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Optimal delegation with a finite number of states

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Online Appendix to Accompany *Optimal Delegation with a Finite Number of States*

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In this online appendix, we provide some details of proofs which are not included in the main text of the paper.

Proof of Observation 2

Let $b \in \left(\frac{1}{2(T-1)}, \frac{1}{(T-1)}\right)$ and $\Delta \in \mathcal{D}_T^*(b)$. From Theorem 1, this implies that $|\Delta| = T$. To compute the moments of the decision taken by the agent when she is offered Δ , we must distinguish between two different cases:

- *Case 1: T is odd.* By Theorem 1(ii),

$$d_t(\Delta, b) = \begin{cases} \ell^e(b, T, T-1) + \frac{t}{T-1} & \text{if } t \text{ is even,} \\ 2b - \ell^e(b, T, T-1) + \frac{t-1}{T-1} & \text{if } t \text{ is odd,} \end{cases}$$

so that the expected value of the agent's decision is

$$\mathbb{E}^E(b) \equiv \frac{1}{T} \sum_{t=0}^{T-1} d_t(\Delta, b) = \frac{1}{T} \left[(T-1)b + \ell^e(b, T, T-1) + \frac{1}{T-1} \sum_{t=0}^{T-1} t \right] = \frac{(1+2b)(T-1)^2}{2T^2}.$$

By definition, the variance of the decision is given by

$$\mathbb{V}^E(b) \equiv \frac{1}{T} \sum_{t=0}^{T-1} [d_t(\Delta, b) - \mathbb{E}^E(b)]^2,$$

so that

$$\begin{aligned} T \frac{d\mathbb{V}^E(b)}{db} &= 2 \sum_{t=0}^{T-1} [d_t(\Delta, b) - \mathbb{E}^E(b)] \cdot \frac{d}{db} [d_t(\Delta, b) - \mathbb{E}^E(b)] \\ &= 2 \sum_{t=0}^{T-1} [d_t(\Delta, b) - \mathbb{E}^E(b)] \cdot \frac{d}{db} [d_t(\Delta, b) - \mathbb{E}^E(b)] , \end{aligned}$$

where

$$\frac{d}{db} [d_t(\Delta, b) - \mathbb{E}^E(b)] = \begin{cases} \frac{T-1}{T^2} & \text{if } t \text{ is even,} \\ \frac{3T-1}{T^2} & \text{if } t \text{ is odd.} \end{cases}$$

Tedious calculations then reveal that

$$\begin{aligned} T \frac{d\mathbb{V}^E(b)}{db} &= \frac{1}{T^3(T-1)} \{T(2T^3 - 4T^2 + T - 1) + 2b(T^3 + T - 1)(T-1) + 1\} \\ &\geq \frac{2T-1}{T} > 0, \end{aligned}$$

where the first inequality comes from $b \geq \frac{1}{2(T-1)}$.

• *Case 2: T is even.* By Theorem 1(ii),

$$d_t(\Delta, b) = \begin{cases} b - \frac{T+1}{2T(T-1)} + \frac{t}{T-1} & \text{if } t \text{ is odd} \\ b - \frac{T+3}{2T(T-1)} + \frac{t}{T-1} & \text{if } t \text{ is even.} \end{cases}$$

As a consequence, the expected value of the agent's decision must be of the form

$$\mathbb{E}^O(b) \equiv \frac{1}{T} \sum_{t=0}^{T-1} d_t(\Delta, b) = b + \text{constant}.$$

This in turn implies that $[d_t(\Delta, b) - \mathbb{E}^O(b)]^2$ must be independent of b for each $t \in \mathbf{T}$, thus proving the variance of the decision does not depend on b .

Detailed proof of Lemma 2

As Δ is top-loaded, the agent takes decision $\delta_{t+\tau}$ in state $t+\tau$, for every $\tau < K$, and takes δ_{t+K} in states $K \leq t+\tau \leq s-1$.

For each $\tau \leq K$, let $\kappa_{t+\tau} \equiv \delta_{t+\tau} - \frac{t+\tau}{T-1}$. By optimality, $(\kappa_{t+\tau})_{\tau=0,\dots,K}$ must solve

$$\min_{\{\kappa_{t+\tau}\}_{\tau=t,\dots,K}} \left\{ \sum_{\tau=0}^{K-1} \kappa_{t+\tau}^2 + \sum_{\tau=K}^{s-1} \left(\kappa_{t+K} - \frac{\tau-K}{T-1} \right)^2 \right\}$$

subject to

$$\kappa_{t+\tau} + \kappa_{t+\tau+1} = 2b - \frac{1}{T-1}, \quad \forall \tau = 0, \dots, K-1; \quad (1)$$

where the equality in (1) must hold at Δ because it satisfies the chain property. It is easy to see that any solution to this problem is also a solution to

$$\min_{\{\kappa_{t+\tau}\}_{\tau=t,\dots,K}} \left\{ \sum_{\tau=0}^{K-1} \kappa_{t+\tau}^2 + (s-K) \left(\kappa_{t+K} - \frac{s-K-1}{T-1} \right) \kappa_{t+K} \right\}, \quad (2)$$

subject to (1). From the incentive constraints (1), we obtain by recursion that

$$\kappa_{t+\tau} = \begin{cases} \kappa_t & \text{if } \tau \text{ is even} \\ 2b - \frac{1}{T-1} - \kappa_t & \text{if } \tau \text{ is odd} \end{cases}, \quad \forall \tau = 0, \dots, K.$$

Substituting into (2), simple convex optimization reveals that, at an optimum:

$$\kappa_t = \begin{cases} \ell^e(b, s, K) & \text{if } K \text{ is even,} \\ \ell^o(b, s, K) & \text{if } K \text{ is odd,} \end{cases}$$

where $\ell^e(b, s, K)$ and $\ell^o(b, s, K)$ are defined before Lemma 2 in Section 2.3.

We end the proof by showing that the principal can reduce his expected loss whenever $b \notin [b^{\min}(K, s), b^{\max}(K, s)]$ — a contradiction to Δ being optimal. More precisely, for every $m \in \mathbb{N}$, let $D^*(m) \equiv \{\delta_t^*(m), \dots, \delta_{t+m-1}^*(m)\}$, where

$$\delta_{t+\tau}^*(m) = \begin{cases} \ell(b, s, m-1) + \frac{t+\tau}{T-1} & \text{if } \tau \text{ is even,} \\ 2b - \ell(b, s, m-1) + \frac{t+\tau-1}{T-1} & \text{if } \tau \text{ is odd.} \end{cases}$$

and ℓ is defined in the text before Lemma 2. We will show that if $b \notin [b^{\min}(K, s), b^{\max}(K, s)]$ then there is a $K' \neq K$ such that the principal can reduce his loss by offering $D^*(K' + 1)$ to the agent.

Suppose that K is even. The change in the principal's loss following a deviation to $D^*(K + 2)$ when $K < s - 1$ can be decomposed as follows:

(i) In each state $t + \tau$, with τ even and no more than K , the change in the principal's loss is:

$$\ell^o(b, s, K + 1)^2 - \ell^e(b, s, K)^2 = \varphi(b, s, K) [\varphi(b, s, K) + 2\ell^e(b, s, K)] ,$$

where

$$\varphi(b, s, K) \equiv \ell^o(b, s, K + 1) - \ell^e(b, s, K) = \frac{s - K - 1}{s} \left(2b - \frac{s - K}{T - 1} \right) .$$

(ii) In each state $t + \tau$, with τ odd and no more than $K - 1$, the change in the principal's loss is:

$$\begin{aligned} & \left[2b - \frac{1}{T-1} - \ell^o(b, s, K + 1) \right]^2 - \left[2b - \frac{1}{T-1} - \ell^e(b, s, K) \right]^2 \\ &= \varphi(b, s, K) \left[\varphi(b, s, K) - 2 \left(2b - \frac{1}{T-1} - \ell^e(b, s, K) \right) \right] . \end{aligned}$$

(iii) In each state $t + K + i$ for $i = 1, \dots, s - K - 1$, the change in the principal's loss is:

$$\begin{aligned} & \left[2b - \ell^o(b, s, K + 1) - \frac{i}{T-1} \right]^2 - \left[\ell^e(b, s, K) - \frac{i}{T-1} \right]^2 \\ &= \xi(b, s, K) \left\{ \xi(b, s, K) + 2 \left[\ell^e(b, s, K) - \frac{i}{T-1} \right] \right\} , \end{aligned}$$

where

$$\xi(b, s, K) \equiv 2b - \ell^o(b, s, K + 1) - \ell^e(b, s, K) = \frac{1}{s} \left(2b + \frac{K}{T-1} \right) .$$

Note that

$$\begin{aligned}
\sum_{i=1}^{s-K-1} \left\{ \xi(b, s, K) + 2 \left[\ell^e(b, s, K) - \frac{i}{T-1} \right] \right\} &= (s-K-1) \left[\xi(b, s, K) + 2\ell^e(b, s, K) - \frac{s-K}{T-1} \right] \\
&= (K+1) \frac{s-K-1}{s} \left(2b - \frac{s-K}{T-1} \right) \\
&= (K+1) \varphi(b, s, K) .
\end{aligned}$$

Thus, summing the changes in the principal's loss over all states, we obtain

$$\begin{aligned}
&(K+1) \varphi(b, s, K) \left[\varphi(b, s, K) + 2\ell^e(b, s, K) - \frac{K}{K+1} \left(2b - \frac{1}{T-1} \right) + \xi(b, s, K) \right] \\
&= (K+1) \varphi(b, s, K) \left[\frac{1}{K+1} \left(2b + \frac{K}{T-1} \right) \right] = \frac{2(s-K-1)}{s} \left(2b + \frac{K}{T-1} \right) [b - b^{\min}(K, s)] .
\end{aligned}$$

This proves that the deviation to $D^*(K+2)$ is unprofitable only if $b \geq b^{\min}(K, s)$. The argument above also implies that the principal can profitably deviate from $D^*(K+2)$ to $D^*(K+1)$ unless $b \leq b^{\max}(K+1, s) = b^{\min}(K, s)$.

The change in the principal's expected loss if he deviates to $D^*(K)$ (assuming $K > 0$) can be decomposed as follows:

(i) In each state $t + \tau$, with τ even and strictly less than $K - 1$, the change in the principal's loss is:

$$\ell^o(b, s, K-1)^2 - \ell^e(b, s, K)^2 = \phi(b, s, K) [\phi(b, s, K) + 2\ell^e(b, s, K)] ,$$

where

$$\phi(b, s, K) \equiv \ell^o(b, s, K-1) - \ell^e(b, s, K) = \frac{s-K}{s} \left(2b - \frac{s-K+1}{T-1} \right) .$$

(ii) In each state $t + \tau$, with τ odd and lower or equal to $K - 1$, the change in the principal's loss is:

$$\begin{aligned}
&\left[2b - \frac{1}{T-1} - \ell^o(b, s, K-1) \right]^2 - \left[2b - \frac{1}{T-1} - \ell^e(b, s, K) \right]^2 \\
&= \phi(b, s, K) \left[\phi(b, s, K) - 2 \left(2b - \frac{1}{T-1} - \ell^e(b, s, K) \right) \right] .
\end{aligned}$$

(iii) In each state $t + K + i$ for $i = 0, \dots, s-K-1$, the change in the principal's loss is:

$$\begin{aligned}
&\left[2b - \ell^o(b, s, K-1) - \frac{i+2}{T-1} \right]^2 - \left[\ell^e(b, s, K) - \frac{i}{T-1} \right]^2 \\
&= \psi(b, s, K) \left\{ \psi(b, s, K) + 2 \left[\ell^e(b, s, K) - \frac{i}{T-1} \right] \right\} ,
\end{aligned}$$

where

$$\psi(b, s, K) \equiv 2 \left(b - \frac{1}{T-1} \right) - \ell^o(b, s, K-1) - \ell^e(b, s, K) = -\frac{K}{s(T-1)} .$$

Note that

$$\begin{aligned} \sum_{i=0}^{s-K-1} \left\{ \psi(b, s, K) + 2 \left[\ell^e(b, s, K) - \frac{i}{T-1} \right] \right\} &= (s-K) \left[\psi(b, s, K) + 2\ell^e(b, s, K) - \frac{s-K-1}{T-1} \right] \\ &= K \frac{s-K}{s} \left(2b - \frac{s-K+1}{T-1} \right) \\ &= K \phi(b, s, K) . \end{aligned}$$

Thus, summing the principal's loss over all states, we obtain

$$\begin{aligned} K \phi(b, s, K) \left[\phi(b, s, K) + 2\ell^e(b, s, K) - \left(2b - \frac{1}{T-1} \right) + \psi(b, s, K) \right] \\ = K \phi(b, s, K) \left(-\frac{1}{T-1} \right) = -\frac{2K}{T-1} \frac{s-K}{s} [b - b^{\max}(K, s)] . \end{aligned}$$

This proves that the deviation to $D^*(K)$ is unprofitable only if $b \leq b^{\max}(K, s)$. The argument above also implies that the principal can profitably deviate from $D^*(K)$ to $D^*(K+1)$ unless $b \geq b^{\min}(K-1, s) = b^{\max}(K, s)$.

We have therefore proved that the principal can reduce his expected loss by choosing $D^*(K+2)$ instead of Δ when $b < b^{\min}(K, s)$, and by choosing $D^*(K)$ instead of Δ when $b > b^{\max}(K, s)$.¹ As $\Delta \in \mathcal{D}_{s,t}^*(b)$, this implies that $b \in [b^{\min}(K, s), b^{\max}(K, s)]$.

Computational details for the proof of Lemma 3

The detailed proof for $\Upsilon(b) \geq 0$ is as follows. We distinguish between four cases:

- **Case 1:** $\min\{K, K'\} > 0$.

In this case, we have both

$$b \in \left[\frac{\tau - K}{2(T-1)}, \frac{\tau - K}{2(T-1)} \right] \text{ and } b \in \left[\frac{s - \tau + 1 - K'}{2(T-1)}, \frac{s - \tau + 2 - K'}{2(T-1)} \right] .$$

As $\tau - K$ and $s - \tau + 1 - K'$ are integers, this in turn implies that

$$\tau - K = s - \tau + 1 - K' . \quad (3)$$

Now, let

$$\begin{aligned} \Upsilon(b) &\equiv 2 \left(b + \frac{t + \tau - 1}{T-1} \right) - \delta_{t+\tau-1} - \delta_{t+\tau} \\ &\geq 2 \left(b + \frac{t + \tau - 1}{T-1} \right) - \left[2b - \ell^o(b, \tau, K) + \ell^e(b, s - \tau + 1, K') + \frac{2t + \tau + K - 1}{T-1} \right] \end{aligned}$$

¹We have ignored the case in which $K = s - 1$ and $b < b^{\min}(K, s)$ because we then have $b < 1/2(T-1)$.

First, note that

$$\frac{d}{db} [2b - \ell^o(b, \tau, K)] = \frac{d}{db} \left[\frac{K+1}{\tau} b + \frac{\tau-1}{2\tau(T-1)} + \frac{(\tau-K)^2}{2\tau(T-1)} \right] = \frac{K+1}{\tau} \leq 1$$

and

$$\frac{d}{db} [\ell^e(b, s - \tau + 1, K')] = \frac{d}{db} \left[\frac{K'}{s - \tau + 1} b \right] = \frac{K'}{s - \tau + 1} \leq 1.$$

As a consequence, $\Upsilon'(b) \geq 0$ and, from (3):

$$\Upsilon(b) \geq \Upsilon\left(\frac{\tau-K}{2(T-1)}\right) = \Upsilon\left(\frac{s-\tau+1-K'}{2(T-1)}\right).$$

At $b = \frac{\tau-K}{2(T-1)}$, we have²

$$\begin{aligned} 2\left(b + \frac{t + \tau - 1}{T-1}\right) - \frac{2t + \tau + K - 1}{T-1} &= \underbrace{\frac{2(\tau-K)}{2(T-1)}}_{\hookrightarrow \Upsilon_1} + \underbrace{\frac{\tau-K-1}{T-1}}_{\hookrightarrow \Upsilon_2}, \\ -[2b - \ell^o(b, \tau, K)] &= -\underbrace{\frac{K+1}{\tau} \frac{\tau-K}{2(T-1)}}_{\hookrightarrow \Upsilon_3} - \underbrace{\frac{\tau-1}{2\tau(T-1)}}_{\hookrightarrow \Upsilon_3} - \underbrace{\frac{(\tau-K)^2}{2\tau(T-1)}}_{\hookrightarrow \Upsilon_1} \end{aligned}$$

and

$$\begin{aligned} -\ell^e(b, s - \tau + 1, K') &= -\frac{K'}{s - \tau + 1} \frac{\tau-K}{2(T-1)} + \frac{1}{2(T-1)} - \frac{(s - \tau + 1 - K')^2}{2(s - \tau + 1)(T-1)} \\ &= \underbrace{-\frac{K'}{s - \tau + 1} \frac{\tau-K}{2(T-1)}}_{\equiv \Upsilon_4} + \underbrace{\frac{1}{2(T-1)}}_{\hookrightarrow \Upsilon_2} - \underbrace{\frac{(\tau-K)^2}{2(s - \tau + 1)(T-1)}}_{\hookrightarrow \Upsilon_1}, \end{aligned}$$

where the second equality follows from (3). Thus,

$$\begin{aligned} \Upsilon_1 &\equiv \frac{2(\tau-K)}{2(T-1)} - \frac{(\tau-K)^2}{2\tau(T-1)} - \frac{(\tau-K)^2}{2(s-\tau+1)(T-1)} \\ &= \frac{\tau-K}{2(T-1)} \left(2 - \frac{\tau-K}{\tau} - \frac{\tau-K}{s-\tau+1} \right), \\ \Upsilon_2 &\equiv \frac{\tau-K-1}{T-1} + \frac{1}{2(T-1)}, \\ \Upsilon_3 &\equiv -\frac{K+1}{\tau} \frac{\tau-K}{2(T-1)} - \frac{\tau-1}{2\tau(T-1)} \\ &= -\frac{1}{2\tau(T-1)} [(K+1)(\tau-K) + \tau-1], \\ \Upsilon_4 &\equiv -\frac{K'}{s-\tau+1} \frac{\tau-K}{2(T-1)}. \end{aligned}$$

²We use “ $\hookrightarrow \Upsilon_i$ ” to mean “is part of Υ_i ”.

Hence,

$$\begin{aligned}
\Upsilon_1 + \Upsilon_4 &= \frac{\tau - K}{2(T-1)} \left(2 - \frac{\tau - K}{\tau} - \frac{\tau - K + K'}{s - \tau + 1} \right) \\
&= \frac{\tau - K}{2(T-1)} \left(2 - \frac{\tau - K}{\tau} - \frac{s - \tau + 1}{s - \tau + 1} \right) \\
&= \frac{\tau - K}{2(T-1)} \left(1 - \frac{\tau - K}{\tau} \right) \\
&= \frac{K(\tau - K)}{2\tau(T-1)} \equiv \Upsilon_5 .
\end{aligned}$$

where the second inequality follows from (3). Moreover,

$$\begin{aligned}
\Upsilon_5 + \Upsilon_3 &= \frac{1}{2\tau(T-1)} [K(\tau - K) - (K+1)(\tau - K) - \tau + 1] \\
&= \frac{K+1}{2\tau(T-1)} - \frac{1}{T-1} \equiv \Upsilon_6 ,
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon_6 + \Upsilon_2 &= \frac{\tau - K - 1}{T-1} + \frac{K+1}{2\tau(T-1)} - \frac{1}{2(T-1)} \\
&= \frac{\tau - K - 1}{T-1} \left(1 - \frac{1}{2\tau} \right) \geq 0
\end{aligned}$$

since $K \leq \tau - 1$.

• **Case 2:** $0 = K < K' \leq s - \tau$.

In this case, $\delta_{t+\tau-1} = \frac{2t+\tau-1}{2(T-1)}$, and $b \geq \frac{s-\tau+1-K'}{2(T-1)}$. If $s' = s - \tau + 1$ then

$$\begin{aligned}
\Upsilon(b) &\geq 2 \left(b + \frac{t + \tau - 1}{T-1} \right) - \left[\frac{2t + \tau - 1}{2(T-1)} + \ell^e(b, s', K') + \frac{t + \tau}{T-1} \right] \\
&= 2b - \ell^e(b, s', K') + \frac{\tau - 3}{2(T-1)} \\
&= \frac{2s' - K'}{s'} b - \frac{(s' - K')^2}{2s'(T-1)} + \frac{\tau - 2}{2(T-1)} \\
&\geq \frac{s' - K' + \tau - 2}{2(T-1)} = \frac{s - K' - 1}{2(T-1)} \geq 0
\end{aligned}$$

since $K' \leq s - \tau \leq s - 1$.

• **Case 3:** $0 = K' < K \leq \tau - 1$.

In this case, $\delta_{t+\tau} = \frac{2t+s+\tau}{2(T-1)}$, and $b \geq \max \left\{ \frac{\tau-K}{2(T-1)}, \frac{s-\tau+1}{2(T-1)} \right\}$; so

$$\begin{aligned}
\Upsilon(b) &\geq 2 \left(b + \frac{t + \tau - 1}{T - 1} \right) - \left[2b - \ell^o(b, \tau, K) + \frac{t + K}{T - 1} + \frac{2t + s + \tau}{2(T - 1)} \right] \\
&\geq b + \frac{t + \tau - 1}{T - 1} - \left[2b - \ell^o(b, \tau, K) + \frac{t + K}{T - 1} \right] - \frac{1}{2(T - 1)} \\
&= b - \left[\frac{K + 1}{\tau} b - \frac{\tau - 1}{2\tau(T - 1)} - \frac{(\tau - K)^2}{2\tau(T - 1)} \right] - \frac{1}{2(T - 1)} + \underbrace{\frac{t + \tau - 1}{T - 1} - \frac{t + K}{T - 1}}_{\geq 0} \\
&\geq \frac{\tau - K - 1}{\tau} \frac{\tau - K}{2(T - 1)} - \frac{1}{2\tau(T - 1)} + \frac{(\tau - K)^2}{2\tau(T - 1)} \\
&= \frac{\tau - K - 1}{2\tau(T - 1)} [2(\tau - K) + 1] \geq 0 .
\end{aligned}$$

• **Case 4:** $K = K' = 0$.

In this case, $\delta_{t+\tau-1} = \frac{2t+\tau-1}{2(T-1)}$, $\delta_{t+\tau} = \frac{2t+s+\tau}{2(T-1)}$, and $b \geq \max \left\{ \frac{\tau}{2(T-1)}, \frac{s-\tau+1}{2(T-1)} \right\}$; so

$$\begin{aligned}
\Upsilon(b) &\geq 2 \left(b + \frac{t + \tau - 1}{T - 1} \right) - \left[\frac{2t + \tau - 1}{2(T - 1)} + \frac{2t + s + \tau}{2(T - 1)} \right] \\
&\geq \frac{2(\tau - 1)}{T - 1} - \frac{2(\tau - 1)}{2(T - 1)} = \frac{\tau - 1}{T - 1} \geq 0 .
\end{aligned}$$

Expertise

Observation 3. *If $b \in \left(\frac{1}{2(T-1)}, \frac{1}{T-1} \right)$ then the expert outperforms an amateur who cannot distinguish between states $T - 2$ and $T - 1$, and takes the same decisions as the expert in all other states.*

Proof: Recall that the principal prefers to appoint this amateur and offer her δ_E if and only if³

$$\left[\frac{2T - K}{T} b - \frac{(T - K)^2 + 3T}{2T(T - 1)} \right]^2 < \left[\frac{K}{T} b + \frac{(T - K)^2 - T}{2T(T - 1)} \right]^2 \quad (4)$$

We need to treat cases in which T is odd and even separately, starting with the former: The left-hand side of (4) is negative because $T > 1$ implies that

$$b < \frac{1}{T - 1} < \frac{3T + 1}{2(T - 1)(T + 1)} .$$

Consequently, the principal prefers not to appoint this amateur and offer her δ_E if $b < \frac{1}{T-1}$.

³See the proof of Proposition 3(ii) in the main text.

As $\delta_E > \delta_{K-2}$, the principal can only satisfy the amateur's incentive constraints by replacing δ_{K-1} and δ_{K-2} with a single decision δ if $\delta \geq \delta_E$. At best, the principal can induce her to take decision $\frac{2K-3}{2(T-1)}$ in E , yielding a loss of $\frac{1}{2(T-1)^2}$ in that event. If $b < \frac{1}{T-1}$ then $K = T - 1$, and the principal would lose $\frac{2(T^2+1)}{T^2} \left[b - \frac{1}{2(T-1)} \right]^2$ in E , which exceeds $\frac{1}{2(T-1)^2}$ if and only if

$$4(T^2 + 1)(T - 1)^2 b^2 - 4T(T^2 + 1)(T - 1)b + 1 > 0 :$$

that is, for

$$\frac{T^2 + 1 + T\sqrt{T^2 + 1}}{2(T^2 + 1)(T - 1)} < b < \frac{1}{T - 1} .$$

Now $\delta_E > \frac{2T-5}{2(T-1)}$ if and only if $b > \frac{2T+1}{2(T^2-1)}$, where the latter term is less than $\frac{1}{T-1}$, in which case the principal cannot improve on the amateur's loss by raising the decision above δ_E . Comparing terms for $b \in \left(\frac{1}{2(T-1)}, \frac{1}{T-1} \right)$: an amateur who is offered a single decision outperforms the expert if and only if

$$\frac{T^2 + 1 + T\sqrt{T^2 + 1}}{2(T^2 + 1)(T - 1)} < b < \frac{2T + 1}{2(T^2 - 1)} .$$

It is easy to confirm that the requisite lower bound exceeds the requisite upper bound on b ; so this amateur cannot outperform the expert if $b < \frac{1}{T-1}$.

Now suppose that T is even. It is easy to confirm that $\delta_{K-2} < \frac{2T-5}{2(T-1)}$; so replacing δ_{T-3} and δ_{T-2} with $\frac{2T-5}{2(T-1)}$ does not violate the amateur's incentive constraints. Substituting $K = T - 1$: the expert outperforms an amateur who takes $\frac{2T-5}{2(T-1)}$ in E (and therefore any such amateur) if and only if $b < \frac{1}{T-1}$; so the expert again outperforms an amateur who takes the same decisions in states outside E if $b \in \left(\frac{1}{2(T-1)}, \frac{1}{T-1} \right)$.

Other amateurs for $T = 4$

We include the ODS and losses for the cases omitted from Appendix C.2 for the four-state case:

If $E = \{2, 3\}$ then

	Δ^{23}	λ^{23}	Λ^{23}
$b < \frac{1}{6}$	$0, \frac{1}{3}, \frac{5}{6}$	$\frac{1}{72}$	$b^2 + \frac{1}{72}$
$\frac{1}{6} < b < \frac{1}{3}$	$b - \frac{1}{6}, b + \frac{1}{6}, \frac{5}{6}$	$\frac{1}{2}b^2 - \frac{1}{6}b + \frac{1}{36}$	$\frac{1}{2}b^2 + \frac{1}{36}$
$\frac{1}{3} < b < \frac{1}{2}$	$\frac{1}{2}b, \frac{3}{2}b, \frac{1}{2}b + \frac{2}{3}$	$\frac{3}{4}b^2 - \frac{1}{3}b + \frac{1}{18}$	$\frac{1}{4}b^2 + \frac{1}{18}$
$\frac{1}{2} < b < \frac{2}{3}$	$\frac{3}{2}b - \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}$	$\frac{3}{4}b^2 - \frac{1}{2}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{0, 1, 2\}$ then

b	Δ^{012}	λ^{012}	Λ^{012}
$b < \frac{1}{3}$	$\frac{1}{3}, 1$	$\frac{1}{18}$	$b^2 + \frac{1}{18}$
$\frac{1}{3} < b < \frac{2}{3}$	$\frac{1}{2}b + \frac{1}{6}, \frac{3}{2}b + \frac{1}{2}$	$\frac{3}{4}b^2 - \frac{1}{2}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

The same details apply when $E = \{0, 2\}$.

If $E = \{1, 2, 3\}$ then

b	Δ^{123}	λ^{123}	Λ^{123}
$\frac{1}{6} < b < \frac{1}{3}$		$\frac{1}{18}$	$b^2 + \frac{1}{18}$
$\frac{1}{3} < b < \frac{2}{3}$	$\frac{3}{2}b - \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}$	$\frac{3}{4}b^2 - \frac{1}{2}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

The same details apply when $E = \{1, 3\}$.

If $E = \{0, 3\}$ then

b	Δ^{03}	λ^{03}	Λ^{03}
$b < \frac{1}{6}$	$b + \frac{1}{4}, b + \frac{5}{12}, b + \frac{7}{12}$	$b^2 - \frac{1}{6}b + \frac{19}{144}$	$\frac{19}{144}$
$\frac{1}{6} < b < \frac{2}{9}$		$\frac{3}{4}b^2 - \frac{1}{6}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$
$b > \frac{2}{9}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{0, 1, 3\}$ then

b	Δ^{013}	λ^{013}	Λ^{013}
$b < \frac{2}{9}$		$\frac{3}{4}b^2 - \frac{1}{6}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$
$b > \frac{2}{9}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{0, 2, 3\}$ then

b	Δ^{023}	λ^{023}	Λ^{023}
$b < \frac{2}{9}$	$\frac{3}{2}b + \frac{1}{6}, \frac{1}{2}b + \frac{1}{2}$	$\frac{3}{4}b^2 - \frac{1}{6}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$
$b > \frac{2}{9}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{0, 1\}, \{2, 3\}$ then

b	$\Delta^{01,23}$	$\lambda^{01,23}$	$\Lambda^{01,23}$
$b < \frac{1}{3}$	$\frac{1}{6}, \frac{5}{6}$	$\frac{1}{36}$	$b^2 + \frac{1}{36}$
$\frac{1}{3} < b < \frac{2}{3}$	$b - \frac{1}{6}, b + \frac{1}{2}$	$b^2 - \frac{2}{3}b + \frac{5}{36}$	$\frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{0, 2\}, \{1, 3\}$ then

b	$\Delta^{02,13}$	$\lambda^{02,13}$	$\Lambda^{02,13}$
$b < \frac{1}{6}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{9}$	$b^2 + \frac{1}{9}$
$\frac{1}{6} < b < \frac{1}{3}$	$b - \frac{1}{6}, b + \frac{1}{2}$	$b^2 - \frac{1}{3}b + \frac{5}{36}$	$\frac{5}{36}$
$b > \frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{0, 3\}, \{1, 2\}$ or $E = \{0, 1, 2, 3\}$ then

b	Δ^E	λ^E	Λ^E
$b > 0$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

Other amateurs for $T = 5$

We include the ODS and losses for the cases omitted from Appendix C.3 for the five-state case:

If $E = \{0, 1\}, \{2, 3, 4\}$ then

b	$\Delta^{01,234}$	$\lambda^{01,234}$	$\Lambda^{01,234}$
$b < \frac{5}{16}$	$\frac{1}{8}, \frac{3}{4}$	$\frac{1}{32}$	$b^2 + \frac{1}{32}$
$\frac{5}{16} < b < \frac{5}{8}$	$\frac{6}{5}b - \frac{1}{4}, \frac{4}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{3}{25}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 1\}, \{3, 4\}$ then

b	$\Delta^{01,34}$	$\lambda^{01,34}$	$\Lambda^{01,34}$
$b < \frac{3}{16}$	$\frac{1}{8}, \frac{1}{2}, \frac{7}{8}$	$\frac{1}{80}$	$b^2 + \frac{1}{80}$
$\frac{3}{16} < b < \frac{3}{8}$	$\frac{2}{5}b + \frac{1}{20}, \frac{8}{5}b + \frac{1}{5}, \frac{2}{5}b + \frac{4}{5}$	$\frac{16}{25}b^2 - \frac{6}{25}b + \frac{7}{200}$	$\frac{9}{25}b^2 + \frac{7}{200}$
$\frac{3}{8} < b < \frac{5}{8}$	$\frac{6}{5}b - \frac{1}{4}, \frac{4}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{3}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 1, 2, 3\}$ or $E = \{0, 3\}, \{1, 2\}$ then

b	Δ^E	λ^E	Λ^E
$b < \frac{5}{16}$	$\frac{3}{8}, 1$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{5}{16} < b < \frac{5}{8}$	$\frac{2}{5}b + \frac{1}{4}, \frac{8}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1, 2\}, \{0, 4\}$ then

b	$\Delta^{12,04}$	$\lambda^{12,04}$	$\Lambda^{12,04}$
$b < \frac{1}{16}$	$\frac{3}{8}, \frac{1}{2}, \frac{3}{4}$	$\frac{17}{160}$	$b^2 + \frac{17}{160}$
$\frac{1}{16} < b < \frac{\sqrt{10}-1}{16}$	$b + \frac{5}{16}, b + \frac{7}{16}, \frac{3}{4}$	$\frac{4}{5}b^2 - \frac{1}{10}b + \frac{7}{64}$	$\frac{1}{5}b^2 + \frac{7}{64}$
$\frac{\sqrt{10}-1}{16} < b < \frac{1}{4}$	$\frac{2}{5}b + \frac{2}{5}, \frac{8}{5}b + \frac{3}{5}$	$\frac{16}{25}b^2 - \frac{3}{25}b + \frac{23}{200}$	$\frac{9}{25}b^2 + \frac{23}{200}$
$b > \frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1, 2\}, \{3, 4\}$ then

b	$\Delta^{12,34}$	$\lambda^{12,34}$	$\Lambda^{12,34}$
$b < \frac{3}{16}$	$0, \frac{3}{8}, \frac{7}{8}$	$\frac{1}{80}$	$b^2 + \frac{1}{80}$
$\frac{3}{16} < b < \frac{9}{32}$	$\frac{4}{3}b + -\frac{1}{4}, \frac{2}{3}b + \frac{1}{4}, \frac{7}{8}$	$\frac{8}{15}b^2 - \frac{1}{5}b + \frac{1}{32}$	$\frac{7}{15}b^2 + \frac{1}{32}$
$\frac{\sqrt{10}-1}{16} < b < \frac{1}{4}$	$\frac{4}{5}b - \frac{1}{10}, \frac{6}{5}b + \frac{1}{10}, \frac{4}{5}b + \frac{13}{20}$	$\frac{24}{25}b^2 - \frac{11}{25}b + \frac{13}{200}$	$\frac{1}{25}b^2 + \frac{13}{200}$
$b > \frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1, 2\}, \{0, 3, 4\}$ then

b	$\Delta^{12,034}$	$\lambda^{12,034}$	$\Lambda^{12,034}$
$b < \frac{5}{48}$	$\frac{3}{8}, \frac{7}{12}$	$\frac{11}{96}$	$b^2 + \frac{11}{96}$
$\frac{5}{48} < b < \frac{5}{24}$	$\frac{6}{5}b + \frac{1}{4}, \frac{4}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{1}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{24}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 4\}, \{2, 3\}$ then

$$\begin{array}{llll}
b & \Delta^{04,23} & \lambda^{04,23} & \Lambda^{04,23} \\
b < \frac{1}{16} & \frac{1}{4}, \frac{1}{2}, \frac{5}{8} & \frac{17}{160} & b^2 + \frac{17}{160} \\
\frac{1}{16} < b < \frac{5(\sqrt{2}-1)}{16} & \frac{1}{4}, b + \frac{7}{16}, b + \frac{9}{16} & \frac{4}{5}b^2 - \frac{1}{10}b + \frac{7}{64} & \frac{1}{5}b^2 + \frac{7}{64} \\
\frac{5(\sqrt{2}-1)}{16} < b < \frac{5}{16} & \left\{ \frac{8}{5}b, \frac{2}{5}b + \frac{1}{2} \right\} & \frac{16}{25}b^2 - \frac{1}{5}b + \frac{1}{8} & \frac{9}{25}b^2 + \frac{1}{8} \\
b > \frac{5}{16} & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$

If $E = \{0, 1, 4\}, \{2, 3\}$ then

$$\begin{array}{llll}
b & \Delta^{014,23} & \lambda^{014,23} & \Lambda^{014,23} \\
b < \frac{5}{48} & \frac{5}{12}, \frac{5}{8} & \frac{11}{96} & b^2 + \frac{11}{96} \\
\frac{5}{48} < b < \frac{5}{24} & \frac{4}{5}b + \frac{1}{3}, \frac{6}{5}b + \frac{1}{2} & \frac{24}{25}b^2 - \frac{1}{5}b + \frac{1}{8} & \frac{1}{25}b^2 + \frac{1}{8} \\
b > \frac{5}{24} & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$

If $E = \{0, 1, 2\}, \{3, 4\}$ then

$$\begin{array}{llll}
b & \Delta^{012,34} & \lambda^{012,34} & \Lambda^{012,34} \\
b < \frac{5}{16} & \frac{1}{4}, \frac{7}{8} & \frac{11}{32} & b^2 + \frac{1}{32} \\
\frac{5}{16} < b < \frac{5}{8} & \frac{4}{5}b, \frac{6}{5}b + \frac{1}{2} & \frac{24}{25}b^2 - \frac{3}{5}b + \frac{1}{8} & \frac{1}{25}b^2 + \frac{1}{8} \\
b > \frac{5}{8} & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$

If $E = \{0, 2\}, \{1, 3\}$ then

$$\begin{array}{llll}
b & \Delta^{02,13} & \lambda^{02,13} & \Lambda^{02,13} \\
b < \frac{1}{8} & \frac{1}{4}, \frac{1}{2}, 1 & \frac{1}{20} & b^2 + \frac{1}{20} \\
\frac{1}{8} < b < \frac{5(\sqrt{2}-1)}{8} & b + \frac{1}{8}, b + \frac{3}{8}, 1 & \frac{4}{5}b^2 - \frac{1}{5}b + \frac{1}{16} & \frac{1}{5}b^2 + \frac{1}{16} \\
\frac{5(\sqrt{2}-1)}{8} < b < \frac{5}{8} & \frac{2}{5}b + \frac{1}{4}, \frac{8}{5}b + \frac{1}{2} & \frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8} & \frac{9}{25}b^2 + \frac{1}{8} \\
b > \frac{5}{8} & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$

If $E = \{0, 3\}, \{1, 4\}$ then

b	$\Delta^{03,14}$	$\lambda^{03,14}$	$\Lambda^{03,14}$
$b < \frac{1}{16}$	$\frac{3}{8}, \frac{1}{2}, \frac{5}{8}$	$\frac{9}{80}$	$b^2 + \frac{9}{80}$
$\frac{1}{16} < b < \frac{1}{4}$	$\frac{1}{4}, \frac{2}{3}b + \frac{7}{12}, \frac{4}{3}b + \frac{2}{3}$	$\frac{8}{15}b^2 - \frac{1}{15}b + \frac{1}{12}$	$\frac{7}{15}b^2 + \frac{1}{12}$
$\frac{1}{4} < b < \frac{5}{8}$	$\frac{6}{5}b, \frac{4}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 3\}, \{2, 4\}$ then

b	$\Delta^{03,24}$	$\lambda^{03,24}$	$\Lambda^{03,24}$
$b < \frac{1}{16}$	$\frac{1}{4}, \frac{3}{8}, \frac{3}{4}$	$\frac{13}{160}$	$b^2 + \frac{13}{160}$
$\frac{1}{16} < b < \frac{1}{8}$	$\frac{4}{3}b + \frac{1}{6}, \frac{2}{3}b + \frac{1}{3}, \frac{3}{4}$	$\frac{8}{15}b^2 - \frac{b}{15} + \frac{1}{12}$	$\frac{7}{15}b^2 + \frac{1}{12}$
$\frac{1}{8} < b < \frac{1}{6}$	$\frac{1}{3}, \frac{3}{4}$	$\frac{1}{12}$	$b^2 + \frac{1}{12}$
$\frac{1}{6} < b < \frac{3}{8}$	$\frac{4}{5}b + \frac{1}{5}, \frac{6}{5}b + \frac{11}{20}$	$\frac{24}{25}b^2 - \frac{8}{25}b + \frac{11}{100}$	$\frac{1}{25}b^2 + \frac{11}{100}$
$b > \frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1, 3\}, \{2, 4\}$ then

b	$\Delta^{13,24}$	$\lambda^{13,24}$	$\Lambda^{13,24}$
$b < \frac{1}{8}$	$0, \frac{1}{2}, \frac{3}{4}$	$\frac{1}{20}$	$b^2 + \frac{1}{20}$
$\frac{1}{8} < b < \frac{1}{4}$	$0, b + \frac{3}{8}, b + \frac{5}{8}$	$\frac{4}{5}b^2 - \frac{b}{5} + \frac{1}{16}$	$\frac{1}{5}b^2 + \frac{1}{16}$
$\frac{1}{4} < b < \frac{5}{16}$	$0, \frac{5}{8}$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{5}{16} < b < \frac{5}{8}$	$\frac{8}{5}b - \frac{1}{2}, \frac{2}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 2, 3\}, \{1, 4\}$ then

b	$\Delta^{023,14}$	$\lambda^{023,14}$	$\Lambda^{023,14}$
$b < \frac{5}{48}$	$\frac{5}{12}, \frac{5}{8}$	$\frac{11}{96}$	$b^2 + \frac{11}{96}$
$\frac{5}{48} < b < \frac{5}{24}$	$\frac{4}{5}b + \frac{1}{3}, \frac{6}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{1}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{24}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 1, 3\}, \{2, 4\}$ then

b	$\Delta^{013,24}$	$\lambda^{013,24}$	$\Lambda^{013,24}$
$b < \frac{5}{24}$	$\frac{1}{3}, \frac{3}{4}$	$\frac{1}{12}$	$b^2 + \frac{1}{12}$
$\frac{5}{24} < b < \frac{5}{12}$	$\frac{4}{5}b + \frac{1}{6}, \frac{6}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 1, 3\}$ then

b	Δ^{013}	λ^{013}	Λ^{013}
$b < \frac{1}{12}$	$\frac{1}{3}, \frac{1}{2}, 1$	$\frac{7}{120}$	$b^2 + \frac{7}{120}$
$\frac{1}{12} < b < \frac{1}{6}$	$\frac{1}{2}b + \frac{7}{24}, \frac{3}{2}b + \frac{3}{8}, 1$	$\frac{3}{5}b^2 - \frac{1}{10}b + \frac{1}{16}$	$\frac{2}{5}b^2 + \frac{1}{16}$
$\frac{1}{6} < b < \frac{3}{16}$	$\frac{3}{8}, 1$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{3}{16} < b < \frac{1}{2}$	$\frac{2}{5}b + \frac{11}{30}, \frac{8}{5}b + \frac{3}{10}, \frac{2}{5}b + \frac{7}{10}$	$\frac{16}{25}b^2 - \frac{4}{25}b + \frac{17}{200}$	$\frac{9}{25}b^2 + \frac{17}{200}$
$b > \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 1, 4\}$ then

b	Δ^{014}	λ^{014}	Λ^{014}
$b < \frac{7+2\sqrt{10}}{48}$	$\frac{2}{5}b + \frac{2}{5}, \frac{8}{5}b + \frac{3}{5}$	$\frac{16}{25}b^2 - \frac{3}{25}b + \frac{23}{200}$	$\frac{9}{25}b^2 + \frac{23}{200}$
$\frac{7+2\sqrt{10}}{48} < b < \frac{3}{8}$	$\frac{b}{2} + \frac{19}{48}, \frac{3}{2}b + \frac{7}{16}, \frac{3}{4}$	$\frac{3}{5}b^2 - \frac{1}{20}b + \frac{7}{64}$	$\frac{2}{5}b^2 + \frac{7}{64}$
$\frac{3}{8} < b < \frac{5}{8}$	$\frac{2}{5}b + \frac{2}{5}, \frac{8}{5}b + \frac{3}{5}$	$\frac{16}{25}b^2 - \frac{3}{25}b + \frac{23}{200}$	$\frac{9}{25}b^2 + \frac{23}{200}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1, 2, 4\}$ then

b	Δ^{124}	λ^{124}	Λ^{124}
$b < \frac{1}{12}$	$0, \frac{7}{12}, \frac{3}{4}$	$\frac{7}{120}$	$b^2 + \frac{7}{120}$
$\frac{1}{12} < b < \frac{1}{6}$	$0, \frac{1}{2}b + \frac{13}{24}, \frac{3}{2}b + \frac{5}{8}$	$\frac{3}{5}b^2 - \frac{1}{10}b + \frac{1}{16}$	$\frac{2}{5}b^2 + \frac{1}{16}$
$\frac{1}{6} < b < \frac{5}{16}$	$0, \frac{5}{8}$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{5}{16} < b < \frac{5}{8}$	$\frac{8}{5}b - \frac{1}{2}, \frac{2}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{2, 3, 4\}$ then

b	Δ^{234}	λ^{234}	Λ^{234}
$b < \frac{1}{8}$	$0, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{40}$	$b^2 + \frac{1}{40}$
$\frac{1}{8} < b < \frac{3}{8}$	$b - \frac{1}{8}, b + \frac{1}{8}, \frac{3}{4}$	$\frac{2}{5}b^2 - \frac{1}{10}b + \frac{1}{32}$	$\frac{2}{5}b^2 + \frac{1}{32}$
$\frac{3}{8} < b < \frac{1}{2}$	$\frac{2}{5}b + \frac{1}{10}, \frac{8}{5}b - \frac{1}{10}, \frac{2}{5}b + \frac{3}{5}$	$\frac{16}{25}b^2 - \frac{7}{25}b + \frac{13}{200}$	$b^2 + \frac{13}{200}$
$\frac{1}{2} < b < \frac{5}{8}$	$\frac{8}{5}b - \frac{1}{2}, \frac{2}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1, 2, 3, 4\}$ then

b	Δ^{1234}	λ^{1234}	Λ^{1234}
$b < \frac{1}{16}$	$0, \frac{5}{8}$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{1}{16} < b < \frac{5}{8}$	$\frac{8}{5}b - \frac{1}{2}, \frac{2}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 2, 3, 4\}$ then

b	Δ^{0234}	λ^{0234}	Λ^{0234}
$b < \frac{5}{32}$	$\frac{1}{4}, \frac{9}{16}$	$\frac{7}{64}$	$b^2 + \frac{7}{64}$
$\frac{5}{32} < b < \frac{5}{16}$	$\frac{8}{5}b, \frac{2}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{1}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{16}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 1, 2, 4\}$ then

b	Δ^{0124}	λ^{0124}	Λ^{0124}
$b < \frac{5}{32}$	$\frac{7}{16}, \frac{3}{4}$	$\frac{7}{64}$	$b^2 + \frac{7}{64}$
$\frac{5}{32} < b < \frac{5}{16}$	$\frac{2}{5}b + \frac{3}{8}, \frac{8}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{1}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{16}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 1, 2, 3\}$ then

b	Δ^{0123}	λ^{0123}	Λ^{0123}
$b < \frac{5}{16}$	$\frac{3}{8}, 1$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{5}{16} < b < \frac{5}{12}$	$\frac{2}{5}b + \frac{1}{4}, \frac{8}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0, 2\}, \{1, 3, 4\}$ then

$$\begin{array}{cccc}
b & \Delta^{02,134} & \lambda^{02,134} & \Lambda^{02,134} \\
b < \frac{5}{24} & \frac{1}{4}, \frac{2}{3} & \frac{1}{12} & b^2 + \frac{1}{12} \\
\frac{5}{24} < b < \frac{5}{12} & \frac{6}{5}b, \frac{4}{5}b + \frac{1}{2} & \frac{24}{25}b^2 - \frac{2}{5}b + \frac{1}{8} & \frac{1}{25}b^2 + \frac{1}{8} \\
b > \frac{5}{12} & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$

If $E = \{0, 2, 4\}$ then

$$\begin{array}{cccc}
b & \Delta^{024} & \lambda^{024} & \Lambda^{024} \\
b < \frac{1}{8} & \frac{1}{4}, \frac{1}{2}, \frac{3}{4} & \frac{1}{10} & b^2 + \frac{1}{10} \\
\frac{1}{8} < b < \frac{1}{4} & \frac{6}{5}b + \frac{1}{10}, \frac{4}{5}b + \frac{2}{5}, \frac{6}{5}b + \frac{3}{5} & \frac{24}{25}b^2 - \frac{6}{25}b + \frac{23}{200} & \frac{1}{25}b^2 + \frac{23}{200} \\
\frac{1}{4} < b < \frac{5}{16} & \frac{8}{5}b, \frac{2}{5}b + \frac{1}{2} & \frac{16}{25}b^2 - \frac{1}{5}b + \frac{1}{8} & \frac{9}{25}b^2 + \frac{1}{8} \\
b > \frac{5}{16} & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$

If $E = \{0, 3, 4\}$ then

$$\begin{array}{cccc}
b & \Delta^{024} & \lambda^{024} & \Lambda^{024} \\
b < \frac{1}{24} & \frac{1}{4}, \frac{1}{2}, \frac{7}{12} & \frac{13}{120} & b^2 + \frac{13}{120} \\
\frac{1}{24} < b < \frac{15-10\sqrt{2}}{8} & \frac{1}{4}, \frac{3}{2}b + \frac{7}{16}, \frac{1}{2}b + \frac{9}{16} & \frac{3}{5}b^2 - \frac{1}{20}b + \frac{7}{64} & \frac{2}{5}b^2 + \frac{7}{64} \\
\frac{15-10\sqrt{2}}{8} < b < \frac{5}{16} & \frac{8}{5}b, \frac{2}{5}b + \frac{1}{2} & \frac{16}{25}b^2 - \frac{1}{5}b + \frac{1}{8} & \frac{9}{25}b^2 + \frac{1}{8} \\
b > \frac{5}{16} & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$

If $E = \{0, 2\}, \{1, 4\}$ then

$$\begin{array}{cccc}
b & \Delta^{02,14} & \lambda^{02,14} & \Lambda^{02,14} \\
b < \frac{1}{16} & \frac{1}{4}, \frac{5}{8}, \frac{3}{4} & \frac{13}{160} & b^2 + \frac{13}{160} \\
\frac{1}{16} < b < \frac{1}{8} & \frac{1}{4}, \frac{2}{3}b + \frac{7}{12}, \frac{4}{3}b + \frac{2}{3} & \frac{8}{15}b^2 - \frac{1}{15}b + \frac{1}{12} & \frac{7}{15}b^2 + \frac{1}{12} \\
\frac{1}{8} < b < \frac{5}{24} & \frac{1}{4}, \frac{2}{3} & \frac{1}{12} & b^2 + \frac{1}{12} \\
\frac{5}{24} < b < \frac{5}{12} & \frac{6}{5}b, \frac{4}{5}b + \frac{1}{2} & \frac{24}{25}b^2 - \frac{2}{5}b + \frac{1}{8} & \frac{1}{25}b^2 + \frac{1}{8} \\
b > \frac{5}{12} & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$

If $E = \{0, 3\}, \{1, 2, 4\}$ then

$$\begin{array}{cccc}
b & \Delta^{03,124} & \lambda^{03,124} & \Lambda^{03,124} \\
b < \frac{5}{48} & \frac{3}{8}, \frac{7}{12} & \frac{11}{96} & b^2 + \frac{11}{96} \\
\frac{5}{48} < b < \frac{5}{24} & \frac{6}{5}b + \frac{1}{4}, \frac{4}{5}b + \frac{1}{2} & \frac{24}{25}b^2 - \frac{1}{5}b + \frac{1}{8} & \frac{1}{25}b^2 + \frac{1}{8} \\
b > \frac{5}{24} & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$

If $E = \{0, 2, 4\}, \{1, 3\}$ or $\{0, 1, 2, 3, 4\}$ then

$$\begin{array}{cccc}
b & \Delta^E & \lambda^E & \Lambda^E \\
b > 0 & \frac{1}{2} & \frac{1}{8} & b^2 + \frac{1}{8}
\end{array}$$