}{\partial P_k} = \frac{(S_1)_k^+ - (S_1)_k^-}{P_k^+ - P_k^-} = \frac{(S_1)_k^+ - (S_1)_k^-}{2\Delta P_k} \\ \Delta_k^{S_5} &= \frac{\partial S_5}{\partial P_k} = \frac{(S_5)_k^+ - (S_5)_k^-}{P_k^+ - P_k^-} = \frac{(S_5)_k^+ - (S_5)_k^-}{2\Delta P_k} \\ \Delta_k^{S_{11}} &= \frac{\partial S_{11}}{\partial P_k} = \frac{(S_{11})_k^+ - (S_{11})_k^-}{P_k^+ - P_k^-} = \frac{(S_{11})_k^+ - (S_{11})_k^-}{2\Delta P_k}\end{aligned}$$

for $k = 1, 2, 3$, and $S()$ is the swap price and $BSwn$ is the Bermudan swaption price, $(.)_k^+$ and $(.)_k^-$ are, respectively, the prices of derivative after the initial forward curve has been bumped up and down by the k th factor. Now if we consider a portfolio, consisting of one long position in 10×1 Bermudan swaption, and some short position in x_{11} units of 11-year swap, x_5 units of 5-year swap and x_1 units of 1-year swap, then total delta mismatch of this portfolio with respect to k^{th} factor, ε_k is

$$\Delta\varepsilon_k = \Delta_k^{BSwn} - x_{11}\Delta_k^{S_{11}} - x_5\Delta_k^{S_5} - x_1\Delta_k^{S_1} \quad (18)$$

assuming that x_{11} , x_5 and x_1 need not to be whole numbers. From equation (18), we can see that when we use more than one hedging instrument in a one-factor model, the hedge ratios would not be unique, and some rule must be applied for constructing the hedge portfolio using the chosen hedge instruments. Here, to obtain the hedge ratios, x_{11} , x_5 and x_1 , we use the basic idea behind the delta hedging. The three hedge ratios x_{11} , x_5 and x_1 are obtained by minimizing the total delta-mismatch of the portfolio with

respect to the first three PCA factors, i.e.

$$\min_{x_{11}, x_5, x_1} \sum_{k=1}^3 (\Delta \varepsilon_k)^2$$

where $\Delta \varepsilon_k$ is given by equation (18).

7.4 Calculating P&L

Once the hedging portfolio has been established on date t , the hedging error can be evaluated one year later at $t + 1$ as follows:

$$\begin{aligned} P\&L_{t+1} = & (BSwn_t - x_{11,t}S_{11,t} - x_{5,t}S_{5,t} - x_{1,t}S_{1,t}) \times (1 + y_{0,t}) \\ & - (BSwn_{t+1} - x_{11,t}S_{10,t+1} - x_{5,t}S_{4,t+1} - x_{1,t}S_{0,t+1}) \end{aligned} \quad (19)$$

where $BSwn_t$ is the value of the Bermudan swaption on day t ; $x_{\tau,t}$ are units of τ -year swap in the hedge portfolio; $S_{\tau,t}$ is the value of τ -year swap on day t ; and $y_{0,t}$ is the current 1-year yield at t .

Since, at the point of initiation, the value of any swap is zero, this means that $S_{11,t} = S_{5,t} = S_{1,t} = 0$. Therefore, equation (19) can be written as

$$P\&L_{t+1} = BSwn_t \times (1 + y_{0,t}) - BSwn_{t+1} + x_{11,t}S_{10,t+1} + x_{5,t}S_{4,t+1} + x_{1,t}S_{0,t+1}. \quad (20)$$

To calculate the price of $BSwn_{t+1}$, which is now a 9×1 Bermudan swaption, we recalibrate at $t + 1$, the interest rate models and use the new calibrated parameter values to calculate the model price of this possibly away-from-money 9×1 Bermudan swaption $BSwn_{t+1}$. Strike rate for this 9×1 Bermudan swaption is kept the same as it was for the 10×1 Bermudan swaption on day t , as the objective is to find the current value of that old swaption. We also use the $t + 1$ term structure to calculate the values of the three swaps at $t + 1$.

The profit and loss of the hedged portfolio is reported in Table 9 and Figure 8. Figure 8 also report the hedging P&L as a percentage of the average Bermudan swaption prices from the four models. The hedging performance of all the short rate models (HW and BK) is better than forward rate models (LMM and SMM) in both Euro and USD market. We also notice that all

the four models perform better in Euro market than in USD market. The reason could be that the interest rate term structure is more volatile in the USD market than in the Euro market during our sample period.

8 Conclusion

The goal of this study is to provide an empirical comparison of four one-factor interest rate term structure models for the purpose of asset liability management in a global bank. In contrast to previous research, we have compared four very different models and we performed the pricing and hedging tests on Bermudan swaptions. We also use a more refined method in delta hedging the Bermudan swaption, used here to mimic the mortgage loan portfolio.

For model calibration, we fix volatility term structure in HW and BK models at three time period and linearly interpolate between fixed points; for the LMM and SMM, we use market standard parametric form for volatility term structure.

Bermudan swaptions are priced via recombining trinomial tree with HW/BK model while for SMM and LMM, Monte Carlo simulation (with 10,000 paths) with predictor-corrector drift approximation is used. Though the modeling approaches are vastly different, the percentage pricing error from calibration results of all four models are quite small and most of the calibrated parameters are relatively stable. Most of the time, the Bermudan prices of all four models are within the bid-ask spread. Thus, we conclude that, for ALM purpose, there is no significant difference between the four popular models though implementation wise, HW model involves less effort and the calibrated parameter values are relatively more stable.

As far as hedging performance is concerned, in the Euro market, the P&L of the hedged portfolio from the four models are very similar to each other. In the USD market, HW and BK models are marginally better than the more complicated SMM and LMM models. This would lead us to conclude that, again from an ALM perspective, we actually could use simple short

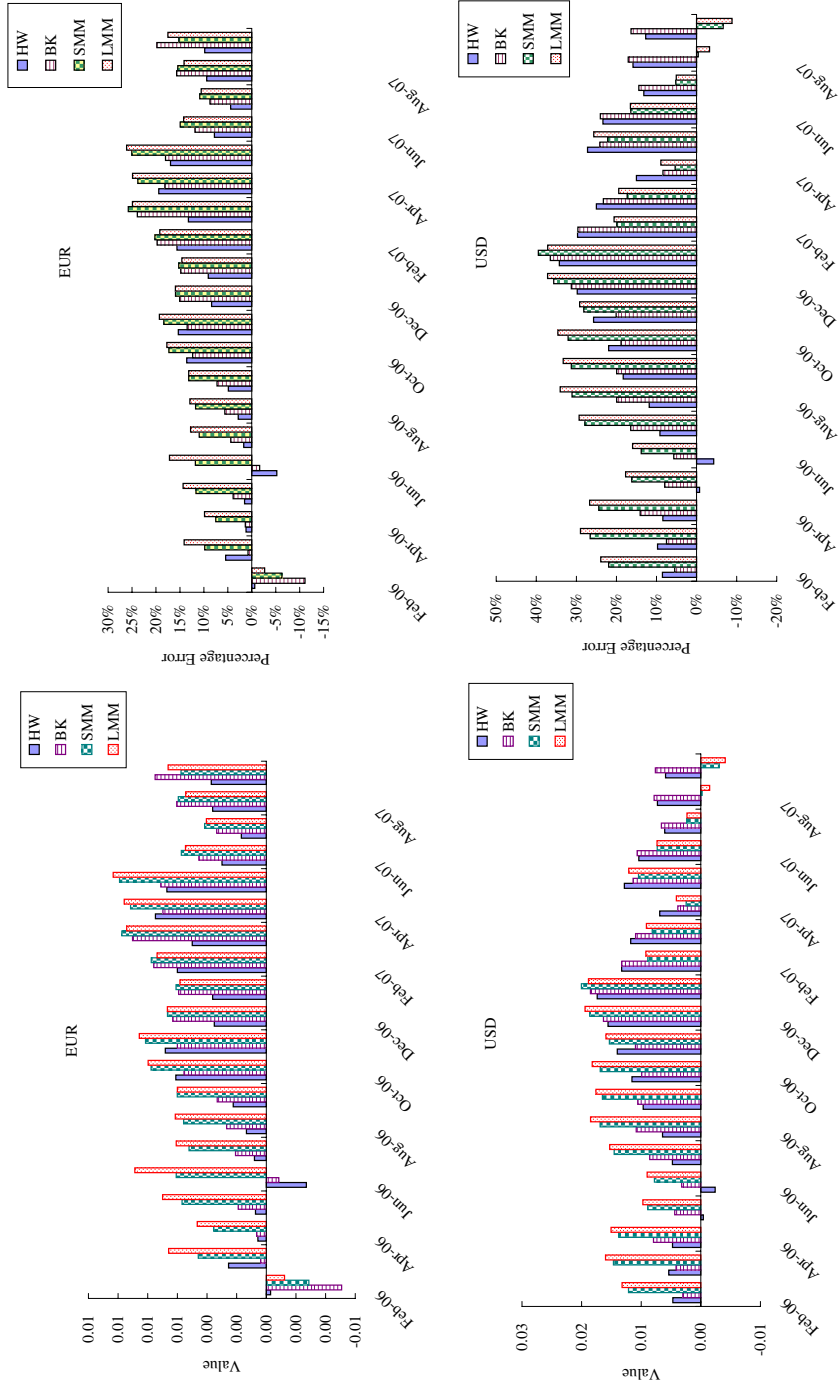
Table 9: Hedging profit and loss in EUR and USD Markets.

EUR	HW	BK	SMM	LMM
2006-02-28	-0.0003	-0.0051	-0.0029	-0.0012
2006-03-31	0.0026	0.0004	0.0046	0.0066
2006-04-28	0.0006	0.0006	0.0036	0.0047
2006-05-31	0.0007	0.0019	0.0057	0.0070
2006-06-30	-0.0027	-0.0008	0.0061	0.0089
2006-07-31	0.0008	0.0021	0.0052	0.0061
2006-08-31	0.0013	0.0027	0.0056	0.0061
2006-09-29	0.0022	0.0033	0.0060	0.0060
2006-10-31	0.0061	0.0056	0.0078	0.0080
2006-11-30	0.0068	0.0060	0.0081	0.0086
2006-12-29	0.0035	0.0063	0.0067	0.0067
2007-01-31	0.0036	0.0059	0.0061	0.0058
2007-02-28	0.0060	0.0076	0.0078	0.0074
2007-03-30	0.0050	0.0090	0.0097	0.0094
2007-04-30	0.0075	0.0070	0.0092	0.0096
2007-05-31	0.0067	0.0071	0.0099	0.0103
2007-06-29	0.0030	0.0045	0.0057	0.0055
2007-07-31	0.0017	0.0033	0.0042	0.0040
2007-08-31	0.0036	0.0060	0.0059	0.0054
2007-09-28	0.0037	0.0075	0.0058	0.0066
RMSS	0.0041	0.0053	0.0066	0.0070

USD	HW	BK	SMM	LMM
2006-02-28	0.0047	0.0030	0.0121	0.0132
2006-03-31	0.0054	0.0041	0.0147	0.0160
2006-04-28	0.0047	0.0079	0.0138	0.0151
2006-05-31	-0.0004	0.0044	0.0089	0.0097
2006-06-30	-0.0024	0.0032	0.0078	0.0090
2006-07-31	0.0048	0.0086	0.0145	0.0153
2006-08-31	0.0064	0.0108	0.0169	0.0185
2006-09-29	0.0097	0.0105	0.0165	0.0176
2006-10-31	0.0115	0.0099	0.0169	0.0182
2006-11-30	0.0140	0.0110	0.0153	0.0159
2006-12-29	0.0156	0.0163	0.0186	0.0194
2007-01-31	0.0174	0.0185	0.0200	0.0188
2007-02-28	0.0133	0.0132	0.0089	0.0092
2007-03-30	0.0118	0.0109	0.0081	0.0091
2007-04-30	0.0069	0.0039	0.0025	0.0041
2007-05-31	0.0128	0.0114	0.0105	0.0121
2007-06-29	0.0104	0.0107	0.0073	0.0074
2007-07-31	0.0060	0.0066	0.0024	0.0023
2007-08-31	0.0073	0.0078	-0.0002	-0.0015
2007-09-28	0.0059	0.0076	-0.0031	-0.0041
RMSS	0.0433	0.0443	0.0553	0.0587

Note: HW stands for Hull-White model, BK stands for Black-Karasinski model, SMM stands for Swap Market Model and LMM stands for Libor Market Model. All numbers are in real value. RMSS is the root mean sum of square of all P&L in each period.

Figure 8: Hedging profit and loss in EUR and USD markets



Note: HW stands for Hull-White model, BK stands for Black-Karasinski model, SMM stands for Swap Market Model and LMM stands for Libor Market Model. All the values are in real monetary value and the percentage error is the hedging P&L in percentage of the Bermudan swaption prices.

rate models, like the HW and BK models, to manage the loan portfolios. The performances of HW and BK are almost indistinguishable.

The results obtained here could have an immediate impact on the ALM departments of global banks and other financial institutions in choosing the right interest rate term structure model to manage their balance sheet. Future research could apply the same test procedure on interest rate from a different economic regime such as that of an emerging market or a developing country.

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