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Prieur, Fabien; Tidball, Mabel; Withagen, Cees A.

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Optimal emission-extraction policy in a world of scarcity and irreversibility

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Optimal Emission-Extraction Policy in a World of Scarcity and Irreversibility

Fabien Prieur
Mabel Tidball
Cees Withagen

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Optimal Emission-Extraction Policy in a World of Scarcity and Irreversibility

Abstract

This paper extends the classical exhaustible-resource/stock-pollution model with the irreversibility of pollution decay. Within this framework, we answer the question how the potential irreversibility of pollution affects the extraction path. We investigate the conditions under which the economy will optimally adopt a reversible policy, and when it is optimal to enter the irreversible region. In the case of irreversibility it may be optimal to leave a positive amount of resource in the ground forever. As far the optimal extraction/emission policy is concerned, several types of solutions may arise, including solutions where the economy stays at the threshold for a while. Given that different programs may satisfy the first order conditions for optimality, we further investigate when each of these is optimal. The analysis is illustrated by means of a numerical example. To sum up, for any pollution level, we can identify a critical resource stock such that there exist multiple optima i.e. a reversible and an irreversible policy that yield exactly the same present value. For any resource stock below this critical value, the optimal policy is reversible whereas with large enough resource, irreversible policies outperform reversible programs. Finally, the comparison between irreversible policies reveals that it is never optimal for the economy to stay at the threshold for a while before entering the irreversible region.

JEL-Code: Q300, Q530, C610.

Keywords: non-renewable resource, irreversible pollution, optimal policy.

Fabien Prieur
LAMETA / University of Montpellier I
& INRA
Montpellier / France
prieur@supagro.inra.fr

Mabel Tidball
LAMETA / University of Montpellier I
& INRA
Montpellier / France
tidball@supagro.inra.fr

Cees Withagen
VU University of Amsterdam
& Tinbergen Institute
Amsterdam / The Netherlands
cwithagen@feweb.vu.nl

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1 Introduction

In the design of optimal climate policy it should be taken into account that most emissions of CO2 result from burning fossil fuel, which originate from non-renewable resources. It is forcefully argued by e.g., D'Arge and Kogiku (1973) that "the 'pure' mining problem must be coupled with the 'pure' pollution problem". Although not really applicable to the climate change problem they also state that questions like these become relevant: which should we run out first, air to breathe or fossil fuels to pollute the air we breathe? These questions have attracted environmental and resources economists' attention for a long time. An early prototype model can be found in Withagen (1994) where utility is derived from consumption of fossil fuel from a non-renewable resource and where the use of the fossil fuel also contributes to the accumulation of CO2. The accumulated stock of CO2 causes damage, represented by a convex damage function. One of the findings is that the extraction path of the fossil fuel becomes flatter than in the absence of environmental damage. In a similar model Sinclair (1994) and Ulph and Ulph (1994) derive the optimal carbon tax needed to implement the first-best optimum. Still focusing on the exhaustible-resource/stock-pollution model, Tahvonen (1997) fully characterizes the properties of the optimal extraction/emission policy in the case of exponential decay of pollution. He notably shows that, since extraction and pollution necessarily converge toward zero, in the long run, the pollution problem does not have an influence on the total amount of resource extracted over the planning period. In that sense, the pollution and the resource management problems are independent of each other.

A widely used alternative to capturing environmental damage is to impose a ceiling on the total accumulated stock of CO2. Examples of this approach are Chakravorty et al. (2006, 2008). This is usually motivated in the following ways. First one could argue that a ceiling is a political reality. International negotiations are indeed aiming at keeping the temperature rise below 2 degrees ^{0}C and it is widely accepted that in order to accomplish this the CO2 concentration should be no more than 450 ppmv. Second, and related to the first argument, taking a damage function rather than a ceiling, may lead to highly undesired outcomes because there is substantial uncertainty surrounding the effects of climate change and catastrophes may occur (see e.g., Tsur and Zemel, 2008). So, with a ceiling one models the relatively safe region and irreversibility.

It is well known that the climate system is extremely complex and that economists' modelling of it is rather rudimentary. The focus of the present contribution is on irreversibility beyond a certain pollution level, but in a different sense. Usually the decay of pollution is modeled as linear, meaning that a constant percentage of the existing stock

is diluted per unit of time. This approach has been criticized by many authors including Dasgupta (1982), Fiedler (1992), and Pethig (1993). Representing decay as a constant fraction of the existing stock is far too simplistic and with a large stock of CO2 the absorption capacity of oceans and forests may be reduced considerably. This is what we want to capture by introducing the ceiling or irreversibility threshold. Indeed, experts of the second working group of the IPCC (2007) have identified positive climate feedbacks due to emissions of greenhouse gases (GHG). There is more and more evidence that increasing emission levels and concentrations of GHG disturb the regeneration capacity of natural ecosystems. Oceans, that form the most important carbon sink, display a buffering capacity that begins to get saturated. At the same time, the assimilation capacity of terrestrial ecosystems (lands and forests form the other important carbon sink) will likely peak by mid-century and then decline to become a net source of carbon by the end of the present century. Therefore, the irreversible degradation of the assimilation capacity of Nature does not seem so distant from today.

In economics alternative specifications have been proposed by e.g., Forster (1975). They usually allow for inverted-U shaped decay with the important feature that there exists a critical threshold of pollution above which the assimilation capacity of Nature becomes permanently exhausted, thereby implying an irreversible concentration of pollution. In this way a ceiling is introduced, not on the allowed stock of pollutants but on the stock of pollutants that allows for decay. The decision maker is then faced with the problem whether it is optimal to stay below the ceiling and benefit from decay or going beyond it, because of higher consumption, and then stay in the irreversible region. Irreversibility of decay has been studied in detail by Tahvonen and Withagen (1996). An important result obtained by Tahvonen and Withagen is the potential multiplicity of equilibria. Their inverted-U shaped decay function introduces a non-convexity that may give rise to multiple paths satisfying the necessary conditions, starting from the same initial stock values.

The present paper adds to the contribution by Tahvonen and Withagen (1996) by introducing the irreversibility of decay in the classical exhaustible-resource/stock-pollution model, without extraction costs and backstop technology. In contrast with Tahvonen and Withagen we take exhaustibility explicitly into account. We also build on Tahvonen (1997) who deals with exhaustibility, but has linear decay. Our approach to modelling irreversibility is to assume that the decay rate is linear for levels of pollution up to some critical level, after which decay is zero and remains zero. This way of modeling decay differs from the quadratic approach adopted by Tahvonen and Withagen (1996). We do so in order to capture the fact that the regenerative capacity may vanish abruptly, and not

in a smooth way. Within this framework, the first question is to know how the potential irreversibility of pollution affects the extraction path. Our aim is also to emphasize the conditions under which the economy will optimally adopt an irreversible or a reversible policy.

Once the situation has turned irreversible, we show that the pollution problem does affect the total amount of resource extracted. In particular, it may be optimal to leave a positive amount of resource in the ground forever. As far the optimal extraction/emission policy is concerned, several types of solutions may arise. We derive a simple condition that guarantees that it is optimal to stay in what is called the reversible region. When this condition does not hold, it is difficult to conclude whether the optimal policy is reversible or irreversible. But we are able to characterize all programs that satisfy the necessary conditions. Moreover we provide some intuition for the optimal choice to be made. We illustrate our findings using a numerical example. We also have a rudimentary non-convexity and it is one of the aims of the present paper to investigate the occurrence of multiplicity in our model.

With our numerical example we are able to divide the plane of initial conditions, for pollution and the resource stock, into several regions, each being associated with one or more optimality candidates. When the initial resource stock is low enough, the upper bound set on the resource stock becoming lower as initial pollution increases, the optimal solution is reversible and may feature a period of time staying at the threshold. For larger resource stock, irreversible and reversible policies co-exist. So, we illustrate the existence of multiple candidates for optimality. Computation of present values reveals that for any pollution level, the reversible policy yields the highest value for low to medium resource stocks, whereas it is optimal to follow an irreversible policy when the initial endowment of the resource is high enough. In addition, among irreversible policies, the ones immediately entering the irreversible region always dominate the others that stay at the threshold for a period of time. More importantly, for any initial pollution level, one can find the corresponding resource stock such that reversible and irreversible policies yield exactly the same present value. Therefore, we show the existence of multiple optima. This result echoes Tahvonen and Withagen (1996)'s findings, in their model with a quadratic decay and abundant resource. It also raises the question of how a policy maker should decide between these two policy alternatives with so distinct features.

We will also pay attention to the type of decay function employed by Tahvonen and Withagen (1996) in order to see what the effect of of exhaustibility in the context of their model. However, since we cannot have sustained extraction at a positive level, it should be clear that, in particular for not too low resource stock, the results differ to a large extent.

The paper is organized as follows. In section 2 we present the formal model and characterize the equilibrium candidates, reversible as well as irreversible. In section 3 we derive the optimum and provide some economic intuition about when irreversibility should play a role. In section 4 compare our results with those obtained by Tahvonen and Withagen (1996). Section 5 concludes.

2 The model

We consider a partial equilibrium representation of the global warming problem. Our carbon economy is described by the following set of assumptions. The economy produces one good y with a technology that uses a nonrenewable natural resource of fossil fuels, whose stock is denoted by $x \geq 0$. Extraction is costless. Emissions are a one-to-one production by-product. So they equal y. The stock of resource follows the usual law:

$$\dot{x}(t) = -y(t) \text{ with } x(0) = x_0 \text{ given. Hence } x(t) = x_0 - \int_0^t y(u)du \tag{1}$$

Let $U(y)$ be the utility derived from consumption of the good. In the same vein as Tahvonen and Withagen (1996) we assume:

Assumption 1. *The utility function is such that:* $U(0) = 0$, $U''(y) < 0$, $0 < U'(0) < \infty$ *and there exists* $\bar{y}$ such that $U'(\bar{y}) = 0$.

Remark. The utility function can also be understood as a profit function: $U(y) = py - c(y)$ with p a constant and exogenous price and $c(y)$ a convex production cost function.

Emissions contribute to the accumulation of a pollution stock, z. Pollution accumulation is not innocuous to ecosystems and, in particular, it affects their capacity to regenerate. We assume that pollution turns irreversible if the stock were to reach a critical threshold $\bar{z} > z_0$. This irreversibility threshold is known to the policy-maker. Therefore, we do not consider any uncertainty surrounding $\bar{z}$. To account for irreversibility, the dynamics of the pollutant are defined piecewise:

$$\dot{z}(t) = \begin{cases} y(t) - \alpha z(t) & \text{if } z(t) \leq \bar{z} \\ y(t) & \text{else} \end{cases} \tag{2}$$

The natural regeneration or assimilation rate α is constant and positive as long as accumulated emissions are not too high, that is, as long as the stock remains below or is at the irreversibility threshold $\bar{z}$. Once the threshold is surpassed, a new stage is entered where the regeneration capacity is completely and permanently vanished. Thus, pollution

becomes irreversible. In section 4 we will consider an alternative specification where decay is not exponential up to the ceiling, but is inverted U-shaped, the case studied by Tahvonen and Withagen (1996), in order to assess the differences. Hereafter, the domain where $z \leq \bar{z}$ is called the reversible region whereas whenever pollution is higher than $\bar{z}$, the economy is said to be in the irreversible region. For the solution, it matters whether the reversible region is an open or a closed set. We assume the latter.

Pollution is damaging to the economy. For any level z, denote the pollution damage as $D(z)$.

Assumption 2. *The damage function is such that:* $D(0) = 0$, $D'(z) > 0$, $D''(z) > 0$, $D'(0) = 0$ *and* $\lim_{z\to\infty} D'(z) = \infty$.

The social welfare function reads

$$\max_{\{y\}} W = \int_0^\infty e^{-\delta t} \left[U(y(t)) - D(z(t))\right] dt \tag{3}$$

with δ the discount rate.

In the sequel we provide a solution to the problem posed. We make a distinction between reversible and irreversible solutions, according to the behavior of the pollution stock. We first deal with reversible solutions.

2.1 Reversible solutions

The first question we address is when an optimal solution is reversible forever? To answer this question we consider the benchmark problem $\mathcal{BP}$ of maximizing the social welfare function subject to (1) and (2) with the additional constraint that $z(t) \leq \bar{z}$ for all $t \geq 0$:

$$\max_y \int_0^\infty e^{-\delta t}(U(y) - D(z))dt$$

subject to

$$\begin{aligned}
&\dot{z} = y - \alpha z,\ z(0) = z_0 \text{ given} \\
&\dot{x} = -y, x(0) = x_0 \text{ given} \\
&y(t) \geq 0, \quad x(t) \geq 0 \quad z(t) \leq \bar{z} \quad \forall t
\end{aligned}$$

Suppose that an optimal solution satisfies $z(t) < \bar{z}$ for all $t \geq 0$. Then the first order conditions read

$$\begin{aligned}
&U'(y) \leq \lambda + \mu,\ y \geq 0, y[U'(y) - \lambda - \mu] = 0 \\
&\dot{\mu} = \delta\mu \\
&\dot{\lambda} = (\delta + \alpha)\lambda - D'(z) \\
&\dot{z} = y - \alpha z \\
&\dot{x} = -y, x \geq 0 \\
&\lim_{t\to\infty} e^{-\delta t}(\lambda(t)z(t) + \mu(t)x(t)) = 0
\end{aligned}$$

For the case where the constraint $z(t) \leq \bar{z}$ is ignored a priori, it has been shown by Tahvonen (1997) that, given x_0, if z_0 is small enough, $z(t)$ is inverted U-shaped. Otherwise z is monotonically decreasing. In both cases, the resource is exhausted in finite time, since it has been assumed that $U'(0) < \infty$. Let us then fix $z_0 < \bar{z}$. There exists $\hat{x}_0$ such that there is a unique $T_z(\hat{x}_0)$ where $z(T_z) = \bar{z}$. Hence for this $\hat{x}_0$ it is optimal to have the Tahvonen program, hitting the threshold for just one instant of time. For all initial resource stocks smaller than $\hat{x}_0$ the threshold will never be hit and we stay in the strictly reversible region forever. The question then arises whether a reversible solution is possible for larger initial resource stocks. The answer is in the affirmative: for any $x_0 > \hat{x}_0$ there exists a program satisfying the necessary conditions. It looks as follows. Initially the rate of extraction is high, pollution approaches the threshold. Then, for a period of time, $(z, y) = (\bar{z}, \alpha\bar{z})$. There is a final phase where pollution is decreasing. The condition $z \leq \bar{z}$ is a pure state constraint, to which we associate a Lagrangian parameter denoted by κ. If in an optimum the constraint is binding for an interval of time, say $[T_z, T_{\bar{z}}]$, then along that interval

$$\dot{\lambda} = (\delta + \alpha)\lambda - D'(z) - \kappa, \quad \kappa \geq 0, \quad \kappa(\bar{z} - z) = 0$$

with continuity of z, x, λ and μ in T_z and $T_{\bar{z}}$. These conditions imply also the continuity of y.

In all the cases considered here, the resource stock is depleted within finite time, extraction from the stock goes to zero as time goes to infinity and the pollution stock vanishes asymptotically. See the path touching the threshold but remaining in the reversible region in figure 1 for an illustration of the kind of solution.

This leads to the following proposition

Proposition 1 *Suppose $z_0 \leq \bar{z}$. For any $x_0 > 0$ there exists a program satisfying the necessary conditions. There exists $\hat{x}_0$ such that the unique optimal program is to have the pollution stock increasing initially, hitting the threshold for an instant of time, and*

decreasing eventually. For $x_0 < \hat{x}$ the unique optimum is to stay in the reversible region forever. For $x_0 > \hat{x}_0$ the program satisfying the necessary conditions has an interval of time along which the pollution stock is at its threshold level. In all cases the resource stock is depleted eventually.

In the next section, we investigate whether with high enough $x_0(> \hat{x}_0)$ and $z_0(\leq \bar{z})$, other optimality candidates, featuring irreversible pollution, may exist.

2.2 Irreversible solutions

It is useful to make a distinction between several types of optimal irreversible programs. One distinction is between staying at the threshold for just one instant of time or for a non degenerate interval of time. The other distinction is between full or partial exhaustion of the resource.

To start with let us consider the possibility of $z = \bar{z}$ only in $t = 0$ and then entering the irreversible region. So the problem is to maximize social welfare (3) subject to

$$\begin{aligned} \dot{z} &= y,\ z(0) = \bar{z} \\ \dot{x} &= -y,\ x(0) = x_0 \end{aligned}$$

Let us assume that

$$U'(0) > D'(\bar{z})/\delta \tag{4}$$

This says that the marginal utility of the first unit of the raw material from the nonrenewable resource is larger than the total discounted marginal damage from the corresponding emission. We can easily show that this is a necessary and sufficient condition for having a solution with the economy entering the irreversible region immediately.

Suppose now that there exists an optimum with $x(\infty) \geq \varepsilon$ for some $\varepsilon > 0$. Hence, exhaustion is only partial. Note that this case cannot occur in Tahvonen's (1997) analysis since the pollution problem has no influence on the extraction policy and exhaustion occurs in finite time in his case. Within our framework with irreversibility, it may however be possible to leave some resource in the ground forever because of the ever increasing environmental damage. Since the shadow value of the resource stock must vanish, necessary conditions for an optimum read

$$\begin{aligned} &U'(y) - \lambda \leqq 0, (U'(y) - \lambda)y = 0, y \geqq 0 \\ &\dot{\lambda} = \delta\lambda - D'(z) \end{aligned} \tag{5}$$

In the limit, for $t \to \infty$, the extraction rate vanishes. This implies that $U'(0) = D'(z(\infty))/\delta$, which uniquely defines $z(\infty)$. Since $z(0) = \bar{z}$ the time path of z is uniquely given, and therefore also the time path of y, including $y(0)$.

Let us now assume that from a given $z_0 < \bar{z}$, it is optimal to go to $\bar{z}$ and that upon arrival there, at T_z, it is optimal to go beyond $\bar{z}$ immediately. Fix $T_z > 0$. Consider the following two problems. The first problem reads

$$\max \int_0^{T_z} e^{-\delta t}(U(y) - D(z))dt$$

subject to

$$\dot{z} = y - \alpha z,\ z(0) = z_0,\ z(T_z) = \bar{z}$$

The second problem reads

$$\max \int_{T_z}^{\infty} e^{-\delta t}(U(y) - D(z))dt$$

subject to

$$\dot{z} = y,\ z(T_z) = \bar{z}$$

The two problems are regular optimal control problems. They should be considered as separate problems however, because the differential equation for z is discontinuous in $\bar{z}$. The current value Hamiltonian for problem 1 reads $H = U(y) - D(z) - \lambda(y - \alpha z)$. The set of necessary optimality conditions includes:

$$\begin{aligned} U'(y) - \lambda &\leqq 0, (U'(y) - \lambda)y = 0, y \geqq 0 \\ \dot{\lambda} &= (\alpha + \delta)\lambda - D'(z) \end{aligned}$$

For the second problem we get the same necessary conditions, with α put equal to zero. Denote the optimum of problem 2 by hats and define the value functions:

$$\begin{aligned} V(T_z) &= \int_0^{T_z} e^{-\delta t}(U(y) - D(z))dt \\ \hat{V}(T_z) &= \int_{T_z}^{\infty} e^{-\delta t}(U(\hat{y}) - D(\hat{z}))dt \end{aligned}$$

all evaluated in the optimum, still for the given T_z. According to Seierstad and Sydsaeter (1987, p. 213) we have

$$\begin{aligned}\frac{\partial V}{\partial T_z} &= e^{-\delta T_z}[U(y(T_z)) - \lambda(T_z)(y(T_z) - \alpha\bar{z}) - D(\bar{z})] \\ &= e^{-\delta T_z}[U(y(T_z)) - U'(y(T_z))y(T_z) + U'(y(T_z))\alpha\bar{z} - D(\bar{z})]\end{aligned}$$

$$\begin{aligned}-\frac{\partial \hat{V}}{\partial T_z} &= e^{-\delta T_z}[U(\hat{y}(T_z)) - \hat{\lambda}(T_z)\hat{y}(T_z) - D(\bar{z})] \\ &= e^{-\delta T_z}[U(\hat{y}(T_z)) - U'(\hat{y}(T_z))\hat{y}(T_z) - D(\bar{z})]\end{aligned}$$

Maximization with respect to T_z requires:

$$U(y(T_z)) - U'(y(T_z))y(T_z) + U'(y(T_z))\alpha\bar{z} = U(\hat{y}(T_z)) - U'(\hat{y}(T_z))\hat{y}(T_z) \tag{6}$$

We have seen that $\hat{y}(T_z)$ is uniquely determined by the boundary conditions $\hat{y}(\infty) = 0$ and $\hat{z}(T_z) = \bar{z}$ and therefore we can calculate $y(T_z)$. We have an upward discontinuity in y at T_z. The economy compensates for the loss of benefits (from pollution decay) by an increase in consumption.

So, for a given z_0 we can now uniquely determine the initial extraction rate $y(0)$ that leads the economy to $y(T_z)$. We can then also find the initial resource stock that makes the proposed path feasible. If the actual initial resource stock is larger than or equal to this critical value, then we can always find a path satisfying the necessary conditions. The higher initial stock will not change social welfare, because the shadow value of the resource stock is zero. Hence, the path satisfying the necessary conditions will not alter. Note that $\hat{y}(T_z)$ does not depend on α and that it should be larger than $\alpha\bar{z}$ because otherwise an upward jump will not prevail. Therefore, and this is intuitively appealing, we should have α small enough for this case to occur. The general conclusion is that with α large it is optimal to choose the reversible program.

As mentioned above, in order to satisfy the transversality condition the shadow price of the resource must be zero (for all t). The dynamical system governing irreversible paths without exhaustion is therefore qualitatively the same as the one we would obtain in the pollution problem alone. In particular the $\dot{y} = 0$ and $\dot{z} = 0$ loci are the same and it is thus possible to illustrate the features of this kind of irreversible policy in the (z, y) plane (see figure 1). An example of reversible policy is also depicted for illustrative purposes only. Actually, it should be clear that for policies featuring exhaustion the shadow price is not

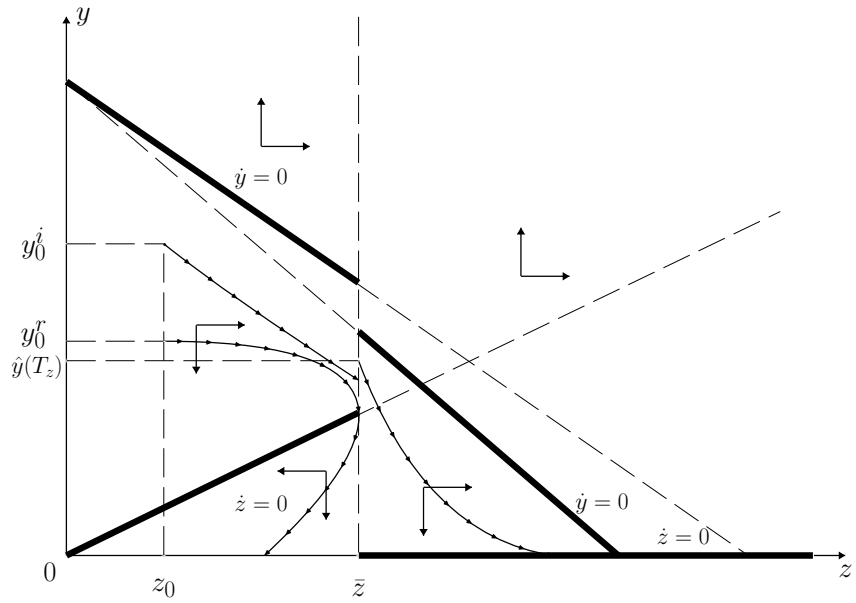

Figure 1: Irreversible path with a discontinuity in extraction.

nil which complicates the dynamics. This notably implies that the $\dot{y} = 0$ locus cannot be represented so simply because its location changes with the evolution of the shadow price.

The next question one can ask is whether it could be optimal to stay in $\bar{z}$ for a while before entering the irreversible region and leaving some of the resource unexploited. The answer is negative, as can easily be seen. Suppose that for $0 \leq t \leq T_1$ we have $z(t) = \bar{z}$ and $y(t) = \alpha\bar{z}$ whereas for $t \geq T_1$ we have the optimal path constructed above, leaving some of the resource unexploited. Since the resource is not completely depleted its shadow price equals zero, meaning that adding to the initial stock doesn't increase social welfare. This implies that the planner is indifferent between the resource stocks at all instants of time between 0 and T_1. Hence, the program followed from time 0 on is welfare equivalent to the program followed from T_1 on. But the welfare values differ. This yields a contradiction. Hence we don't have to worry about programs that do not exhaust the stock and stay in $\bar{z}$ for a while.

Finally, we consider the possibility of entering the irreversible region with full exhaustion of the resource. There are two irreversible candidates leading to exhaustion: one where the system stays for a while in $\bar{z}$ and one where the system passes through $\bar{z}$ at just

one instant of time. To characterize the paths we study the second problem formulated above once more but we include the resource constraint.

$$\max \int_{T_z}^{\infty} e^{-\delta t}(U(y) - D(z))dt$$

subject to

$$\begin{aligned} \dot{z} &= y, z(T_z) = \bar{z} \\ \dot{x} &= -y,\ y(t) \geq 0,\ x(t) \geq 0,\ x(T_z) = \bar{x} \end{aligned}$$

where $\bar{x}$ is a given positive number. The necessary conditions for an interior solution **for** $y > 0$ read

$$\begin{aligned} U'(y) &= \lambda + \mu \\ \dot{\mu} &= \delta\mu \\ \dot{\lambda} &= \delta\lambda - D'(z) \end{aligned}$$

From this we have a second order differential equation for $z:\ U''(\dot{z})\ddot{z} = \delta U'(\dot{z}) - D'(z)$. Let us denote the moment of exhaustion of the resource stock by T_x. Then the boundary conditions relevant for the solution of the second order differential equation are $z(T_z) = \bar{z}$, $z(T_x) = \bar{z} + \bar{x}$, and, because at the moment of exhaustion there will be no extraction, $\dot{z}(T_x) = 0$.

In addition, considering the possibility of staying in $\bar{z}$ for a period of time before entering the irreversible region, we have $z(t) = \bar{z}$ and $y(t) = \alpha\bar{z}$ for $T_{\underline{z}} \leq t \leq T_z$, with $T_{\underline{z}}$ the instant when the economy reaches the threshold. Also now we should optimize over T_z which gives a boundary condition as in (6). This then implies that, given we are in $\bar{z}$, there exists a unique resource stock $\bar{x}$ for which we enter in the irreversible region. There are several ways to get to $\bar{z}$ and one may wonder what is the role of the initial stock of the resource. Clearly, the initial condition has no influence on the value of its shadow price in this case since μ_0 is determined by the solution of the problem starting at T_z. It will only affect the length of the period spent at the threshold, before entering the irreversible region. Indeed, the higher x_0, the higher the extraction rates in the first phase with reversible pollution and the sooner the threshold $\bar{z}$ is hit. It also means that the larger the initial resource endowment, the longer the period spent at the threshold (before reaching $\bar{x}$ and entering in the irreversible region).

Another possibility is to never stay in $\bar{z}$ that is, when touching $\bar{z}$, to switch immediately from the reversible to the irreversible region. For this kind of solution we can no longer use

the boundary condition $y(T_z) = \alpha\bar{z}$ but we still need a boundary condition equivalent to (6). This means that in this case the scarcity rent is dependent on the initial resource stock. With a low resource stock (still compatible with the existence of irreversible policies) this latter candidate will exist whereas the former will not.

We can summarize the analysis above with the following proposition

Proposition 2 *Suppose $z_0 \leq \bar{z}$ and $x_0 > \hat{x}_0$. Then there exist three irreversible optimality candidates:*

i/ Two irreversible candidates with exhaustion of the resource in finite time. One directly reaching the irreversible region whereas along the other, the economy stays at threshold for a non-degenerate period of time.

ii/ One irreversible candidate with some amount of resource left in the ground in the long run.

The existence of the latter candidate requires the initial resource stock to be high enough.

3 Optimality

In the previous section we have given a full account of programs that satisfy the necessary conditions. However, in some cases we have multiple optimality candidates. For example we always have a feasible program in the reversible region, but at the same time we could have an irreversible solution. Nevertheless some preliminary conclusions can be drawn. We take the initial pollution stock as given and vary the initial resource stock. Clearly, with a very small initial resource stock, it is optimal to stay in the reversible region, possibly hitting the threshold for a single instant of time. For high initial resource stocks, three optimality candidates arise. One reversible, one irreversible with exhaustion within finite time, after staying at the threshold level for a nondegenerate period of time, and one with inexhaustibility and never staying at the threshold. The comparison is based on the following considerations. Along the reversible path the pollution stock is low, but so is the level of consumption from the nonrenewable resource. In the irreversible solutions extraction, and therefore instantaneous utility can be considerably higher. And, when deciding between the two, the program staying at the threshold for a while also has lower consumption, but has lower pollution as well. Clearly, the ultimate comparison is complicated, as it depends most likely on all model parameters involved. Therefore, we resort to a numerical analysis for that purpose. Most of the following will be illustrated

using a simple quadratic example.

$$\begin{cases} U(y) = \theta y(\bar{y} - y), \ \theta > 0 \\ D(z) = \frac{\gamma z^2}{2}, \ \gamma > 0 \end{cases} \tag{7}$$

In the sequel we consider the following set of baseline parameters: $\delta = 0.2$, $\alpha = 0.03$, $\gamma = 0.02$, $\theta = 2$, $\bar{y} = 50$, $\bar{z} = 200$.

We first provide an illustration of the multiplicity of trajectories satisfying all the necessary optimality conditions and originating from the same initial condition $z_0 = 100$ and $x_0 = 264.36$.

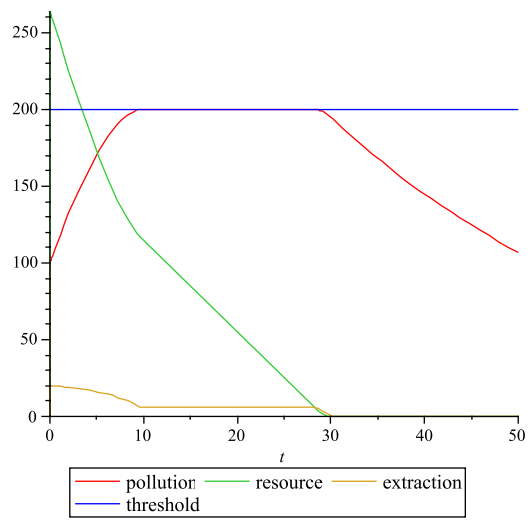

Figure 2: A reversible path staying in $\bar{z}$ for a period of time

In figure 2, the evolution of all variables is depicted when the pollution stock remains at the threshold for a while. After this period of time the economy goes back to the reversible region, with full exhaustion of the resource. In figure 3, left, we have another path, differing from the previous one in that at some instant of time we enter the strictly irreversible region, without staying at the threshold for a nondegenerate period of time. When the economy follows this solution, the resource is also exhausted in finite time. Finally, a third candidate for optimality is characterized again by a period of time where the system remains in $\bar{z}$ for a while, before entering the irreversible region (see figure 3, right). Again, we observe exhaustion of the resource.

The comparison between the three candidates gives some insight into their similarities and differences. Regarding extraction rates, the reversible candidate is characterized by

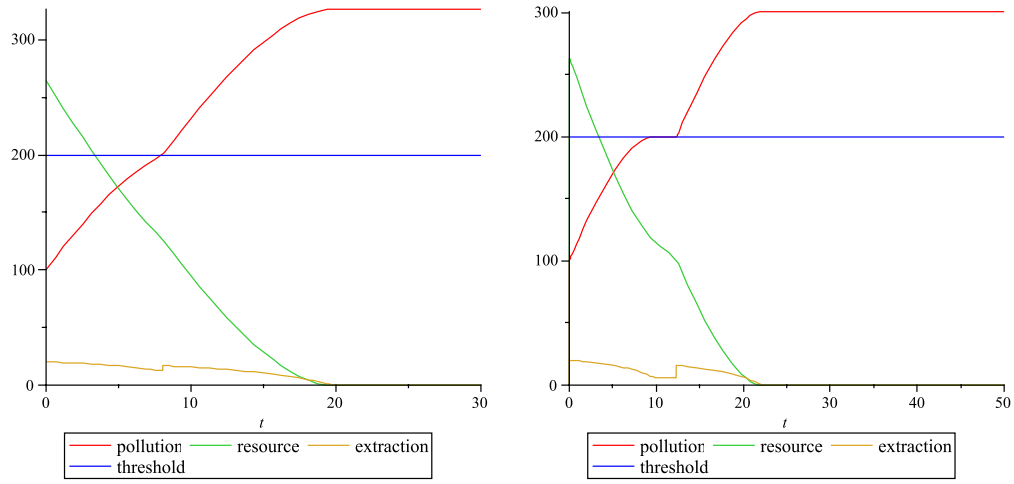

Figure 3: Optimality candidates featuring irreversibility and exhaustion

a first phase of relatively rapid extraction until the threshold is hit. Then, the extraction sets to a low level (the one needed to maintain pollution to the threshold) and finally starts to decline just before the resource is exhausted. Moreover, exhaustion occurs within finite time. Along irreversible policies, extraction paths have different features. The extraction rates are larger which implies that exhaustion arises sooner. Remaining for a while in $\bar{z}$ translates into later exhaustion and lower pollution in the long run. But, we observe that the extraction path is smoother (and the jump less important) when going directly to the irreversible region.

By varying the initial pollution stock and the resource stock we are able to divide the (z_0, x_0) plane into five regions, delimited by four frontiers (see figure 4). The meaning of these frontiers is as follows.

The "reversible" frontier is the set of initial stocks (z_0, x_0) such that starting from a point in this set, there exists a path satisfying the necessary conditions, that stays in the reversible region and hits the threshold only once. From any initial condition strictly below this frontier, the unique solution is purely reversible $(z(t) < \bar{z}, \forall t)$ whereas starting on the frontier, the optimal path is still unique and reversible but hits the threshold at just one date.

Let T_x be the instant when the resource is exhausted. The "irreversible $T_x < \infty$" and "$T_x = \infty$" frontiers are the lower and upper bound of existence of irreversible solutions crossing $z = \bar{z}$ with exhaustion of the resource. Originating from any point between the reversible and the "irreversible $T_x < \infty$" frontiers, the optimal solution is still reversible but has a period of time when the system remains in $\bar{z}$ for a while (the pollution stock has

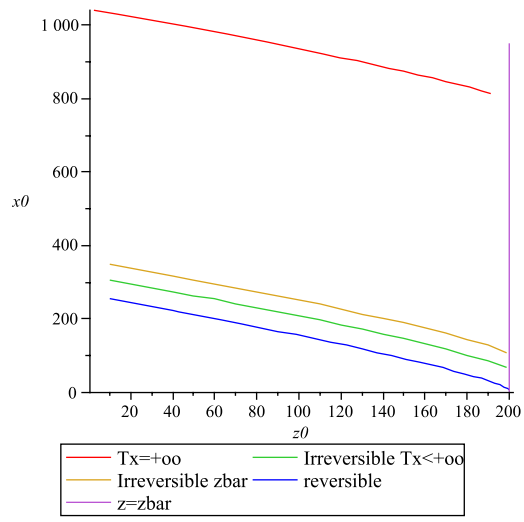

Figure 4: Frontiers

the same features as the ones depicted in figure 2). It is worth noting that this kind of trajectory exists as long as the economy starts above or on the reversible frontier. A third frontier is called "irreversible $\bar{z}$". For any initial point lying between "irreversible $T_x < \infty$" and "irreversible $\bar{z}$", there are two optimality candidates: the reversible path staying a period in $\bar{z}$ and an irreversible path that passes through the threshold for just an instant of time, both featuring exhaustion in finite time.

Above this frontier there exists another candidate along which the economy stays in $\bar{z}$ for a while before entering the irreversible region and exhausting the resource. In other words, within the region between "irreversible $\bar{z}$" and "$T_x = \infty$" we have three optimality candidates.[1] Finally, starting with a very high stock of resource, that is above "$T_x = \infty$", there still exist three optimality candidates: reversible and irreversible paths with a period of time in $\bar{z}$ and a third irreversible path along which the resource is not exhausted.

Next we have to make the welfare comparison for the cases where there are multiple equilibrium candidates. For that purpose, we depict for three different values of the initial pollution stock $z_0 = 20, 100, 180$, the present values associated with three optimality candidates, namely the reversible, and the two irreversible with exhaustion differing by the fact that one of them stays in $\bar{z}$ for a while and the other doesn't. Figure 5 illustrates how the values evolve when varying x_0.[2]

[1]Figures 2 and 3 have been drawn for an initial point within this area.

[2]This also means that we restrict our attention to the most interesting case where x_0 is below the

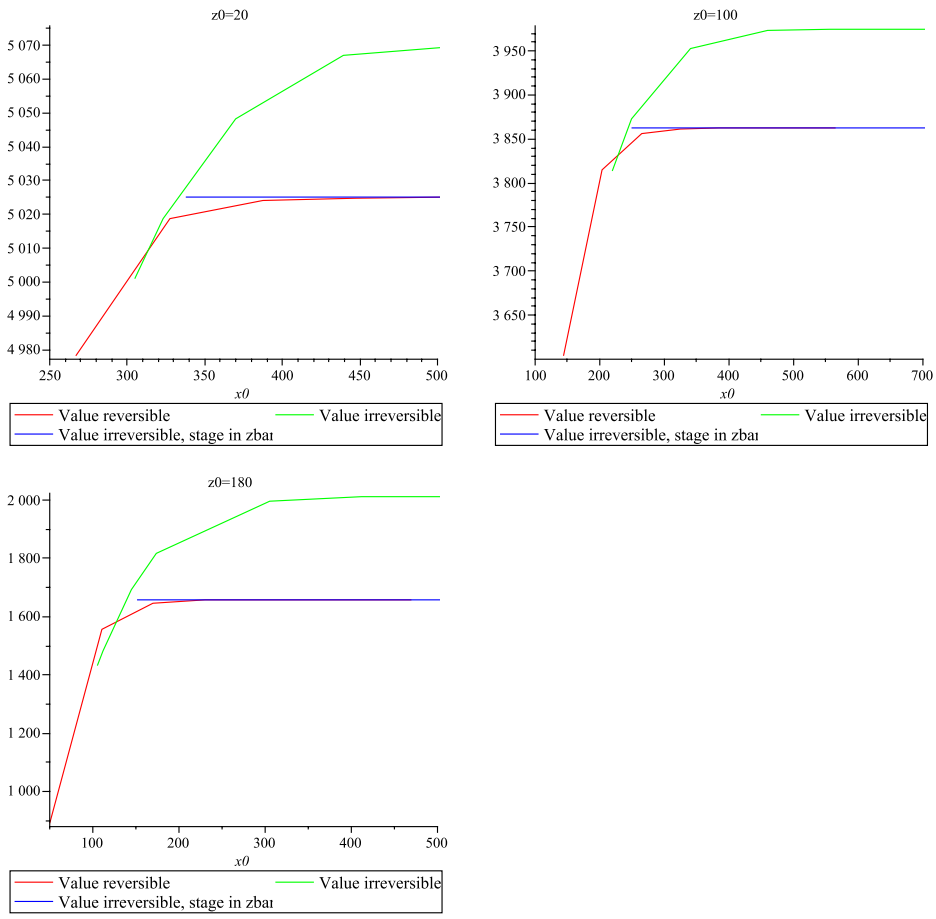

Figure 5: Optimality candidates with exhaustion: comparison of present values

Two important lessons can be learned from this exercise. First, in all three cases, we can find a critical initial condition for which we have multiple optima (and not simply multiple candidates for optimality): the reversible policy staying for a while in $\bar{z}$ and the policy entering directly the irreversible region. This is a striking feature that echoes Tahvonen and Withagen (1996)'s finding in their model with quadratic decay but without exhaustibility. The reversible policy is the optimum for low enough stock of resources. But, once the resource endowment is above a critical x_0 the optimum becomes the irreversible one. The second noticeable feature pertains to the fact that the irreversible path with the economy not staying in $\bar{z}$ always dominates the other candidate with a stage in $\bar{z}$. Actually, it is clear that these two irreversible candidates with exhaustion share some similarities (but they are not identical) but the latter proves to be very constraining with its unique

upper frontier.

$\bar{x}$ determining the instant at which the system enters the irreversible region and the little role given to the initial resource stock. We can easily prove that this feature holds in the simple quadratic example (7) with quadratic functions. Within our general framework, a formal proof of the dominance is subject of present research.

4 Quadratic decay

Tahvonen and Withagen (1996), T&W hereafter, study a decay function that is zero at zero pollution, increases, reaches a maximum and then declines towards zero, reaching zero at some point $\bar{z}$. This introduces a nonconvexity in the problem that leads to the possibility of multiple equilibrium candidates. T&W don't include exhaustibility. It is the purpose of this section to see what the effect is in their model of being constrained by a nonrenewable resource. Like in T&W it cannot be expected that we reach many general conclusions from a purely theoretical perspective. Some of their results, however, do go through if we include exhaustibility. First, if the discounted marginal damage at $\bar{z}$ is larger than the marginal utility of the first drop of oil, then we will only have reversible solutions. Second, there exists a critical value of pollution $\tilde{z}_0$ such that for $z_0 < \tilde{z}_0$ the only solution is a solution with reversibility. Third, another obvious result is that due to exhaustibility we will never have a positive steady state extraction rate, whereas in T&W convergence to a steady state is a possibility. Fourth, in the case of quadratic decay it is not optimal to stay at the threshold for longer than just an instant of time. This implies that the set of equilibrium candidates is much more easy to depict. To illustrate this we provide a numerical exercise using the same specification and parameter values as T&W. In particular we assume $\alpha(z) = \alpha z$ for $z \in [0, \frac{\bar{z}}{4}], = \frac{\alpha}{3}(\bar{z} - z)$ for $z \in [\frac{\bar{z}}{4}, \bar{z}]$ and we take $\{\delta, \alpha, \gamma, \theta, \bar{y}, \bar{z}\} = \{0.2; 0.3; 0.012; 0.5; 44; 200\}$. The critical value $\tilde{z}_0$ we obtain is similar to the one of T&W ($\tilde{z}_0 \approx 170$). A full comparison with T&W would include a larger range of parameters, but it is our aim here to see how the introduction of exhaustibility affects the optimum without exhaustion.

For $z_0 \geq \tilde{z}_0$, the subsets of the (z_0, x_0) plane that delineate the optimality candidates look much simpler that in the previous sections. As explained above, the reason is that it is never optimal now to stay at the threshold for longer than an instant of time. Actually, the threshold here essentially differs from the threshold in the previous section. Here there is almost no decay close to the threshold, whereas earlier decay is maximal at the threshold. So, now there is no benefit to be found in staying at the threshold.

Generally, there are three regions that can be distinguished, for a given initial pollution

stock z_0. For a small initial resource stock it is optimal to stay in the reversible region and to deplete the resource stock. For higher resource stocks it is optimal to enter the irreversible region, with exhaustion in finite time. With still higher resource stock we are back in the T&W model with only partial exhaustion. See figure 6.

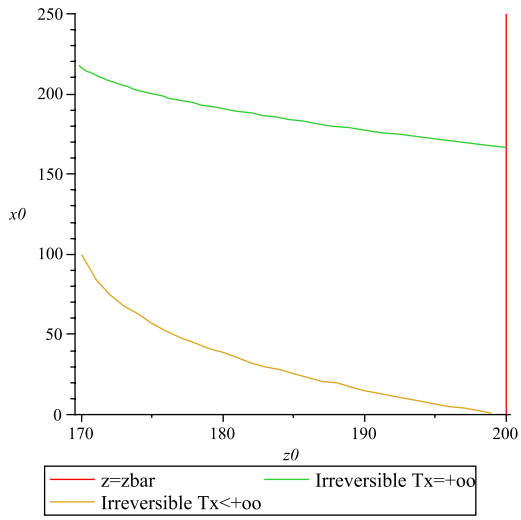

Figure 6: Zones of existence of the different optimality candidates

If for a given initial pollution stock, we increase the initial resource stock is small, the optimum is unambiguously reversible, with exhaustion within finite time. If we increase the initial resource stock, we arrive in a region where there are two trajectories that satisfy the necessary conditions. One reversible and one irreversible, with exhaustion within finite time. A choice has to be made between these possibilities. It is to be expected that reversibility is optimal if the initial pollution stock is close to the critical value $\tilde{z}_0$, and irreversibility is preferred if the initial pollution stock is far from this value. For very high initial resource stock, three trajectories may satisfy the necessary conditions: a reversible one, and two irreversible paths, one exhausting the resource and the other not exhausting the resource.

Starting from $\tilde{z}_0$, figure 7 illustrates the time paths of the extraction rates and pollution for each candidate. Extraction paths, for irreversible policies, have the same qualitative features as in T&W, with a peak reached at the instant when pollution hits the threshold. However, with only partial exhaustion, this threshold is reached sooner due to initially higher extraction rates. Once in the irreversible region, extraction monotonically decreases for both policies but now extraction rates are higher when the resource is exhausted in

finite time (see figure 7, right). The pollution level is always higher when the resource is conserved and the difference becomes tiny in the long run (see figure 7, left). By contrast, with the reversible candidate, extraction starts at a lower level, which is a means to avoid the crossing of the threshold. After the initial phase of increase, extraction finally sharply decreases with the resource stock approaching zero. For this initial condition, we find that the reversible path is optimal. However, if we start with a high stock of pollutants and a high initial resource stock, irreversibility turns out to be optimal.

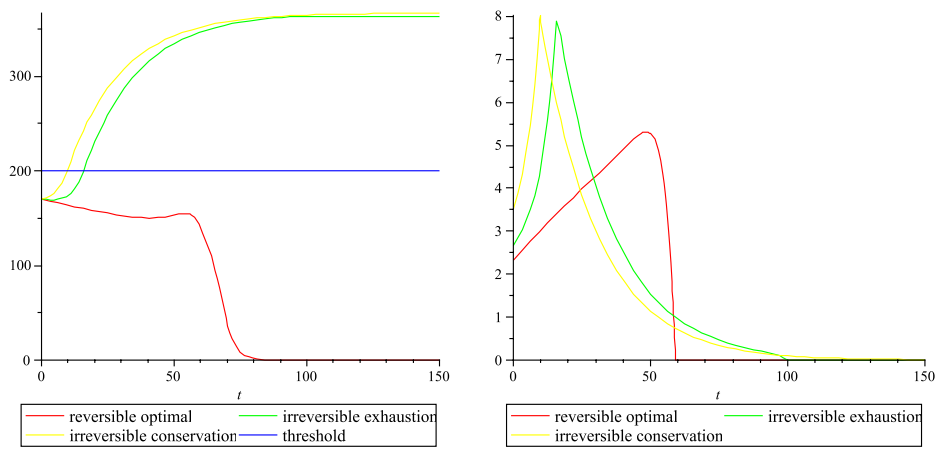

Figure 7: Time paths of pollution (right) and extraction (left)

5 Conclusion

This paper introduced the irreversibility of pollution decay in the classical exhaustible-resource/stock-pollution model. Within this framework, we studied how the potential irreversibility of pollution affected the extraction path. Several results have been obtained. For a small marginal utility of the first unit of the raw material from the non-renewable resource compared to total discounted marginal damage at the threshold level, the economy will never enter the irreversible region. It is then optimal to deplete the entire resource stock. For a resource stock not too small it is optimal to stay at the threshold level for a while. With increasing resource stock it may become optimal to enter the irreversible region, possible after staying at the threshold for a while. These results indicate that staying at the threshold, in order to benefit from the high decay rate, is profitable. However, there is a danger in doing so. With the introduction of uncertainty on e.g., the threshold level, or the actual emissions, one might easily enter the irreversible region, whereas this

is suboptimal, or premature (see Ayong et al., 2011, who introduce uncertainty about the threshold). Hence, the equilibrium during the phase at the threshold is really a knife-edge. There is also evidence that available and existing stock of fossil fuels are not well known. This suggests a first extension of our work, which is motivated by the observation that a lot of uncertainty surrounds both the extent of fossil fuels reserves present in the ground and the concentration of GHG that will initiate irreversible phenomena. Another extension of the present paper naturally comes into mind. The introduction of a backstop technology would be a means to examine how the optimal timing of the backstop adoption and how the optimal combination of technologies are affected by irreversibility.

We have also investigated the introduction of exhaustibility in a model with quadratic decay. It turns out that qualitatively the results are not very sensitive, except for the obvious fact that no steady state extraction will occur. An important outcome is, however, that with quadratic decay the set of optimality candidates looks much simpler, which also makes it easier to determine the optimum. Hence, this strengthens our conclusion that knowledge on actual decay is crucial for determining the best extraction-pollution program. An ongoing development of this work investigates more deeply the quadratic decay case.

For future research it would be important to investigate in more detail and for larger parameters sets the differences in welfare arising from different specifications of decay, and the seriousness of making mistakes in estimating the decay schedule.

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