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## Portfolio and consumption decisions under ambiguity for regime switching mean returns

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# Working Paper Series

## Portfolio and Consumption Decisions under Ambiguity for Regime Switching Mean Returns

Hening Liu

Manchester Business School Working Paper No 582

## Manchester Business School

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## **Keywords**

Ambiguity, Malliavin derivative, regime switching, portfolio choice

## **Abstract**

This paper examines a continuous-time intertemporal consumption and portfolio choice problem for an ambiguity-averse investor with multiple priors when the expected return of a risky asset is unobservable and follows a hidden Markov chain. The investor's beliefs over investment opportunities are represented by a set of priors over the process governing the dynamics of the conditional estimates of the unobservable state. The investor is assumed to have Chen and Epstein's (2002) recursive multiple priors utility preferences. Using the Malliavin calculus technique, we characterize the optimal consumption and portfolio rules explicitly in terms of the Malliavin derivatives and stochastic integrals. We find that continuous Bayesian revisions under incomplete information can generate an ambiguity-driven hedging demand that mitigates the intertemporal hedging demand for the risky asset. In addition, ambiguity aversion magnifies the importance of the intertemporal hedging demand.

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# Portfolio and Consumption Decisions under Ambiguity for Regime Switching Mean Returns

Hening Liu<sup>†</sup>

October 13, 2009

## Abstract

This paper examines a continuous-time intertemporal consumption and portfolio choice problem for an ambiguity-averse investor with multiple priors when the expected return of a risky asset is unobservable and follows a hidden Markov chain. The investor's beliefs over investment opportunities are represented by a set of priors over the process governing the dynamics of the conditional estimates of the unobservable state. The investor is assumed to have Chen and Epstein's (2002) recursive multiple priors utility preferences. Using the Malliavin calculus technique, we characterize the optimal consumption and portfolio rules explicitly in terms of the Malliavin derivatives and stochastic integrals. We find that continuous Bayesian revisions under incomplete information can generate an ambiguity-driven hedging demand that mitigates the intertemporal hedging demand for the risky asset. In addition, ambiguity aversion magnifies the importance of the intertemporal hedging demand.

*JEL:* G11, D81, C61

*Keywords:* Ambiguity, Malliavin derivative, regime switching, portfolio choice

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# 1 Introduction

Recently, a large body of literature has examined dynamic consumption and portfolio choice problems in which investment opportunities are time-varying.<sup>1</sup> Merton (1971) first shows that the intertemporal hedging demand, which arises to hedge against stochastic variation in investment opportunities, is important to portfolio decision-making. This bulk of works assumes that investment opportunities, i.e., moments of distributions of returns, are fully observable to investors. In the real world, however, these moments are often unobservable to investors and thus must be estimated from observed market signals. Following this logic, a number of papers have examined dynamic portfolio choice and asset prices in the context of incomplete information economies. Dothan and Feldman (1986) and Detemple (1986) were the first to study asset prices under incomplete information in general equilibrium, followed by David (1997), Veronesi (1999, 2000), Lundtofte (2008) and Ai (2009).<sup>2</sup> Brennan (1998), Lakner (1998), Xia (2001), Honda (2003) and others analyze portfolio choice and the intertemporal hedging demand with continuous Bayesian revisions in partial equilibrium settings. Feldman (2007) provides an elaborate review of this literature and related discussions. In a partially observable economy, investors form estimates of moments conditional on available information. In most cases, they use recursive-filtering methods to construct stochastic processes describing the dynamics of the estimated moments. One can treat the processes of conditional moments as perfectly known and then solve for optimal consumption and portfolio decision rules and asset prices using techniques for solving complete information economies. These papers generally postulate that an investor's beliefs are represented by a subjective prior; that is, he completely trusts the probability model of conditional moments obtained by applying the Bayes rule in continuous time. Following this assumption, optimal consumption and portfolio decisions and asset prices in equilibrium can be derived in the expected utility framework.

In this paper, we follow Honda (2003) and consider a continuous-time incomplete information economy with two assets in which the expected return of a risky asset is unobservable to an investor and follows a hidden Markov chain. For the sake of analytical convenience, we assume that the hidden Markov chain is governed by two different regimes, as in Honda (2003). Based on the observed asset prices, the investor updates his belief over the unobservable state as consistent with the Bayes rule. However, different from the standard literature on incomplete information economies in the expected utility framework, we employ Chen and Epstein's (2002) recursive multiple priors utility (hereafter RMPU) framework to incorporate ambiguity and ambiguity aversion into the partially observable economy with regime switching mean returns and then analyze dynamic consumption and portfolio choice in a partial equilibrium setting. Ambiguity and ambiguity aversion are concepts used

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<sup>1</sup> An incomplete list of those studies include Barberis (2000), Campbell and Viceira (1999), Chacko and Viceira (2005), Kim and Omberg (1996) and Liu (2007).

<sup>2</sup> David (1997) investigates portfolio choice and asset pricing when investment opportunities are unobservable and regime switching. In a general equilibrium setting, Lundtofte (2008) analyzes expected life-time utility and hedging demands in a partially observable economy in which the endowment process is allowed to have imperfect correlation with changes in the unobservable growth rate of endowments.

to describe a decision maker who is uncertain over which probability distribution describes the state variables in an economy and is also averse to such uncertainty. In the incomplete information economy considered here, the investor is ambiguous over which probability model governs the stochastic process of the conditional estimates of the unobservable state.

Ambiguity and ambiguity aversion are embedded into the problem using Chen and Epstein's continuous-time formulation of RMPU.<sup>3</sup> With RMPU preferences, the investor endogenously chooses the worst-case alternative model among a set of candidate models that governs the dynamics of the investor's beliefs. Such an ambiguity-aversion specification features max-min decision-making. In making consumption and portfolio decisions, the investor takes into account not only incomplete information risk that results from time-varying precision of the conditional estimates of the unobservable state but also ambiguity about the process that governs the dynamics of the filtered probabilities of the underlying state. Thus, the specification employed here naturally implies that the effect of ambiguity does not wear away as the investor engages continuous Bayesian revisions over time. This approach of modeling ambiguity is similar to that in Cagetti et al. (2002) and builds on the work of Miao (2001) in which a model with incomplete information and ambiguity is presented without being specialized to a specific investment opportunity set. Honda (2003) examines optimal consumption and portfolio choice in the expected utility framework, assuming that the expected return is unobservable and regime switching. The RMPU preferences adopted here nest the expected utility framework used in Honda (2003) as a special case in which there is no ambiguity.

The investor's ambiguity toward his beliefs about the investment opportunity set stems from uncertainty in estimating expected returns. Merton (1980) documents the difficulty in estimating the expected return of stock prices and argues that a very long time period is required to efficiently estimate the expected return, even when it is constant. When expected returns follow a regime-switching process, it becomes much more difficult to obtain a precise estimate of the expected return at each point in time.<sup>4</sup> Due to this apparent difficulty, it is reasonable to argue that the processes describing the dynamics of beliefs over investment opportunities are subject to a substantial degree of uncertainty. This uncertainty makes it difficult to justify treating the probability model that underlies belief dynamics as perfectly known and fixed. By allowing multiple priors over the probability model, we can take into consideration investors' lack of confidence due to concerns about model uncertainty. Distinguishing ambiguity from risk also has a well-grounded theoretic basis. The Ellsberg Paradox and related evidence have demonstrated that such a distinction is behaviorally meaningful. For these reasons, it is both reasonable and desirable to introduce ambiguity over belief dynamics into an incomplete information economy when studying dynamic consumption and portfolio decisions.

Using the Malliavin calculus technique and the Clark-Ocone formula, we show that the optimal

<sup>3</sup> Gilboa and Schmeidler (1989) provide an axiomatic foundation for multiple priors utility. Epstein and Schneider (2003) axiomatize the discrete-time version of recursive multiple priors utility. Epstein and Miao (2003) study asset pricing in a general equilibrium exchange economy using Chen and Epstein's RMPU.

<sup>4</sup> Guidolin and Timmermann (2007) study a wide range of model specifications to characterize regime switching in asset returns. They find that none of those models passes all specification tests that they consider. They then choose a parsimonious four-state model with regime-dependent mean and covariance matrix and analyze asset allocation decisions.

consumption and portfolio rules can be explicitly characterized in terms of the Malliavin derivatives and stochastic integrals. The solutions derived here contrast with Honda's (2003) study in which the partial differential equation approach is employed but fails to deliver explicit solutions for general coefficients of risk aversion. In this paper, we further distinguish incomplete information risk from ambiguity and obtain explicit solutions to the optimal consumption and portfolio choice. Our solutions are based on the martingale method of Cox and Huang (1989). Karatzas and Xue (1991) consider the application of the martingale method to incomplete information economies; and Dybvig, Rogers and Back (1999) use the method to examine consumption and portfolio decisions for stochastic investment opportunity sets. The numerical calculations in this paper are implemented using the Malliavin derivative Monte Carlo (MDMC) method developed by Detemple, Garcia and Rindisbacher (2003).

In an incomplete information economy, the optimal allocation to the risky asset has a myopic component that depends on the current risk-return trade-off as well as an intertemporal hedging component that takes into account time variation in the conditional estimates of the unobservable state. We find that continuous Bayesian revisions together with time-invariant ambiguity aversion yield an ambiguity-driven hedging component that is state- and horizon- dependent. This component does not wear off over the horizon because Bayesian updating does not resolve ambiguity and in addition, the precision of the conditional estimates fluctuates stochastically throughout the horizon. The impact of ambiguity on the intertemporal hedging demand is strong when the role incomplete information risk becomes dominant. Furthermore, ambiguity magnifies the relative importance of the intertemporal hedging demand in the optimal portfolio choice.

The problem analyzed here is different from the problems addressed by models on learning under ambiguity in discrete time (Epstein and Schneider, 2007, 2008) and continuous time (Leippold, Trojani and Vanini, 2008). These papers assume that information on fundamentals is ambiguous and thus take into account the updating of beliefs through multiple priors and likelihoods. In the problem analyzed in this paper, however, the investor treats the model of filtered probabilities as ambiguous and has multiple beliefs with respect to the model resulting from continuous Bayesian revisions. Schroder and Skiadas (2003) analyze optimal consumption and portfolio choice for generalized recursive utility preferences that incorporate RMPU as a special case. They show that the solution can be characterized up to the solution to a single constrained backward stochastic differential equation (BSDE). But they did not consider the role of incomplete information. Sbuelz and Trojani (2008) examine asset prices in a continuous-time exchange equilibrium with locally-constrained-entropy RMPU (LCE-RMPU). They exogenously posit that the local bound on the size of ambiguity is some function of state variables and thus is time-varying. Without deriving explicit solutions to the optimal consumption and portfolio policies, Sbuelz and Trojani (2008) identify that the impact of ambiguity on the optimal equity demand is state-dependent in a non-standard way. In this paper, we show that even with a constant local bound on the size of ambiguity, the model can still generate some form of non-standard state-dependence through endogenous Bayesian updating.

This paper contributes to a limited but growing body of literature that examines the implications of ambiguity for portfolio choice and asset pricing. Maenhout (2004) employs the robust control approach developed by Anderson, Hansen and Sargent (2000) to derive optimal portfolio rules for i.i.d. returns. Uppal and Wang (2003) extend the work of Maenhout (2004) to a multi-assets setting. Maenhout (2006) assumes time-varying investment opportunities and derives closed-form solutions for an investor maximizing expected power utility over terminal wealth. Gagliardini, Porchia, and Trojani (2009) study the term structure of interest rates with ambiguity aversion. Trojani and Vanini (2002, 2004) employ RMPU with time-invariant ambiguity aversion to study portfolio choice and asset pricing in general equilibrium under complete information.

The rest of this paper is organized as follows: Section 2 presents the model and derives optimal consumption and portfolio rules using the martingale method developed by Cox and Huang (1989); Section 3 describes the numerical simulations and discusses the results. Finally, Section 4 concludes. Proofs are collected in the Appendix.

## 2 The Model

In this section, we present the investment opportunity set with unobservable regime-switching mean returns. The investor obtains the conditional estimates of the unobservable state by observing past and current asset prices. In the benchmark case, the investor employs a non-linear recursive filter to extract filtered probabilities that are updated according to the Bayes rule. This benchmark model of conditional estimates serves to represent an approximate description of the investor's beliefs. Using the  $\kappa$ -ignorance specification in Chen and Epstein (2002), we consider a constrained set of alternative models surrounding the reference model. This prescribed set of priors and RMPU reflect the investor's lack of confidence in the reference model and thus ambiguity and ambiguity aversion. Then we describe the dynamic optimization problem for the ambiguity-averse investor. Finally, we solve for the optimal consumption and portfolio rules using the martingale method.

### 2.1 The investment opportunity set

We assume that there are two assets available for investment, namely, a riskless short-term bond paying an instantaneous return  $r$  and a risky asset with the following price dynamics:

$$dS_t = S_t\mu_t dt + S_t\sigma_S dB_t$$

where  $\sigma_S$  is a constant and the expected return  $\mu_t$  follows a continuous-time Markov chain with two states  $\mu_H$  and  $\mu_L$ . It is assumed that  $\mu_H > \mu_L$ , meaning that regime  $H$  represents the high-mean-return regime. The infinitesimal generating matrix of  $\mu_t$  is

$$\Lambda \equiv \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}$$



where  $\lambda_0, \lambda_1 > 0$ . Suppose the prior distribution of the two regimes is as follows. At the initial time  $t = 0$ , the economy is in the high-mean-return regime  $\mu_H$  with probability  $\pi_0$ ; the probability of the state being in the low-mean-return regime is then  $1 - \pi_0$ . Upon the arrival at state  $i$  ( $i = \mu_H$  or  $\mu_L$ ), the process  $\mu_t$  remains there for an exponentially-distributed time duration, and then jumps to state  $j$  ( $j \neq i$ ). The transition density parameters  $\lambda_0$  and  $\lambda_1$  are the parameters of the exponential density functions. The transition probabilities over any interval of time can be constructed from the infinitesimal matrix via the exponential formula  $T_s = \exp(s\Lambda)$ .

We assume that the investor can observe neither the expected return  $\mu_t$  nor the Brownian motion  $B$ . Instead, he can only observe the asset prices  $S$ . The parameters  $\mu_H, \mu_L, \lambda_0, \lambda_1$  and  $\sigma_S$  are assumed to be known constants. Given an initial prior over the regimes, the investor estimates the unobservable state, i.e., the probability of the current state being in the high-mean-return regime, based on the observed asset prices. The investor updates his beliefs as more prices are observed. As in previous works (e.g., Dothan and Feldman, 1986; Feldman, 1989), we can identify a  $\sigma$ -algebra equivalent economy in which the state variable is the filtered probability of the underlying state. This economy is a Markovian representation of the original economy.<sup>5</sup> The optimal consumption and portfolio rules obtained in this economy are also optimal in the original economy. Note that the characterization of the investment opportunity set is similar to that in Honda (2003).

Define the filtered probability  $\pi_t$  as the posterior probability that the current state is in the high-mean-return regime, that is

$$\pi_t = \Pr(\mu_t = \mu_H \mid \mathcal{F}_t^S)$$

with  $\pi_0$  given, where  $\{\mathcal{F}_t^S\}$  is the information filtration generated by the asset price process  $S$ . It follows from Theorem 9.1 in Liptser and Shiryaev (2001) that  $\pi$  satisfies the stochastic differential equation (SDE)

$$d\pi_t = [\lambda_0 - (\lambda_0 + \lambda_1)\pi_t] dt + \pi_t(1 - \pi_t) \frac{\mu_H - \mu_L}{\sigma_S} d\hat{B}_t \quad (1)$$

where  $\hat{B}$  is a standard Brownian motion with respect to the price filtration  $\{\mathcal{F}_t^S\}$  and is defined by

$$\hat{B}_t = \int_0^t \frac{dS_\tau - S_\tau \hat{\mu}_\tau d\tau}{S_\tau \sigma_S}$$

with

$$\hat{\mu}_\tau = \mu_H \pi_\tau + \mu_L (1 - \pi_\tau). \quad (2)$$

That is,  $\hat{\mu}_t$  is the conditional expected return with respect to  $\{\mathcal{F}_t^S\}$ . In the non-linear filtering equation (1), the precision of the conditional estimates of the unobservable state depends on the filtered probability and fluctuates stochastically. To explore more properties of (1), it is useful to rewrite it in terms of the original standard Brownian motion.<sup>6</sup> Suppose the investor knows that

<sup>5</sup> See Feldman (2007) for a discussion of the irrelevance of the Separation Principle for solving the optimization problems in dynamic incomplete information economies.

<sup>6</sup> David (1997) also provides a detailed discussion of several properties of this filtering process.

during some time interval  $[t_1, t_2]$ , the true expected return is  $\mu_t$ . Then  $\pi$  satisfies the following SDE

$$d\pi_t = \left[ \lambda_0 - (\lambda_0 + \lambda_1) \pi_t + \frac{\pi_t (\mu_H - \hat{\mu}_t) (\mu_t - \hat{\mu}_t)}{\sigma_S^2} \right] dt + \frac{\pi_t (\mu_H - \hat{\mu}_t)}{\sigma_S} dB. \quad (3)$$

During  $[t_1, t_2]$ , if the true expected return  $\mu_t = \mu_H$ , then the second term in the drift of (3) tends to trend  $\pi_t$  toward 1. When  $\pi_t$  approaches 1, both the second term in the drift and the diffusion term converge to zero, making the first term in the drift dominant. This term prevents  $\pi_t$  from converging to 1 and drives  $\pi_t$  toward the inside of  $[0, 1]$ . If the true expected return  $\mu_t = \mu_L$ , then the second term in the drift is negative and tends to trend  $\pi_t$  toward 0. When  $\pi_t$  approaches 0, the first term in the drift becomes large in magnitude and drives  $\pi_t$  back inside the interval  $[0, 1]$ . Thus, the drift term in (1) has the effect of mean reversion. It is worth noting that a large  $\lambda_1$  implies a strong tendency to trend  $\pi_t$  away from 1 while a large  $\lambda_0$  implies a strong prevention of  $\pi_t$  from converging to 0. In the special case of non-switching regimes, that is  $\lambda_0 = \lambda_1 = 0$ , there is no mean-reverting effect, where the precision of the estimate should increase as more returns are observed. This case is similar to that considered in Brennan (1998).

The volatility term in (1) quantifies a risk arising in an incomplete information economy, which is termed *incomplete information risk*<sup>7</sup>. This risk stems from time-varying precision of the conditional estimates of the unobservable state. When  $\pi_t$  takes values near the middle of the interval  $[0, 1]$ , the investor is barely confident about the current estimate, leading to a sizable incomplete information risk. In this case, the investor put much weight on innovations in returns when updating beliefs. When the investor is fairly confident about the current estimate, that is, when  $\pi_t$  takes values near the boundaries of the interval  $[0, 1]$ , the magnitude of incomplete information risk is small. In this case, innovations in returns have little effect on revisions in beliefs. In addition, if  $\mu_H$  is significantly different from  $\mu_L$ , the signal-to-noise ratio  $(\mu_H - \mu_L)/\sigma_S$  is large. As a result, a small innovation in returns can lead to a large revision in beliefs because much information can be revealed. A high return volatility  $\sigma_S$  implies that signals are very noisy and hardly useful in updating beliefs. Thus, the scope of incomplete information risk is limited.

## 2.2 Ambiguity and recursive multiple priors utility

### 2.2.1 The set of alternative models

Denote the planning horizon as  $T$ , the wealth process as  $\{W_t\}_{t=0}^T$  and the consumption process  $C$  as  $\{C_t\}_{t=0}^T$ . Given an initial wealth endowment  $W_0 > 0$ , the wealth dynamics can be described by the following SDE:

$$dW_t = [W_t (r + \alpha_t (\hat{\mu}_t - r)) - C_t] dt + W_t \alpha_t \sigma_S d\hat{B}_t \quad (4)$$

where  $\alpha_t$  is the proportion of wealth invested in the risky asset and  $\hat{\mu}_t$  is the mean return estimate given in (2). The investor makes consumption and portfolio decisions based on the conditional estimates of the unobservable state. Thus far, the original incomplete information economy has been converted

<sup>7</sup> We thank David Feldman for suggesting this terminology.

into a Markovian equivalent economy with state dynamics (1) and (4).

Denote the state vector  $Y_t \equiv (W_t, \pi_t)$ . The reference model can be written as

$$dY_t = \boldsymbol{\mu}(Y_t) dt + \boldsymbol{\sigma}(Y_t) d\hat{B}_t \quad (5)$$

where  $\boldsymbol{\mu}$  and  $\boldsymbol{\sigma}$  have the appropriate functional forms inherited from SDEs (1) and (4). The reference model serves as a benchmark model among all the models that the investor is willing to consider. The investor has lack of confidence about the full usefulness of the reference model (5) and thus deems the state vector dynamics (5) as only an approximation of his beliefs. He wishes to consider a family of alternative models that are close to the reference model and are difficult to distinguish from it.

The set of alternative models on which RMPU is defined is constructed from  $\{\mathcal{F}_t^S\}_{t=0}^T$ -adapted *density generators* defined by  $\theta \equiv \{\theta_t\}_{t=0}^T \in \Theta$  satisfying  $\sup |\theta_t| \leq \kappa$  with  $\kappa \geq 0$ . This specification is referred to as  $\kappa$ -ignorance in Chen and Epstein (2002). Each density generator  $\theta$  delivers a local distortion to the reference model. Suppose  $\mathbb{P}$  is the subjective probability measure under the reference model, that is, the probability measure with respect to the Brownian motion  $\hat{B}$ . Each density generator  $\theta$  generates a martingale  $z^\theta$  under  $\mathbb{P}$ :

$$z_t^\theta = \exp\left(-\frac{1}{2} \int_0^t \theta_s^2 ds - \int_0^t \theta_s d\hat{B}_s\right), \quad 0 \leq t \leq T.$$

The set of alternative models  $\mathcal{P}$  is specified in terms of the Radon-Nikodym derivatives of the alternative models with respect to the reference model:

$$\mathcal{P} \equiv \left\{ \mathbb{Q}^\theta : \theta \in \Theta, \frac{d\mathbb{Q}^\theta}{d\mathbb{P}} = z_T^\theta \right\}.$$

Alternatively, the set  $\mathcal{P}$  is also referred to as the *set of priors*. It is obvious that the size of the ambiguity set increases with the parameter  $\kappa$ . A higher value of  $\kappa$  implies that more alternative models would be considered by the investor and that he is less confident about the reference model. In the special case of  $\kappa = 0$ , all alternative models coincide with the reference model, in which case the ambiguity set collapses to a singleton  $\mathbb{P}$ . The investor has complete confidence in the reference model.

It follows from Girsanov's theorem that the distorted law of motion of the state vector implied by an alternative model  $\mathbb{Q}^\theta$  is

$$dY_t = \boldsymbol{\mu}(Y_t) dt + \boldsymbol{\sigma}(Y_t) \left( d\hat{B}_t^{\mathbb{Q}^\theta} - \theta_t dt \right) \quad (6)$$

where  $\hat{B}^{\mathbb{Q}^\theta}$  is the Brownian motion under  $\mathbb{Q}^\theta$ . The interpretation is that the investor is ambiguous about whether  $\hat{B}$  is a Brownian motion with respect to his information filtration. Thus, the investor considers the alternative models that are absolutely continuous with respect to the reference model, in which case ambiguity indeed concerns uncertainty about the drift functions of the state processes. In particular, under the alternative model  $\mathbb{Q}^\theta$ , the distorted law of motion of the estimate  $\pi_t$  can be

explicitly written as <sup>8</sup>

$$d\pi_t = [\lambda_0 - (\lambda_0 + \lambda_1)\pi_t] dt + \pi_t(1 - \pi_t) \frac{\mu_H - \mu_L}{\sigma_S} \left( d\hat{B}_t^{\mathbb{Q}^\theta} - \theta_t dt \right). \quad (7)$$

It is worth noting that the distorted drift function in (7) depends on the term quantifying incomplete information risk. When the precision of an estimate is low, which usually occurs when the estimation of the expected return is difficult, the magnitude of the drift distortion is large, and thus, the investor has low confidence in the estimate. As shown below, this drift distortion term can generate an ambiguity-driven hedging demand.

For time-invariant ambiguity, that is,  $\theta_t = \kappa$  for all  $t \in [0, T]$ , one can write (6) as

$$dY_t = \boldsymbol{\mu}(Y_t) dt + \boldsymbol{\sigma}(Y_t) \left( d\hat{B}_t^{\mathbb{Q}} - \kappa dt \right)$$

where for brevity  $\mathbb{Q}$  denotes the alternative model with  $\theta = \kappa$ . Indeed, we verify that time-invariant ambiguity can be supported as an optimum in the optimization problem with RMPU described below.

## 2.2.2 The RMPU preferences

The investor has a time preference rate  $\rho$  and a constant relative risk aversion (CRRA) utility function of the following form

$$u(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma}, & \gamma > 0, \neq 1 \\ \log C, & \gamma = 1 \end{cases}$$

The investor's objective is to choose consumption and portfolio policies to maximize the expected utility under the worst-case alternative model. Among all those prescribed models induced by the allowed set of priors, the worst-case alternative model delivers the minimum of expected utility given a consumption process and terminal wealth. Put formally, the investor's value function is given by

$$V_0(C, W_T) = \max_{C_t, \alpha_t} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}^\theta} \left[ \int_0^T e^{-\rho t} u(C_t) dt + e^{-\rho T} u(W_T) \right] \quad (8)$$

subject to the state dynamics (1) and (4), or compactly, the reference model (5). The minimization operator in (8) captures the concern that an alternative model may have an adverse effect on the continuation value. As a result, the investor would like to take into account the worst-case alternative model. The worst-case model is to be endogenously selected from the set of priors  $\mathcal{P}$ , which contains a prescribed family of alternative models generated from locally distorting the reference model. The multiplicity of the set of priors  $\mathcal{P}$  represents the investor's lack of confidence in the reference model, and the minimization operator reflects his aversion to such ambiguity. Since each alternative probability measure  $\mathbb{Q}^\theta$  is associated with a corresponding density generator  $\theta$ , the minimization operator is also taken with respect to the process  $\theta$ .

<sup>8</sup> Cagetti et al. (2002) consider a similar distorted law of motion to describe the dynamics of the estimates of a hidden Markov chain in the robust control framework. They use the distorted state processes to derive asset pricing implications in a production economy with uncertain growth rates.

Chen and Epstein (2002) derive the following backward stochastic differential equation (BSDE) representation of the utility process  $V_t$

$$dV_t = \left[ -u(C_t) + \rho V_t + \max_{\theta \in \Theta} \theta_t \sigma_t^V \right] dt + \sigma_t^V d\hat{B}_t, \quad V_T = u(W_T).$$

where the volatility term  $\sigma_t^V$  is endogenous and is part of the complete solution to the above BSDE.<sup>9</sup> For the  $\kappa$ -ignorance specification, Chen and Epstein (2002) show that the endogenous drift distortion is given by

$$\max_{\theta \in \Theta} \theta_t \sigma_t^V = \theta_t^* \sigma_t^V = \kappa |\sigma_t^V|, \quad \text{with } \theta_t^* = \kappa \times \mathbf{sgn}(\sigma_t^V) \quad (9)$$

where  $\mathbf{sgn}(\sigma_t^V) = |\sigma_t^V|/\sigma_t^V$  if  $\sigma_t^V \neq 0$  and  $\mathbf{sgn}(\sigma_t^V) = 0$  otherwise. The term  $\kappa |\sigma_t^V|$  is entirely attributed to ambiguity aversion rather than risk aversion. The parameter  $\kappa$  is also interpreted as an ambiguity aversion parameter. It is worth noting in (9) that the worst-case probability law associated with the drift distortion  $\theta^*$  actually depends on the sign of the volatility of the utility process. The volatility of  $V_t$  as a part of the complete solution to the problem (8) relies on the optimal consumption path, which is in turn a function of the endogenous drift distortion  $\theta^*$  (see the proof of Proposition 1). Thus, unlike other studies (e.g., Gagliardini, Porchia and Trojani, 2008; Trojani and Vanini, 2002) in which one can first solve out the worst-case drift distortion explicitly and then tackle the maximization problem under the resulting worst-case model, here in this paper the solutions to the optimal consumption and portfolio policies together with the endogenous worst-case model must be jointly determined from solving the optimization problem (8). In the analysis below, we employ the guess-and-verify method to show that both conditions  $\sigma_t^V > 0$  and  $\theta_t^* = \kappa$  for all  $t \in [0, T]$  can be supported together in the optimum. That is, the size of ambiguity or ambiguity aversion is time-invariant throughout the horizon. As a result, the optimal consumption and portfolio policies can be derived conditioning on time-invariant ambiguity  $\theta^* = \kappa$ .

### 2.3 Optimal consumption and portfolio choice

Define the conditional market price of risk  $\hat{\nu}_t$  as

$$\hat{\nu}_t = \frac{\hat{\mu}_t - r}{\sigma_S}.$$

Since  $\pi_t$  is progressively measurable and always bounded between 0 and 1, the conditional market price of risk is also progressively measurable and bounded given that  $\mu_H$ ,  $\mu_L$  and  $\sigma_S$  are all finite constants. Because  $\hat{\nu}$  is a bounded process, Novikov's condition holds; that is,  $\mathbb{E}^{\mathbb{P}} \left( \exp \left\{ \frac{1}{2} \int_0^T \hat{\nu}_t^2 dt \right\} \right) < \infty$ . In addition, the market is complete in the Markovian equivalent economy. Thus, there exists a unique

<sup>9</sup> Chen and Epstein (2002) shows that the ambiguity set formulated by  $\kappa$ -ignorance satisfies a property called ‘‘rectangularity’’. Rectangularity together with other technical conditions ensures that the utility process under multiple priors is indeed recursive, and hence, time consistent (in the sense of Johnsen and Donaldson 1985).

equivalent martingale measure  $\tilde{\mathbb{P}}$  given by  $d\tilde{\mathbb{P}} = \zeta_T d\mathbb{P}$  where

$$\zeta_t = \exp\left(-\frac{1}{2} \int_0^t \hat{\nu}_s^2 ds - \int_0^t \hat{\nu}_s d\hat{B}_s\right)$$

with  $\zeta_0 = 1$ . Under  $\tilde{\mathbb{P}}$ , the process  $\tilde{B}_t = \hat{B}_t + \int_0^t \hat{\nu}_s ds$  is a Brownian motion. The state price density, denoted as  $\xi$ , is given by

$$\xi_t = \exp\left(-\int_0^t r ds - \frac{1}{2} \int_0^t \hat{\nu}_s^2 ds - \int_0^t \hat{\nu}_s d\hat{B}_s\right).$$

The boundedness of the conditional market price of risk implies that all moments of the state price density are finite, which is a sufficient condition that guarantees the applicability of the martingale method for stochastic investment opportunity sets (Dybvig, Rogers and Back, 1999). This condition is also referred to as the ‘‘DRB condition’’ in Korn and Kraft (2004). As a consequence, the optimization problem is well-defined and the pitfalls raised by Korn and Kraft (2004) can be naturally avoided.

Hereafter, we assume that the lower bound of the conditional market price of risk adjusted for ambiguity is nonnegative; namely,

$$\frac{\mu_L - r}{\sigma_S} - \kappa \geq 0. \quad (10)$$

This assumption guarantees that in all states of the economy, the conditional market price of risk adjusted for ambiguity, or the *effective* conditional market price of risk, is strictly positive unless  $\pi_t$  exactly reaches 0. This assumption plays a key role in proving that time-invariant ambiguity indeed obtains in the optimum (see the proof of Proposition 1 in the Appendix).

The static variational problem corresponding to the optimization problem (8) is formulated by standard arguments as follows:

$$\max_{C_t, W_T} \min_{\mathbb{Q}^\theta \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}^\theta} \left[ \int_0^T e^{-\rho t} u(C_t) dt + e^{-\rho T} u(W_T) \right] \quad (11)$$

$$\text{s.t. } W_0 = \mathbb{E}^{\mathbb{P}} \left[ \int_0^T \xi_t C_t dt + \xi_T W_T \right]. \quad (12)$$

Suppose the model  $\mathbb{Q}^{\theta^*}$  solves the inner minimization problem, then the Lagrangian of problem (11) subject to (12) is given by

$$\mathcal{L} = \mathbb{E}^{\mathbb{Q}^{\theta^*}} \left[ \int_0^T e^{-\rho t} u(C_t) dt + e^{-\rho T} u(W_T) \right] + y \left\{ W_0 - \mathbb{E}^{\mathbb{P}} \left[ \int_0^T \xi_t C_t dt + \xi_T W_T \right] \right\} \quad (13)$$

where  $\theta^*$  satisfies the condition (9) and  $y$  a scalar Lagrange multiplier. It is worth noting that different from the standard martingale formulation with expected utility, the endogenous probability law under which the expectation of discounted future utility is taken deviates from the probability law governing the static budget constraint. The deviation reflects the investor’s distrust of the reference model that dictates a particular consumption process in the martingale formulation. To solve the static variational problem, a change of measure is applied to convert the worst probability measure

$\mathbb{Q}^{\theta^*}$  to the reference measure  $\mathbb{P}$  using the Radon-Nikodym derivative between the two measures. The first-order conditions (shown in the Appendix) can then be derived under the reference measure. The solutions to the optimal consumption and portfolio policies together with the endogenous probability law  $\mathbb{Q}^{\theta^*}$  are given in the following proposition.

**Proposition 1** *Suppose  $\gamma \neq 1$  and the following condition holds<sup>10</sup>:*

$$\mathbb{E}_t^{\tilde{\mathbb{P}}} \left[ \int_t^T e^{-rs} C_s^* \int_t^s \left( D_t \hat{\nu}_\tau d\tilde{B}_\tau \right) ds + e^{-rT} W_T^* \int_t^T D_t \hat{\nu}_s d\tilde{B}_s \right] > 0 \quad \text{for all } t \in [0, T] \quad (14)$$

where  $C_s^*$  and  $W_T^*$  are given in (ii). The following solutions characterize an optimum:

- (i) The endogenous probability law is given by  $\mathbb{Q}^{\theta^*}$  with  $\theta_t^* = \kappa$  for all  $t \in [0, T]$
- (ii) The optimal consumption  $C_t^*$  and terminal wealth  $W_T^*$  are

$$C_t^* = \left( \frac{e^{-\rho t} z_t^\kappa}{y \xi_t} \right)^{\frac{1}{\gamma}} \quad \text{and} \quad W_T^* = \left( \frac{e^{-\rho T} z_T^\kappa}{y \xi_T} \right)^{\frac{1}{\gamma}} \quad (15)$$

where the Lagrange multiplier  $y$  satisfies

$$y = \left( \mathbb{E}^{\mathbb{P}} \left[ \int_0^T (\xi_t)^{\frac{\gamma-1}{\gamma}} (e^{-\rho t} z_t^\kappa)^{\frac{1}{\gamma}} dt + (\xi_T)^{\frac{\gamma-1}{\gamma}} (e^{-\rho T} z_T^\kappa)^{\frac{1}{\gamma}} \right] / W_0 \right)^\gamma \quad (16)$$

- (iii) The optimal portfolio  $\alpha_t^*$  is

$$\alpha_t^* = \frac{\hat{\mu}_t - r}{\gamma \sigma_S^2} - \frac{\kappa}{\gamma \sigma_S} + \frac{1 - \gamma}{\gamma} \frac{e^{rt}}{\sigma_S W_t^*} \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[ \int_t^T e^{-rs} C_s^* \int_t^s \left( D_t \hat{\nu}_\tau d\tilde{B}_\tau \right) ds + e^{-rT} W_T^* \int_t^T D_t \hat{\nu}_s d\tilde{B}_s \right] \quad (17)$$

where the conditional expectation is taken under the equivalent martingale measure  $\tilde{\mathbb{P}}$ . The Malliavin derivative of the conditional market price of risk  $D_t \hat{\nu}_s$  is given by  $D_t \hat{\nu}_s = \frac{(\mu_H - \mu_L)}{\sigma_S} D_t \pi_s$  in which the Malliavin derivative of  $\pi$ ,  $D_t \pi_s$ , satisfies the following SDE:

$$d(D_t \pi_s) = -(\lambda_0 + \lambda_1) D_t \pi_s ds + \frac{\mu_H - \mu_L}{\sigma} (1 - 2\pi_s) D_t \pi_s d\hat{B}_s$$

subject to the boundary condition  $\lim_{s \rightarrow t} D_t \pi_s = \pi_t (1 - \pi_t) \frac{\mu_H - \mu_L}{\sigma_S}$ .

**Proof.** See Appendix. ■

**Corollary 1** *Suppose  $\pi_0 > 0$ . The solutions in Proposition 1 nest the case of logarithmic utility with  $\gamma = 1$ .*

**Proof.** See Appendix. ■

In the Appendix, we show that both conditions (14) and (10) ensure that the time-invariant density generator  $\theta^* = \kappa$  delivers the endogenous worst-case alternative model in Lagrangian (13). That is, in the optimum, the density generator does not switch between different regimes.<sup>11</sup> This implies that

<sup>10</sup> Ideally, we would like to be able to show analytically that this condition holds for the solution in (ii). Unfortunately, we have been unable to do so, since the condition involves the Malliavin derivatives and stochastic integrals, both of which have to be computed numerically. Nevertheless, in the numerical simulations below, we have verified that this condition does hold on a fine grid of the state variable  $\pi \in [0, 1]$  and time  $t \in [0, T]$ .

<sup>11</sup> In general, the case of switching density generator is intractable primarily due to two reasons. First, the worst-case model jumps across different regimes, rendering a state vector solution difficult to obtain. Second, the volatility of the

there are no sudden jumps in the drift distortion associated with the worst-case model. The investor's attitude toward multiple beliefs is therefore state-independent and exhibit stationarity. This simplified result provides analytical convenience, namely, delivers explicit solutions to the optimal consumption and portfolio policies conditioning on the time-invariant density generator.

In the optimal portfolio formula, the first term is the myopic demand for the risky asset, which is instantaneously mean-variance efficient and depends on the current estimate of the unobservable state. The second term reflects the effect of ambiguity on the myopic component, which relies on the magnitude of ambiguity. Under the condition (10), a larger  $\kappa$  implies that the investor allocates a smaller proportion of wealth to the risky asset when he behaves myopically by ignoring time variation of the conditional estimates. Together, the first two terms are called *ambiguity-adjusted myopic demand* hereafter. When returns are i.i.d. and expected returns are fully observable, the optimal portfolio policy is given by the ambiguity-adjusted myopic component, as shown by Chen and Epstein (2002). Their results hence give rise to a form of observational equivalence; i.e., with respect to the effect on consumption and portfolio choice, an increase in the size of ambiguity is observationally equivalent to a decline in the effective market price of risk. Under incomplete information, this form of observational equivalence cannot be sustained because ambiguity also has an impact on the intertemporal hedging demand in a non-standard way.

The third term quantifies the intertemporal hedging demand, which is induced to hedge against the future time variation of the conditional estimates of the unobservable state. In the optimal portfolio formula (17), the Malliavin derivative  $D_t\pi_s$  captures the effect of an innovation in the Brownian motion  $\hat{B}$  at time  $t$  on the state variable  $\pi$  at time  $s$ . A notable difference between the solution derived here and those derived in the expected utility framework without ambiguity (Brennan, 1998; Honda, 2003; Lundtofte, 2008) is that the intertemporal hedging demand is driven not only by incomplete information risk but also by ambiguity. In particular, ambiguity affects hedging demand through the intermediate consumption decisions, as seen in (17) and (15). The hedging term in (17) can be further decomposed into two terms  $hedge^{IIR}$  and  $hedge^{ambiguity}$ , and the optimal portfolio rule can be rewritten as

$$\alpha_t^* = \frac{\hat{\mu}_t - r}{\gamma\sigma_S^2} - \frac{\kappa}{\gamma\sigma_S} + hedge^{IIR} + hedge^{ambiguity}$$

where  $hedge^{IIR}$  is obtained from the hedging term in (17) by setting  $\kappa = 0$ , and  $hedge^{ambiguity}$  accounts for the difference between  $hedge^{IIR}$  and the hedging term. In this way,  $hedge^{IIR}$  is solely attributed to the intertemporal hedging of incomplete information risk, while  $hedge^{ambiguity}$  is purely driven by ambiguity. The hedging term  $hedge^{IIR}$  exists in an expected utility model with incomplete information, but  $hedge^{ambiguity}$  does not. Although the effect of ambiguity on myopic demand does not depend on the state variable,  $hedge^{ambiguity}$  is state-dependent in a non-standard way. In the

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utility process, which is the key determinant of the instantaneous regime of the density generator, is endogenous and depends on the decision variables when investment opportunities are time-varying. This adds enormous difficulty in solving the optimization problem in the presence of jumps in the density generator.



special case of  $\gamma = 1$ , the investor is uninterested in hedging, and ambiguity only affects the myopic demand for the risky asset.

### 3 Numerical Simulations and Results

In this section, we perform numerical simulations to further study the impacts of incomplete information risk and ambiguity. The baseline parameter values for numerical simulations are:

$$\mu_H = 0.20, \mu_L = 0.06, \sigma_S = 0.10, r = 0.02, \lambda_0 = 2.00, \lambda_1 = 1.00, \rho = 0.05, T = 10.$$

We use the Malliavin derivative Monte Carlo (MDMC) method developed by Detemple, Garcia and Rindisbacher (2003) to compute the optimal portfolios and the consumption-to-wealth ratios. As shown by Detemple, Garcia and Rindisbacher (2003), simulating optimal portfolios that are in explicit forms can improve the efficiency and accuracy of numerical approximations relative to the standard numerical stochastic programming with backward iterations (Barberis, 2000; Guidolin and Timmermann, 2007). We run a random number generator to simulate a large number of sample paths and numerically evaluate the stochastic integrals in the expressions of the optimal portfolio and consumption-to-wealth ratio. The state variable  $\pi$  and its Malliavin derivative are simulated using a variance-stabilizing transformation to minimize approximation error as suggested by Detemple, Garcia and Rindisbacher (2003). The reformulation allows us to adopt a change of variables that normalizes the volatility of the process of the filtered probabilities to a constant. This makes the numerical calculation of the Malliavin derivative  $D_t\pi_s$  of the same complexity as the numerical solution of an ordinary differential equation (ODE).<sup>12</sup> The number of Monte Carlo replications is set to  $N = 20,000$ . The initial wealth  $W_0$  is set to 1 such that the optimal consumption  $C_0^*$  can be interpreted as the consumption-to-wealth ratio.

Table 1 summarizes the solution to expected utility optimization problem with and without intermediate consumption assuming no ambiguity. The ambiguity aversion parameter  $\kappa$  is set to 0. The state variable  $\pi$  takes values ranging from 0 to 1.<sup>13</sup> Our results based on the MDMC method stand in contrast to the findings in Honda (2003) in the following respects. First, our results highlight the importance of the intertemporal hedging demand arising due to incomplete information risk. For example, in Table 1, when the coefficient of relative risk aversion  $\gamma$  is 8 and  $\pi$  is 0.5, the ratio of hedging demand to the optimal demand for the risky asset is 20%.

**[Insert Table 1 here]**

Second, we find that the sign of the intertemporal hedging demand is negative for various coefficients of risk aversion greater than 1, including very high coefficients (e.g.,  $\gamma = 20$ ), while Honda (2003)

<sup>12</sup> See Detemple, Garcia and Rindisbacher (2003) for further details.

<sup>13</sup> To implement the variance-stabilizing transformation in Detemple, Garcia and Rindisbacher (2003), we use a value of  $\pi$  close to 0 ( $\pi = 0.002$ ) to approximate  $\pi = 0$  and a value of  $\pi$  close to 1 ( $\pi = 0.999$ ) to approximate  $\pi = 1$ . We do this because the transformation entails the calculation of the inverse function of the volatility term in the state process  $\pi_t$ , which is equal to 0 when  $\pi$  takes a value of either 0 or 1.

finds that the sign of hedging demand turns out to be positive for high levels of risk aversion. Our findings are in line with those in Brennan (1998) and Lundtofte (2008), though Lundtofte (2008) also takes into account an additional hedging component due to the correlation between the endowment process and changes in the unobservable growth rate. An investors who is more risk averse than a logarithmic investor has the utility function with its shape unbounded from below and bounded from above. As a result, he behaves more conservatively than the logarithmic investor and prefers to hold a portfolio less risky than the myopic portfolio. In the context of the model, continuous Bayesian revisions create a perfect positive correlation between innovations to asset returns and revisions in the conditional mean return estimates. The risky asset tends to have low (high) returns when its mean returns are expected to be low (high). Due to the investor's desire to smooth utility across states, a portfolio is deemed less risky if it can deliver wealth when investment opportunities are poor or when the mean return estimates are low. Thus, an investor with high coefficients of relative risk aversion (i.e., higher than 1) optimally decreases the investment in the risky asset, which implies that the intertemporal hedging demand has a negative sign. For an investor maximizing utility over consumption, the negative sign of hedging demand can also be explained by the desire to smooth consumption across states. As shown in Table 2, the consumption-to-wealth ratio increases with the state belief  $\pi$  when the coefficient of relative risk aversion is greater than 1. An increase in the state belief causes the income effect and the substitution effect to arise. The income effect increases current consumption because the investor can entertain a higher level of consumption for a given quantity of wealth. The substitution effect decreases consumption because the investment opportunity set is more attractive. When the intertemporal elasticity of substitution is less than 1, which implies that  $\gamma > 1$  for the CRRA utility, the income effect dominates over the substitution effect.<sup>14</sup> To smooth consumption across states, the investor desires a portfolio that can generate wealth when state beliefs are bad, that is, a portfolio with negative hedging demand.

An investors who is less risk averse than a logarithmic investor has the utility function with its shape bounded from below and unbounded from above. He therefore behaves more aggressively and is willing to hold a more risky portfolio. As a result, hedging demand has a positive sign. The consumption-to-wealth ratio decreases with the state belief because the elasticity of intertemporal substitution is greater than 1 (that is,  $\gamma < 1$ ) in which case the substitution effect dominates the income effect.

**[Insert Table 2 here]**

Third, the size of hedging demand is non-monotonic in risk aversion. As shown in Table 1, when an investor maximizes utility over consumption, the magnitude of hedging demand increases and then decreases relative to risk aversion. This result is similar to the findings of Campbell and Viceira (1999)

<sup>14</sup>For the CRRA utility function, the elasticity of intertemporal substitution and the coefficient of relative risk aversion are inversely related. Due to this relationship, the consumption-to-wealth ratio first increases and then decreases with risk aversion. See Campbell and Viceira (1999) for a related discussion.

but contrasts with those of Honda (2003), who finds that the hedging component against incomplete information risk is monotonically decreasing in risk aversion. When an investor becomes more risk averse, he has a stronger desire to smooth utility and/or consumption, which calls for a less risky portfolio and a larger size of hedging demand. When the level of risk aversion further increases, the investor wants to limit his exposure to the risky asset in all states of his beliefs about the underlying state, in which case the hedging motives wear off.

Table 1 also shows that the presence of intermediate consumption mainly reduces the impact of incomplete information risk on hedging demand for a given investment horizon. This effect is especially strong for investors with  $\gamma > 1$ . Suppose there is a negative shock to returns, this implies that the future estimates of the unobservable state are low, and thus, expected future returns are low. An investor maximizing utility over consumption can adjust both his consumption and portfolio decisions, while an investor with utility over terminal wealth can only adapt his portfolio decisions. Thus, the latter investor has to allocate a large proportion of his wealth to hedge against incomplete information risk. It is also interesting to note that in the case of utility over terminal wealth, the size of hedging demand increases with risk aversion for relatively low state beliefs. This is because an investor possessing those bad beliefs worries about the occurrences of even worse states and thus has very strong hedging motives, especially when he is highly risk averse.

Turning to the effect of ambiguity, Table 3 summarizes the consumption-to-wealth ratio for various values of the ambiguity-aversion parameter  $\kappa$ . An ambiguity-averse investor believes that the investment opportunity set is represented by the worst-case alternative model, under which the mean return estimates are perceived to be lower than those under the reference model; see Equation (6). Two competing effects arise in the presence of the worst-case alternative model. On the one hand, the investor lowers consumption because a given level of wealth can deliver a smaller flow of consumption due to perceived deterioration in investment opportunities. On the other hand, the investor is less interested in investing in the risky asset and tends to increase consumption. When  $\gamma > 1$ , the former effect dominates, and ambiguity aversion decreases the consumption-to-wealth ratio. When  $\gamma < 1$ , the latter effect becomes significant, and ambiguity aversion increases the ratio.

For  $\gamma > 1$ , Table 3 shows that when the current mean return estimate rises (i.e.,  $\pi$  is higher), an ambiguity-averse investor increases the consumption-to-wealth ratio less sharply than an expected-utility investor without ambiguity. This result can be attributed to the precautionary savings effect, which strengthens the relative importance of the substitution effect associated with a change in the investment opportunity set. Filtering the unobservable state under incomplete information creates a second channel for ambiguity to unfavorably affect how an ambiguity-averse investor perceives the investment opportunity set apart from its effect on price dynamics. Given an increase in the current mean return estimate, the investor still fears the adverse impact of ambiguity on the future variation of the mean return estimates and thus responds conservatively by increasing consumption less notably. This stands in contrast to an investor with expected utility who is not ambiguity-averse and has complete confidence in the reference model. Similarly, due to the precautionary savings effect, an

ambiguity-averse investor with  $\gamma < 1$  reduces the consumption-to-wealth ratio more significantly in response to an increase in the state belief.

[Insert Table 3 here]

Table 4 examines the intertemporal hedging demand and the ambiguity-adjusted myopic demand at different magnitudes of ambiguity. An ambiguity-averse investor not only decreases myopic demand but also shrinks hedging demand. Along with the change in the optimal consumption-to-wealth ratio in response to ambiguity, the incentive to hold a less risky portfolio and that to hold a more risky portfolio are mitigated, respectively, for investors with  $\gamma > 1$  and those with  $\gamma < 1$ . Figure 1 plots the proportion of wealth allocated to hedge against incomplete information risk ( $hedge^{IIR}$ ) and the proportion of wealth allocated to hedge against ambiguity ( $hedge^{ambiguity}$ ) as functions of the current estimate of the unobservable state  $\pi$ . When the investor becomes more ambiguity-averse, the size of  $hedge^{ambiguity}$  increases. In addition,  $hedge^{ambiguity}$  is non-monotonic and displays a mildly humped, that is, increasing-decreasing, shape with respect to the estimate of the unobservable state. This non-standard non-monotonic dependence can be explained by the drift distortion term in the worst-case alternative model governing the dynamics of the conditional estimates; see equation (7). The drift distortion in state dynamics implied by the worst-case prior indeed drives the the hedging component against ambiguity. The magnitude of this distortion term is large when  $\pi$  takes values near 1/2, that is, when incomplete information risk becomes sizable. This term imputes a negative drift distortion that tends to trend the estimate  $\pi$  away from 1. As a result, the hedging motive against stochastic variation in the estimates shrinks. This can also explain the effect of ambiguity on hedging demand when utility is defined over terminal wealth. A comparison between Panel B and Panel C shows that the impact of ambiguity turns out to be stronger in the case of utility over intermediate consumption than in the case of utility over terminal wealth. This is because ambiguity influences the intertemporal hedging demand mainly through intermediate consumption, as shown in the optimal portfolio formula (17). Interestingly, for a highly risk-averse investor with utility over terminal wealth, ambiguity slightly increases hedging demand when  $\pi$  is very low. As discussed above, a highly risk-averse investor fears even worse states when the current estimate of the unobservable state is already low. Ambiguity further strengthens this concern and thus enhances the hedging motives.

[Insert Table 4 here]

Table 5 reports the fraction of the optimal allocation to the risky asset due to hedging concerns. Obviously, ambiguity increases the relative importance of the intertemporal hedging demand. For example, the ratio of hedging demand to the optimal demand rises from 15% to 20% for  $\gamma = 5$  and  $\pi = 0.5$  when utility is defined over terminal wealth and from 26% to 42% when utility is over consumption. Within the ambiguity-adjusted myopic component, the ambiguity aversion parameter appears additively to the conditional estimate of the market price of risk. This implies a first-order

effect of ambiguity on myopic demand. The effect on hedging demand, however, is of second order as indicated in both Table 4 and Figure 1. As a result, an ambiguity-averse investor steers his portfolio composition toward the intertemporal hedging component while away from the myopic component.

[Insert Table 5 here]

Figure 2 summarizes the horizon dependence of the optimal allocation to the risky asset for various estimates of the unobservable state. The coefficient of relative risk aversion is 5. In each graph, the cases of both utility over consumption and utility over terminal wealth are plotted for the horizon  $T$  ranging from 1 to 20. Furthermore, the optimal demand for the risky asset for an expected utility investor without ambiguity and that for an ambiguity-averse investor with  $\kappa = 0.4$  are also plotted in each graph. Figure 2 shows that the horizon effect largely relies on the unobservable-state estimate. When the current estimate approaches the boundaries of the interval  $[0, 1]$ , the optimal allocation to the risky asset varies less in response to changes in the horizon. The reason is that investors are fairly sure about the unobservable-state estimate and thus are reluctant to engage in hedging even with long horizons. When the estimate is near the middle of the interval  $[0, 1]$ , the horizon effect is sizable because the role of incomplete information risk becomes significant. As the horizon increases, the investor allocates significantly more of his wealth to hedge against stochastic variation in the estimates of the unobservable state. The horizon profile is flatter for utility over consumption than for utility over terminal wealth. This is because when the horizon increases, an investor maximizing utility over consumption can adjust consumption decisions in response to an increase in the amount of remaining fluctuations in the investment opportunity set. As a result, hedging demand responds less drastically to changes in the horizon. Relative to the expected-utility investor without ambiguity, the ambiguity-averse investor has a similar pattern of horizon dependence, except that the total allocations to the risky asset decline to lower levels.

[Insert Figure 2 here]

Finally, Table 6 and Table 7 summarize the impacts of parameters  $\mu_H, \mu_L$  and  $\sigma_S$ , where the coefficient of risk aversion is 5. Table 6 shows that as the disparity between the low-regime mean return and the high-regime mean return shrinks, both the size of hedging demand and its relative importance decrease. The reason is that an innovation in returns yields less useful information about the unobservable state when the low regime and the high regime become more difficult to distinguish from each other, that is, when the signal-to-noise ratio declines. As a consequence, the scope of incomplete information risk is limited, resulting in lower hedging demand. In addition, an increase in the return volatility parameter leads to a smaller size of hedging demand and also lowers its relative importance. A high  $\sigma_S$  implies that information quality is poor. In this case, an investor relies less on innovations in returns when updating beliefs because the signals are noisy. The role of incomplete information risk is therefore underplayed. The mean-reverting effect in filtering the unobservable state then becomes dominant, thereby dampening hedging demand. Nevertheless, none of those

effects is comparable to the effect of ambiguity on a similar basis. The parameters  $\mu_H, \mu_L$  and  $\sigma_S$  affect the intertemporal hedging demand by altering the magnitude of incomplete information risk, while ambiguity influences hedging demand by changing the probability law governing the dynamics of the unobservable-state estimates. This also constitutes an important difference from the i.i.d. setting in which the observational equivalence holds and the effect of ambiguity is similar to that of changing the mean return and/or return volatility.

[Insert Table 6 and Table 7 here]

## 4 Conclusion

In this paper, we have examined a continuous-time intertemporal consumption and portfolio choice problem for an ambiguity-averse investor when the expected returns of a risky asset are unobservable and follow a continuous-time Markov chain. The investor lacks confidence in the reference model that characterizes a Markovian representation of the original incomplete information economy. Facing an optimization problem with RMPU preferences, the investor considers the worst-case model among a family of alternative models surrounding the reference model. We find that ambiguity with respect to the reference model lowers the optimal demand for the risky asset and also generates an ambiguity-driven hedging demand that acts in the opposite direction relative to the hedging component against incomplete information risk. Although the intertemporal hedging demand is mitigated, its relative importance in the optimal portfolio increases.

There are several ways to extend this paper. For example, future research can take into account return predictability and study dynamic asset allocation problems in which an ambiguity-averse investor learns about regime switching predictability. In addition, future works can also examine the implications of ambiguity in incomplete information economies with the recursive preferences in Epstein and Zin (1989) and Duffie and Epstein (1992) that further allow for the separation between risk aversion and intertemporal substitution.

## Appendix

**The Clark-Ocone Formula** The space of random variables for which Malliavin derivatives are defined is called  $\mathbb{D}^{1,2}$ .<sup>15</sup> Any random variable  $F \in \mathbb{D}^{1,2}$  can be decomposed as

$$F = \mathbb{E}(F) + \int_t^T \mathbb{E}[D_t F | \mathcal{F}_t] dB_t$$

where  $\mathcal{F}_t$  represents the information filtration generated by the Brownian motion  $B$  up to time  $t$ .

**Proof of Proposition 1** The proof employs the guess-and-verify method to show that in the optimum, the density generator  $\theta$  is given by  $\theta_t^* = \kappa$  for all  $t \in [0, T]$  and to derive the optimal consumption and portfolio rules. First, we conjecture  $\theta_t^* = \kappa$  for all  $t \in [0, T]$  and proceed to derive the solution to the volatility of the utility process  $\sigma_t^V$ . Then we verify that  $\sigma_t^V > 0$  holds for all  $t \in [0, T]$ , which is also a verification of  $\theta_t^* = \kappa$  for all  $t \in [0, T]$ , in accordance with (9). The explicit solutions to the optimal consumption and portfolio choice conditioning on  $\theta^* = \kappa$  follow naturally.

The first-order conditions of the Lagrangian problem (13) are

$$e^{-\rho t} z_t^\kappa (C_t^*)^{-\gamma} = y \xi_t \quad \text{and} \quad e^{-\rho t} z_t^\kappa (W_T^*)^{-\gamma} = y \xi_T \quad (18)$$

where  $z_t^\kappa$  is given by

$$z_t^\kappa = \exp\left(-\frac{1}{2} \int_0^t \kappa^2 ds - \int_0^t \kappa d\hat{B}_s\right).$$

The optimal consumption and terminal wealth in (15) can be obtained from the first-order conditions. Substituting (18) into the budget constraint (12) yields the Lagrange multiplier  $y$  in (16).

Applying Ito's lemma to the optimal consumption in (15) yields the following dynamics of the consumption process

$$\frac{dC_t^*}{C_t^*} = \mu_t^C dt + \sigma_t^C d\hat{B}_t$$

where

$$\begin{aligned} \mu_t^C &= \frac{1}{\gamma} (r - \rho) + \frac{1}{2} (1 + \gamma) (\sigma_t^C)^2 + \sigma_t^C \kappa \\ \sigma_t^C &= \frac{1}{\gamma} (\hat{\nu}_t - \kappa). \end{aligned}$$

By homogeneity, we conjecture that the utility process along the optimal path has the following form

$$V_t = \frac{(C_t^*)^{1-\gamma}}{1-\gamma} A_t$$

with  $V_T = \frac{(W_T^*)^{1-\gamma}}{1-\gamma} A_T$ , where  $A_t$  satisfies the BSDE:

$$\frac{dA_t}{A_t} = \mu_t^A dt + \sigma_t^A d\hat{B}_t, \quad A_T = 1.$$

<sup>15</sup> Oksendal(1997) provides a concise introduction to Malliavin calculus. Interested readers can also refer to Nualart (1995) for a full treatment.

It follows from Ito's lemma that  $V_t$  satisfies the BSDE:

$$dV_t = \mu_t^V dt + \sigma_t^V d\hat{B}_t, \quad V_T = \frac{(W_T^*)^{1-\gamma}}{1-\gamma}$$

with  $\sigma_t^V = V_t [(1-\gamma)\sigma_t^C + \sigma_t^A]$ . The derivation of  $\mu_t^V$  is irrelevant to the proof and thus omitted. To obtain  $\sigma_t^V$  and  $\sigma_t^A$  in explicit forms, we first derive the optimal portfolio rule  $\alpha_t^*$  as a function of both  $\sigma_t^C$  and  $\sigma_t^A$  and then characterize  $\alpha_t^*$  in terms of the Malliavin derivatives and stochastic integrals. The explicit expression of  $\sigma_t^A$  can be immediately obtained. Then,  $\sigma_t^V$  can be solved explicitly, and its sign can be readily determined.

Multiply both sides of the first-order condition for consumption in (18) by  $C_t^*$  and integrate over the product space  $dt \otimes d\mathbb{P}$  to obtain

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T e^{-\rho t} (C_t^*)^{1-\gamma} z_t^\kappa dt \right] = y \mathbb{E}^{\mathbb{P}} \left[ \int_0^T \xi_t C_t^* dt \right].$$

Similarly, one can obtain  $\mathbb{E}^{\mathbb{P}} \left[ e^{-\rho T} (W_T^*)^{1-\gamma} z_T^\kappa \right] = y \mathbb{E}^{\mathbb{P}} [\xi_T W_T^*]$ . It follows from the complementary slackness condition (12), the equality  $y = (C_0^*)^{-\gamma}$  and the definition of RMPU (8) that the following equality holds

$$W_0 = (1-\gamma)(C_0^*)^{-\gamma} V_0.$$

In the same way, one can deduce for all  $t \in [0, T]$

$$W_t^* = (1-\gamma)(C_t^*)^{-\gamma} V_t. \quad (19)$$

Applying Ito's Lemma to (19) and matching the volatility term with that in the budget constraint (4) yield an expression for the optimal portfolio  $\alpha_t^*$

$$\begin{aligned} \alpha_t^* &= \frac{\sigma_t^C + \sigma_t^A}{\sigma_S} \\ &= \frac{\hat{\mu}_t - r}{\gamma \sigma_S^2} - \frac{1}{\gamma} \frac{\kappa}{\sigma_S} + \frac{\sigma_t^A}{\sigma_S}. \end{aligned} \quad (20)$$

The martingale representation theorem implies that wealth at time  $t$ ,  $W_t$ , is given by

$$\xi_t W_t^* = \mathbb{E}_t^{\mathbb{P}} \left[ \int_t^T \xi_s C_s^* ds + \xi_T W_T^* \right]. \quad (21)$$

By Ito's lemma, the volatility of the left-hand side of (21) is  $-\xi_t W_t^* \hat{\nu}_t + \xi_t W_t^* \alpha_t^* \sigma_S$ . By the Clark-Ocone formula, the volatility of the right-hand side is given by  $\mathbb{E}_t^{\mathbb{P}} \left[ D_t \left( \int_t^T \xi_s C_s^* ds + \xi_T W_T^* \right) \right]$ . The two volatilities must be equal, leading to the following equality

$$\xi_t W_t^* \alpha_t^* = \frac{1}{\sigma_S} \xi_t W_t^* \hat{\nu}_t + \frac{1}{\sigma_S} \mathbb{E}_t^{\mathbb{P}} \left[ D_t \left( \int_t^T \xi_s C_s^* ds + \xi_T W_T^* \right) \right]. \quad (22)$$

In (22), the Malliavin derivative  $D_t \left( \int_t^T \xi_s C_s^* ds + \xi_T W_T^* \right)$  can be computed as follows. First, by linearity and exchangeability between the Malliavin derivative and the ordinary Lebesgue integral,



we have

$$D_t \left( \int_t^T \xi_s C_s^* ds + \xi_T W_T^* \right) = \int_t^T D_t (\xi_s C_s^*) ds + D_t (\xi_T W_T^*). \quad (23)$$

The second term on the right-hand side of (23) can be written as  $D_t (\xi_T W_T^*) = W_T^* D_t \xi_T + \xi_T D_t W_T^*$  where  $D_t W_T^*$  is computed by the chain rule of Malliavin calculus:

$$D_t W_T^* = \frac{W_T^*}{\gamma} \left( \frac{z_T^\kappa}{\xi_T} \right)^{-1} D_t \left( \frac{z_T^\kappa}{\xi_T} \right) = \frac{W_T^*}{\gamma} \left( \frac{z_T^\kappa}{\xi_T} \right)^{-1} \left( \frac{1}{\xi_T} D_t z_T^\kappa - \frac{z_T^\kappa}{\xi_T^2} D_t \xi_T \right).$$

The term  $D_t z_T^\kappa$  is further computed as

$$\begin{aligned} D_t z_T^\kappa &= D_t \exp \left( -\frac{1}{2} \int_0^T \kappa^2 ds - \int_0^T \kappa d\hat{B}_s \right) \\ &= z_T^\kappa D_t \left( -\frac{1}{2} \int_0^T \kappa^2 ds - \int_0^T \kappa d\hat{B}_s \right) = -\kappa z_T^\kappa 1_{t \leq T} \end{aligned}$$

where  $1_{t \leq T}$  is an indicator function. Rearranging the terms and assuming  $t \leq T$  yield

$$D_t (\xi_T W_T^*) = -\frac{\kappa}{\gamma} \xi_T W_T^* + \frac{\gamma-1}{\gamma} W_T^* D_t \xi_T. \quad (24)$$

Similarly, for  $t \leq s$ , one can derive

$$D_t (\xi_s C_s^*) = -\frac{\kappa}{\gamma} \xi_s C_s^* + \frac{\gamma-1}{\gamma} W_s^* D_t \xi_s. \quad (25)$$

Substituting (25) and (24) into (23), rearranging terms and applying the equality (22) give us

$$\alpha_t^* = \frac{\hat{\nu}_t}{\sigma_S} - \frac{\kappa}{\gamma \sigma_S} + \frac{\gamma-1}{\gamma} \frac{1}{\sigma_S W_t^*} \mathbb{E}_t^{\mathbb{P}} \left[ \int_t^T C_s^* \frac{D_t \xi_s}{\xi_t} ds + W_T^* \frac{D_t \xi_T}{\xi_t} \right]$$

where  $D_t \xi_s, t \leq s$  is computed as

$$D_t \xi_s = -\xi_s \left( \hat{\nu}_t + \int_t^s (d\hat{B}_\tau + \hat{\nu}_\tau d\tau) D_t \hat{\nu}_\tau \right).$$

The optimal portfolio  $\alpha_t^*$  is given by

$$\begin{aligned} \alpha_t^* &= \frac{\hat{\nu}_t - \kappa}{\gamma \sigma_S} \\ &+ \frac{1-\gamma}{\gamma} \frac{1}{\sigma_S W_t^*} \mathbb{E}_t^{\mathbb{P}} \left[ \int_t^T \frac{\xi_s}{\xi_t} C_s^* \left( \int_t^s D_t \hat{\nu}_\tau (d\hat{B}_\tau + \hat{\nu}_\tau d\tau) \right) ds + \frac{\xi_T}{\xi_t} W_T^* \int_t^T (d\hat{B}_s + \hat{\nu}_s ds) D_t \hat{\nu}_s \right] \end{aligned}$$

which can be rewritten using the equivalent martingale measure  $\tilde{\mathbb{P}}$  as

$$\alpha_t^* = \frac{\hat{\mu}_t - r}{\gamma \sigma_S^2} - \frac{\kappa}{\gamma \sigma_S} + \frac{1-\gamma}{\gamma} \frac{e^{rt}}{\sigma_S W_t^*} \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[ \int_t^T e^{-rs} C_s^* \int_t^s (D_t \hat{\nu}_\tau d\tilde{B}_\tau) ds + e^{-rT} W_T^* \int_t^T D_t \hat{\nu}_s d\tilde{B}_s \right] \quad (26)$$

where  $\tilde{B}_t = \hat{B}_t + \int_0^t \hat{\nu}_s ds$ . Comparing (26) to (20), we have

$$\sigma_t^A = \frac{1-\gamma}{\gamma} \frac{e^{rt}}{W_t^*} \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[ \int_t^T e^{-rs} C_s^* \int_t^s (D_t \hat{\nu}_\tau d\tilde{B}_\tau) ds + e^{-rT} W_T^* \int_t^T D_t \hat{\nu}_s d\tilde{B}_s \right].$$

Thus,  $\sigma_t^V$  is given by

$$\sigma_t^V = \frac{(1-\gamma)V_t}{\gamma} \left[ (\hat{\nu}_t - \kappa) + \frac{e^{rt}}{W_t^*} \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[ \int_t^T e^{-rs} C_s^* \int_t^s (D_t \hat{\nu}_\tau d\tilde{B}_\tau) ds + e^{-rT} W_T^* \int_t^T D_t \hat{\nu}_s d\tilde{B}_s \right] \right].$$

Because  $\mathbb{E}_t^{\tilde{\mathbb{P}}} \left[ \int_t^T e^{-rs} C_s^* \int_t^s (D_t \hat{\nu}_\tau d\tilde{B}_\tau) ds + e^{-rT} W_T^* \int_t^T D_t \hat{\nu}_s d\tilde{B}_s \right] > 0$  and  $W_t^* > 0$  for all  $t \in [0, T]$  and the condition (10) implies that  $\nu_t - \kappa \geq 0$  always holds true,  $\sigma_t^V$  is strictly positive for all  $t \in [0, T]$  when either  $\gamma > 1$  or  $0 < \gamma < 1$  holds. As a result, one can verify by (9) that  $\theta_t^* = \kappa$  for all  $t \in [0, T]$  and the optimal consumption and portfolio choice are given by (15) and (17). This completes the proof.  $\square$

**Proof of Corollary 1** For  $\gamma = 1$ , the utility process along the optimal path has the form  $V_t = \log C_t^* + \log A_t$ . By Ito's lemma,  $\sigma_t^V$  is given by  $\sigma_t^V = \sigma_t^C + \sigma_t^A$ , where  $\sigma_t^C = \hat{\nu}_t - \kappa$ . The optimal portfolio  $\alpha_t^*$  can be expressed as

$$\alpha_t^* = \frac{\hat{\mu}_t - r}{\sigma_S^2} - \frac{\kappa}{\sigma_S} + \frac{\sigma_t^A}{\sigma_S}.$$

It follows from  $\gamma = 1$  that

$$D_t(\xi_T W_T^*) = -\kappa \xi_T W_T^* \quad \text{and} \quad D_t(\xi_s C_s^*) = -\kappa \xi_s C_s^*$$

and that the optimal portfolio  $\alpha_t^*$  is given by

$$\alpha_t^* = \frac{\hat{\mu}_t - r}{\sigma_S^2} - \frac{\kappa}{\sigma_S}.$$

Thus,  $\sigma_t^A = 0$  and  $\sigma_t^V = \hat{\nu}_t - \kappa$ . Since  $\Pr(\pi_t > 0) = 1$  for  $0 \leq t \leq T$  under the condition  $\pi_0 > 0$  (see Lemma 9.3, Liptser and Shiryaev 2001), it follows that  $\sigma_t^V > 0$  for  $0 \leq t \leq T$ . Thus,  $\theta_t^* = \kappa$  for all  $t \in [0, T]$  and the optimal consumption and portfolio rules are those given in (15) and (17) by setting  $\gamma = 1$ . This completes the proof.  $\square$

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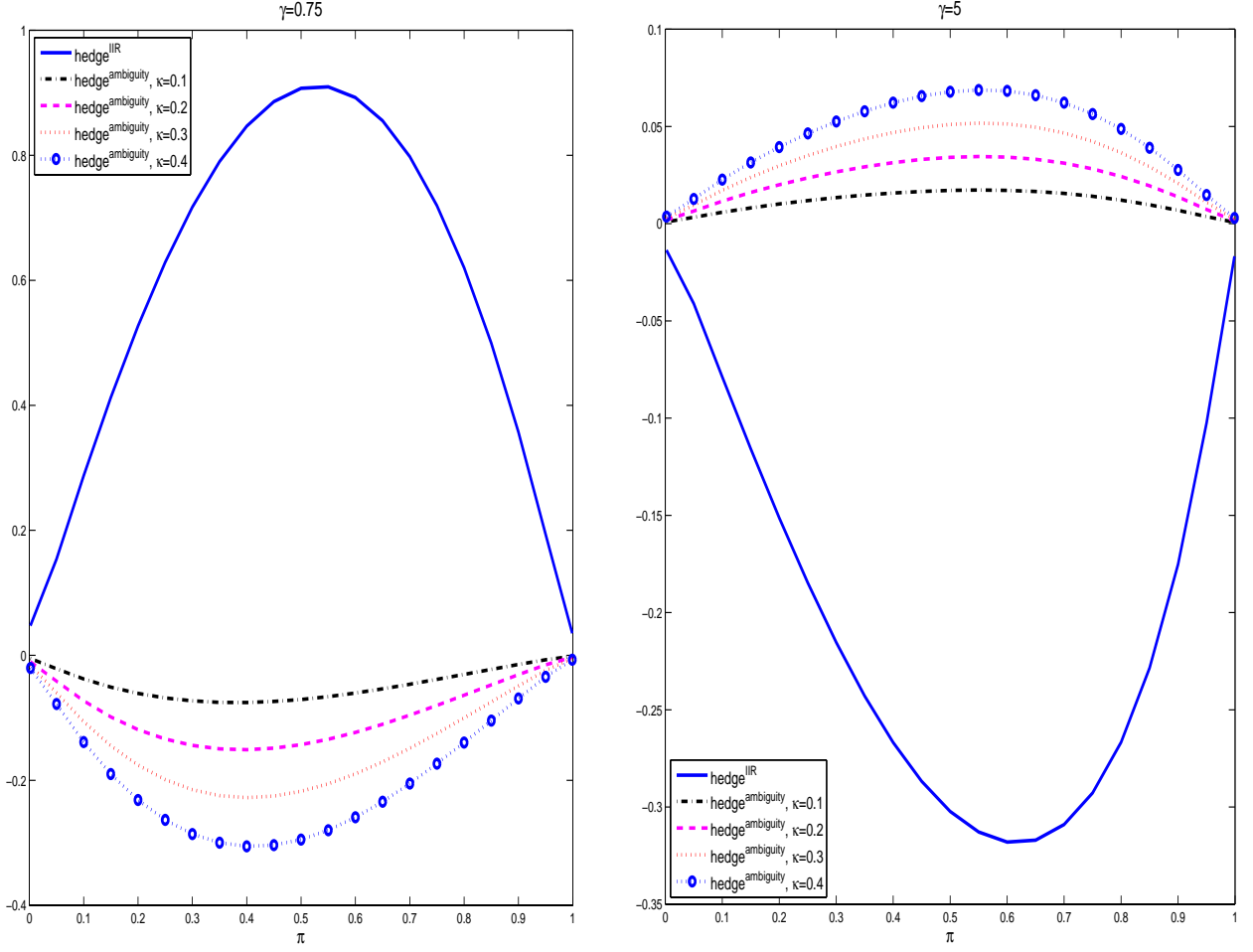


Figure 1: This figure plots the hedging allocation to the risky asset against incomplete information risk,  $hedge^{IRR}$ , and the hedging allocation against ambiguity,  $hedge^{ambiguity}$ , as functions of the state belief  $\pi$ , which is the probability of the current state being in the regime with mean return  $\mu_H$ .  $hedge^{IRR}$  is calculated from the hedging term in the optimal portfolio formula by assuming  $\kappa = 0$ .  $hedge^{ambiguity}$  is calculated by applying the optimal portfolio formula with a value of  $\kappa$  and then subtracting  $hedge^{IRR}$  from the hedging term in the formula. The investment horizon  $T$  is 10. The coefficients of risk aversion  $\gamma$  are 0.75 (the left panel) and 5 (the right panel). Other parameters are  $\mu_H = 0.20$ ,  $\mu_L = 0.06$ ,  $\sigma_S = 0.10$ ,  $r = 0.02$ ,  $\lambda_0 = 2.00$ ,  $\lambda_1 = 1.00$  and  $\rho = 0.05$ .

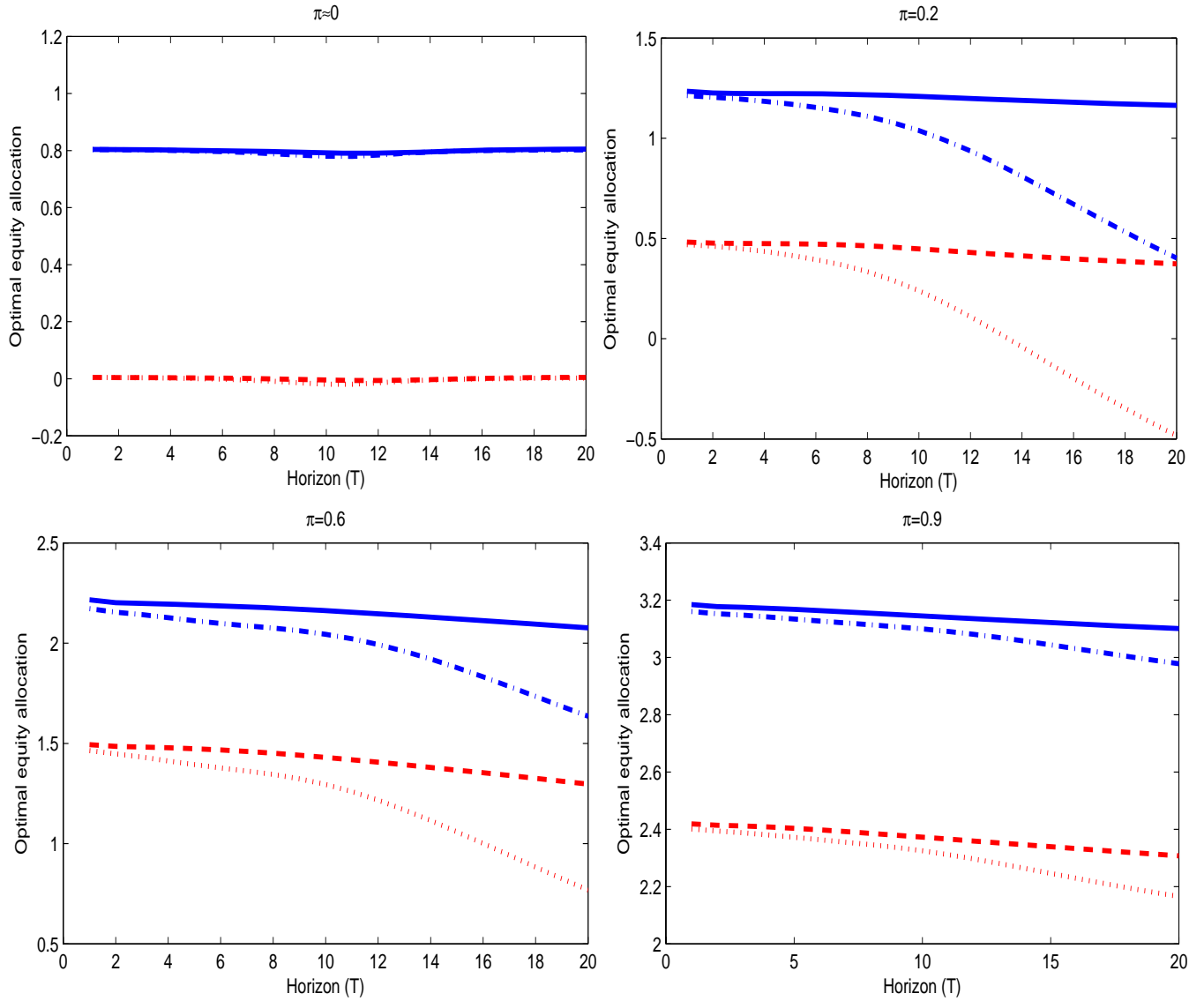


Figure 2: This figure plots the optimal proportion of wealth allocated to the risky asset for horizons ranging from  $T = 1$  to  $T = 20$ . The coefficient of risk aversion is 5. Four cases are plotted:  $\pi \approx 0$ ,  $\pi = 0.2$ ,  $\pi = 0.6$  and  $\pi = 0.9$ . Figure legend: interim consumption and  $\kappa = 0$ : solid line; terminal wealth and  $\kappa = 0$ : dash-dot line; interim consumption and  $\kappa = 0.4$ : dashed line; terminal wealth and  $\kappa = 0.4$ : dotted line.



Table 1: This table summarizes myopic demand for the risky asset, hedging demand with interim consumption and hedging demand with terminal wealth for various coefficients of relative risk aversion when the ambiguity aversion parameter  $\kappa$  is set equal to 0. The results are shown for different values of  $\pi$  ranging from 0 to 1, where  $\pi \approx 0$  is approximated with  $\pi = 0.002$  and  $\pi \approx 1$  is approximated with  $\pi = 0.999$ . The investment horizon  $T$  is 10. The values of other parameters are  $\mu_H = 0.20, \mu_L = 0.06, \sigma_S = 0.10, r = 0.02, \lambda_0 = 2.00, \lambda_1 = 1.00$  and  $\rho = 0.05$ . The myopic demand for the risky asset is defined by  $\alpha_{myopic}^* = \frac{\hat{\mu}-r}{\gamma\sigma_S^2} - \frac{\kappa}{\gamma\sigma_S}$  where  $\hat{\mu} = \mu_H\pi + \mu_L(1-\pi)$ . Hedging demand with interim consumption is given by  $\alpha_{hedge}^* = \frac{1-\gamma}{\gamma} \frac{1}{\sigma_S W_0^*} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T e^{-rs} C_s^* \int_0^s (D_0 \hat{\nu}_\tau d\tilde{B}_\tau) ds + e^{-rT} W_T^* \int_0^T D_0 \hat{\nu}_s d\tilde{B}_s \right]$ , and hedging demand with terminal wealth is given by  $\alpha_{hedge}^* = \frac{1-\gamma}{\gamma} \frac{1}{\sigma_S W_0^*} \mathbb{E}^{\mathbb{P}} \left[ e^{-rT} W_T^* \int_0^T D_0 \hat{\nu}_s d\tilde{B}_s \right]$  where  $W_0 = 1$ . The results are calculated based on the MDMC method with the number of simulations  $N = 20,000$ .

$\pi_H \backslash$	$\gamma = 0.75$	$\gamma = 1$	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 5$	$\gamma = 8$	$\gamma = 20$
Panel A: Myopic demand							
$\approx 0$	5.371	4.028	2.685	2.014	0.806	0.504	0.201
0.1	7.200	5.400	3.600	2.700	1.080	0.675	0.270
0.2	9.067	6.800	4.533	3.400	1.360	0.850	0.340
0.3	10.933	8.200	5.467	4.100	1.640	1.025	0.410
0.4	12.800	9.600	6.400	4.800	1.920	1.200	0.480
0.5	14.667	11.000	7.333	5.500	2.200	1.375	0.550
0.6	16.533	12.400	8.267	6.200	2.480	1.550	0.620
0.7	18.400	13.800	9.200	6.900	2.760	1.725	0.690
0.8	20.267	15.200	10.133	7.600	3.040	1.900	0.760
0.9	22.133	16.600	11.067	8.300	3.320	2.075	0.830
$\approx 1$	23.981	17.986	11.991	8.993	3.597	2.248	0.899
Panel B: Hedging demand (interim consumption)							
$\approx 0$	0.047	0.000	-0.019	-0.020	-0.014	-0.011	-0.008
0.1	0.288	0.000	-0.108	-0.116	-0.079	-0.064	-0.051
0.2	0.527	0.000	-0.204	-0.220	-0.151	-0.124	-0.097
0.3	0.717	0.000	-0.287	-0.312	-0.215	-0.175	-0.134
0.4	0.847	0.000	-0.352	-0.384	-0.267	-0.214	-0.159
0.5	0.907	0.000	-0.393	-0.433	-0.302	-0.241	-0.174
0.6	0.893	0.000	-0.406	-0.451	-0.318	-0.252	-0.179
0.7	0.798	0.000	-0.382	-0.429	-0.309	-0.246	-0.175
0.8	0.620	0.000	-0.315	-0.359	-0.267	-0.216	-0.157
0.9	0.357	0.000	-0.193	-0.224	-0.175	-0.145	-0.111
$\approx 1$	0.035	0.000	-0.019	-0.022	-0.017	-0.014	-0.010
Panel C: Hedging demand (terminal wealth)							
$\approx 0$	0.047	0.000	-0.021	-0.024	-0.026	-0.026	-0.026
0.1	0.288	0.000	-0.121	-0.143	-0.184	-0.204	-0.230
0.2	0.527	0.000	-0.228	-0.268	-0.322	-0.346	-0.376
0.3	0.717	0.000	-0.319	-0.374	-0.410	-0.421	-0.436
0.4	0.847	0.000	-0.390	-0.455	-0.452	-0.438	-0.424
0.5	0.908	0.000	-0.435	-0.506	-0.457	-0.416	-0.369
0.6	0.893	0.000	-0.447	-0.520	-0.436	-0.375	-0.304
0.7	0.799	0.000	-0.420	-0.491	-0.395	-0.330	-0.252
0.8	0.621	0.000	-0.345	-0.407	-0.332	-0.278	-0.215
0.9	0.358	0.000	-0.212	-0.254	-0.220	-0.192	-0.160
$\approx 1$	0.035	0.000	-0.021	-0.025	-0.019	-0.015	-0.010

Table 2: This table summarizes the consumption-to-wealth ratio for various coefficients of relative risk aversion when the ambiguity aversion parameter  $\kappa$  is set equal to 0. The results are shown for different values of  $\pi$  ranging from 0 to 1. The investment horizon  $T$  is 10 and initial wealth is  $W_0 = 1$ . Other parameters are  $\mu_H = 0.20, \mu_L = 0.06, \sigma_S = 0.10, r = 0.02, \lambda_0 = 2.00, \lambda_1 = 1.00$  and  $\rho = 0.05$ . The optimal consumption  $C_t^*$  is given by  $C_t^* = \left( \frac{e^{-\rho t} z_t^\kappa}{y \xi_t} \right)^{\frac{1}{\gamma}}$  where  $y$  is the Lagrange multiplier. The results are calculated based on the MDMC method with the number of simulations  $N = 20,000$ .

$\pi_H \backslash$	$\gamma = 0.75$	$\gamma = 1$	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 5$	$\gamma = 8$	$\gamma = 20$
$\approx 0$	0.007	0.118	0.236	0.243	0.178	0.151	0.124
0.1	0.007	0.118	0.233	0.240	0.176	0.150	0.123
0.2	0.007	0.118	0.235	0.242	0.177	0.151	0.124
0.3	0.007	0.118	0.237	0.245	0.179	0.152	0.124
0.4	0.007	0.118	0.240	0.248	0.180	0.153	0.125
0.5	0.007	0.118	0.242	0.251	0.181	0.154	0.125
0.6	0.006	0.118	0.245	0.254	0.183	0.155	0.126
0.7	0.006	0.118	0.248	0.257	0.185	0.156	0.127
0.8	0.006	0.118	0.252	0.261	0.187	0.157	0.127
0.9	0.006	0.118	0.255	0.266	0.189	0.159	0.128
$\approx 1$	0.006	0.118	0.254	0.265	0.189	0.159	0.128

Table 3: This table summarizes the consumption-to-wealth ratio for various coefficients of relative risk aversion and various values of the ambiguity aversion parameter  $\kappa$ . The investment horizon  $T$  is 10, and initial wealth is  $W_0 = 1$ . Other parameters are  $\mu_H = 0.20, \mu_L = 0.06, \sigma_S = 0.10, r = 0.02, \lambda_0 = 2.00, \lambda_1 = 1.00$  and  $\rho = 0.05$ . The results are calculated based on the MDMC method with the number of simulations  $N = 20,000$ .

$\pi_H \backslash \kappa$	$\gamma = 0.75$			$\gamma = 2$			$\gamma = 5$		
	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4
$\approx 0$	0.007	0.020	0.043	0.243	0.197	0.163	0.178	0.154	0.136
0.1	0.007	0.020	0.043	0.240	0.195	0.162	0.176	0.153	0.135
0.2	0.007	0.020	0.043	0.242	0.197	0.163	0.177	0.153	0.136
0.3	0.007	0.020	0.042	0.245	0.198	0.164	0.179	0.154	0.136
0.4	0.007	0.019	0.042	0.248	0.200	0.165	0.180	0.155	0.137
0.5	0.007	0.019	0.041	0.251	0.202	0.166	0.181	0.157	0.138
0.6	0.006	0.018	0.040	0.254	0.204	0.168	0.183	0.158	0.139
0.7	0.006	0.018	0.039	0.257	0.207	0.170	0.185	0.159	0.140
0.8	0.006	0.018	0.039	0.261	0.210	0.172	0.187	0.161	0.141
0.9	0.006	0.017	0.038	0.266	0.213	0.174	0.189	0.163	0.143
$\approx 1$	0.006	0.017	0.038	0.265	0.212	0.173	0.189	0.162	0.143

Table 4: This table summarizes myopic demand for the risky asset, hedging demand with interim consumption and hedging demand with terminal wealth for various coefficients of relative risk aversion and various values of the ambiguity aversion parameter  $\kappa$ . The investment horizon  $T$  is 10. Other parameters are  $\mu_H = 0.20$ ,  $\mu_L = 0.06$ ,  $\sigma_S = 0.10$ ,  $r = 0.02$ ,  $\lambda_0 = 2.00$ ,  $\lambda_1 = 1.00$  and  $\rho = 0.05$ . The results are calculated based on the MDMC method with the number of simulations  $N = 20,000$ .

$\pi_H \setminus \kappa$	$\gamma = 0.75$			$\gamma = 2$			$\gamma = 5$		
	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4
Panel A: Myopic demand									
$\approx 0$	5.371	2.704	0.037	2.014	1.014	0.014	0.806	0.406	0.006
0.1	7.200	4.533	1.867	2.700	1.700	0.700	1.080	0.680	0.280
0.2	9.067	6.400	3.733	3.400	2.400	1.400	1.360	0.960	0.560
0.3	10.933	8.267	5.600	4.100	3.100	2.100	1.640	1.240	0.840
0.4	12.800	10.133	7.467	4.800	3.800	2.800	1.920	1.520	1.120
0.5	14.667	12.000	9.333	5.500	4.500	3.500	2.200	1.800	1.400
0.6	16.533	13.867	11.200	6.200	5.200	4.200	2.480	2.080	1.680
0.7	18.400	15.733	13.067	6.900	5.900	4.900	2.760	2.360	1.960
0.8	20.267	17.600	14.933	7.600	6.600	5.600	3.040	2.640	2.240
0.9	22.133	19.467	16.800	8.300	7.300	6.300	3.320	2.920	2.520
$\approx 1$	23.981	21.315	18.648	8.993	7.993	6.993	3.597	3.197	2.797
Panel B: Hedging demand (interim consumption)									
$\approx 0$	0.047	0.037	0.027	-0.020	-0.016	-0.012	-0.014	-0.012	-0.010
0.1	0.288	0.215	0.149	-0.116	-0.091	-0.067	-0.079	-0.067	-0.056
0.2	0.527	0.408	0.296	-0.220	-0.177	-0.135	-0.151	-0.131	-0.112
0.3	0.717	0.573	0.431	-0.312	-0.256	-0.201	-0.215	-0.189	-0.163
0.4	0.847	0.696	0.541	-0.384	-0.321	-0.258	-0.267	-0.235	-0.205
0.5	0.907	0.764	0.613	-0.433	-0.367	-0.301	-0.302	-0.268	-0.234
0.6	0.893	0.769	0.633	-0.451	-0.388	-0.323	-0.318	-0.284	-0.250
0.7	0.798	0.702	0.593	-0.429	-0.374	-0.317	-0.309	-0.278	-0.247
0.8	0.620	0.556	0.481	-0.359	-0.317	-0.273	-0.267	-0.242	-0.218
0.9	0.357	0.326	0.288	-0.224	-0.201	-0.176	-0.175	-0.161	-0.147
$\approx 1$	0.035	0.032	0.028	-0.022	-0.020	-0.017	-0.017	-0.015	-0.014
Panel C: Hedging demand (terminal wealth)									
$\approx 0$	0.047	0.037	0.028	-0.024	-0.021	-0.019	-0.026	-0.025	-0.025
0.1	0.288	0.216	0.152	-0.143	-0.125	-0.112	-0.184	-0.184	-0.186
0.2	0.527	0.411	0.301	-0.268	-0.235	-0.212	-0.322	-0.320	-0.321
0.3	0.717	0.577	0.439	-0.374	-0.329	-0.296	-0.410	-0.403	-0.400
0.4	0.847	0.700	0.550	-0.455	-0.401	-0.358	-0.452	-0.437	-0.427
0.5	0.908	0.769	0.622	-0.506	-0.446	-0.396	-0.457	-0.434	-0.417
0.6	0.893	0.774	0.643	-0.520	-0.459	-0.406	-0.436	-0.408	-0.385
0.7	0.799	0.706	0.601	-0.491	-0.435	-0.385	-0.395	-0.368	-0.343
0.8	0.621	0.559	0.487	-0.407	-0.364	-0.324	-0.332	-0.309	-0.288
0.9	0.358	0.328	0.292	-0.254	-0.230	-0.207	-0.220	-0.207	-0.195
$\approx 1$	0.035	0.032	0.029	-0.025	-0.022	-0.020	-0.019	-0.017	-0.016

Table 5: This table summarizes the ratios of hedging demand with utility over interim consumption and hedging demand with utility over terminal wealth to the optimal proportion of wealth allocated to the risky asset for various coefficients of relative risk aversion and various values of the ambiguity aversion parameter  $\kappa$ . The investment horizon  $T$  is 10. Other parameters are  $\mu_H = 0.20, \mu_L = 0.06, \sigma_S = 0.10, r = 0.02, \lambda_0 = 2.00, \lambda_1 = 1.00$  and  $\rho = 0.05$ . The ratio is defined by  $\alpha_{hedge}^* / (\alpha_{myopic}^* + \alpha_{hedge}^*)$  where  $\alpha_{hedge}^*$  represents hedging demand with utility defined either over interim consumption or over terminal wealth. The results are calculated based on the MDMC method with the number of simulations  $N = 20,000$ .

$\pi_H \setminus \kappa$	$\gamma = 0.75$			$\gamma = 2$			$\gamma = 5$		
	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4
Panel A: Ratio of hedging demand (interim consumption) to optimal demand (%)									
$\approx 0$	0.88	1.35	42.10	-1.00	-1.60	-600.00	-1.77	-3.05	250.00
0.1	3.84	4.52	7.38	-4.49	-5.66	-10.58	-7.89	-10.93	-25.00
0.2	5.49	5.99	7.34	-6.92	-7.96	-10.67	-12.49	-15.80	-25.00
0.3	6.16	6.48	7.15	-8.24	-9.00	-10.58	-15.09	-17.98	-24.08
0.4	6.20	6.42	6.76	-8.70	-9.23	-10.15	-16.15	-18.29	-22.40
0.5	5.83	5.99	6.16	-8.55	-8.88	-9.41	-15.91	-17.49	-20.07
0.6	5.12	5.26	5.35	-7.84	-8.06	-8.33	-14.71	-15.81	-17.48
0.7	4.16	4.27	4.34	-6.63	-6.77	-6.92	-12.61	-13.35	-14.42
0.8	2.97	3.06	3.12	-4.96	-5.05	-5.12	-9.63	-10.09	-10.78
0.9	1.59	1.65	1.69	-2.77	-2.83	-2.87	-5.56	-5.84	-6.19
$\approx 1$	0.15	0.15	0.15	-0.25	-0.25	-0.24	-0.47	-0.47	-0.50
Panel B: Ratio of hedging demand (terminal wealth) to optimal demand (%)									
$\approx 0$	0.88	1.36	42.55	-1.21	-2.11	380.00	-3.33	-6.56	131.58
0.1	3.84	4.55	7.52	-5.59	-7.94	-19.05	-20.54	-37.10	-197.87
0.2	5.49	6.03	7.46	-8.56	-10.85	-17.85	-31.02	-50.00	-134.31
0.3	6.16	6.52	7.27	-10.04	-11.87	-16.41	-33.33	-48.15	-90.91
0.4	6.21	6.46	6.86	-10.47	-11.80	-14.66	-30.79	-40.35	-61.62
0.5	5.83	6.02	6.25	-10.13	-11.00	-12.76	-26.22	-31.77	-42.42
0.6	5.13	5.28	5.43	-9.15	-9.68	-10.70	-21.33	-24.40	-29.73
0.7	4.16	4.30	4.40	-7.66	-7.96	-8.53	-16.70	-18.47	-21.21
0.8	2.97	3.08	3.16	-5.66	-5.84	-6.14	-12.26	-13.26	-14.75
0.9	1.59	1.66	1.71	-3.16	-3.25	-3.40	-7.10	-7.63	-8.39
$\approx 1$	0.15	0.15	0.15	-0.28	-0.28	-0.29	-0.53	-0.53	-0.58

Table 6: This table summarizes myopic demand for the risky asset, hedging demand and the ratio of hedging demand with utility over interim consumption to the optimal proportion of wealth allocated to the risky asset for various mean returns of the two regimes. The risk aversion coefficient  $\gamma$  is 5 and the ambiguity aversion parameter  $\kappa$  is set equal to 0. The investment horizon  $T$  is 10. Other parameters are  $\sigma_S = 0.10$ ,  $r = 0.02$ ,  $\lambda_0 = 2.00$ ,  $\lambda_1 = 1.00$  and  $\rho = 0.05$ . The results are calculated based on the MDMC method with the number of simulations  $N = 20,000$ .

$\pi_H \setminus$	$\mu_H = 0.20, \mu_L = 0.03$	$\mu_H = 0.20, \mu_L = 0.06$	$\mu_H = 0.15, \mu_L = 0.06$
Panel A: Myopic demand			
$\approx 0$	0.207	0.806	0.804
0.1	0.540	1.080	0.980
0.2	0.880	1.360	1.160
0.3	1.220	1.640	1.340
0.4	1.560	1.920	1.520
0.5	1.900	2.200	1.700
0.6	2.240	2.480	1.880
0.7	2.580	2.760	2.060
0.8	2.920	3.040	2.240
0.9	3.260	3.320	2.420
$\approx 1$	3.597	3.597	2.598
Panel B: Hedging demand			
$\approx 0$	-0.015	-0.014	-0.005
0.1	-0.084	-0.079	-0.029
0.2	-0.169	-0.151	-0.054
0.3	-0.248	-0.215	-0.075
0.4	-0.314	-0.267	-0.092
0.5	-0.361	-0.302	-0.103
0.6	-0.385	-0.318	-0.106
0.7	-0.380	-0.309	-0.099
0.8	-0.336	-0.267	-0.082
0.9	-0.230	-0.175	-0.051
$\approx 1$	-0.022	-0.017	-0.005
Panel C: Ratio of hedging demand to optimal demand (%)			
$\approx 0$	-7.99	-1.72	-0.61
0.1	-18.52	-7.87	-3.02
0.2	-23.80	-12.52	-4.88
0.3	-25.54	-15.12	-5.97
0.4	-25.20	-16.14	-6.45
0.5	-23.46	-15.93	-6.43
0.6	-20.75	-14.71	-5.95
0.7	-17.28	-12.61	-5.07
0.8	-13.01	-9.62	-3.80
0.9	-7.58	-5.57	-2.13
$\approx 1$	-0.61	-0.47	-0.19

Table 7: This table summarizes the myopic demand for the risky asset, hedging demand and the ratio of hedging demand with utility over interim consumption to the optimal proportion of wealth allocated to the risky asset for different values of the return volatility parameter  $\sigma_S$ . The risk aversion coefficient  $\gamma$  is 5 and the ambiguity aversion parameter  $\kappa$  is set equal to 0. The investment horizon  $T$  is 10. Other parameters are  $\mu_H = 0.20, \mu_L = 0.06, r = 0.02, \lambda_0 = 2.00, \lambda_1 = 1.00$  and  $\rho = 0.05$ . The results are calculated based on the MDMC method with the number of simulations  $N = 20,000$ .

$\pi_H \setminus$	$\sigma = 0.05$	$\sigma = 0.10$	$\sigma = 0.20$
Panel A: Myopic demand			
$\approx 0$	3.222	0.806	0.201
0.1	4.320	1.080	0.270
0.2	5.440	1.360	0.340
0.3	6.560	1.640	0.410
0.4	7.680	1.920	0.480
0.5	8.800	2.200	0.550
0.6	9.920	2.480	0.620
0.7	11.040	2.760	0.690
0.8	12.160	3.040	0.760
0.9	13.280	3.320	0.830
$\approx 1$	14.389	3.597	0.899
Panel B: Hedging demand			
$\approx 0$	-0.128	-0.014	-0.001
0.1	-0.769	-0.079	-0.006
0.2	-1.562	-0.151	-0.011
0.3	-2.313	-0.215	-0.015
0.4	-2.978	-0.267	-0.019
0.5	-3.562	-0.302	-0.021
0.6	-4.084	-0.318	-0.022
0.7	-4.545	-0.309	-0.020
0.8	-4.759	-0.267	-0.016
0.9	-3.859	-0.175	-0.010
$\approx 1$	-0.342	-0.017	-0.001
Panel C: Ratio of hedging demand to optimal demand (%)			
$\approx 0$	-4.14	-1.72	-0.50
0.1	-21.66	-7.87	-2.20
0.2	-40.28	-12.52	-3.33
0.3	-54.46	-15.12	-3.90
0.4	-63.33	-16.14	-4.08
0.5	-67.99	-15.93	-3.97
0.6	-69.98	-14.71	-3.60
0.7	-69.97	-12.61	-3.01
0.8	-64.31	-9.62	-2.21
0.9	-40.96	-5.57	-1.22
$\approx 1$	-2.43	-0.47	-0.11