

Conrad, Christian

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for the Hyperbolic GARCH Model

Christian Conrad

KOF

ETH Zurich
KOF Swiss Economic Institute
WEH D 4
Weinbergstrasse 35
8092 Zurich
Switzerland

Phone +41 44 632 42 39
Fax +41 44 632 12 18
www.kof.ethz.ch
kof@kof.ethz.ch

NON-NEGATIVITY CONDITIONS FOR THE HYPERBOLIC GARCH MODEL

Christian Conrad

ETH Zurich – KOF Swiss Economic Institute

WEH D4, Weinbergstrasse 35, 8092 Zurich, Switzerland

E-mail: conrad@kof.ethz.ch

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Abstract

In this article we derive conditions which ensure the non-negativity of the conditional variance in the Hyperbolic GARCH(p, d, q) (HYGARCH) model of Davidson (2004). The conditions are necessary and sufficient for $p \leq 2$ and sufficient for $p > 2$ and emerge as natural extensions of the inequality constraints derived in Nelson and Cao (1992) for the GARCH model and in Conrad and Haag (2006) for the FIGARCH model. As a by-product we obtain a representation of the ARCH(∞) coefficients which allows computationally efficient multi-step-ahead forecasting of the conditional variance of a HYGARCH process. We also relate the necessary and sufficient parameter set of the HYGARCH to the necessary and sufficient parameter sets of its GARCH and FIGARCH components. Finally, we analyze the effects of erroneously fitting a FIGARCH model to a data sample which was truly generated by a HYGARCH process. An empirical application of the HYGARCH(1, d , 1) model to daily NYSE data illustrates the importance of our results.

Keywords: Inequality constraints, fractional integration, long memory GARCH processes.

JEL Classification: C22, C52, C53.

1 Introduction

Long-range dependence in the absolute or squared observations of many macro and financial time series – first reported in the seminal articles of Ding et al. (1993) and Ding and Granger (1996) – has become a widely documented stylized fact. For a recent review article on long memory processes and their applications in economics and finance see Henry and Zaffaroni (2003). The Bollerslev (1986) stationary GARCH model implies an exponentially decaying autocorrelation function (ACF) for the squared innovations and hence can not reproduce the observed persistence in the conditional variance of such time series. To overcome this shortcoming Baillie et al. (1996) suggested the fractionally integrated GARCH (FIGARCH) model which allows for a hyperbolic decay of shocks to the conditional variance and so explicitly captures the empirical observation. However, by construction the FIGARCH does not specify a covariance stationary process. Consequently, Davidson (2004) proposed the hyperbolic GARCH (HYGARCH) model which nests both GARCH and FIGARCH as special cases. The HYGARCH shares with the GARCH model the desired property of covariance stationarity while at the same time it obeys hyperbolically decaying impulse response coefficients as the FIGARCH. Moreover, it provides a natural framework for testing geometric versus hyperbolic decay.

Although proposed only recently, the HYGARCH model has proven to be successful in modeling the long-run dynamics in the second conditional moment of several financial time series. For the sake of brevity, we concentrate on a few examples from the literature. Davidson (2004) shows that the HYGARCH is capable of modeling the volatility dynamics in three Asian currencies during the crisis period 1997–1998. While it is usually believed that the modeling of such a period with its extreme changes from low to high volatility and vice versa requires some sort of external switching mechanism, it is the advantage of the HYGARCH that it can explain such behavior by a simple endogenous mechanism entirely driven by its innovation process. *Ñíguez und Rubia* (2006) apply the HYGARCH model to a portfolio of five exchange rates and report that it clearly outperforms simpler GARCH variants in terms of out-of-sample forecasting. Their analysis reveals that the correct modeling of the volatility’s long-run component is important even if one is only interested in short-run volatility forecasting. Finally, Tang and Shieh (2006) compare the performance of FIGARCH and HYGARCH models in predicting Value-at-Risk for three stock index futures markets. Based on Kupiec LR tests their results show that the HYGARCH model dominates over the FIGARCH.

In all three models – GARCH, FIGARCH and HYGARCH – the conditional variance can be expressed as an infinite sum of weighted lagged squared residuals. A necessary condition for such a specification to define a valid non-negative conditional variance process is that all the weights in this sum – the so-called ARCH(∞) coefficients – are non-negative. To ensure this, conditions have to be placed on the parameters of the process. Such conditions have been derived by Nelson and Cao (1992) and Tsai and Chan (2007) for the GARCH model and by Conrad and Haag (2006) for the FIGARCH model. Davidson (2004) simply assumes that the parameters of the HYGARCH process are a priori chosen such that the

non-negativity of all the ARCH(∞) coefficients is satisfied without further investigating the issue.

In this article we first show that the ARCH(∞) coefficients of the HYGARCH model obey a recursive representation very similar to the one obtained by Conrad and Haag (2006) for the ARCH(∞) coefficients of the FIGARCH model. Second, we establish necessary and sufficient conditions for the non-negativity of the conditional variance in the HYGARCH(p, d, q) with $p \leq 2$ and sufficient conditions for $p > 2$. The availability of such conditions is of importance for any practitioner estimating HYGARCH models, since they are a first inevitable check of model validation. In particular, our results are of interest for those who use the HYGARCH model for forecasting because a misspecified HYGARCH will generate negative conditional variance forecasts out-of-sample with positive probability even if this did not occur in-sample. Our recursive representation provides an efficient way of calculating multi-step-ahead volatility forecasts, while the inequality constraints guarantee the non-negativity of the forecasted conditional variances. Moreover, although the HYGARCH can be thought of as a linear combination of a GARCH and FIGARCH component we show that the necessary and sufficient HYGARCH set does not simply coincide with the intersection of the necessary and sufficient sets of its components.

We investigate a further central question for the empirical researcher: what are the effects of erroneously applying a FIGARCH model to a data sample which was truly generated by a HYGARCH process? It is shown that the persistence parameter which will be estimated for the misspecified FIGARCH model, can be expressed as a function of the persistence parameter of the underlying HYGARCH model and the weight on the FIGARCH component. The two persistence parameters will coincide only in the trivial case, namely if the weight on the FIGARCH component is one. We illustrate this point by means of a Monte-Carlo simulation.

Finally, an empirical application of the HYGARCH model to NYSE data highlights the practical importance of our theoretical results.

The remainder of the article is organized as follows. Section 2 defines the HYGARCH model, presents its properties, and sets out assumptions and notation. In Section 3 we derive the inequality constraints which ensure the non-negativity of the conditional variance of the HYGARCH(p, d, q) process. A discussion of the relationship between the persistence parameters in the HYGARCH and the FIGARCH model is provided in Section 4. Section 5 presents an empirical example and Section 6 concludes the article. All proofs are deferred to the appendix.

2 The Hyperbolic GARCH Model

In this section we present the HYGARCH model as a natural extension of the GARCH and FIGARCH models and introduce some notation which makes the derivations in the subsequent sections more tractable.

2.1 From GARCH and FIGARCH to HYGARCH

Following Bollerslev (1986) we define a GARCH(p, q^*) process $\{\varepsilon_t, t \in \mathbb{Z}\}$ by the equations

$$\varepsilon_t = Z_t \sqrt{h_t}, \quad (1)$$

where $\{Z_t, t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed random variables with $\mathbf{E}[Z_t] = \mathbf{E}[Z_t^2 - 1] = 0$, and

$$B(L)h_t = \omega + \alpha(L)\varepsilon_t^2 \quad (2)$$

for some $\omega \in \mathbb{R}^+$ and lag polynomials $\alpha(L)$, $B(L)$ given by $\alpha(L) = \sum_{i=1}^{q^*} \alpha_i L^i$ and $B(L) = 1 - \sum_{i=1}^p \beta_i L^i$ with L being the lag operator. Equation (2) is often rewritten as

$$\Phi(L)\varepsilon_t^2 = \omega + B(L)v_t \quad (3)$$

the so-called “ARMA in squares” representation, where $v_t = \varepsilon_t^2 - h_t$ and $\Phi(L) = 1 - \sum_{i=1}^q \phi_i L^i$ with $q = \max\{p, q^*\}$ and $\phi_i = \alpha_i + \beta_i$ for $i = 1, \dots, q$. By construction v_t is a martingale difference sequence with respect to the filtration \mathcal{F}_t generated by $\{\varepsilon_s, s \leq t\}$.

Under the assumption that the roots of the polynomial $B(L)$ lie outside the unit circle, the GARCH(p, q^*) process obeys an ARCH(∞) representation of the form

$$h_t = \frac{\omega}{B(1)} + \Psi^{GA}(L)\varepsilon_t^2 = \frac{\omega}{B(1)} + \sum_{i=1}^{\infty} \psi_i^{GA} \varepsilon_{t-i}^2 \quad (4)$$

with

$$\Psi^{GA}(L) = \frac{B(L) - \Phi(L)}{B(L)} = \frac{\alpha(L)}{B(L)}. \quad (5)$$

For example, for a simple GARCH(1, 1) the ψ_i^{GA} coefficients are defined recursively by $\psi_1^{GA} = \phi_1 - \beta_1 = \alpha_1$ and $\psi_i^{GA} = \beta_1 \psi_{i-1}^{GA}$ for $i \geq 2$. Obviously, for a GARCH model to be well defined restrictions on the parameters $(\beta_1, \dots, \beta_p, \alpha_1, \dots, \alpha_{q^*})$ have to be imposed ensuring that $h_t \geq 0$ for all t almost surely (a.s.). Bollerslev (1986) simply assumed the non-negativity of all the α and β coefficients which (together with non-negative starting values, say, h_{-1}, \dots, h_{-p} and $\varepsilon_{-1}, \dots, \varepsilon_{-q^*}$) is by equation (2) of course a sufficient condition for the non-negativity of h_t for $t \geq 0$. However, as can be seen from equation (4) the non-negativity of all the α and β coefficients is not necessary. Instead a necessary and sufficient condition must only require the weaker condition that $\psi_i^{GA} \geq 0$ for $i = 1, 2, \dots$, whereby the ARCH(∞) coefficients can be directly expressed in terms of the α and β coefficients. Consequently, Nelson and Cao (1992) derived necessary and sufficient conditions for $p \leq 2$ and sufficient conditions for $p > 2$. E.g. the conditions for the GARCH(1, 2) are $\psi_1 = \alpha_1 \geq 0$, $\psi_2 = \beta_1 \alpha_1 + \alpha_2 \geq 0$ and $0 \leq \beta_1 < 1$. Note, that these conditions allow for $\alpha_2 < 0$. The condition $\beta_1 < 1$ is required to guarantee invertibility.¹ Recently, Tsai and Chan (2007) have provided the surprising result that the conditions stated in Nelson and Cao (1992) for $p > 2$ are not only sufficient, but also necessary.

In a more general setting equation (1) together with $h_t = \omega^* + \Psi(L)\varepsilon_t^2$, $\Psi(L) = \sum_{j=1}^{\infty} \psi_j L^j$ with $\psi_j \geq 0$ for all j , defines what is called an ARCH(∞) process (see Zaffaroni, 2004). From equation (1) we have $\mathbf{E}[\varepsilon_t] = 0$ and $\mathbf{Cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$ for $j \geq 1$. Hence, covariance stationarity of such a process ε_t requires that $\Psi(1) < 1$ which implies $\mathbf{E}[\varepsilon_t^2] = \omega^*/(1 - \Psi(1)) < \infty$. The GARCH model can be considered as a specific case of an ARCH(∞) with a finite parametrization $\omega^* = \omega/B(1)$ and $\Psi(L) = \Psi^{GA}(L)$. For the GARCH model the condition $\Psi^{GA}(L) < 1$ is equivalent to the well-known condition that $\Phi(1) > 0$. If the polynomial $\Phi(L)$ has a unit root and hence can be factored as $\Phi(L) = \tilde{\Phi}(L)(1-L)$ the process ε_t is referred to as an integrated GARCH (IGARCH) model. Although possibly strictly stationary, the IGARCH process is not covariance stationary since its unconditional second moment does not exist.² While the stable GARCH is characterized by an exponentially decaying impulse response function (IRF), the IRF of the IGARCH is a constant indicating complete persistence of shocks to the conditional variance. Thus, the behavior of the IRFs of the GARCH and IGARCH models are very much the same as those of ARMA and ARIMA models for the mean which are said to be integrated of order zero and one, respectively. In sharp contrast, shocks to the conditional variance of many financial times series are empirically found to be highly persistent, i.e. decaying at a slow hyperbolic rate, but nevertheless transitory. A model that can deal with such long-range dependence in the conditional second moment of a time series is the FIGARCH model by Baillie et al. (1996). The FIGARCH allows for fractional orders of integration between zero and one, and implies hyperbolically decaying impulse response weights. Baillie et al. (1996) define the FIGARCH via

$$(1-L)^d \Phi(L) \varepsilon_t^2 = \omega + B(L) v_t, \quad (6)$$

for some ω , $\Phi(L)$ and $B(L)$ as before and $0 \leq d \leq 1$ being the fractional differencing parameter. The FIGARCH implies the ARCH(∞) representation

$$h_t = \frac{\omega}{B(1)} + \Psi^{FI}(L) \varepsilon_t^2 = \frac{\omega}{B(1)} + \sum_{i=1}^{\infty} \psi_i^{FI} \varepsilon_{t-i}^2$$

with

$$\Psi^{FI}(L) = 1 - \frac{(1-L)^d \Phi(L)}{B(L)}. \quad (7)$$

For example, the FIGARCH(1, d , 1) coefficients ψ_i are given by $\psi_1^{FI} = d + \phi_1 - \beta_1$ and $\psi_i^{FI} = \beta_1 \psi_{i-1}^{FI} + (f_i - \phi_1)(-g_{i-1})$ for $i \geq 2$, where f_i and g_i are functions of the fractional differencing parameter d and will be defined in the next subsection. Similarly, as in the GARCH model conditions on the parameters $(\beta_1, \dots, \beta_p, d, \phi_1, \dots, \phi_q)$ of the FIGARCH process have to be imposed to guarantee the non-negativity of the conditional variance. Conrad and Haag (2006) extend the results of Nelson and Cao (1992) to the FIGARCH set-up and establish necessary and sufficient conditions for $p \leq 2$ and sufficient conditions for $p > 2$. Endowed with the additional flexibility of the fractional differencing parameter the FIGARCH proved to be successful in modeling the long-run features in the volatility of many time series such as

stock market returns, exchange rates and inflation (see e.g. Bollerslev and Mikkelsen, 1996, Tse, 1998, or Conrad and Karanasos, 2005). In contrast to the exponentially decaying IRF of the GARCH model the FIGARCH (with $0 < d < 1$) is characterized by hyperbolically decaying impulse response weights π_j , $j = 1, 2, \dots$, of the order $\pi_j = O(j^{d-1})$ (for details see Conrad and Karanasos, 2006). The FIGARCH nests the GARCH model for $d = 0$ and the IGARCH model for $d = 1$. However, the major drawback of the FIGARCH is that for any $0 < d < 1$ we have $\Psi^{FI}(1) = 1$ and hence the unconditional variance of ε_t does not exist. As in the IGARCH case the FIGARCH process is not covariance stationary.

Therefore, Davidson (2004) constructed the hyperbolic GARCH (HYGARCH) model in a way to overcome this drawback. The HYGARCH is obtained by modifying equation (6) to

$$\Phi(L)\left((1-\tau) + \tau(1-L)^d\right)\varepsilon_t^2 = \omega + B(L)v_t, \quad (8)$$

by incorporating the additional parameter $\tau \geq 0$. Clearly, the HYGARCH nests the GARCH model under the restriction $\tau = 0$ (or $d = 0$) and the FIGARCH model under the restriction $\tau = 1$. When $d = 1$ the parameter τ becomes an autoregressive root and the HYGARCH reduces to either a stationary GARCH ($\tau < 1$), an IGARCH ($\tau = 1$) or an explosive GARCH ($\tau > 1$). Rewriting equation (8) as

$$h_t = \frac{\omega}{B(1)} + \Psi^{HY}(L)\varepsilon_t^2 = \frac{\omega}{B(1)} + \sum_{i=1}^{\infty} \psi_i^{HY} \varepsilon_{t-i}^2$$

with

$$\Psi^{HY}(L) = \tau\Psi^{FI}(L) + (1-\tau)\Psi^{GA}(L)$$

shows that the HYGARCH is constructed such that its ARCH(∞) coefficients are a linear combination of the ARCH(∞) coefficients from a FIGARCH and a GARCH model with weights τ and $(1-\tau)$, respectively, i.e. can be written as

$$\psi_i^{HY} = \tau\psi_i^{FI} + (1-\tau)\psi_i^{GA} \quad \text{for } i = 1, 2, \dots \quad (9)$$

In the following, we will refer to the “FIGARCH component” and the “GARCH component” of the HYGARCH process meaning the processes which generate the ψ_i^{FI} and ψ_i^{GA} coefficients. To check whether the HYGARCH is covariance stationary or not, the behavior of $\Psi^{HY}(1)$ has to be investigated. The trivial case arises when $d = 0$. Then the HYGARCH reduces to a GARCH model with the corresponding covariance stationarity condition $\Phi(1) > 0$. The interesting case $0 < d \leq 1$ leads to $\Psi^{HY}(1) = \tau + (1-\tau)\Psi^{GA}(1)$. Checking the condition $\Psi^{HY}(1) < 1$ reveals that the HYGARCH process will be covariance stationary provided that $(1-\tau)\Phi(1) > 0$. Hence, for any $0 \leq \tau < 1$ the HYGARCH will be a covariance stationary process, if the GARCH component fulfills the condition $\Phi(1) > 0$. In contrast to the FIGARCH model, the HYGARCH allows to combine the desired properties of hyperbolically decaying impulse response coefficients and covariance stationarity.³

Finally, we would like to point out that – under a small modification – it is possible to obtain a covariance stationary HYGARCH process even if $\tau > 1$. We call the new specification a generalized

HYGARCH model. The modification is to replace the assumption $\mathbf{E}[Z_t^2] = 1$ by $\mathbf{E}[Z_t^2] = \sigma_Z^2 < 1$. Hence, instead of equation (8) we define the generalized HYGARCH by

$$\left(1 - \sigma_Z^2 \Psi^{HY}(L)\right) \varepsilon_t^2 = \sigma_Z^2 \omega / B(1) + v_t. \quad (10)$$

For the generalized HYGARCH the covariance stationarity condition can be written as $\sigma_Z^2 \Psi^{HY}(1) < 1$. Now, rearranging terms shows that this condition is equivalent to

$$\sigma_Z^2(1 - \tau)\Phi(1) > (\sigma_Z^2 - 1)B(1) \quad (11)$$

which can be satisfied although $\tau > 1$. Note, that the ARCH(∞) representation of the generalized HYGARCH is still the same as for the Davidson (2004) HYGARCH.

As for the GARCH and the FIGARCH models restrictions on the parameters of the HYGARCH have to be imposed to ensure the non-negativity of the conditional variance. The remainder of this article will be devoted to deriving such conditions. In the following derivations we always assume that $0 < d < 1$ and $0 < \tau \neq 1$, since otherwise the inequality constraints of Nelson and Cao (1992) or Conrad and Haag (2006) can be directly applied. The following subsection introduces some more notation and assumptions.

2.2 Assumptions and Notation

We closely follow the notation used in Conrad and Haag (2006). We assume that the inverse roots λ_i , $i = 1, \dots, p$, of the polynomial $B(L)$ are real and $0 \neq |\lambda_i| < 1$ for $i = 1, \dots, p$. By $(\lambda_{(1)}, \lambda_{(2)}, \dots, \lambda_{(p)})$ we denote a certain ordering of those inverse roots. Additionally, we assume that the roots of $\Phi(L)$ lie outside the unit circle and $\Phi(L)$ and $B(L)$ have no common roots. Note, that these assumptions on the roots of $\Phi(L)$ and $B(L)$ imply that $\Phi(1) > 0$ and $B(1) > 0$, i.e. they guarantee the covariance stationarity of the GARCH component of the HYGARCH process and invertibility. The fractional differencing operator $(1 - L)^d$ can be expanded as $(1 - L)^d = \sum_{j=0}^{\infty} g_j L^j$, where the coefficients g_j are given by

$$g_j = f_j \cdot g_{j-1} = \prod_{i=1}^j f_i \quad \text{with} \quad f_j = \frac{j-1-d}{j} \quad \text{for } j = 1, 2, \dots$$

and $g_0 = 1$. Note, that $f_1 = -d < 0$, $f_2 = (1-d)/2 > 0$ and $f_j > 0$ for all $j > 2$ and hence $g_j < 0$ for all $j \geq 1$. It is easy to see that $f_j < f_{j+1}$ and $f_j \rightarrow 1$ as $j \rightarrow \infty$. Moreover, for $i > q \geq 0$ we define $F_i = -\sum_{l=0}^q \phi_l \prod_{j=l}^{i-1} f_{i-j}$ with $\phi_0 = -1$ and $\prod_{j=0}^{-1} = 1$, then $F_i < F_{i+1}$ and $F_i \rightarrow 1 - \phi_1 - \dots - \phi_q > 0$ as $i \rightarrow \infty$. Let $\Lambda_r = \sum_{i=1}^r \lambda_{(i)}$, $r \leq p$. It follows that $F_i^{(r)} = \Lambda_r F_{i-1} + F_i f_{i-q} \rightarrow (\Lambda_r + 1)(1 - \phi_1 - \dots - \phi_q)$ and the limit is positive provided that $\Lambda_r > -1$.

3 Non-negativity Conditions for HYGARCH(p, d, q)

From equation (9) it directly follows, that for $0 < \tau < 1$ a sufficient condition for the non-negativity of the ψ_i^{HY} coefficients is the non-negativity of all the ψ_i^{FI} and ψ_i^{GA} coefficients which can be easily guaranteed

by applying the results of Nelson and Cao (1992) and Conrad and Haag (2006) to the GARCH and the FIGARCH component separately. However, imposing such a sufficient condition will be by far too restrictive. Intuitively, this is because the non-negativity of the ARCH(∞) coefficients of the HYGARCH does not preclude the case that some or even all ARCH(∞) coefficients of one of the components are negative, as long as the other component's coefficients are positive and dominating. Additionally, in the case where $\tau > 1$ it is not clear how to proceed, because the weight on the ψ_i^{GA} coefficients will be negative and hence not even a sufficient condition is readily available.

In this section we focus on deriving necessary and sufficient conditions which do not necessarily require the non-negativity of all the ψ_i^{FI} and ψ_i^{GA} coefficients. We begin by stating a recursive representation for the ARCH(∞) coefficients of the HYGARCH(p, d, q).

Lemma 1. *In the HYGARCH(p, d, q) the sequence $\{\psi_i^{HY}, i = 1, 2, \dots\}$ can be written as*

$$\psi_i^{HY} = \psi_i^{HY(p)} \quad \text{where} \quad (12)$$

$$\psi_i^{HY(r)} = \lambda_{(r)} \psi_{i-1}^{HY(r)} + \psi_i^{HY(r-1)} \quad 1 < r \leq p, \quad i \geq 1, \quad (13)$$

with starting values $\psi_0^{HY(r)} = -\tau$, $r = 2, \dots, p$, and $\{\psi_i^{HY(1)}\}$ given by

$$\psi_i^{HY(1)} = \tau \left(-c_i + \sum_{j=1}^i \phi_j c_{i-j} \right) + (1 - \tau) \sum_{j=1}^i \lambda_{(1)}^{i-j} (\phi_j - \beta_j) \quad \text{for } i = 1, \dots, q \quad (14)$$

$$= \lambda_{(1)} \psi_{i-1}^{HY(1)} + \tau F_i(-g_{i-q}) \quad \text{for } i \geq q + 1 \quad (15)$$

where $c_i = \sum_{j=0}^i \lambda_{(1)}^{i-j} g_j$.

The representation given by Lemma 1 has an interesting and intuitive interpretation. For this, recall that an ARMA(p, q) can be rewritten as $(p-1)$ AR(1) processes with autoregressive parameters equal to the inverse roots of the autoregressive polynomial, whereby the first $(p-2)$ of those AR(1) processes have an innovation term which is itself an AR(1) and the last one an innovation term which is ARMA(1, q). Similarly, Lemma 1 expresses the ARCH(∞) coefficients of a HYGARCH(p, d, q) model as $(p-1)$ “AR(1)-type” expressions $\psi_i^{HY(r)}$ with autoregressive parameters $\lambda_{(r)}$ and “innovations” $\psi_i^{HY(r-1)}$ which are themselves “AR(1)-type”, apart from the last ones, $\psi_i^{HY(1)}$, which are the ARCH(∞) coefficients of a HYGARCH(1, d, q).

It is important to observe the analogy between the recursive structure of the ARCH(∞) coefficients in the HYGARCH model and the corresponding representation of the FIGARCH coefficients derived in Conrad and Haag (2006). The main difference comes from equations (14) and (15) which combine the ARCH(∞) coefficients of a GARCH(1, q) and a FIGARCH(1, d, q) to the ARCH(∞) coefficients of a HYGARCH(1, d, q) (with $\beta_1 = \lambda_{(1)}$). Clearly, for $\tau = 1$ the recursions coincide with those for the FIGARCH(p, d, q) (see Conrad and Haag, 2006).

The recursive representation given by Lemma 1 can be directly used for multi-step-ahead volatility

forecasting. The K -step-ahead volatility forecast can be constructed by recursive substitution in

$$\hat{h}_{t+k|t} = \mathbf{E}(h_{t+k}|\mathcal{F}_t) = \frac{\omega}{B(1)} + \Psi^{HY}(L)\mathbf{E}(\varepsilon_{t+k}^2|\mathcal{F}_t) = \frac{\omega}{B(1)} + \sum_{j=1}^{k-1} \psi_j^{HY} \hat{h}_{t+k-j|t} + \sum_{j=0}^{\infty} \psi_{k+j}^{HY} \varepsilon_{t-j}^2$$

for $k = 1, \dots, K$ and an appropriately chosen truncation lag in the second sum. Andersen et al. (2006, p. 805) present the corresponding recursions for the FIGARCH(1, d , 1) and argue that recursions for “higher order FIGARCH models or volatility forecast filters, may be defined in an analogous fashion”. Lemma 1 states a computationally efficient and exact representation of the ψ_i^{HY} coefficients for the general HYGARCH(p, d, q) volatility forecast filter, which can be directly inserted into the forecast formula.

In the following two subsections we exploit the recursive representation of Lemma 1 for deriving inequality constraints first for the empirically most relevant HYGARCH(1, d , 1) and then for the general HYGARCH(p, d, q).

3.1 HYGARCH(1, d , 1)

Recall that throughout this section we assume stationarity of the GARCH component ($-1 < \phi_1 < 1$), invertibility ($-1 < \beta_1 < 1$) and non-reducibility ($\phi_1 \neq \beta_1$). For the HYGARCH(1, d , 1) model the recursive representation of the ARCH(∞) coefficients stated above simplifies to

$$\psi_1^{HY} = \tau d + \phi_1 - \beta_1 \quad (16)$$

$$\psi_i^{HY} = \beta_1 \psi_{i-1}^{HY} + \tau(f_i - \phi_1)(-g_{i-1}) \quad \text{for } i \geq 2 \quad (17)$$

$$\text{and alternatively, } \psi_i^{HY} = \beta_1^2 \psi_{i-2}^{HY} + \tau(\beta_1(f_{i-1} - \phi_1) + (f_i - \phi_1)f_{i-1})(-g_{i-2}) \text{ for all } i \geq 3. \quad (18)$$

Equation (16) directly follows from equation (14) for $i = 1$. Equation (17) follows from equation (15) by realizing that $F_i = (f_i - \phi_1)$ when $q = 1$. Finally, equation (18) is an iterated version of equation (17).

Theorem 1. *The conditional variance of the HYGARCH(1, d , 1) is non-negative a.s. iff*

Case 1: $0 < \beta_1 < 1$

either $\psi_1^{HY} \geq 0$ and $\phi_1 \leq f_2$ or for $k > 2$ with $f_{k-1} < \phi_1 \leq f_k$ it holds that $\psi_{k-1}^{HY} \geq 0$.

Case 2: $-1 < \beta_1 < 0$

either $\psi_1^{HY} \geq 0$, $\psi_2^{HY} \geq 0$ and $\phi_1 \leq f_2(\beta_1 + f_3)/(\beta_1 + f_2)$ or for $k > 3$ with $f_{k-2}(\beta_1 + f_{k-1})/(\beta_1 + f_{k-2}) < \phi_1 \leq f_{k-1}(\beta_1 + f_k)/(\beta_1 + f_{k-1})$ it holds that $\psi_{k-1}^{HY} \geq 0$ and $\psi_{k-2}^{HY} \geq 0$.

The proof of Theorem 1 – which is presented in the appendix – follows the same arguments as the proof of Theorem 1 in Conrad and Haag (2006). The parameter τ in equations (17) and (18) simply serves as a scaling parameter but does not have an effect on the central argument used in the proof, namely that there always exists a k such that $f_{k-1} < \phi_1 \leq f_k$ or $f_{k-2}(\beta_1 + f_{k-1})/(\beta_1 + f_{k-2}) < \phi_1 \leq f_{k-1}(\beta_1 + f_k)/(\beta_1 + f_{k-1})$, respectively. Although the ARCH(∞) coefficients are complicated functions

of the underlying parameters, it turns out that checking two (Case 1) or three (Case 2) conditions is sufficient for ensuring the non-negativity of the conditional variance for all t .

With the following Proposition we relate the necessary and sufficient parameter set of a HYGARCH(1, d , 1), denoted by \mathcal{NS}^{HY} , to the necessary and sufficient parameter sets of its FIGARCH(1, d , 1) and GARCH(1, 1) components, denoted by \mathcal{NS}^{FI} and \mathcal{NS}^{GA} .⁴ Analyzing this relationship is of interest because – as discussed earlier – it is not a priori clear how these sets are related. Recall that even if we know that for a HYGARCH parameter combination $(\beta_1, d, \phi_1, \tau)$, (β_1, d, ϕ_1) is not an element of \mathcal{NS}^{FI} and/or (β_1, ϕ_1) not an element of \mathcal{NS}^{GA} this means only that there is at least one coefficient which is negative in the ψ_i^{FI} and/or ψ_i^{GA} sequence. What this implies for $\psi_i^{HY} = \tau\psi_i^{FI} + (1 - \tau)\psi_i^{GA}$ with $0 < \tau < 1$ or $\tau > 1$ has to be investigated.

Proposition 1. \mathcal{NS}^{HY} is related to \mathcal{NS}^{FI} and \mathcal{NS}^{GA} as follows.

Case 1: For $0 < \tau < 1$ there do not exist parameter combinations $(\beta_1, d, \phi_1, \tau) \in \mathcal{NS}^{HY}$ for which neither (β_1, d, ϕ_1) is an element of \mathcal{NS}^{FI} nor (β_1, ϕ_1) an element of \mathcal{NS}^{GA} .

Case 2: For $\tau > 1$ there do not exist parameter combinations $(\beta_1, d, \phi_1, \tau) \in \mathcal{NS}^{HY}$ for which (β_1, ϕ_1) is an element of \mathcal{NS}^{GA} but (β_1, d, ϕ_1) is not an element of \mathcal{NS}^{FI} .

Case 1 tells us that either all ψ_i^{FI} , all ψ_i^{GA} or both have to be non-negative to guarantee that $\psi_i^{HY} \geq 0$ for all i . Since \mathcal{NS}^{HY} is non empty, Case 1 implies that for $0 < \tau < 1$ and any $0 < d < 1$ the necessary and sufficient set for the HYGARCH(1, d , 1) model represented in the (ϕ_1, β_1) space lies entirely inside the union of the necessary and sufficient sets of its FIGARCH(1, d , 1) and GARCH(1, 1) components.

The intuition for the result in Case 2 is that for $\tau > 1$, the positive ψ_i^{GA} coefficients have negative weight and hence their negative effect on ψ_i^{HY} must be compensated by positive ψ_i^{FI} coefficients. However, note that when $\tau > 1$ it may well be that parameter combinations $(\beta_1, d, \phi_1, \tau) \in \mathcal{NS}^{HY}$ exist for which neither (β_1, d, ϕ_1) is an element of \mathcal{NS}^{FI} nor (β_1, ϕ_1) of \mathcal{NS}^{GA} .

Next, we provide a graphical representation of the admissible parameter set described by Theorem 1 and illustrate the result of Proposition 1. The left panel of Figure 1 plots the necessary and sufficient set for a HYGARCH(1, d , 1) model (solid) with $d = 0.3$ and $\tau = 0.4$. The dashed line characterizes the necessary and sufficient set for $d = 0.3$ but $\tau = 1$, i.e. for the FIGARCH(1, d , 1) component only. The necessary and sufficient set for the GARCH(1, 1) component is the lower triangular in the first quadrant. As can be seen, the HYGARCH set is a subset of the FIGARCH set in the second, third and fourth quadrant, while the HYGARCH set lies partly inside the FIGARCH set and partly inside the GARCH set in the first quadrant. It also shows the restrictiveness of the sufficient condition mentioned above which requires that both the inequality constraints for the GARCH and the FIGARCH are satisfied and thereby would limit \mathcal{NS}^{HY} to the first quadrant. On the contrary, the HYGARCH necessary and sufficient set in quadrants two, three and four is based on positive ψ_i^{FI} coefficients combined with ψ_i^{GA} coefficients with negative or alternating sign. For $\tau \rightarrow 0$ the HYGARCH set approaches the GARCH(1, 1) set while for $\tau = 1$ it is identical with the FIGARCH set.

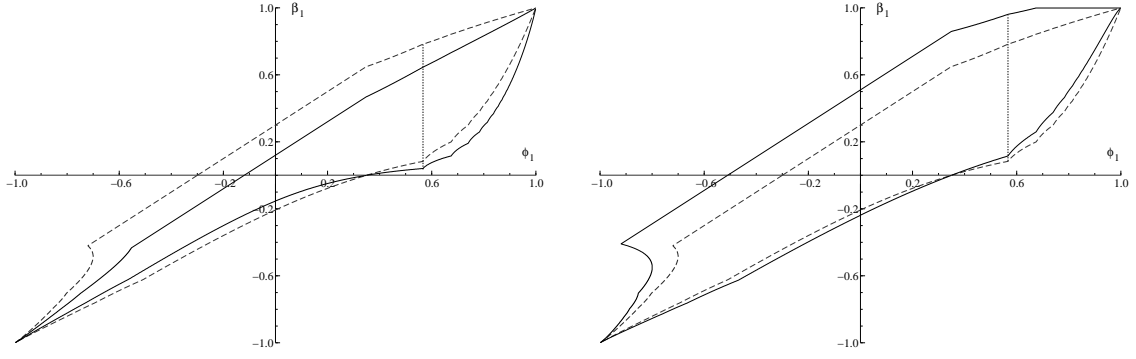


Figure 1: Necessary and sufficient parameter sets for HYGARCH models (solid) with $d = 0.3$ and $\tau = 0.4$ (left panel) and $\tau = 1.7$ (right panel) and their FIGARCH components (dashed). The dotted line corresponds to $\phi_1 = f_3$.

The right panel of Figure 1 shows a situation in which $\tau = 1.7$, i.e. the weight on the FIGARCH component is 1.7 while the weight on the GARCH component is -0.7. Now, the HYGARCH set encloses the FIGARCH set in the second, third and fourth quadrant. In the first quadrant the HYGARCH set lies partly inside and partly outside the FIGARCH set. In particular, note that in a major area the HYGARCH set lies outside both the FIGARCH set and the GARCH set.

Finally, the dotted vertical lines in Figure 1 bound a sufficient set which is given by the conditions

$$\beta_1 > 0, \quad \beta_1 - \tau d \leq \phi_1 \leq \frac{2-d}{3} \quad \text{and} \quad \tau d \left(\phi_1 - \frac{1-d}{2} \right) \leq \beta_1 (\phi_1 - \beta_1 + \tau d).$$

These conditions are a modified version of the sufficient conditions for the FIGARCH(1, d , 1) derived by Bollerslev and Mikkelsen (1996) and are applied, e.g., by Dark (2005). As in the FIGARCH case such sufficient conditions are very restrictive since they arbitrarily assume $\phi_1 \leq f_3$ and $\beta_1 > 0$ and thereby exclude a wide range of the necessary and sufficient set given by Theorem 1. In particular, applications to high-frequency data have shown that $\phi_1 > f_3$ arises as a very natural result (see e.g., Baillie et al., 2004, Tables 9 to 12).

We conclude this section by stating two results for popular submodels of the HYGARCH(1, d , 1) which follow directly from Theorem 1.

Corollary 1. *The conditional variance of the HYGARCH(0, d , 1) is non-negative a.s. iff*

1. $\psi_1^{HY} \geq 0 \Leftrightarrow \tau d + \phi_1 \geq 0$
2. $F_2 \geq 0 \Leftrightarrow (1-d)/2 - \phi_1 \geq 0$

Corollary 2. *The conditional variance of the HYGARCH(1, d , 0) is non-negative a.s. iff⁵*

Case 1: $0 < \beta_1 < 1$

$$\psi_1^{HY} \geq 0 \Leftrightarrow \tau d - \beta_1 \geq 0$$

Case 2: $-1 < \beta_1 < 0$

$$\psi_2^{HY} \geq 0 \Leftrightarrow (\tau d - \sqrt{\tau d(2 - d(2 - \tau))})/2 \leq \beta_1$$

3.2 Higher order HYGARCH models

From the recursions derived in Lemma 1 and the proof of Theorem 1 it is clear that the inequality constraints derived in Conrad and Haag (2006) for the FIGARCH(p, d, q) likewise extend to the HYGARCH(p, d, q). Here we present only a necessary and sufficient condition for the HYGARCH($1, d, q$) which is a direct extension of the $(1, d, 1)$ case and a sufficient condition for the HYGARCH(p, d, q). In particular, the HYGARCH($2, d, q$) case is not presented for reasons of brevity but the conditions in Conrad and Haag (2006) can be directly applied with ψ_i replaced by ψ_i^{HY} .

Theorem 2. *The conditional variance of the HYGARCH($1, d, q$) is non-negative a.s. iff*

Case 1: $0 < \beta_1 < 1$

1. $\psi_1^{HY}, \dots, \psi_{q-1}^{HY} \geq 0$ and
2. either $\psi_q^{HY} \geq 0$ and $F_{q+1} \geq 0$ or for $k > q + 1$ with $F_{k-1} < 0 \leq F_k$ it holds that $\psi_{k-1}^{HY} \geq 0$.

Case 2: $-1 < \beta_1 < 0$

1. $\psi_1^{HY}, \dots, \psi_{q-1}^{HY} \geq 0$ and
2. either $\psi_q^{HY} \geq 0$, $\psi_{q+1}^{HY} \geq 0$ and $F_{q+2}^{(1)} \geq 0$ or for $k > q + 2$ with $F_{k-1}^{(1)} < 0 \leq F_k^{(1)}$ it holds that $\psi_{k-1}^{HY} \geq 0$ and $\psi_{k-2}^{HY} \geq 0$.

Clearly, in the HYGARCH($1, d, q$) it suffices to check $q + 1$ (Case 1) or $q + 2$ (Case 2) conditions to ensure the non-negativity of the conditional variance for all t .

Finally, we present a general sufficient condition for the HYGARCH(p, d, q). For this condition we have to find $0 \leq p_1 \leq p_2 \leq p$ with $p_2 - p_1$ even, such that the ordering of the p inverse roots of $B(L)$ is in the following way

$$\begin{aligned} \lambda_{(1)} &\leq \dots \leq \lambda_{(p_1)} < 0 \\ \lambda_{(p_1+1)} &> 0, \lambda_{(p_1+2)} < 0, \dots, \lambda_{(p_2-1)} > 0, \lambda_{(p_2)} < 0 \\ &\text{with } \lambda_{(p_1+2i-1)} + \lambda_{(p_1+2i)} \geq 0, \quad i = 1, \dots, (p_2 - p_1)/2 \\ \lambda_{(p_2+1)} &\geq \dots \geq \lambda_{(p)} > 0 \end{aligned}$$

It should be noted that this ordering is not unique as it is always possible to combine positive and negative roots to form a new pair as well as pairs can be separated, such that p_1 and p_2 differ. But it is always possible to find such an ordering.

Theorem 3. *If in the HYGARCH(p, d, q) there exists an ordering of the roots such that $\Lambda_{p_1} > -1$ then there exists a k such that $\psi_i^{HY} \geq 0$ for all $i > k$.*

Note, that Theorem 3 imposes only weak conditions on the parameters of the HYGARCH process to derive a strong result. By applying the computationally efficient algorithm stated in Lemma 1 the above condition can be easily validated for a given set of estimated parameters. Tsai and Chan (2007) conjecture that by using their approach it may be shown that the sufficient condition for the FIGARCH(p, d, q) stated in Conrad and Haag (2006) is also necessary. If this conjecture can be verified then the same should be true for the above Theorem 3. This, however, is beyond the scope of the present article and we leave the issue as an interesting future research problem.

4 HYGARCH vs. FIGARCH

The last section illustrated the close connection between the FIGARCH and the HYGARCH model. The central feature of both models is that they can deal with data that is highly persistent in its second conditional moment. The persistence or memory of a particular model specification is best characterized by the rate of decay of the corresponding impulse response coefficients. Conrad and Karanasos (2006) derive convenient representations for the IRF of the FIGARCH(p, d, q) model. The impulse response coefficients π_k are given by the optimal forecast of the future conditional variance h_{t+k} as a function of the current surprise innovation v_t , i.e. by the difference between ε_t^2 and its conditional expectation h_t . Likewise, the impulse response coefficients of the HYGARCH(p, d, q) model can be obtained from the expansion of the following polynomial:

$$\Pi(L) = \frac{B(L)}{\Phi(L)} \left((1 - \tau) + \tau(1 - L)^d \right)^{-1} - 1$$

From the expression of $\Pi(L)$ it is clear that for $\tau = 0$ the impulse response coefficients will behave like those of a GARCH model, i.e. they will be governed by an exponential rate of decay. On the other hand, for $\tau = 1$ the coefficients will decay with a slow hyperbolic rate characterizing a FIGARCH process. The HYGARCH coefficients will be a mixture of both components, dominated by the hyperbolic rate of decay of the FIGARCH component.

We will now investigate the following question: what is the effect of estimating a FIGARCH model to a data set which was truly generated by a HYGARCH process? The answer is closely related to the fact, that estimating a FIGARCH model to such a data set is equivalent to the problem of finding the best approximation $\tilde{\Pi}(L)$ of $\Pi(L)$ under the restriction $\tau = 1$.

For simplicity we abstract from the term $B(L)/\Phi(L)$. This is reasonable since the $\Phi(L)$ and $B(L)$ polynomials will determine only the short-run behavior of π_k , while the long-run behavior is governed by the persistence parameter d and the parameter τ . Alternatively, we could think of the true data generating process (DGP) being a HYGARCH($0, d, 0$). If the DGP is HYGARCH, but one estimates a FIGARCH model to the data, then this can be thought of as choosing a persistence parameter \tilde{d} such that π_k is approximated best by $\tilde{\pi}_k$ coming from the expansion of $(1 - L)^{-\tilde{d}}$, where \tilde{d} is the persistence parameter of the FIGARCH process. Since π_k can be numerically evaluated and it is well known that

$\tilde{\pi}_k \sim 1/\Gamma(\tilde{d}) \cdot k^{\tilde{d}-1}$, where $\Gamma(\cdot)$ is the Gamma function, we can approximate \tilde{d} by running the restricted OLS regression

$$\ln(\pi_k) = \gamma_0 + \gamma_1 \ln(k) + \Delta_k \quad \text{s.t.} \quad \gamma_0 = \ln(1/\Gamma(\gamma_1 + 1))$$

with Δ_k being the approximation error. The persistence parameter \tilde{d} is then given by $\hat{\gamma}_1 + 1$. Figure 2 plots \tilde{d} as a function of the true d , given a fixed value of $\tau \in \{0.85, 1.15\}$. Clearly, \tilde{d} is smaller than d if $\tau < 1$ and \tilde{d} is bigger than d if $\tau > 1$. For a fixed value of τ , the effect is the stronger the larger d . Also, for a fixed value of d , the effect will be the stronger the further τ is away from one.

The intuition for this is straightforward. In the case of a HYGARCH the persistence parameter d is associated with the FIGARCH component, which is multiplied by a factor $\tau < 1$ ($\tau > 1$). Hence, when a pure FIGARCH is estimated to the data the persistence parameter \tilde{d} which best approximates the degree of dependence in the data is smaller (larger) than the one in the FIGARCH component of the HYGARCH.

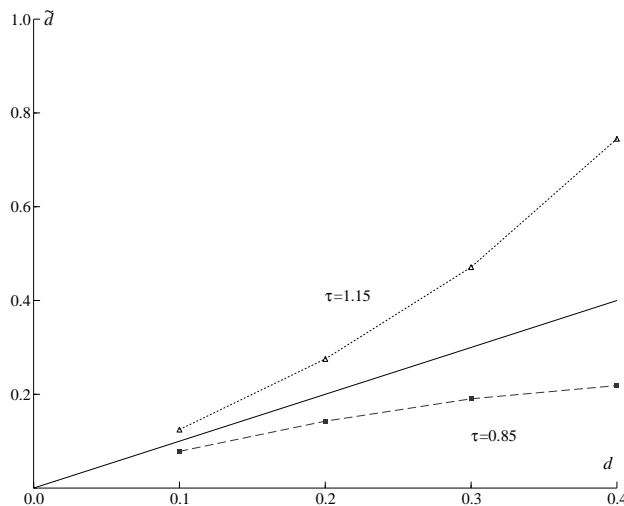


Figure 2: The figure plots \tilde{d} as a function of d given $\tau = 0.85$ or $\tau = 1.15$.

Next, we perform a Monte-Carlo simulation in order to confirm our theoretical considerations. This will allow us to gain further insights into the relation between FIGARCH and HYGARCH models. The true DGPs are a FIGARCH, a HYGARCH with $\tau = 0.85$ and a HYGARCH with $\tau = 1.15$. The true parameter values are chosen such that they are typical for empirical applications and satisfy the non-negativity conditions derived in the last section. The innovations Z_t are drawn from a standard normal distribution. In order to avoid start-up problems we delete the first 10000 realizations of each replication. All simulations are performed $M = 1000$ times for a sample size of $T = 5000$. For the practical estimations we employ the G@RCH package for Ox developed by Laurent and Peters (2006). Finally, all estimated parameter combinations which do not satisfy the non-negativity conditions are discarded from the simulation. Table 1 provides the simulation results.

Table 1: Monte-Carlo estimates for FIGARCH(1, d , 1) and HYGARCH(1, d , 1) models.

DGP: $\omega = 0.05$, $\phi_1 = 0.6$, $\beta_1 = 0.35$, $d = 0.3$ and						
	$\tau = 1$, i.e. $\ln(\tau) = 0$		$\tau = 0.85$, i.e. $\ln(\tau) = -0.16$		$\tau = 1.15$, i.e. $\ln(\tau) = 0.14$	
	FIGARCH	HYGARCH	FIGARCH	HYGARCH	FIGARCH	HYGARCH
ω	0.052	0.052	0.046	0.050	0.091	0.054
	[0.047, 0.059]	[0.043, 0.062]	[0.041, 0.053]	[0.044, 0.057]	[0.072, 0.120]	[0.036, 0.077]
ϕ_1	0.593	0.597	0.623	0.602	0.529	0.595
	[0.554, 0.633]	[0.545, 0.635]	[0.590, 0.657]	[0.555, 0.637]	[0.482, 0.579]	[0.547, 0.633]
β_1	0.342	0.335	0.316	0.339	0.338	0.343
	[0.305, 0.383]	[0.296, 0.380]	[0.282, 0.351]	[0.300, 0.389]	[0.295, 0.384]	[0.307, 0.383]
d	0.304	0.306	0.198	0.305	0.370	0.305
	[0.264, 0.345]	[0.243, 0.369]	[0.168, 0.233]	[0.225, 0.385]	[0.341, 0.401]	[0.253, 0.357]
$\ln(\tau)$	-	0.014	-	-0.143	-	0.147
		[-0.057, 0.087]		[-0.218, -0.056]		[0.101, 0.220]
Notes: The entries are the median of the estimated parameters over the $M = 1000$ replications. The entries in brackets are the 25% and 75% quantiles over the $M = 1000$ replications.						

Clearly, if the true DGP is a FIGARCH the median parameter estimates from both models – the correctly specified FIGARCH and the misspecified HYGARCH – are close to the true parameter values. The estimated median parameter value for τ in the HYGARCH model is indistinguishable from one and hence one would correctly reject the overparameterized HYGARCH in favor of the FIGARCH model.⁶ As expected in a situation in which the additional flexibility provided by the HYGARCH is not needed, the estimates of ϕ_1 , β_1 and d are very similar for both models. If the true DGP is a HYGARCH with $\tau < 1$, the HYGARCH parameter estimates are again close to their true values. Also, the FIGARCH parameter estimates for β_1 and ϕ_1 are close to the true values. In contrast, the persistence parameter d is estimated considerably lower than the value specified for the FIGARCH component of the underlying HYGARCH. In particular, the interquartile reported for the estimated FIGARCH persistence parameter does not cover the value of the persistence parameter of the true HYGARCH. This result is completely in line with the above considerations. The last two columns of Table 1 report the results for the case $\tau > 1$. The true DGP is very well recovered by the HYGARCH parameter estimates. As expected, the FIGARCH parameter estimate for d is higher than the one of the underlying HYGARCH process.

Interestingly, our simulation results are in accordance with an observation made by Dark (2005) who applies several variants of the HYGARCH model to daily Nikkei 225 returns. First, he reports a spectral density estimate of 0.3 for the degree of persistence in the volatility. Then, estimating a FIGARCH model to the same data he obtains a considerably higher estimate of $\hat{d} = 0.52$. Finally, he estimates a

HYGARCH and a HYAPARCH model.⁷ The latter model results in a τ parameter estimate of 1.16 in combination with a persistence parameter of 0.31. Hence, the HYGARCH estimate of the persistence parameter is much closer to the spectral density estimate than the FIGARCH estimate of d .

Similar findings are reported by Tang and Shieh (2006) in their Tables 3 and 4. For example, for the NASDAQ 100 the FIGARCH estimate of d is 0.38, while the HYGARCH estimate of d is 0.20 in combination with a τ of 1.25. All these results are in perfectly in line with the above considerations.

Moreover, in the literature it is often argued that the estimated persistence parameters from FIGARCH models are “too high”. In the light of our results, a likely explanation for this finding could be that the true DGP is closer to a HYGARCH with lower d , but τ greater than one.

5 Empirical Application

In this section we illustrate the usefulness of our results in an empirical application of the HYGARCH model. For this purpose we employ daily stock index data on the NYSE obtained from the Datastream Database for the periods 02.09.85 – 30.04.93 and 21.06.89 – 25.08.05. The main intention of this section is to illustrate that the phenomena discussed in the last two sections are not only of theoretical interest but arise naturally in empirical applications.

Davidson (2004) applies the HYGARCH model to ten major dollar exchange rates and to three Asian currencies during the crisis 1997–1998. For the ten major series the estimate of τ is close to and in most of the times not significantly different from one. Hence, for those ten currencies the FIGARCH model explains the data sufficiently well. In sharp contrast, for the three crisis currencies the estimate of τ is considerably larger than and significantly different from one. Although the crisis currencies seem to be characterized by two entirely different periods, a tranquil one before the crisis and one with large fluctuations afterwards, the HYGARCH does a very good job in modeling the series.

Motivated by these findings we have chosen our samples such that they correspond to “extreme periods” for the NYSE. The first sample includes the “Black Monday” in October 1987 with its enormous daily movements, while the second sample fully covers the “Dot-Com Bubble” which burst in 2000.

Table 2 presents MA(1)-FIGARCH(1, d , 1) and MA(1)-HYGARCH(1, d , 1) parameter estimates on the NYSE return series for the two selected time periods. Instead of reporting all the parameters we focus only on those which are relevant in our context. Clearly, according to the Ljung-Box statistics the null hypothesis of uncorrelated standardized and squared standardized residuals cannot be rejected for all four models.

The parameter estimate for $\ln(\tau)$ in the HYGARCH model is significantly negative in the first period, implying a parameter $\hat{\tau} = 0.811$. The corresponding estimate for the fractional differencing parameter is $\hat{d} = 0.582$. The pure FIGARCH model leads to an estimate $\hat{d} = 0.114$, which is considerably smaller than the one from the HYGARCH model applied to the same data. Also the ϕ_1 and β_1 parameters

Table 2: HY/FIGARCH models for daily NYSE returns.

	02.09.85 – 30.04.93		21.06.89 – 25.08.05	
	FIGARCH	HYGARCH	FIGARCH	HYGARCH
ϕ_1	0.852 (0.054)	0.410 (0.071)	0.223 (0.027)	0.177 (0.050)
β_1	0.807 (0.061)	0.693 (0.073)	0.514 (0.040)	0.374 (0.066)
d	0.114 (0.018)	0.582 (0.109)	0.344 (0.028)	0.202 (0.038)
$\ln(\tau)$	-	-0.209 (0.071)	-	0.210 (0.087)
$Q(25)$	13.38 [0.96]	12.63 [0.97]	31.66 [0.14]	31.87 [0.13]
$Q^2(25)$	7.16 [0.99]	7.03 [0.99]	11.69 [0.97]	11.62 [0.98]

Notes: The table reports the parameter estimates for a MA(1)-FI/HYGARCH(1, d , 1) model. Standard errors are given in parenthesis, p -values in brackets.

are very different for the two models, i.e. restricting τ to be one severely affects the estimates of the remaining parameters. For the second period the picture is the exact opposite. For the HYGARCH model a significantly positive parameter $\ln(\tau)$ is estimated, implying $\hat{\tau} = 1.234$, combined with a fractional differencing parameter of $\hat{d} = 0.202$. The fractional differencing parameter of the pure FIGARCH model is now considerably higher, namely $\hat{d} = 0.344$. Both results are perfectly in line with the outcome of the Monte-Carlo simulations in the last section.

Next, we check whether the estimated parameters satisfy the non-negativity conditions derived in Section 3. Figure 3 shows in the ϕ_1, β_1 space the necessary and sufficient sets for the FIGARCH (left panel, dashed line) and HYGARCH (right panel, solid line) models with parameters d and τ as reported in Table 2 for the first period. The dots represent the estimated parameter combinations. In both cases, the estimated parameters clearly lie inside the necessary and sufficient sets. Interestingly, the parameter values estimated for the FIGARCH model violate the Bollerslev and Mikkelsen (1996) conditions, but satisfy the Conrad and Haag (2006) conditions. Hence, relying on the Bollerslev and Mikkelsen (1996) conditions would have led to the erroneous rejection of a valid model specification. The example nicely illustrates, that checking the necessary and sufficient conditions derived in this article is an essential step in validating a particular HYGARCH specification. The parameters estimated for the second period also satisfy the conditions derived in Section 3, but we omit the figure for reasons of brevity.

In order to compare the persistence properties of the estimated FIGARCH and HYGARCH models from Table 2 we compute the corresponding IRFs. Figure 4 shows the IRFs for the FIGARCH (dotted) and HYGARCH (solid) models presented in Table 2. The dashed lines are the IRFs for the FIGARCH components corresponding to the HYGARCH models.

For the first period (Figure 4, left panel) the solid line is considerably below the dashed line. This is because in the first period we estimated a τ of less than one and hence the dashed line does not take

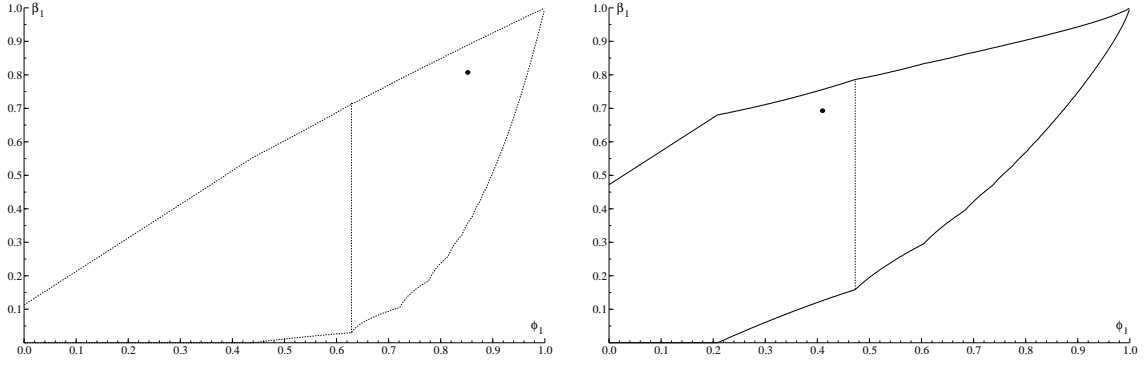


Figure 3: Necessary and sufficient parameter sets for FIGARCH (left panel) and HYGARCH (right panel) models as estimated in Table 2 columns 2 and 3. The dots represent the estimated parameter combinations.

into account the exponentially decaying GARCH component of the HYGARCH process. For the second period (Figure 4, right panel) we have τ greater than one and therefore obtain a IRF of the HYGARCH which is strictly above the dashed line. The dotted lines are the IRFs for the pure FIGARCH models estimated for the two periods in columns 2 and 4 of Table 2. Interestingly, the dotted lines and the solid lines are very similar. By choosing persistence parameters d in the pure FIGARCH models which are lower (higher) than the ones in the HYGARCH models, the implied IRFs of the FIGARCH are very much the same as the ones of the HYGARCH. Again, this finding is in line with our simulation results. The pure FIGARCH model leads to a higher (lower) estimate of d in comparison to the persistence parameter of the FIGARCH component of the HYGARCH model if $\tau > 1$ ($\tau < 1$).

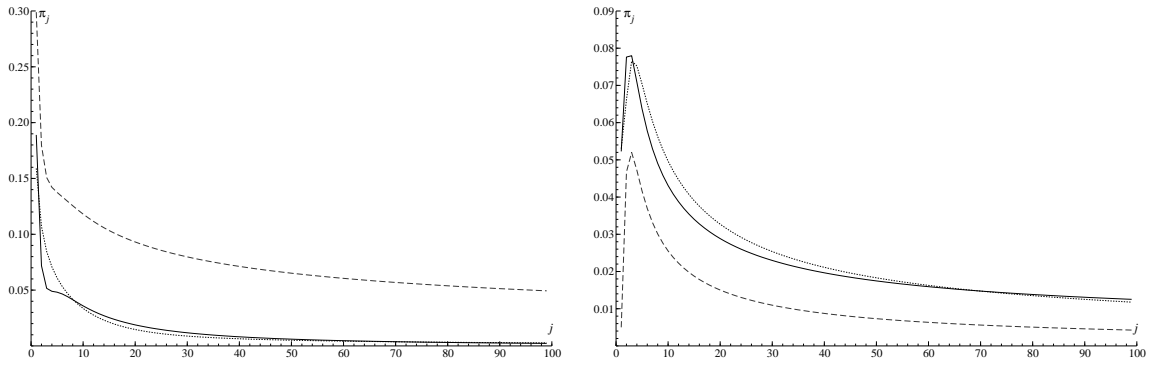


Figure 4: IRFs for HYGARCH models as estimated in Table 2 columns 3 (left panel) and 5 (right panel). The solid line represents the IRF with τ as estimated and the dashed line represents the IRF when τ is restricted to being one. The dotted IRFs represent the FIGARCH models estimated in Table 2 columns 2 (left panel) and 4 (right panel).

6 Conclusions

We extend the results of Nelson and Cao (1992) and Conrad and Haag (2006) by deriving necessary and sufficient conditions which ensure the non-negativity of the conditional variance in the HYGARCH model of the order $p \leq 2$ and sufficient conditions for the general model. The starting point for our derivations is the idea to exploit the fact that the ARCH(∞) coefficients of the HYGARCH process are a linear combination of the ARCH(∞) coefficients of its GARCH and FIGARCH components. Recursive representations for the ARCH(∞) coefficients of those components are well known in the literature and can be directly plugged in. It turns out that the coefficients of the HYGARCH obey a structure very similar to the FIGARCH coefficients. The only difference is due to the additional parameter τ which allows the HYGARCH to constitute a covariance stationary process. But this parameter appears simply as a multiplicative constant and therefore the arguments used in Conrad and Haag (2006) to establish non-negativity constraints for the FIGARCH can be extended to the HYGARCH process. It is important to notice that our recursive representation of the ARCH(∞) coefficients has a powerful direct application in constructing multi-step-ahead volatility forecasts for e.g. option pricing or Value-at-Risk computations as suggested by Andersen et al. (2006).

Non-negativity conditions for the HYGARCH process are important since their validity is a necessary condition for the parameters of a particular HYGARCH model to represent a well defined non-negative conditional variance process. Similarly, as in the FIGARCH case one can not deduce the non-negativity of the conditional variance only from the signs of the estimated parameters which means that non-negativity conditions should always be checked. Moreover, it is shown that an extension of the Bollerslev and Mikkelsen (1996) sufficient condition to the HYGARCH(1, d , 1) is overly restrictive and, therefore, clearly the necessary and sufficient conditions presented here are preferable. For the (1, d , 1) model we provide an analytical and graphical comparison of the necessary and sufficient parameter set of the HYGARCH with the corresponding sets of its GARCH and FIGARCH components.

Moreover, we provide some further insights into the mechanics of the HYGARCH and FIGARCH models. In the literature it is often reported as surprising, that both models applied to the same data set result in quite different parameter estimates for the persistence parameter d . We explain that this is a natural consequence if the true DGP is best described by a HYGARCH model.

Finally, the conditions derived in this article can be directly applied to multivariate models such as the constant correlation HYGARCH, the orthogonal HYGARCH or possibly to a hyperbolic autoregressive conditional duration (HYACD) model for the conditional duration time.

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Endnotes

¹Recall, that throughout the article we always assume that $\omega > 0$. Hence, this condition is not explicitly restated here.

²The IGARCH specification implies $\Phi(1) = 0$ and thus violates the covariance stationarity condition. This however does not preclude strict stationarity. E.g. the IGARCH(1, 1) process will be strictly stationary and ergodic provided that $\mathbf{E}[\ln(\alpha_1 Z_t^2 + \beta_1)] < 0$ (see Nelson, 1990).

³Karanasos et al. (2004) point to an alternative modification of the FIGARCH which also specifies a covariance stationary ε_t process. In analogy to the ARFIMA model for the mean, they construct their long memory GARCH (LMGARCH) process such that the fractional differencing operator is applied to the “demeaned” squared innovations, i.e. instead of equation (6) they assume $(1-L)^d \Phi(L)(\varepsilon_t^2 - \omega) = B(L)v_t$, which makes ω the unconditional variance of ε_t .

⁴The necessary and sufficient parameter set for the FIGARCH(1, d , 1) component is obtained from Theorem 1 by setting $\tau = 1$ and replacing ψ_i^{HY} by ψ_i^{FI} . The necessary and sufficient parameter set for the GARCH(1, 1) component is given by $0 < \beta_1 < \phi_1 < 1$.

⁵Please note the typo in Case 2 of Corollary 3, p. 427, in Conrad and Haag (2006) which writes $(d - \sqrt{2(2-d)})/2 \leq \beta_1$ instead of $(d - \sqrt{d(2-d)})/2 \leq \beta_1$.

⁶In its output G@RCH provides estimates for $\ln(\tau)$ instead of τ .

⁷The term HYAPARCH refers to hyperbolic asymmetric power GARCH. This model allows for asymmetric effects of positive and negative shocks to the volatility and estimates a flexible power transformation of the conditional variance.

Appendix:

Proof of Lemma 1.

Recall from equation (9) that for $i = 1, 2, \dots$ we have that $\psi_i^{HY} = \tau \psi_i^{FI} + (1 - \tau) \psi_i^{GA}$. Additionally, Conrad and Haag (2006) provide recursions for the ARCH(∞) coefficients of the FIGARCH(p, d, q) (see Lemma 1 in Conrad and Haag, 2006) and the GARCH(p, q) (see Remark 3 in Conrad and Haag, 2006). Following the notation in Conrad and Haag (2006) we write $\psi_i^{HY} = \psi_i^{HY(p)} = \tau \psi_i^{FI(p)} + (1 - \tau) \psi_i^{GA(p)}$.

By application of the corresponding recursions we obtain for $1 < r \leq p$, $i \geq 1$

$$\begin{aligned}\psi_i^{HY(r)} &= \tau(\lambda_{(r)}\psi_{i-1}^{FI(r)} + \psi_i^{FI(r-1)}) + (1-\tau)(\lambda_{(r)}\psi_{i-1}^{GA(r)} + \psi_i^{GA(r-1)}) \\ &= \lambda_{(r)}(\tau\psi_{i-1}^{FI(r)} + (1-\tau)\psi_{i-1}^{GA(r)}) + \tau\psi_i^{FI(r-1)} + (1-\tau)\psi_i^{GA(r-1)} \\ &= \lambda_{(r)}\psi_{i-1}^{HY(r)} + \psi_i^{HY(r-1)}.\end{aligned}$$

The $\psi_i^{HY(1)}$ sequence of coefficients is given as a linear combination of the coefficients from a FIGARCH(1, d , q) and a GARCH(1, q), i.e. we have for $i = 1, \dots, q$

$$\psi_i^{HY(1)} = \tau(-c_i + \sum_{j=1}^i \phi_j c_{i-j}) + (1-\tau) \sum_{j=1}^i \lambda_{(1)}^{i-j} (\phi_j - \beta_j)$$

and for $i \geq q+1$

$$\begin{aligned}\psi_i^{HY(1)} &= \tau(\lambda_{(1)}\psi_{i-1}^{FI(1)} + F_i(-g_{i-q})) + (1-\tau)\lambda_{(1)}\psi_{i-1}^{GA(1)} \\ &= \lambda_{(1)}(\tau\psi_{i-1}^{FI(1)} + (1-\tau)\psi_{i-1}^{GA(1)}) + \tau F_i(-g_{i-q}) \\ &= \lambda_{(1)}\psi_{i-1}^{HY(1)} + \tau F_i(-g_{i-q}).\end{aligned}$$

Note, that an iterated version of the last equation for $i \geq q+2$ would be

$$\psi_i^{HY(1)} = \beta^2 \psi_{i-2}^{HY} + \tau F_i^{(2)}(-g_{i-2}).$$

□

Proof of Theorem 1.

Case 1: $0 < \beta_1 < 1$

" \Leftarrow "

1. If $\psi_1^{HY} \geq 0$ and $\phi_1 \leq f_2$ this ensures that $\psi_i^{HY} = \beta_1 \psi_{i-1}^{HY} + \tau(f_i - \phi_1)(-g_{i-1}) \geq 0$ for all $i \geq 2$, since f_i is increasing and $\tau \geq 0$.
2. If $\phi_1 > f_2$, then there exists a k such that $\phi_1 \leq f_k$. Then for ψ_i^{HY} with $1 < i < k$ it holds that

$$\psi_i^{HY} = \beta_1 \psi_{i-1}^{HY} + \tau(f_i - \phi_1)(-g_{i-1}) \geq 0 \Leftrightarrow \beta_1 \psi_{i-1}^{HY} \geq \tau(f_i - \phi_1)g_{i-1} > 0 \Rightarrow \psi_{i-1} \geq 0.$$

Thus, $\psi_i \geq 0$ implies $\psi_{i-1} \geq 0$. As $\psi_{k-1} \geq 0$ it follows recursively that $\psi_i \geq 0$ for all $1 \leq i < k$. For $i \geq k$ we have $\psi_i^{HY} = \beta_1 \psi_{i-1}^{HY} + \tau(f_i - \phi_1)(-g_{i-1})$ and hence, from $\psi_{k-1}^{HY} \geq 0$ follows $\psi_k^{HY} \geq 0$ since $\phi_1 \leq f_k \Rightarrow \psi_i \geq 0$ for all $i > k$ by induction.

" \Rightarrow "

1. $\psi_1^{HY}, \psi_{k-1}^{HY} \geq 0$ are trivially fulfilled.
2. Either $\phi_1 \leq f_2$ or $\phi_1 > f_2$, but since $f_{i-1} \leq f_i$ and $f_i \rightarrow 1$ there exists a k s.t. $\phi_1 \leq f_i$ for all $i \geq k$.

Case 2: $-1 < \beta_1 < 0$

" \Leftarrow "

We make use of the recursion

$$\psi_i^{HY} = \beta_1^2 \psi_{i-2}^{HY} + \tau(\beta_1(f_{i-1} - \phi_1) + (f_i - \phi_1)f_{i-1})(-g_{i-2}) \text{ for } i \geq 3. \quad (19)$$

1. If $\phi_1 \leq f_2(\beta_1 + f_3)/(\beta_1 + f_2)$ then $\psi_1^{HY} \geq 0$ and $\psi_2^{HY} \geq 0$ ensure that $\psi_i^{HY} \geq 0$ for all $i \geq q + 2$.
2. If $\phi_1 > f_2(\beta_1 + f_3)/(\beta_1 + f_2)$ then for ψ_i^{HY} with $3 < i < k$ it holds that

$$\begin{aligned} \psi_i^{HY} &= \beta_1^2 \psi_{i-2}^{HY} + \tau(\beta_1(f_{i-1} - \phi_1) + (f_i - \phi_1)f_{i-1})(-g_{i-2}) \geq 0 \\ \Leftrightarrow \beta_1^2 \psi_{i-2}^{HY} &\geq \tau(\beta_1(f_{i-1} - \phi_1) + (f_i - \phi_1)f_{i-1})(g_{i-2}) \geq 0 \\ \Rightarrow \psi_{i-2}^{HY} &\geq 0 \end{aligned}$$

Thus, $\psi_i^{HY} \geq 0$ implies $\psi_{i-2}^{HY} \geq 0$. As $\psi_{k-1}^{HY} \geq 0$ and $\psi_{k-2}^{HY} \geq 0$ it follows recursively that $\psi_i^{HY} \geq 0$ for all $1 \leq i < k$. For $i \geq k$ we use equation (19) and hence, from $\psi_{k-1}^{HY} \geq 0$ and $\psi_{k-2}^{HY} \geq 0$ it follows that $\psi_i^{HY} \geq 0$ for all $i > k$ by induction.

" \Rightarrow "

1. $\psi_1^{HY}, \psi_2^{HY}, \psi_{k-2}^{HY}, \psi_{k-1}^{HY} \geq 0$ are trivially fulfilled.
2. Either $\phi_1 \leq f_2(\beta_1 + f_3)/(\beta_1 + f_2)$ or $\phi_1 > f_2(\beta_1 + f_3)/(\beta_1 + f_2)$, but since $f_{i-1} \leq f_i$ and $f_i \rightarrow 1$ there exists a k s.t. $\phi_1 \leq f_2(\beta_1 + f_3)/(\beta_1 + f_2)$ for all $i \geq k$.

□

Proof of Proposition 1.

Case 1: $0 < \tau < 1$

Consider the behavior of ψ_i^{GA} if the condition $0 < \beta_1 < \phi_1 < 1$ is violated. It is straightforward to show that in this case either $\psi_i^{GA} < 0$ for all i (which happens e.g. when $\phi_1 < 0$ and $\beta_1 > 0$) or ψ_i^{GA} has alternating sign (which happens when $\beta_1 < 0$). For any $0 < d < 1$, the former case immediately leads to at least one $\psi_i^{HY} < 0$ if not all $\psi_i^{FI} \geq 0$. The latter case could lead to a non-negative ψ_i^{HY} sequence if the ψ_i^{FI} would be of changing sign themselves. Obviously, this is not the case. Hence, it follows that for any $(\beta_1, \phi_1) \notin \mathcal{NS}^{GA}$ and $(\beta_1, d, \phi_1) \notin \mathcal{NS}^{FI}$ that also $(\beta_1, d, \phi_1, \tau) \notin \mathcal{NS}^{HY}$.

Case 2: $\tau > 1$

In this situation, the weight $(1 - \tau)$ on the ψ_i^{GA} coefficients is negative. Hence, for all parameter combinations $(\beta_1, \phi_1) \in \mathcal{NS}^{GA}$, the relation $\psi_i^{HY} = \tau \psi_i^{FI} + (1 - \tau) \psi_i^{GA}$ implies that if $(\beta_1, d, \phi_1) \notin \mathcal{NS}^{FI}$ it must be that there is a $\psi_i^{HY} < 0$ and hence $(\beta_1, d, \phi_1, \tau) \notin \mathcal{NS}^{HY}$.

□

Proof of Theorem 2.

As the proof of Theorem 1 in Conrad and Haag (2006).

□

Proof of Theorem 3.

As the proof of Theorem 4 in Conrad and Haag (2006).

□

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