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**Working Paper**

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KOF Working Papers, No. 189

**Provided in Cooperation with:**

KOF Swiss Economic Institute, ETH Zurich

*Suggested Citation:* Conrad, Christian; Karanasos, Menelaos (2008) : Negative volatility spillovers in the unrestricted ECCG-GARCH model, KOF Working Papers, No. 189, ETH Zurich, KOF Swiss Economic Institute, Zurich,  
<https://doi.org/10.3929/ethz-a-005552237>

This Version is available at:

<https://hdl.handle.net/10419/50443>

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# KOF Working Papers

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# Negative Volatility Spillovers in the Unrestricted ECCC-GARCH Model\*

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February 2008

## Abstract

This paper considers a formulation of the extended constant or time-varying conditional correlation GARCH model which allows for volatility feedback of either sign, i.e., positive or negative. In the previous literature, negative volatility spillovers were ruled out by the assumption that all the coefficients of the model are non-negative, which is a sufficient condition for ensuring the positive definiteness of the conditional covariance matrix. In order to allow for negative feedback, we show that the positive definiteness of the conditional covariance matrix can be guaranteed even if some of the parameters are negative. Thus, we extend the results of Nelson and Cao (1992) and Tsai and Chan (2008) to a multivariate setting. For the bivariate case of order one we look into the consequences of adopting these less severe restrictions and find that the flexibility of the process is substantially increased. Our results are helpful for the model-builder, who can consider the unrestricted formulation as a tool for testing various economic theories.

**Keywords:** Inequality constraints, multivariate GARCH processes, volatility feedback.

**JEL Classification:** C32, C51, C52, C53.

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\*We would like to thank James Davidson, Martin Gassebner and Michael J. Lamla for their valuable suggestions. We have also benefited from the comments received from participants at the Seminar held in the Department of Economics at the University of Exeter.

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# 1 Introduction

The availability of multivariate GARCH (MGARCH) models is essential for enhancing our understanding of the relationships between the (co)volatilities of economic and financial time series. For recent surveys on MGARCH specifications and their practical importance in various areas such as asset pricing, portfolio selection and risk management see, e.g., Bauwens et al. (2006) and Silvennoinen and Teräsvirta (2007a). There are two major problems in specifying a valid multivariate model. First, in many formulations the number of parameters increases quickly with the dimensionality of the model and hence there is a need for parsimonious parameterizations. Second, multivariate specifications have to be parameterized in a way that guarantees the positive definiteness of the conditional covariance matrix almost surely at all points in time. In this article, our focus is on the latter issue. In most of the currently available specifications the positive definiteness is achieved by making rather restrictive assumptions. Either feedback between the conditional variances is completely ruled out, or it is a priori assumed to be positive. For a general class of MGARCH formulations, we derive the necessary and sufficient conditions for the positive definiteness of the covariance matrix and show that these weaker conditions allow for negative volatility feedback.

Bollerslev's (1990) diagonal constant conditional correlation (DICCC) GARCH specification is among the most commonly employed multivariate models and serves as a benchmark against which other formulations can be compared.<sup>1</sup> The diagonal structure implies that each variance behaves as a univariate GARCH process. Hence, the positivity of each conditional variance can be achieved by simply assuming that the parameters of each equation satisfy the conditions derived in Nelson and Cao (1992). However, the main drawback of the diagonal specification is that it rules out potential volatility feedback by assumption. As a consequence, the autocorrelation function of each of the squared observations of the multivariate formulation is no more flexible than that of a univariate GARCH process.

A generalized version of the DICCC model is defined by Jeantheau (1998) and termed extended CCC (ECCC)-GARCH by He and Teräsvirta (2004).<sup>2</sup> In this new formulation the off-diagonal elements of the matrices are allowed to take positive values (see also Ling and McAleer, 2003). Clearly, under this assumption positive volatility feedback is incorporated into the model. The results of He and Teräsvirta (2004) show that the squared observations of the extended specification have a remarkably richer correlation structure than those of the diagonal one and, hence, are more suited for replicating the manifold features of empirical autocorrelation functions that are observed in practice. In addition, Nakatani and Teräsvirta (2007) suggest a procedure for testing the hypothesis of a diagonal structure against the hypothesis of volatility feedback within the extended framework. Although their test does not place any restriction on the sign of the feedback

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<sup>1</sup>Other multivariate GARCH models are the CCC-ARCH process that appears in Cecchetti et al. (1988) or the VECH model of Bollerslev et al. (1988). A multivariate specification that can be viewed as a restricted version of the VECH model is the BEKK formulation defined in Engle and Kroner (1995). Scherrer and Ribaritis (2007) deal with issues of structure and parametrization of VECH and BEKK models.

<sup>2</sup>For empirical applications of the extended version see Nakatani and Teräsvirta (2007).

under the alternative, they impose the non-negativity constraints during the estimation and, hence, test the null of no volatility spillovers against the alternative of positive feedback. The assumption that only positive feedback is allowed for is tempting because positive constants and parameter matrices with non-negative coefficients are a sufficient condition for the positive definiteness of the conditional covariance matrix in the extended formulation. Since Bollerslev (1988) was the first to impose the non-negativity restrictions on the parameters of the univariate process, we will refer to the corresponding constraints for the multivariate model as Bollerslev’s conditions. He and Teräsvirta (2004) point out that despite the appealing theoretical properties of the extended formulation, more work is needed to find out how useful it is in practice.

In summary, the currently available specifications are somewhat extreme. That is, at the one extreme, the diagonal model assumes that there is no causal link between the volatilities, whereas, at the other extreme, the extended version (in which the parameters are restricted to being non-negative) only allows for a positive variance relationship. Leboit et al. (2003) point out that “although it can be useful to impose sensible restrictions for forecasting purposes, there is also the danger of employing restrictions that are strongly violated by the data”.

At this point one alternative process suggests itself. That is, we consider a formulation of the extended model which allows for feedback effects between the volatilities, which can be of either sign, positive or negative. We will term this specification the unrestricted ECCC (UECCC)-GARCH. A crucial problem concerns the identification of necessary and sufficient conditions for the unrestricted model to have positive definite conditional covariance matrix.

Nelson and Cao (1992) derived necessary and sufficient conditions for the positivity of the conditional variance of a univariate GARCH( $p, q$ ) model with  $p \leq 2$  and a sufficient one when  $p > 2$ . Recently, Tsai and Chan (2008) have shown that the latter condition is also necessary.<sup>3</sup> In this paper we show that the methodologies developed in Nelson and Cao (1992) and Tsai and Chan (2008) can be applied to the  $N$ -dimensional UECCC-GARCH. We do so by expressing each of the  $N$  conditional variances as a ‘univariate’ ARCH( $\infty$ ) specification. That is, each variance admits an infinite moving-average representation in terms of the  $N$  convolutions of the GARCH kernels and the corresponding squared errors. Hence all the  $N$  conditional variances are always non-negative if all the  $N^2$  kernels are non-negative. For practical implementation we suggest estimating first the unrestricted model and, in case some of the estimated parameters are negative, validating the necessary and sufficient conditions ex-post.<sup>4</sup>

He and Teräsvirta (2004) investigate the properties of the auto- and cross-correlations of the squared errors in the extended model of order (2, 2) under the assumption that all the parameters are constrained to be positive. The results in this research suggest that relaxing these constraints allows for more flexibility in the shape of the correlation

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<sup>3</sup>Non-negativity conditions for the fractionally integrated GARCH (FIGARCH) and the hyperbolic GARCH model can be found in Conrad and Haag (2006) and Conrad (2007) respectively. Finally, non-negativity conditions for ARMA processes are provided by Tsai and Chan (2006).

<sup>4</sup>In a recent article, Gouriéroux (2007) also deals with the problem of deriving positivity conditions for multivariate volatility processes. However, his results apply to bivariate autoregressive volatility specifications only.

functions. It thus appears that the unrestricted model (the one with possibly negative parameters) may characterize some features of the series that are not adequately captured by the restricted one.<sup>5</sup>

While empirical violations of the Bollerslev constraints might be thought of as resulting either from sampling error or model misspecification, we show that this is not necessarily the case. Interestingly, they may be in line with economic theory. For example, several theories predict either a positive or a negative association between the variability of inflation and output growth uncertainty (for more details and a review of the literature, see, Fountas et al., 2006).<sup>6</sup> Similarly, Caporin (2007) argues that an increase in stock return volatility may lead to a reduction in the variance of volume.<sup>7</sup>

The outline of the paper is as follows. Section 2 summarizes some basics concerning the notation used throughout the paper and introduces the unrestricted specification. The main results are stated in Section 3. Section 4 contains two empirical examples and the conclusions can be found in Section 5. Appendix A briefly discusses the second and fourth moment structure of the model, while Appendix B contains all the proofs.

## 2 The Model

### 2.1 Notation

Throughout the paper we will adhere to the following conventions. In order to distinguish matrices(vectors) from scalars the former are denoted by upper(lower)-case boldface symbols.  $\mathbf{X}(L) = [X_{ij}(L)]_{i,j=1,\dots,N}$  denotes an  $N \times N$  matrix polynomial in the lag operator  $L$ , i.e., with  $ij$ th element  $X_{ij}(L)$  being a polynomial of order  $p$ . Using standard notation  $\det[\mathbf{X}(L)]$  denotes the determinant,  $\mathbf{X}^{ji}(L)$  the  $\mathbf{X}(L)$  matrix without its  $j$ th row and  $i$ th column, and  $\text{adj}[\mathbf{X}(L)] = [X_{ij}^{\{a\}}(L)]_{i,j=1,\dots,N}$  denotes the adjoint of the  $\mathbf{X}(L)$  matrix. That is,  $X_{ij}^{\{a\}}(L) = (-1)^{i+j} \det[\mathbf{X}^{ji}(L)]$  is a scalar polynomial of order  $(N-1) \times p$ .

Furthermore, for square matrices  $\mathbf{X} = [x_{ij}]_{i,j=1,\dots,N} \in \mathbb{R}^{N \times N}$  we define  $\text{vec}(\mathbf{X})$  as the  $N^2 \times 1$  vector in which the columns of the square matrix  $\mathbf{X}$  are stacked one underneath the other. The symbols  $\odot$  and  $\otimes$  denote the Hadamard and Kronecker products respectively.

Moreover,  $\mathbf{I}_N$  denotes the  $N \times N$  identity matrix. The transpose and inverse of a matrix are denoted by  $\mathbf{X}'$  and  $\mathbf{X}^{-1}$  respectively. Column vectors will be denoted by lower-case letters, i.e.,  $\mathbf{x} = [x_i]_{i=1,\dots,N}$  (unless otherwise indicated) and a diagonal matrix with elements  $\{x_1, \dots, x_N\}$  will be denoted by  $\text{diag}\{\mathbf{x}\}$ . Also let  $\wedge$  and  $\mathbb{E}$  denote the elementwise exponentiation and expectation operator respectively. That is,  $\mathbf{X}^{\wedge k} = [x_{ij}^k]_{i,j=1,\dots,N}$ , and  $\mathbb{E}(\mathbf{X}) = [\mathbb{E}(x_{ij})]_{i,j=1,\dots,N}$ . Finally, the matrix (vector) inequality sign  $\mathbf{X} > 0$  ( $\mathbf{x} > 0$ ) represents element-by-element inequality.

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<sup>5</sup>Similar results were obtained by He and Teräsvirta (1999) and Conrad and Haag (2006) for the univariate GARCH and FIGARCH models respectively. Both show that the potential shapes of the autocorrelation functions are considerably more flexible under the necessary and sufficient conditions than under the ones which impose non-negative parameters.

<sup>6</sup>We will use the terms variance, variability, uncertainty and volatility interchangeably in the remainder of the text.

<sup>7</sup>Caporin (2007) suggests the exponential causality GARCH model, which allows for negative volatility feedback.

## 2.2 Conditional Variances

In this section we introduce the UECCC-GARCH( $p, q$ ) model. Consider the  $N$ -dimensional weakly stationary vector process:

$$\mathbf{y}_t = \mathbb{E}(\mathbf{y}_t | \mathcal{F}_{t-1}) + \boldsymbol{\varepsilon}_t, \quad (1)$$

where  $\mathcal{F}_{t-1} = \sigma(\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \dots)$  is the filtration generated by all the available information up through time  $t - 1$ .

We assume that the noise vector  $\boldsymbol{\varepsilon}_t$  is characterized by the relation:

$$\boldsymbol{\varepsilon}_t = \mathbf{z}_t \odot \mathbf{h}_t^{\wedge 1/2}, \quad (2)$$

where  $\mathbf{h}_t = [h_{it}]_{i=1, \dots, N}$  is  $\mathcal{F}_{t-1}$  measurable and the stochastic vector  $\mathbf{z}_t = [z_{it}]_{i=1, \dots, N}$  is independent and identically distributed (*i.i.d.*) with mean zero, finite second moments, and correlation matrix  $\mathbf{R} = [\rho_{ij}]_{i,j=1, \dots, N}$  such that  $\rho_{ij} = 1$  for  $i = j$ , and  $|\rho_{ij}| < 1$  for  $i \neq j$ . From the above equation it follows that  $\mathbb{E}(\boldsymbol{\varepsilon}_t | \mathcal{F}_{t-1}) = \mathbf{0}$  and  $\mathbf{H}_t = \mathbb{E}(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' | \mathcal{F}_{t-1}) = \text{diag} \left\{ \mathbf{h}_t^{\wedge 1/2} \right\} \mathbf{R} \text{diag} \left\{ \mathbf{h}_t^{\wedge 1/2} \right\}$ .

A major problem in specifying a valid multivariate process lies in choosing appropriate parametric specifications for  $\mathbf{h}_t$  such that  $\mathbf{H}_t$  is positive definite almost surely for all  $t$ . Positive definiteness of  $\mathbf{H}_t$  follows if, in addition to the correlation matrix  $\mathbf{R}$  being positive definite, the conditional variances  $h_{it}$ ,  $i = 1, \dots, N$ , are positive as well.

Next, we specify the parametric structure of  $\mathbf{h}_t$ . Let  $\boldsymbol{\mu} = [\mu_i]_{i=1, \dots, N}$  be a column vector with finite elements,  $\mathbf{B}(L) = \mathbf{I}_N - \sum_{l=1}^p \mathbf{B}^{(l)} L^l$  with  $\mathbf{B}^{(l)} = [b_{ij}^{(l)}]_{i,j=1, \dots, N}$  and  $\mathbf{A}(L) = \sum_{r=1}^q \mathbf{A}^{(r)} L^r$  with  $\mathbf{A}^{(r)} = [a_{ij}^{(r)}]_{i,j=1, \dots, N}$ .<sup>8</sup>

We define the vector GARCH ( $p, q$ ) process as follows:

$$\mathbf{B}(L)\mathbf{h}_t = \boldsymbol{\mu} + \mathbf{A}(L)\boldsymbol{\varepsilon}_t^{\wedge 2}. \quad (3)$$

Obviously, the above process nests Bollerslev's diagonal specification as a special case if we assume that  $\mathbf{A}(L)$  and  $\mathbf{B}(L)$  are diagonal matrices. Moreover, if it is assumed that all the parameters in expression (3) are positive, then the process corresponds to the ECCC-GARCH model. This assumption guarantees that the conditional variances  $h_{it}$ ,  $i = 1, \dots, N$ , are positive almost surely for all  $t$ . Although this condition is sufficient for ensuring the positivity of all the conditional variances, the results of Nelson and Cao (1992) and Tsai and Chan (2008) suggest that it is not necessary. In the following we will investigate whether it is possible to relax this condition by allowing some of the parameters in equation (3) to take negative values while the positivity of all the conditional variances is still ensured. As mentioned in the introduction, the resulting process is termed UECCC-GARCH.

**Remark 1** *Although in what follows we will focus our attention on the CCC process, our results hold for the time varying CC specification as well. This model (termed TVCC) differs only in allowing the correlation matrix to be time varying:  $\mathbf{R}_t = [\rho_{ij,t}]_{i,j=1, \dots, N}$ . For examples of such specifications see Engle (2002), Tse and Tsui (2002), Silvennoinen and Teräsvirta (2007b) and Bai and Chen (2008).*

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<sup>8</sup>For simplicity and without loss of generality, we assume that  $b_{ij}^{(p)} \neq 0$  and  $a_{ij}^{(q)} \neq 0$  for  $i, j = 1, \dots, N$ .



**Assumption 1 (Identifiability)** *The formulation of the  $N$ -dimensional vector UECCC-GARCH( $p, q$ ) model at the true values of the parameters is minimal if  $\mathbf{A}(L)$  and  $\mathbf{B}(L)$  satisfy the following conditions:*

1.  $\det[\mathbf{A}(L)] \neq 0$  and  $\det[\mathbf{B}(L)] \neq 0$ .
2.  $\mathbf{A}(L)$  and  $\mathbf{B}(L)$  are coprime. That is, any of the greatest common left divisors of  $\mathbf{A}(L)$  and  $\mathbf{B}(L)$  are unimodular.
3.  $\mathbf{A}(L)$  or  $\mathbf{B}(L)$  is column reduced. That is,  $\det[\mathbf{A}^{(q)}] \neq 0$  or  $\det[\mathbf{B}^{(p)}] \neq 0$ .

Assumption (1) guarantees that the model in equation (3) is identifiable (see Proposition 3.4 in Jeantheau, 1998).

In Appendix A we i) state the covariance stationarity assumption, ii) present expressions for the unconditional second moment of the squared errors and iii) review the main theoretical results of He and Teräsvirta (2004) on the fourth-moment structure of the bivariate GARCH(1, 1) process. However, the stationarity assumption is not needed in the following theoretical development.

### 3 Non-negativity Constraints

We now derive the necessary and sufficient conditions for the positivity of the conditional variances in the  $N$ -dimensional UECCC-GARCH( $p, q$ ) model. In the first step, we show that each variance admits a ‘univariate’ representation. From this formulation, we obtain an ARCH( $\infty$ ) expansion of each conditional variance in terms of convolutions of GARCH kernels and corresponding squared errors. The non-negativity of the variances is guaranteed if and only if all the kernels are non-negative, i.e., if the *infinite* number of coefficients in the ARCH( $\infty$ ) expansions of the  $N^2$  kernels are non-negative. For this, we express these coefficients as functions of the parameters of the original process. It is then shown that checking a *finite* number of inequality constraints on these parameters ensures the non-negativity of all GARCH kernels. Special attention is given to the bivariate case, which is most relevant for empirical applications.

#### 3.1 ‘Univariate’ Representations

In order to simplify the description of our analysis we will introduce the following notation. Set  $\beta(L) = 1 - \sum_{l=1}^{N \times p} \beta_l L^l = \det[\mathbf{B}(L)]$ . Recall, that we have assumed  $\beta_{N \times p} \neq 0$ , hence  $\beta(L)$  is a scalar polynomial of order  $N \times p$ . Moreover, denote by  $\phi_n$ ,  $n = 1, \dots, N \times p$ , the inverse of the roots of  $\beta(z)$ . Define  $\boldsymbol{\omega} = [\omega_i]_{i=1, \dots, N} = \text{adj}[\mathbf{B}(L)]\boldsymbol{\mu}$  and  $\boldsymbol{\alpha}(L) = \sum_{r=1}^{(N-1) \times p + q} \boldsymbol{\alpha}^{(r)} L^r = \text{adj}[\mathbf{B}(L)]\mathbf{A}(L)$  with  $\boldsymbol{\alpha}^{(r)} = [\alpha_{ij}^{(r)}]_{i,j=1, \dots, N}$ , i.e.,  $\boldsymbol{\alpha}(L)$  is a square matrix polynomial. We can also express it as  $\boldsymbol{\alpha}(L) = [\alpha_{ij}(L)]_{i,j=1, \dots, N}$  with  $\alpha_{ij}(L) = \sum_{r=1}^{(N-1) \times p + q} \alpha_{ij}^{(r)} L^r$ . Since we have assumed that  $\alpha_{ij}^{(N-1) \times p + q} \neq 0$  for all  $i, j = 1, \dots, N$ , the scalar polynomials  $\alpha_{ij}(L)$  are of the order  $(N-1) \times p + q$ .

**Assumption 2 (Invertibility)** *The inverse roots  $\phi_n$ ,  $n = 1, \dots, N \times p$ , of  $\beta(z)$  lie inside the unit circle and without loss of generality are ordered as follows:  $|\phi_1| \geq |\phi_2| \geq \dots \geq |\phi_{N \times p}|$ .*

**Lemma 1** Under Assumptions (A1) and (A2) the ‘univariate’ representation of the  $N$ -dimensional vector UECCC-GARCH  $(p, q)$  process  $\mathbf{h}_t$  is given by

$$\beta(L)\mathbf{h}_t = \boldsymbol{\omega} + \boldsymbol{\alpha}(L)\boldsymbol{\varepsilon}_t^{\wedge 2}. \quad (4)$$

Lemma 1 states that we can write each conditional variance  $h_{it}$ ,  $i = 1, \dots, N$ , as being linear in a constant  $\omega_i$ , its own lags  $h_{it-l}$ ,  $l = 1, \dots, N \times p$ , its own lagged squared residuals  $\varepsilon_{it-r}^2$  as well as the lagged squared errors  $\varepsilon_{jt-r}^2$  from the other equations,  $j \neq i$ ,  $r = 1, \dots, (N-1) \times p + q$ . Most importantly, in the ‘univariate’ representation  $h_{it}$  no longer depends on lagged values of  $h_{jt}$ .

Before presenting the general results, we will discuss a specific model in order to make our analysis more concise. Consider the bivariate process of order  $(1, 1)$ :

$$\left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} L \right] \begin{pmatrix} h_{1t} \\ h_{2t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} L \begin{pmatrix} \varepsilon_{1t}^2 \\ \varepsilon_{2t}^2 \end{pmatrix}, \quad (5)$$

where for typographical convenience we have set  $b_{ij} = b_{ij}^{(1)}$ ,  $a_{ij} = a_{ij}^{(1)}$ ,  $i, j = 1, 2$  and  $\rho_{12} = \rho$ . In matrix form we have  $(\mathbf{I}_2 - \mathbf{B}L)\mathbf{h}_t = \boldsymbol{\mu} + \mathbf{A}L\boldsymbol{\varepsilon}_t^2$  with  $\mathbf{A} = [a_{ij}]_{i,j=1,2}$  and  $\mathbf{B} = [b_{ij}]_{i,j=1,2}$ .

**Corollary 1** The ‘univariate’ representation of the bivariate UECCC-GARCH(1, 1) process is given by

$$(1 - \beta_1 L - \beta_2 L^2) \begin{pmatrix} h_{1t} \\ h_{2t} \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \left[ \begin{pmatrix} \alpha_{11}^{(1)} & \alpha_{12}^{(1)} \\ \alpha_{21}^{(1)} & \alpha_{22}^{(1)} \end{pmatrix} L + \begin{pmatrix} \alpha_{11}^{(2)} & \alpha_{12}^{(2)} \\ \alpha_{21}^{(2)} & \alpha_{22}^{(2)} \end{pmatrix} L^2 \right] \begin{pmatrix} \varepsilon_{1t}^2 \\ \varepsilon_{2t}^2 \end{pmatrix}, \quad (6)$$

with  $\beta_1 = b_{11} + b_{22}$ ,  $\beta_2 = b_{12}b_{21} - b_{11}b_{22}$ ,

$$\boldsymbol{\omega} = \begin{pmatrix} (1 - b_{22})\mu_1 + b_{12}\mu_2 \\ (1 - b_{11})\mu_2 + b_{21}\mu_1 \end{pmatrix}, \quad \boldsymbol{\alpha}^{(1)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and  $\boldsymbol{\alpha}^{(2)} = \begin{pmatrix} a_{21}b_{12} - a_{11}b_{22} & a_{22}b_{12} - a_{12}b_{22} \\ a_{11}b_{21} - a_{21}b_{11} & a_{12}b_{21} - a_{22}b_{11} \end{pmatrix}.$

### 3.2 The N-dimensional Process

Recall that Jeanteau (1998) assumes that all the coefficients of the  $\mathbf{A}^{(r)}$  and  $\mathbf{B}^{(l)}$ , ( $r = 1, \dots, q$ ,  $l = 1, \dots, p$ ) matrices are positive. He and Teräsvirta (2004) point out that a sufficient condition for  $\mathbf{h}_t > 0$  for all  $t$  is that all elements in  $\boldsymbol{\mu}$  are positive and all elements in the  $\mathbf{A}^{(r)}$  and  $\mathbf{B}^{(l)}$  matrices are non-negative for each  $r$  and  $l$ . In addition, by referring to the results of Nelson and Cao (1992), they conjecture that this condition is not necessary, at least not if  $p > 1$  and/or  $q > 1$  (see Remark 1 in He and Teräsvirta, 2004). By investigating the ARCH( $\infty$ ) representation of the univariate GARCH( $p, q$ ) process, Nelson and Cao (1992) derive necessary and sufficient conditions for the non-negativity

of the conditional variances which is ensured if and only if all the ARCH( $\infty$ ) coefficients are non-negative. This, however, does not necessarily mean that all the parameters of the process have to be positive. Next, we derive the ARCH( $\infty$ ) expansion of the vector UECCC-GARCH( $p, q$ ) model.

**Lemma 2** *Let Assumptions (A1) and (A2) be satisfied. Then, equation (4) can be rewritten in the ARCH( $\infty$ ) form:*

$$\mathbf{h}_t = \boldsymbol{\omega}/\beta(1) + \boldsymbol{\Psi}(L)\boldsymbol{\varepsilon}_t^{\wedge 2}, \quad (7)$$

where  $\boldsymbol{\Psi}(L) = [\Psi_{ij}(L)]_{i,j=1,\dots,N} = \boldsymbol{\alpha}(L)/\beta(L)$  with  $\Psi_{ij}(L) = \sum_{k=1}^{\infty} \psi_{ij}^{(k)} L^k = \alpha_{ij}(L)/\beta(L)$ .

Here, each  $\Psi_{ij}(L)$  can be thought of as an ARCH( $\infty$ ) kernel of a GARCH model of the order  $(N \times p, (N - 1) \times p + q)$ .

For illustrative purposes, consider again the bivariate process of order (1, 1). If Assumptions (A1) and (A2) hold, then from equation (6) it follows that the ARCH( $\infty$ ) representation of the process exists and is given by

$$\begin{pmatrix} h_{1t} \\ h_{2t} \end{pmatrix} = \begin{pmatrix} \omega_1/(1 - \beta_1 - \beta_2) \\ \omega_2/(1 - \beta_1 - \beta_2) \end{pmatrix} + \begin{pmatrix} \Psi_{11}(L) & \Psi_{12}(L) \\ \Psi_{21}(L) & \Psi_{22}(L) \end{pmatrix} \begin{pmatrix} \varepsilon_{1t}^2 \\ \varepsilon_{2t}^2 \end{pmatrix}, \quad (8)$$

where each of the four kernels  $\Psi_{ij}(L)$  corresponds to an ARCH( $\infty$ ) kernel of a univariate GARCH(2, 2) process:

$$\Psi_{ij}(L) = \frac{\alpha_{ij}^{(1)} L + \alpha_{ij}^{(2)} L^2}{(1 - \beta_1 L - \beta_2 L^2)} = \sum_{k=0}^{\infty} \psi_{ij}^{(k)} L^k \quad \text{for } i, j = 1, 2.$$

Following the proof of Proposition 1 in Conrad and Haag (2006), we can recursively express each  $\psi_{ij}^{(k)}$  sequence as

$$\psi_{ij}^{(k)} = \beta_1 \psi_{ij}^{(k-1)} + \beta_2 \psi_{ij}^{(k-2)} \quad \text{for } k \geq 3,$$

where  $\psi_{ij}^{(1)} = \alpha_{ij}^{(1)}$  and  $\psi_{ij}^{(2)} = \beta_1 \alpha_{ij}^{(1)} + \alpha_{ij}^{(2)}$ . Obviously, the  $\psi_{ij}^{(k)}$ 's can now be expressed in terms of the  $a$ 's and  $b$ 's using Corollary 1. For example,  $\psi_{11}^{(1)} = a_{11}$  and  $\psi_{11}^{(2)} = a_{11}b_{11} + a_{21}b_{12}$ .

Clearly, for the  $N$ -dimensional process in equation (3) to be well-defined and the  $N$  conditional variances to be positive almost surely for all  $t$ , all the constants  $\omega_i$  must be positive and all the  $\psi_{ij}^{(k)}$  coefficients in the ARCH( $\infty$ ) representation, that is equation (7), must be non-negative:  $\psi_{ij}^{(k)} \geq 0$ ,  $i, j = 1, \dots, N$ , for  $k = 1, 2, \dots$

In practice, given a particular set of parameters, checking the non-negativity of  $\{\psi_{ij}^{(k)}\}_{k=1}^{\infty}$ ,  $i, j = 1, \dots, N$ , may be a numerically infeasible task. In the following theorem we show that under some conditions, the non-negativity of  $\{\psi_{ij}^{(k)}\}_{k=1}^{k_{ij}^*}$  for some tractable integers  $k_{ij}^*$  is necessary and sufficient for the non-negativity of  $\{\psi_{ij}^{(k)}\}_{k=1}^{\infty}$ .

**Theorem 1** *Consider the  $N$ -dimensional vector UECCC-GARCH( $p, q$ ) model in equation (3) and let Assumptions (A1), (A2) be satisfied and all the inverse roots be distinct. Then, the following conditions are necessary and sufficient for  $h_{it} > 0$ ,  $i = 1, \dots, N$ , for all  $t$ :*

(a)  $\omega_i > 0$  for all  $i = 1, \dots, N$ .

(b)  $\phi_1$  is real, and  $\phi_1 > 0$ , (C1)

$\alpha_{ij}(\phi_1^{-1}) > 0$ , for  $i, j = 1, \dots, N$ , (C2)

$\psi_{ij}^{(k)} \geq 0$ , for  $i, j = 1, \dots, N$  and  $k = 1, \dots, k_{ij}^*$ , (C3)

where  $k_{ij}^*$  is the smallest integer greater than or equal to  $\max\{0, \varphi\}$  with

$$\varphi = \{\log(\eta_{ij}^{(1)}) - \log(Np - 1)\eta_{ij}^*\} / \{\log(|\phi_2|) - \log(\phi_1)\},$$

$$\eta_{ij}^* = \max_{2 \leq n \leq N \times p} |\eta_{ij}^{(n)}|, \text{ and } \eta_{ij}^{(n)} = -\frac{\alpha_{ij}(\phi_n^{-1})}{\beta'(\phi_n^{-1})}, 1 \leq n \leq N \times p,$$

where  $\beta'(z)$  denotes the first derivative of  $\beta(z)$ .

The proof of Theorem 1 relies on the observation that the results of Tsai and Chan (2008) can be applied separately to each of the  $N^2$  GARCH kernels  $\Psi_{ij}(L)$ . Hence, we reduce an infinite number of inequality constraints on the ARCH( $\infty$ ) coefficients to a finite number of conditions on the parameters of the process.

As mentioned in Remark 1, the results for the UECCC model hold also for any parametrization of the UETVCC formulation. In addition, it can be shown that they hold for the asymmetric power version of it as well.

### 3.3 The Bivariate Process of Order (1, 1)

Because the bivariate model of order (1, 1) is definitely the most often applied specification, we intensively discuss the corresponding inequalities and their interpretation.

**Proposition 1** *Let Assumptions (A1), (A2) be satisfied and  $\phi_1 \neq \phi_2$ . The following conditions are necessary and sufficient for  $h_{it} > 0$ ,  $i = 1, 2$ , for all  $t$  in the bivariate UECCC-GARCH(1, 1) model:*

(a) For the two constants we require

$$\omega_1 = (1 - b_{22})\mu_1 + b_{12}\mu_2 > 0, \quad \text{and} \quad \omega_2 = (1 - b_{11})\mu_2 + b_{21}\mu_1 > 0. \quad (9)$$

(b) Condition (C1) in Theorem 1 which reduces to

$$(b_{11} - b_{22})^2 > -4b_{12}b_{21}, \quad \text{and} \quad \phi_1 > 0. \quad (C1')$$

Condition (C2) which becomes

$$\begin{aligned} \alpha_{11}(\phi_1^{-1}) &= (b_{11} - \phi_2)a_{11} + b_{12}a_{21} > 0, & \alpha_{12}(\phi_1^{-1}) &= (b_{11} - \phi_2)a_{12} + b_{12}a_{22} > 0, \\ \alpha_{21}(\phi_1^{-1}) &= b_{21}a_{11} + (b_{22} - \phi_2)a_{21} > 0, & \alpha_{22}(\phi_1^{-1}) &= b_{21}a_{12} + (b_{22} - \phi_2)a_{22} > 0. \end{aligned} \quad (C2')$$

Condition (C3) which, since  $k_{ij}^* = 2$ , amounts to

$$\begin{aligned} \psi_{11}^{(1)} = a_{11} \geq 0, & \quad \psi_{12}^{(1)} = a_{12} \geq 0, \\ \psi_{21}^{(1)} = a_{21} \geq 0, & \quad \psi_{22}^{(1)} = a_{22} \geq 0, \end{aligned} \tag{C3'a}$$

and

$$\begin{aligned} \psi_{11}^{(2)} = b_{11}a_{11} + b_{12}a_{21} \geq 0, & \quad \psi_{12}^{(2)} = b_{11}a_{12} + b_{12}a_{22} \geq 0, \\ \psi_{21}^{(2)} = b_{21}a_{11} + b_{22}a_{21} \geq 0, & \quad \psi_{22}^{(2)} = b_{21}a_{12} + b_{22}a_{22} \geq 0. \end{aligned} \tag{C3'b}$$

Note that if  $\phi_2 > 0$  then  $\alpha_{ij}(\phi_1^{-1}) > 0$ , that is condition (C2'), directly implies  $\psi_{ij}^{(2)} \geq 0$ , that is condition (C3'b), and vice versa if  $\phi_2 < 0$ .

In the bivariate model both conditional variances are always positive if the two constants  $\omega_1$  and  $\omega_2$  are positive and all four GARCH(2, 2) kernels are non-negative. Note that the conditions which Proposition 1 places on each of the four kernels are equivalent to the ones derived in Nelson and Cao (1992) and He and Teräsvirta (1999) (see Assumption (A23) in their paper) for the univariate GARCH(2, 2) model. However, the four kernels are functions of the same underlying parameters of the bivariate process and the conditions on the four kernels have to be satisfied simultaneously. It is clearly apparent that the conditions of Proposition 1 are satisfied if all parameters are assumed to be positive. Finally, it should be noted that conditions (C2'), (C3'a) and (C3'b) can be written compactly in a matrix form as:  $(\mathbf{B} - \phi_2 \mathbf{I}_2)\mathbf{A} > 0$ ,  $\mathbf{A} \geq 0$  and  $\mathbf{BA} \geq 0$  respectively.

**Remark 2** Nakatani and Teräsvirta (2006) mention that for the conditional variances in the bivariate GARCH(1, 1) process to be positive almost surely for all  $t$ , all the GARCH kernels  $\Psi_{ij}(L)$  in the ARCH( $\infty$ ) representation must be non-negative and the vector of constants must be positive. The authors, however, do not elaborate on how to obtain necessary and sufficient conditions for the positivity of  $\mathbf{h}_t$ .

Nelson and Cao (1992) have shown that under the necessary and sufficient conditions for the univariate GARCH(2, 2) process with parameters  $\omega$ ,  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$  and  $\beta_2$ , at most two out of the five parameters may be negative. These are the second order coefficients, i.e.,  $\alpha_2$  and  $\beta_2$ . With the following corollaries we will investigate which restrictions Proposition 1 imposes on the parameters of the bivariate model.

First, note from Condition (C3'a) that all four elements in  $\mathbf{A}$  must be non-negative. This is not surprising, since the coefficients of  $\mathbf{A}$  are the first order ARCH parameters,  $\alpha_{ij}^{(1)}$ , of the GARCH(2, 2) kernels in the 'univariate' representations (see Corollary 1) and, as proved in Nelson and Cao (1992), those should be non-negative. Hence, only coefficients in  $\boldsymbol{\mu}$  and  $\mathbf{B}$  may be allowed to be negative.

**Corollary 2** In the bivariate UECC-GARCH(1, 1) model both diagonal elements of  $B$  cannot be negative simultaneously.

Again, Corollary 2 does not come as a surprise, because  $\beta_1 = b_{11} + b_{22}$  is the first order GARCH parameter of the four kernels in the univariate representations and according to Nelson and Cao (1992) has to be non-negative.

It is worth noting that while a negative value of, say,  $b_{11}$  might suggest that the higher  $h_{1t-1}$  the lower  $h_{1t}$ , this, however, is only a 'partial effect'. The 'univariate' representation implies  $\beta_1 = b_{11} + b_{22} > 0$  and hence a 'overall' positive relation between  $h_{1t-1}$  and  $h_{1t}$ .

**Corollary 3** *In the bivariate UECCC-GARCH(1,1) process both elements of one row of  $\mathbf{B}$  cannot be negative simultaneously.*

**Corollary 4** *In the bivariate UECCC-GARCH(1,1) specification negative volatility feedback in both directions ( $b_{12} < 0$  and  $b_{21} < 0$ ) is ruled out.*

Corollary 4 states an important result, because it shows the model's limitations: economic theories which imply *negative* volatility spillovers in both equations can not be tested within the bivariate framework.

**Example 1** *Tsay (2002, p. 369) applied the model in equation (5) to monthly returns of the IBM stock and the S&P 500 index from January 1926 to December 1999. He obtained the following estimates for the entries of the  $\mathbf{B}$  matrix:  $b_{11} = 0.873$ ,  $b_{12} = -0.031$ ,  $b_{21} = -0.066$  and  $b_{22} = 0.913$ . Since both off-diagonal elements are negative, the estimated model does not guarantee the positive definiteness of  $\mathbf{H}_t$ .*

Suppose  $b_{12}$  and  $b_{21}$  were negative, while  $b_{11}$  and  $b_{22}$  as well as the elements in  $\mathbf{A}$  are positive. Such a constellation would imply that all second order ARCH coefficients in the four GARCH(2,2) kernels are negative, that is  $\alpha^{(2)} < 0$  in Corollary 1, but it is ruled out by Corollary 4. However, the case that all four second order ARCH coefficients are negative simultaneously is not ruled out in general, because it can arise even if all parameters are positive.

**Corollary 5** *In the bivariate UECCC-GARCH(1,1) model at most two elements of  $\mathbf{B}$  can be negative, if and only if they belong to the same column.*

More specifically, and without loss of generality, let  $b_{11}, b_{21} > 0$  and one or both elements in the second column of the  $\mathbf{B}$  matrix be negative. Then under (C1') and (C3'a), conditions (C2') and (C3'b) reduce to  $a_{1i}b_{j1} \geq |b_{j2} - \phi_2| a_{2i}$  and  $a_{1i}b_{j1} \geq |b_{j2}| a_{2i}$ ,  $i, j = 1, 2$ , respectively.

A particularly interesting case arises when one of the off-diagonal elements of  $\mathbf{B}$  is negative while all other parameters are positive. Such a case implies that  $\beta_2 = b_{12}b_{21} - b_{11}b_{22} < 0$  and both second order ARCH coefficients in one row of  $\alpha^{(2)}$  are negative, e.g., if  $b_{12} < 0$  this implies that  $\alpha_{11}^{(2)}$  and  $\alpha_{12}^{(2)}$  in equation (6) are negative.

**Example 2** *Bai and Chen (2008) apply the bivariate UETVCC-GARCH(1,1) model to the data set used in Tsay (2002, p.374). They use the following Cholesky decomposition:*

$\mathbf{H}_t = \mathbf{L}_t \mathbf{G}_t \mathbf{L}_t'$  where  $\mathbf{L}_t = \begin{pmatrix} 1 & 0 \\ q_t & 1 \end{pmatrix}$ ,  $\mathbf{G}_t = \begin{pmatrix} h_{1t} & 0 \\ 0 & g_{2t} \end{pmatrix}$  with  $q_t = \rho_t \sqrt{h_{2t}} / \sqrt{h_{1t}}$  and  $g_{2t} = h_{2t}(1 - \rho_t^2)$ . Then the bivariate process:  $[\mathbf{I}_2 - \mathbf{B}\mathbf{L}]\mathbf{g}_t = \boldsymbol{\mu} + \mathbf{A}\boldsymbol{\eta}_{t-1}^2$  where  $\mathbf{g}_t = [\text{diag}\{\mathbf{G}_t\}]'$ ,  $\boldsymbol{\eta}_t' = (\varepsilon_{1t} \quad \eta_{2t})$  with  $\eta_{2t} = \varepsilon_{2t} - q_t \varepsilon_{1t}$  is estimated.<sup>9</sup> The following estimates for the entries of the  $\mathbf{A}$  and  $\mathbf{B}$  matrices are obtained:  $\mathbf{A} = \begin{pmatrix} 0.113 & - \\ 0.021 & 0.052 \end{pmatrix}$ ,  $\mathbf{B} =$

<sup>9</sup>For the time-varying conditional correlation they employ the equation:  $(1 - \gamma_1 L)q_t = \gamma_0 + \gamma_2 L \varepsilon_{2t}$  (see also Tsay, 2002, p.374).

$\begin{pmatrix} 0.804 & - \\ -0.040 & 0.937 \end{pmatrix}$ . Since  $b_{12} = 0$  we have  $\phi_1 = b_{22} = 0.937$  and  $\phi_2 = b_{11} = 0.804 > 0$ . Most importantly, condition (C2') is violated because  $b_{21}a_{11} + (b_{22} - \phi_2)a_{21} = -0.115 < 0$ . Therefore the estimated model does not guarantee the positive definiteness of either  $\mathbf{G}_t$  or  $\mathbf{H}_t$ .

To summarize our results concerning the elements of  $\mathbf{B}$  figuratively we may say that at most two out of the four coefficients of the  $\mathbf{B}$  matrix can be negative as long as they are elements of the same column.

Further, unlike the univariate GARCH(2, 2) model, the bivariate one can produce, under some conditions, negative values for the constants. The case that one of the two is negative is straightforward and requires no further discussion. We now consider the case of both constants being negative. First note that the condition  $\phi_i < 1$  for  $i = 1, 2$  implies that  $b_{11}$  and  $b_{22}$  cannot be greater than one simultaneously, because  $b_{11} + b_{22} = \phi_1 + \phi_2$ .

**Corollary 6** *In the bivariate UECCC-GARCH(1, 1) model the constants  $\mu_1$  and  $\mu_2$  can be negative simultaneously if and only if one of the two diagonal elements of  $\mathbf{B}$  is greater than one.*

Finally, we should emphasize that the set of conditions provided by the above proposition is weaker than those in Jeanteau (1998), He and Teräsvirta (2004) and Nakatani and Teräsvirta (2007). That is, although all four ARCH coefficients must be non-negative, the two constants and two out of the four GARCH parameters, under some conditions, can be negative.

These results require more discussion. For illustrative purposes we examine in the next subsection a few numerical examples.

### 3.4 Numerical Examples

In what follows we graphically illustrate the necessary and sufficient parameter set for the bivariate UECCC-GARCH(1, 1) model. This will provide a better understanding of the results presented in the previous subsection. We discuss four examples.

In the first example we allow the two off-diagonal elements of  $\mathbf{B}$ , to vary. The parameters chosen for Example 1 are rather standard, except that we assume  $a_{21} = 0.2$ , which implies that the squared innovations  $\varepsilon_{1t-1}^2$  have a strong impact on  $h_{2t}$ . In Example 2 we examine the situation where  $b_{11}$  and  $b_{21}$ , i.e., two coefficients in the first column of  $\mathbf{B}$ , vary. In Example 3 the two constants  $\mu_1$  and  $\mu_2$  vary freely, while the two off-diagonal elements of  $\mathbf{A}$  vary in Example 4.

In the following figures the bold solid lines show which combinations of the two freely varying parameters do satisfy the necessary and sufficient conditions of Proposition 1 when the other parameters are fixed as in the Examples of Table 1. Additionally, they show which combinations satisfy the conditions for the existence of the unconditional second (dotted lines) and fourth moments (dashed lines), which are presented in Appendix A.

We begin by discussing the implications of Example 1, which is presented in Figure 1, left panel. First, all combinations of  $b_{12}$  and  $b_{21}$  which are bounded by the three bold solid

Table 1: Data generating processes (DGP) for Examples 1 to 4.

	DGP Ex. 1	DGP Ex. 2	DGP Ex. 3	DGP Ex. 4
$\boldsymbol{\mu}'$	$( 0.10 \ 0.20 )$	$( 0.10 \ 0.20 )$	$( \mu_1 \ \mu_2 )$	$( 0.20 \ 0.10 )$
$\mathbf{A}$	$\begin{pmatrix} 0.03 & 0.02 \\ 0.20 & 0.05 \end{pmatrix}$	$\begin{pmatrix} 0.03 & 0.02 \\ 0.40 & 0.05 \end{pmatrix}$	$\begin{pmatrix} 0.07 & 0.03 \\ 0.01 & 0.002 \end{pmatrix}$	$\begin{pmatrix} 0.03 & a_{12} \\ a_{21} & 0.05 \end{pmatrix}$
$\mathbf{B}$	$\begin{pmatrix} 0.30 & b_{12} \\ b_{21} & 0.80 \end{pmatrix}$	$\begin{pmatrix} b_{11} & 0.30 \\ b_{21} & 0.80 \end{pmatrix}$	$\begin{pmatrix} 1.2 & -0.50 \\ 0.50 & 0.15 \end{pmatrix}$	$\begin{pmatrix} 0.10 & 0.30 \\ -0.35 & 0.80 \end{pmatrix}$

lines satisfy the conditions of Proposition 1. The line in quadrant one represents the invertibility Assumption (A2), which is satisfied when  $b_{12} < (1 - b_{11})(1 - b_{22})/b_{21} = 0.14/b_{21}$ . In quadrant four the line stands for the condition that  $\phi_1$  is real:  $b_{12} < -(b_{11} - b_{22})^2/(4b_{21}) = -0.062/b_{21}$ . The requirement that  $b_{12}$  is non-negative is triggered by condition (C2'):  $b_{12} > a_{11} |b_{11} - \phi_2|/a_{21} = |0.30 - \phi_2|/6.67$ . In line with Corollary 4 this ensures that not both off-diagonal elements of  $\mathbf{B}$  can be negative simultaneously. Finally, it is evident that the Bollerslev conditions which would restrict both  $b_{12}$  and  $b_{21}$  to being positive are too strong, since they exclude the area below the bold solid line in quadrant four where  $b_{21} < 0$ . Second, the combinations of  $b_{12}$  and  $b_{21}$ , which are bounded by the dotted lines, satisfy the conditions for the existence of the second moments. The dotted lines in quadrant one and four represent the covariance stationarity condition (see Assumption (3) in Appendix A), while the straight dotted line in quadrants two and three is triggered by the restriction that the unconditional variances are non-negative (see equation (A2) in Appendix A). Similarly, the constraints for the existence of the fourth moments imply the dashed lines (see equations (A3) and (A4) in Appendix A). In the first quadrant the restrictions implied by the conditions for the existence of the second and fourth moments are more restrictive than the one implied by the invertibility Assumption (A2). In summary, the parameter set which satisfies all conditions simultaneously is given by the area which is below the dashed line in the first and fourth quadrant and below the bold solid line in the fourth quadrant.

Next, we consider the case where both elements of the first column of  $\mathbf{B}$  vary. The bold solid lines in Figure 1, right panel, show the necessary and sufficient set for Example 2. As suggested by Corollary 5, we can now have both elements in a column being negative. The line in quadrants one and two represents the invertibility Assumption (A2):  $b_{11} < 1 - b_{12}b_{21}/(1 - b_{22}) = 1 - 1.50b_{21}$ , while the one in quadrants two and three is triggered by condition (C3'b):  $b_{11} \geq -b_{12}a_{22}/a_{12} = -0.75$ . The line in quadrant four represents condition (C2'):  $b_{21} > -(b_{22} - \phi_2)a_{22}/a_{12} = -2.50(0.80 - \phi_2)$ .

Figure 2, left panel, shows that for the parameters in Example 3 we can have both constants being negative. All parameter combinations above the bold solid lines in quadrants one and three are valid:  $\mu_1 > |b_{12}|\mu_2/(1 - b_{22}) = 0.59\mu_2$  and  $\mu_1 > |1 - b_{11}|\mu_2/b_{21} = 0.40\mu_2$ ,



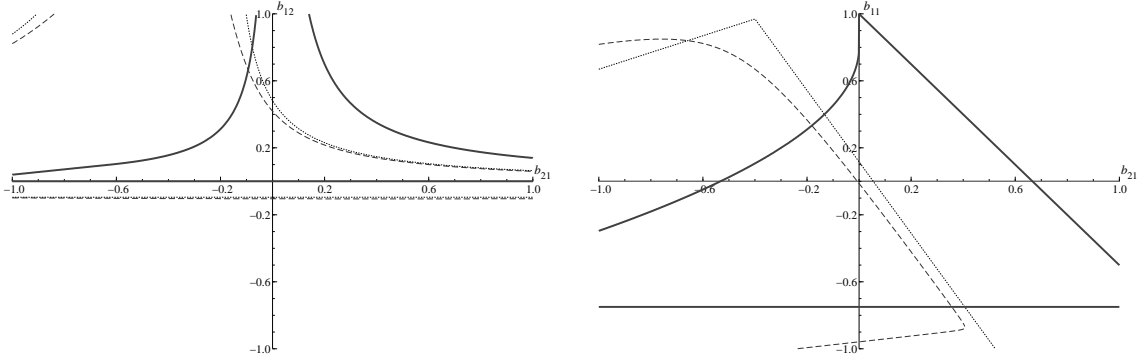


Figure 1: Necessary and sufficient parameter sets for bivariate UECCC-GARCH(1, 1) from Examples 1 (left panel) and 2 (right panel). The bold solid lines represent the restrictions implied by Proposition 1. The dotted and dashed lines stand for the restrictions for the existence of the unconditional second and fourth moments.

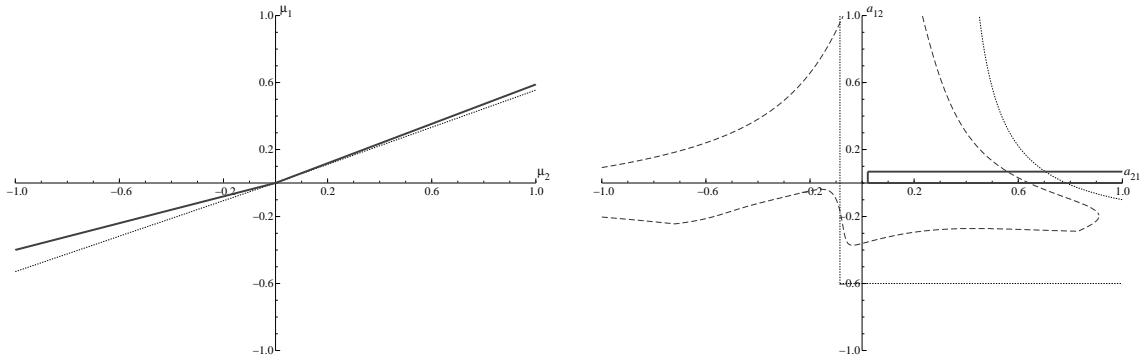


Figure 2: Necessary and sufficient parameter sets for bivariate UECCC-GARCH(1, 1) from Examples 3 (left panel) and 4 (right panel). The bold solid lines represent the restrictions implied by Proposition 1. The dotted and dashed lines stand for the restrictions for the existence of the unconditional second and fourth moments.

respectively. Interestingly, combinations with  $\mu_1 < 0$ ,  $\mu_2 > 0$  are ruled out whereas in sharp contrast all possible combinations with  $\mu_1 > 0$ ,  $\mu_2 < 0$  are permitted. Example 3 is in line with Corollary 6, which requires for this case that one of the diagonal elements of  $\mathbf{B}$  must be greater than one ( $b_{11} = 1.2$ ). Example 4 is visualized in Figure 2, right panel. The negative volatility feedback from equation one to equation two does not violate the conditions of Proposition 1 only if  $a_{12}$  takes rather small values. In particular, since  $\phi_2 = 0.32 > 0$ , the two off-diagonal elements of  $\mathbf{A}$  should satisfy condition (C2'):  $a_{21} > a_{11} |b_{11} - \phi_2| / b_{12} = 0.02$  and  $a_{12} < a_{22} b_{12} / |b_{11} - \phi_2| = 0.07$ .

The examples above impressively show that the weaker conditions of Proposition 1 substantially enlarge the necessary and sufficient parameter set in comparison to the Bollerslev condition. Having a wider admissible parameter set should increase the flexibility of the UECCC-GARCH(1, 1) model. For Example 2 we illustrate this by plotting

the auto- and cross-correlations of the squared errors.<sup>10</sup> For this, we set  $b_{21} = -0.10$  and vary  $b_{11}$  from  $-0.70$  to  $0$  by steps of  $0.10$ . Since  $b_{22} = 0.80$  the restriction that  $\beta_1 = b_{11} + b_{22} > 0$  is always satisfied. When  $b_{11}$  takes large negative values we observe an oscillating behaviour in the auto- and cross-correlations  $r_{11}(m)$  and  $r_{12}(m)$ ,  $m = 1, 2, \dots$ , which disappears as  $b_{11}$  gets close to zero.

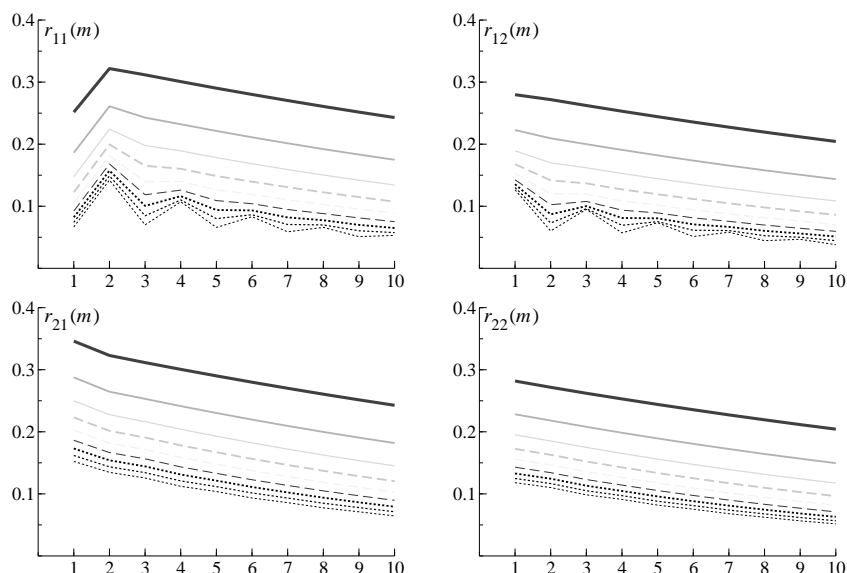


Figure 3: The figure shows the  $m$ -th order auto- and cross-correlations,  $r_{ij}(m)$ , between  $\varepsilon_{it}^2$  and  $\varepsilon_{jt-m}^2$  for the bivariate UECCC-GARCH(1, 1) from Example 2. We set  $b_{21} = -0.10$  and vary  $b_{11}$  from  $-0.70$  (lowest, dotted line) to  $0$  (highest, solid line) by steps of  $0.10$ .

## 4 Testing Economic Theory

### 4.1 The Stock Returns-Volume Link

Caporin (2007) argues that highly volatile stock prices induce a decrease in trading variations, stabilizing the average volume of trades. While such a negative effect from stock volatility to volume variability is untestable under the Bollerslev conditions, the necessary and sufficient conditions of Proposition 1 allow for such a scenario. We employ daily log returns ( $y_{1t}$ ) and de-trended log volume ( $y_{2t}$ ) for the BASF stock which is traded on the DAX30 stock index. The data set is available at *Yahoo!Finance* and spans the period January 2000 to August 2004. Because log volume appears to be fractionally integrated (see, for example, Karanasos and Kartsaklas, 2007) we estimate a bivariate ARFIMA(1,  $d$ , 0)

<sup>10</sup>The analytic forms for the auto- and cross-correlations of the squared errors of the UECCC-GARCH(1,1) model are given in Corollary 3 in He and Teräsvirta (2004). It is interesting to note that one can express the  $N$ -dimensional vector UECCC-GARCH( $p, q$ ) model in an ARMA representation and then apply the methodology in Karanasos (1999b, 2007) to obtain the correlation structure of the squared errors (see, Conrad and Karanasos, 2008).

process with residuals following a UECCC-GARCH(1, 1) specification and normally distributed innovations  $z_{it}$ . For reasons of brevity we omit the parameter estimates for the mean equation and directly discuss the results for the conditional variances:

$$\mathbf{h}_t = \begin{pmatrix} 0.095 \\ (0.016) \\ 0.107 \\ (0.022) \end{pmatrix} + \begin{pmatrix} 0.206 & 0.048 \\ (0.026) & (0.018) \\ 0.016 & 0.020 \\ (0.006) & (0.006) \end{pmatrix} \varepsilon_{t-1}^{\wedge 2} + \begin{pmatrix} 0.703 & - \\ (0.022) & \\ -0.030 & 0.852 \\ (0.008) & (0.034) \end{pmatrix} \mathbf{h}_{t-1}, \quad (10)$$

with  $\rho = -0.072$  (0.027); the numbers in parentheses are robust standard errors. Since  $b_{12}$  is insignificant we set it to zero. All other parameters were found to be highly significant. Most importantly, we find evidence that the variability of stock returns affects volume volatility negatively. The conditions for the existence of the unconditional second and fourth moments are satisfied. Figure 4 highlights the negative unidirectional feedback by plotting the auto- and cross-correlation,  $r_{ij}(m)$ , of the squared errors. The behaviour of  $r_{21}$  reflects the theoretical prediction that the variance of volume tends to decrease when stock volatility increases.

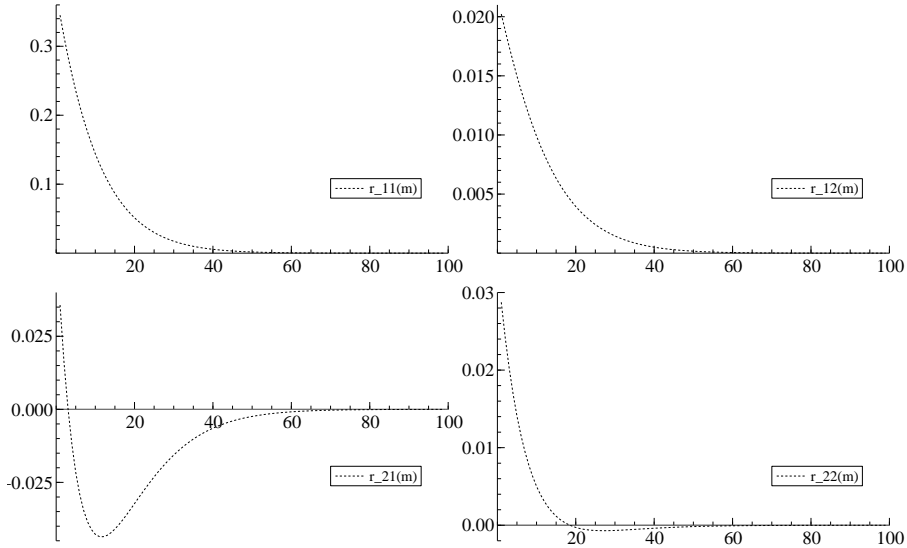


Figure 4: The figure shows the  $m$ -th order auto- and cross-correlations,  $r_{ij}(m)$ , between  $\varepsilon_{it}^2$  and  $\varepsilon_{jt-m}^2$  for a bivariate process of order (1, 1) with parameter estimates as presented in equation (10).

The example above highlights the empirical relevance of the unrestricted model, which allows for negative volatility feedback. However, when we check the conditions of Proposition 1 it turns out that the estimated parameter combination violates condition (C2'). In particular, the condition  $\alpha_{21}(\phi_1^{-1}) = b_{21}a_{11} + (b_{22} - \phi_2)a_{12} > 0$  is not satisfied. Hence, there is a positive probability that the conditional covariance matrix is not positive definite for all  $t$ . When employed for volatility forecasting, the estimated model may produce negative out-of-sample predictions. Interestingly, when we fix four of the parameters in  $\alpha_{21}(\phi_1^{-1})$  at the estimated values and choose the remaining one such that  $\alpha_{21}(\phi_1^{-1}) > 0$  is

just satisfied, then all auto- and cross-correlations take positive values only.<sup>11</sup> Hence, it appears that the bivariate UECCC-GARCH(1, 1) process, under the necessary and sufficient conditions of Proposition 1, allows for negative unidirectional feedback but cannot generate negative auto- and cross-correlations. Analyzing whether this is true in general is an interesting topic for future research.

## 4.2 The Inflation-Output Growth Relationship

The debate about the inflation-growth interaction is linked to another ongoing dispute, that of the existence or absence of a variance relationship. As Fuhrer (1997) puts it: ‘It is difficult to imagine a policy that embraces targets for the level of inflation or output growth without caring about their variability around their target levels. The more concerned the monetary policy is about maintaining the level of an objective at its target, the more it will care about the variability of that objective around its target’.

There are many controversies in the theoretical literature on the relationship between the four variables (see Fountas et al., 2006, and the references therein). The extent to which there is an interaction of either sign between the two variances is an issue that cannot be resolved on merely theoretical grounds. Not only that, the models regarding the ‘uncertainty link’ that do exist are often ambiguous in their predictions. These considerations reinforce a widespread awareness of the need for more empirical evidence, but also make clear that a good empirical framework is lacking. In what follows we will employ the bivariate process to examine how US nominal and real uncertainties are interrelated.

Monthly US data for the period 1960:01 to 2007:11 are obtained from *Datastream* to provide a reasonable number of observations. The inflation rate ( $y_{1t}$ ) and output growth ( $y_{2t}$ ) series are calculated as the monthly difference in the natural log of the Consumer Price Index and Industrial Production Index respectively.

Assuming conditional normality we estimate an ARMA(12, 0)-UECCC-GARCH(1, 1) specification. Again, we omit the results for the mean equation. The estimation results for the conditional variances are:

$$\mathbf{h}_t = \begin{pmatrix} 0.482 \\ (0.140) \\ 5.028 \\ (3.262) \end{pmatrix} + \begin{pmatrix} 0.086 & 0.009 \\ (0.018) & (0.002) \\ - & 0.257 \\ & (0.054) \end{pmatrix} \varepsilon_{t-1}^{\wedge 2} + \begin{pmatrix} 0.888 & -0.010 \\ (0.042) & (0.004) \\ 1.296 & 0.512 \\ (0.491) & (0.065) \end{pmatrix} \mathbf{h}_{t-1}, \quad (11)$$

with  $\rho = -0.026$  (0.045); the numbers in parentheses are robust standard errors.

The parameter  $a_{21}$  was set to zero because it turned out to be insignificant. Interestingly, there is a bidirectional feedback between the two variables. In particular, there is strong evidence supporting the Logue and Sweeney (1981) theory that inflation uncertainty has a positive impact on the volatility of growth. In sharp contrast real variability affects nominal uncertainty negatively as predicted by, among others, Fuhrer (1997). Clearly, the negative coefficient  $b_{12}$  would have been ruled out by the sufficient Bollerslev conditions. On the other hand, it is easy to check that in this example the conditions of

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<sup>11</sup>Choosing such values is possible for all parameters with the exception of  $b_{22}$ . The value that has to be chosen for  $b_{22}$  leads the larger of the inverse roots of  $\beta(z)$  to being greater than one, and hence to another violation of Proposition 1.

Proposition 1 are satisfied for the given parameter combination. Moreover, the conditions for the existence of the unconditional second and fourth moments are satisfied as well.

Figure 5 shows the auto- and cross-correlations of the squared errors. As expected, we observe first an increase in the cross-correlation,  $r_{21}(m)$ , between the squared residuals of output and inflation before it starts to decline to zero. This is driven by the strong positive effect from inflation uncertainty to output variability captured by the coefficient  $b_{21}$ .

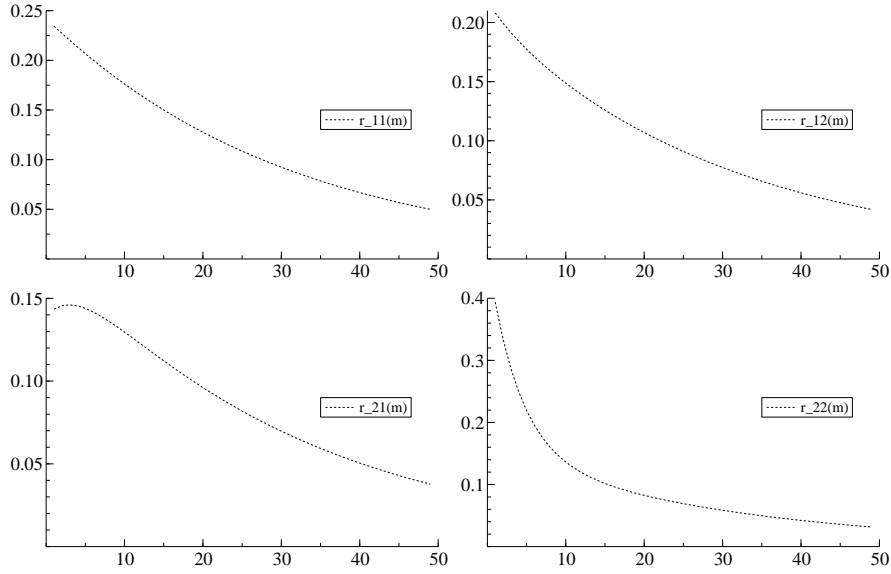


Figure 5: The figure shows the  $m$ -th order auto- and cross-correlations,  $r_{ij}(m)$ , between  $\varepsilon_{it}^2$  and  $\varepsilon_{jt-m}^2$  for a bivariate process of order  $(1, 1)$  with parameter estimates as presented in equation (11).

## 5 Conclusions

We have derived necessary and sufficient conditions which ensure the positive definiteness of the conditional covariance matrix in the  $N$ -dimensional UECCC-GARCH model almost surely for all  $t$ . For this, we have shown that each variance admits an ARCH( $\infty$ ) representation in terms of the  $N$  convolutions of the GARCH kernels and the corresponding squared errors. All the  $N$  conditional variances are almost surely non-negative for all  $t$  if all the  $N^2$  kernels are simultaneously non-negative. It is then straightforward to apply the methodologies developed in Nelson and Cao (1992) and Tsai and Chan (2008) to each of these kernels. In contrast to the sufficient Bollerslev condition, the necessary and sufficient conditions do not rule out the possibility that some of the parameters of the model take negative values. In particular, negative volatility feedback is allowed. We have shown that this substantially increases the permissible parameter space and thereby the flexibility of the model in capturing the stylized facts of economic and financial data.

The availability of the necessary and sufficient conditions allows us to test economic theories within the unrestricted framework which were, by construction, excluded in the restricted version. On the other hand, our results also highlight the limitations of the model, e.g. the bivariate process of order  $(1, 1)$  does not make it possible to test theories which imply negative volatility spillovers in both directions.

Finally, we should highlight that our results do not only hold for the CCC process but also for the various parameterizations of the TVCC formulation.

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## A Second and Fourth Moments

### A.1 Unconditional Variances

In this Appendix, we present expressions for the unconditional second moments of the squared errors.

Define  $p^* = \max(p, q)$ , and set  $\mathbf{C}_t^{(l)} = [c_{ij,t}^{(l)}]_{i,j=1,\dots,N} = \mathbf{A}^{(l)}\mathbf{Z}_t + \mathbf{B}^{(l)}$ ,  $l = 1, \dots, p^*$ , where  $\mathbf{Z}_t = \text{diag}\{\mathbf{z}_t^{\wedge 2}\}$ ,  $\mathbf{A}^{(l)} = 0$  if  $l > q$  and  $\mathbf{B}^{(l)} = 0$  if  $l > p$ . Note that  $\{\mathbf{C}_t^{(l)}\}$  is a sequence of *i.i.d.* random matrices (for  $\mathbf{A}^{(l)} \neq 0$ ) such that  $\mathbf{C}_t^{(l)}$  is independent of  $\mathbf{h}_t$ .

By equation (2) we may rewrite equation (3) as

$$\mathbf{h}_t = \boldsymbol{\mu} + \sum_{l=1}^{p^*} \mathbf{C}_{t-l}^{(l)} \mathbf{h}_{t-l}. \quad (\text{A1})$$

Next we shall make use of the following notation. Let  $\boldsymbol{\Gamma}(L) = \mathbf{I}_N - \sum_{l=1}^{p^*} \boldsymbol{\Gamma}^{(l)} L^l$  where  $\boldsymbol{\Gamma}^{(l)} = [\gamma_{ij}^{(l)}]_{i,j=1,\dots,N} = \mathbb{E}(\mathbf{C}_t^{(l)})$ , with  $\gamma_{ij}^{(l)} = a_{ij}^{(l)} E(z_{it}^2) + b_{ij}^{(l)}$ . Set  $\gamma(L) = 1 - \sum_{l=1}^{N \times p^*} \gamma_l L^l = \det[\boldsymbol{\Gamma}(L)]$ . We have assumed  $\gamma_{N \times p^*} \neq 0$ , that is  $\gamma(L)$  is a scalar polynomial of order  $N \times p^*$ . Define also  $\tilde{\boldsymbol{\omega}} = [\tilde{\omega}_i]_{i=1,\dots,N} = \text{adj}[\boldsymbol{\Gamma}(1)]\boldsymbol{\mu}$ .



**Assumption 3 (Stationarity)** *The roots of  $\det[\Gamma(z)]$  lie outside the unit circle.*

Notice that when  $\mathbf{A}^{(r)} \geq 0$ ,  $r = 1, \dots, q$ , the stationarity assumption implies the invertibility condition.

**Lemma 3** *When Assumption (A3) holds, the unconditional variances of the elements of  $\boldsymbol{\varepsilon}_t$ ,  $\boldsymbol{\mu}_2 = [\mu_{2,i}]_{i=1, \dots, N} = \mathbb{E}(\mathbf{h}_t)$ , exist if  $\tilde{\omega} > 0$ , and are given by*

$$\boldsymbol{\mu}_2 = \frac{1}{\gamma(1)} \tilde{\omega}. \quad (\text{A2})$$

Let  $\tilde{\omega} > 0$  hold. Then under Assumption (A3) the vector UECCC-GARCH( $p, q$ ) model has a weakly stationary solution. Moreover, this solution is unique and is also strictly stationary and ergodic (see Proposition 3.1 in Jeantheau, 1998).

**Remark 3** *Recall that He and Teräsvirta (2004) assume all the parameters in  $\boldsymbol{\mu}$  to be positive, and in  $\mathbf{A}(L)$  and  $\mathbf{B}(L)$  to be non-negative. In this situation and under Assumption (A3) the positivity of  $\tilde{\omega}$  is guaranteed by construction (see Remark 3 in their paper). In sharp contrast in the unrestricted model, we allow some of these parameters to be negative and hence the condition  $\tilde{\omega} > 0$  has to be checked.*

## A.2 Fourth-Moment Structure of the process of order (1, 1)

To keep this article relatively self-contained, we briefly review the main theoretical results of He and Teräsvirta (2004) on the fourth-moment structure of the bivariate UECCC-GARCH(1, 1) process defined in equation (5).<sup>12</sup> Equation (A1) becomes

$$\mathbf{h}_t = \boldsymbol{\mu} + \mathbf{C}_{t-1} \mathbf{h}_{t-1},$$

where for typographical convenience we have set  $\mathbf{C}_{t-1} = \mathbf{C}_{t-1}^{(1)}$ .

Assume  $E(z_{it}^2 z_{jt}^2) < \infty$ ,  $i, j = 1, 2$ , and let  $\boldsymbol{\Gamma}_C = \mathbb{E}(\mathbf{C}_t)$ ,  $\boldsymbol{\Gamma}_{Z \otimes Z} = \mathbb{E}(\mathbf{Z}_t \otimes \mathbf{Z}_t)$  and  $\boldsymbol{\Gamma}_{C \otimes C} = \mathbb{E}(\mathbf{C}_t \otimes \mathbf{C}_t)$ . Notice that under the assumption of conditional normality we have  $\boldsymbol{\Gamma}_C = \mathbf{A} + \mathbf{B}$  and  $\boldsymbol{\Gamma}_{Z \otimes Z} = \text{diag}\{3, 1 + 2\rho^2, 1 + 2\rho^2, 3\}$ .

Moreover, let  $\lambda(\boldsymbol{\Gamma}_{C \otimes C})$  denote the modulus of the largest eigenvalue of  $\boldsymbol{\Gamma}_{C \otimes C}$ . Then the matrix  $\mathbb{E}[\boldsymbol{\varepsilon}_t^{\wedge 2} (\boldsymbol{\varepsilon}_t^{\wedge 2})']$  of the fourth moments of  $\{\boldsymbol{\varepsilon}_t\}$  exists if

$$\lambda(\boldsymbol{\Gamma}_{C \otimes C}) < 1, \quad (\text{A3})$$

and

$$\text{adj}(\tilde{\boldsymbol{\Gamma}}_{C \otimes C}) [\text{vec}(\boldsymbol{\mu} \boldsymbol{\mu}') + \boldsymbol{\Gamma}_C^\mu] > 0, \quad (\text{A4})$$

where

$$\begin{aligned} \tilde{\boldsymbol{\Gamma}}_{C \otimes C} &= \mathbf{I}_4 - \boldsymbol{\Gamma}_{C \otimes C}, \\ \boldsymbol{\Gamma}_C^\mu &= (\boldsymbol{\Gamma}_C \otimes \boldsymbol{\mu} + \boldsymbol{\mu} \otimes \boldsymbol{\Gamma}_C) (\boldsymbol{\mu}' \otimes \mathbf{I}_2) \text{vec}(\mathbf{I}_2 - \boldsymbol{\Gamma}_C)^{-1}. \end{aligned}$$

<sup>12</sup>The papers by Karanasos (1999b) and Hafner (2003) also contain results on fourth moments of MGARCH models.

Denote  $\boldsymbol{\mu}_4 = [\mu_{4,ij}]_{i,j=1,2} = \mathbb{E}[\boldsymbol{\varepsilon}_t^{\wedge 2}(\boldsymbol{\varepsilon}_t^{\wedge 2})']$ . Notice that  $\boldsymbol{\mu}_4$  is a square matrix. Under equations (A3)-(A4),

$$\boldsymbol{\mu}_4 = \boldsymbol{\Gamma}_{Z \otimes Z} \{(\tilde{\boldsymbol{\Gamma}}_{C \otimes C})^{-1} [\text{vec}(\boldsymbol{\mu}\boldsymbol{\mu}') + \boldsymbol{\Gamma}_C^\mu]\}.$$

(see Corollary 2 in He and Teräsvirta, 2004).

**Remark 4** Condition (A4) is needed because by Proposition 1 we allow some of the parameters in the vector  $\boldsymbol{\mu}$  and the matrix  $\mathbf{B}$  to take negative values. Under the stricter assumption made by He and Teräsvirta (2004), namely that all parameters of the process are non-negative, condition (A4) is directly satisfied.

## B Proofs

**Proof of Lemma 1.** Multiply both sides of equation (3) by  $\text{adj}[\mathbf{B}(L)] = \beta(L)[\mathbf{B}(L)]^{-1}$ .

■

**Proof of Corollary 1.** The proof follows immediately from Lemma 1. ■

**Proof of Theorem 1.** From the ARCH( $\infty$ ) expansion, that is equation (7), it follows that each conditional variance,  $h_{it}$ , admits an infinite moving-average representation in terms of the  $N$  convolutions of the GARCH kernels ( $\Psi_{ij}(L)$ ,  $j = 1, \dots, N$ ) and the corresponding squared errors. Thus the proof follows by applying Theorem 1 in Tsai and Chan (2008) to each of the  $N^2$  kernels. ■

**Proof of Proposition 1.** Initially we obtain the conditions in terms of the  $\omega$ 's,  $\alpha$ 's and  $\beta$ 's. That is (C2'):  $\phi_1 \alpha_{ij}^{(1)} + \alpha_{ij}^{(2)} > 0$ ; (C3'a):  $\psi_{ij}^{(1)} = \alpha_{ij}^{(1)} \geq 0$  and (C3'b):  $\psi_{ij}^{(2)} = \beta_1 \alpha_{ij}^{(1)} + \alpha_{ij}^{(2)} \geq 0$ . Note that since  $\beta_1 = \phi_1 + \phi_2$ , if  $\phi_2 > 0$  then  $\alpha_{ij}^{(1)} \phi_1 + \alpha_{ij}^{(2)} > 0$  directly implies  $\psi_{ij}^{(2)} \geq 0$  and vice versa if  $\phi_2 < 0$ . Finally, using the expressions in equation (6) and the fact that  $\phi_1 = \beta_1 - \phi_2 = b_{11} + b_{22} - \phi_2$  the conditions are rewritten in terms of the  $\mu$ 's,  $a$ 's and  $b$ 's. ■

**Proof of Corollary 2.** The assumption that  $\phi_1 > |\phi_2|$  implies  $\beta_1 = \phi_1 + \phi_2 = b_{11} + b_{22} \geq 0$ . Hence,  $b_{11}$  and  $b_{22}$  cannot be negative simultaneously. ■

**Proof of Corollary 3.** Conditions (C3'a) and (C2') (when  $\phi_2 > 0$ ) or (C3'b) (when  $\phi_2 < 0$ ) imply that both elements of the same row ( $b_{i1}, b_{i2}$ ,  $i = 1, 2$ ) cannot take negative values. ■

**Proof of Corollary 4.** Let the two off-diagonal elements of the  $\mathbf{B}$  matrix be negative and the other two positive. We will show that if conditions (C1') and (C3'a) in Proposition 1 are satisfied, conditions (C2') and (C3'b) are violated. First we will show that the case  $\phi_2 < 0$  or (since  $\phi_1 > 0$ )  $\beta_2 = -\phi_1 \phi_2 = b_{12} b_{21} - b_{11} b_{22} > 0$  violates the constraints of condition (C3'b). Under condition (C3'a) the latter condition amounts to

$$\begin{aligned} b_{11} a_{11} &\geq |b_{12}| a_{21}, & b_{11} a_{12} &\geq |b_{12}| a_{22}, \\ b_{22} a_{21} &\geq |b_{21}| a_{11}, & b_{22} a_{22} &\geq |b_{21}| a_{12}. \end{aligned}$$

Then if we multiply the two inequalities of the first column we get  $b_{11}b_{22} \geq b_{12}b_{21}$  which contradicts  $\phi_2 < 0$ . Next, we will focus our attention on the case  $\phi_2 > 0$ . Notice that  $\phi_1, \phi_2 > 0$  implies that  $\beta_2 < 0$  or  $b_{12}b_{21} < b_{11}b_{22}$ . Since  $\phi_2 > 0$  condition (C3'b) is redundant. Under condition (C3'a) and the fact that  $\phi_2 < b_{11}, b_{22} < \phi_1$  condition (C2') amounts to

$$\begin{aligned} (b_{11} - \phi_2)a_{11} &> |b_{12}|a_{21}, & (b_{11} - \phi_2)a_{12} &> |b_{12}|a_{22}, \\ (b_{22} - \phi_2)a_{21} &> |b_{21}|a_{11}, & (b_{22} - \phi_2)a_{22} &> |b_{21}|a_{12}. \end{aligned}$$

Notice that the two inequalities of the first column imply that  $\frac{|b_{12}|}{(b_{11}-\phi_2)} < \frac{a_{11}}{a_{21}} < \frac{(b_{22}-\phi_2)}{|b_{21}|}$ , while those of the second column imply that  $\frac{|b_{21}|}{(b_{22}-\phi_2)} < \frac{a_{22}}{a_{12}} < \frac{(b_{11}-\phi_2)}{|b_{12}|}$ . Since  $\frac{b_{21}}{(b_{11}-\phi_2)} = \frac{(b_{22}-\phi_2)}{b_{12}}$  the four inequalities cannot be satisfied simultaneously. ■

**Proof of Corollary 5.** From the previous corollaries we know that not more than two entries of  $\mathbf{B}$  can be negative. That is, neither the two diagonal, the two off-diagonal nor the two entries in one row can be negative. Further, assume that conditions (C1') and (C3'a) hold. Let also, without loss of generality,  $b_{11}, b_{21} > 0$ , and one or both elements in the second column of the  $\mathbf{B}$  matrix be negative. We will examine the two cases where  $\phi_2 \leq 0$ . Notice that when only  $b_{12} < 0$  then  $\phi_2 > 0$ , whereas when only  $b_{22} < 0$  then  $\phi_2 < 0$ . Moreover, when  $b_{12}, b_{22} < 0$  then  $\phi_2 \geq 0$  if  $|b_{12}b_{21}| \geq |b_{11}b_{22}|$ . If  $\phi_2 < 0$  condition (C2') is redundant and condition (C3'b) becomes

$$a_{11}b_{21} \geq a_{21}|b_{22}|, \quad a_{12}b_{21} \geq a_{22}|b_{22}|.$$

The above inequalities hold when i) only  $b_{22} < 0$  and ii)  $b_{12}, b_{22} < 0$  and  $|b_{12}b_{21}| < |b_{11}b_{22}|$ . Notice that the last inequality implies that  $\frac{|b_{12}|}{b_{11}} < \frac{|b_{22}|}{b_{21}}$ . If  $\phi_2 > 0$  we only need to check condition (C2'). This condition reduces to

$$a_{11}b_{21} > a_{21}|b_{22} - \phi_2|, \quad a_{12}b_{21} > a_{22}|b_{22} - \phi_2|,$$

since  $\frac{b_{12}}{(b_{11}-\phi_2)} = \frac{b_{22}-\phi_2}{b_{21}}$  and hence  $b_{22} < \phi_2$ . The above inequalities hold when i) only  $b_{12} < 0$  and ii)  $b_{12}, b_{22} < 0$  and  $|b_{12}b_{21}| > |b_{11}b_{22}|$ . ■

**Proof of Corollary 6.** Let condition (C1') in Proposition 1 hold. We will examine the two cases where  $b_{11} \leq 1$ . First, assume  $b_{11} < 1$ . If  $b_{12} < 0, b_{22} \leq 0$  and the other two parameters in  $\mathbf{B}$  are positive then the two inequalities in equation (9) rule out the possibility that  $\mu_1 \leq 0$ . Thus equation (9) becomes

$$\begin{aligned} \mu_1 &> 0, \mu_2 \leq 0, \text{ and } \frac{|\mu_2|}{\mu_1} < \frac{b_{21}}{(1-b_{11})} \text{ or} \\ \mu_1 &> 0, \mu_2 \geq 0, \text{ and } \frac{\mu_2}{\mu_1} < \frac{(1-b_{22})}{|b_{12}|}. \end{aligned}$$

Note that the inequality in the first expression is more binding than the one in the second expression since from the invertibility condition  $\frac{(1-b_{22})}{|b_{12}|} > \frac{b_{21}}{(1-b_{11})}$ . Further, if  $b_{22} \leq 0$  and the other three parameters in  $\mathbf{B}$  are positive then the two inequalities in equation (9) rule

out the possibility that  $\mu_1, \mu_2 \leq 0$ . Thus equation (9) reduces to

$$\begin{aligned} \mu_1 &> 0, \mu_2 \geq 0, \text{ or } \mu_1 \geq 0, \mu_2 > 0, \text{ or} \\ \mu_1 &\leq 0, \mu_2 > 0, \text{ and } \frac{|\mu_1|}{\mu_2} < \frac{b_{12}}{(1-b_{22})}, \text{ or} \\ \mu_1 &> 0, \mu_2 \leq 0, \text{ and } \frac{\mu_2}{|\mu_1|} < \frac{(1-b_{22})}{b_{12}}, \end{aligned}$$

since from the invertibility condition  $\frac{b_{12}}{(1-b_{22})} < \frac{(1-b_{11})}{b_{21}}$ . Second, let  $b_{11} > 1$ . If  $b_{12} < 0$ ,  $b_{22} \leq 0$  and  $b_{21} > 0$  then the two inequalities in equation (9) rule out the possibility that  $\mu_1 \leq 0$  and  $\mu_2 \geq 0$ . In this case part (a) of Proposition 1 becomes

$$\begin{aligned} \mu_1 &\leq 0, \mu_2 < 0, \text{ and } \frac{|\mu_1|}{|\mu_2|} < \frac{|1-b_{11}|}{b_{21}} \text{ or} \\ \mu_1 &> 0, \mu_2 \geq 0, \text{ and } \frac{\mu_2}{\mu_1} < \frac{(1-b_{22})}{|b_{12}|} \text{ or} \\ \mu_1 &> 0, \mu_2 \leq 0, \end{aligned}$$

since from the invertibility condition  $\frac{(1-b_{22})}{|b_{12}|} < \frac{b_{21}}{|1-b_{11}|}$ . ■

**Proof of Lemma 3.** Taking expectations on both sides of equation (A1) yields  $\Gamma(1)\mathbb{E}(\mathbf{h}_t) = \boldsymbol{\mu}$ . Multiply both sides of this expression by  $\text{adj}[\Gamma(1)] = \gamma(1)[\Gamma(1)]^{-1}$  to get equation (A2). ■