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## Measuring market liquidity risk - which model works best?

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**Working Paper 2009 No. 01**

**Measuring Market Liquidity Risk – Which Model Works Best?**

**CORNELIA ERNST  
SEBASTIAN STANGE  
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**WORKING PAPER SERIES**



**Center for Entrepreneurial and  
Financial Studies**



# Measuring Market Liquidity Risk - Which Model Works Best?

Cornelia Ernst\*, Sebastian Stange<sup>†</sup>, Christoph Kaserer<sup>‡</sup>

February 3, 2009

Market liquidity risk, the difficulty or cost of trading assets in crises, has been recognized as an important factor in risk management. Literature has already proposed several models to include liquidity risk in the standard Value-at-Risk framework. While theoretical comparisons between those models have been conducted, their empirical performance has never been benchmarked. This paper performs comparative back-tests of daily risk forecasts for a large selection of traceable liquidity risk models. In a 5.5 year stock sample we show which model provides most accurate results and provide detailed recommendations which model is most suitable in a specific situation.

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*JEL classification:* G11, G12, G18, G32

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# 1 Introduction

That assets cannot be liquidated as previously thought seems to frequently come as surprise in crises situations. The discussion always flourishes after stock market crashes. But liquidity is a continuous problem of financial institutions. Trading strategies and investments that yield high profits, often invest in less liquid assets like private equity, emerging markets or low capitalization stocks. In crash situations, those asset positions can often not be traded anywhere close to fair prices, because scarce liquidity is consumed by the concerted sales of many market participants. Yet, market liquidity risk often remains unattended in many risk management systems.

Several liquidity risk models have already been proposed in the literature. While overarching theoretical discussions and summaries already exist,<sup>1</sup> empirical testing is indispensable. All models must necessarily use simplifying assumption, but which are most distorting for the overall result? Only the empirical evaluation of model preciseness and relative performance will clarify which simplification is most detrimental. So far, comparative tests have not been conducted.

In this paper, we provide a comprehensive overview on existing, traceable models and conduct extensive back-tests in a large stock data set of daily data. We examine Bangia et al. (1999), Berkowitz (2000a), Cosandey (2001), Francois-Heude and Van Wynendale (2001), Giot and Grammig (2005), Stange and Kaserer (2008c) and Ernst et al. (2008). We provide recommendations which model is most suitable in practice. Theoretical models without obvious empirical specifications as well as models requiring intraday data generally remained outside the scope of this analysis.

We proceed as follows: Section 2 defines liquidity risk in a general framework, outlines liquidity risk models in detail and sketches our implementation approach. In section 3, we evaluate and compare all liquidity risk models based on the precision of their risk forecasts. Section 4 summarizes and concludes.

## 2 Model descriptions and implementation

### 2.1 General remarks

#### 2.1.1 Definition of market liquidity risk

We define market liquidity as the cost of trading an asset relative to its fair value.<sup>2</sup> Fair value is defined as the mid-price  $P_{mid,t}$ , the middle of the bid-ask-spread, which

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<sup>1</sup>Cp. Mahadevan (2001); Erzegovesi (2002); Loebnitz (2006); Bervas (2006); Jorion (2007); Stange and Kaserer (2008b, 2009).

<sup>2</sup>Cp. also discussion in Stange and Kaserer (2008b).

is least subjective. Relative liquidity costs  $L_t(q)$  in percent of the mid-price for an order of quantity  $q$  at time  $t$  can be split into three components

$$L_t(q) = T(q) + PI_t(q) + D_t(q) \quad (1)$$

$T(q)$  are constant direct trading costs including exchange fees, brokerage commissions, and transaction taxes,  $PI_t(q)$  are price impact costs of order quantity  $q$  at time  $t$  as the difference between the transaction price and the fair price,  $D_t(q)$  are delay costs if a position cannot be instantly traded. For this analysis, direct trading costs  $T(q)$  and delay costs  $D_t(q)$  are neglected. The former is negligible for most institutional investors, the latter is negligible in fairly liquid markets.<sup>3</sup> Price impact costs  $PI_t(q)$  amount to the bid-ask-spread for small positions, but can rise with position size, if positions larger than the spread depth are traded.

In this framework, liquidity risk is defined as the potential loss due to time-varying liquidity costs. In the following we describe different approaches to measure these liquidity costs and different assumption with respect to their distribution.

### 2.1.2 Selection of models

We sort liquidity risk models into two broad categories: Traceable and theoretical. A large stream of literature has developed theoretical modeling approaches, where implementation procedures are still missing and not obvious. These include Lawrence and Robinson (1995), Almgren and Chriss (2000) and Almgren (2003), Subramanian and Jarrow (2001), Hisata and Yamai (2000), Dubil (2003) and Engle and Ferstenberg (2007).<sup>4</sup> These models generally use optimal trading strategies to minimize the Value-at-Risk of a position including liquidity. However, empirical estimation techniques for the large range of parameters of these models still need to be developed.

Among those liquidity risk models that are empirically traceable, several work on intraday or transaction data only. Berkowitz (2000a), Jarrow and Protter (2005) and Angelidis and Benos (2006) belong to this class. In order not to completely neglect these, we choose Berkowitz (2000a), which seemed most promising to adapt for daily data. We include all traceable models available for daily data: Bangia et al. (1999), Cosandey (2001), Francois-Heude and Van Wynendaele (2001), Giot and Grammig (2005), Stange and Kaserer (2008c) and Ernst et al. (2008). For all models we choose a straight forward implementation for daily stock data. We group

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<sup>3</sup>Liquidity risk models explicitly treating delay are still under development, cp. discussion in Stange and Kaserer (2008b).

<sup>4</sup>For a more detailed discussion please refer to Stange and Kaserer (2008b).

the chosen models by the type of data required for their estimation: bid-ask-spread models, transaction or volume data models, and models requiring limit order book data.

### 2.1.3 General implementation specifications

For all models, we calculate a standard, daily, relative Value-at-Risk (VaR) at a 99 % confidence level. In general, we tried to keep the implementation procedure as straight-forward as possible to allow for best comparisons.

Means, including those of liquidity costs, are generally calculated with a 20-day rolling procedure. If mid-price return is separately estimated in a normal-distribution framework, we set its mean to zero, as is common practice. We account for volatility clustering with the standard exponential weighted average (EWMA) model over 20 days by JP Morgan (1996) using a weight  $\delta$  of 0.94 defined as

$$\sigma_t^2 = (1 - \delta) \sum_{i=1}^{20} \delta^{i-1} r_{t-i}^2 + \delta^{20} r_{t-20}^2 \quad (2)$$

Where applicable we estimate skewness and excess-kurtosis with a simple non-weighted rolling procedure. The skewness of  $y$  is computed from historical data rolling over the last 500 days as  $\gamma = \frac{1}{500} \sum_{t=1}^{500} (y_t - \mu_y)^3 / \sigma_y^3$  with  $\mu_y$  and  $\sigma_y$  as mean and volatility of  $y$ . The excess kurtosis for  $y$  is  $\kappa = \frac{1}{500} \sum_{t=1}^{500} (y_t - \mu_y)^4 / \sigma_y^4 - 3$ .<sup>5</sup>

To allow for best comparison, we use ten standardized order size classes to calculate the the liquidity risk for a stock position of a specific size. In the following, we describe the individual risk models and any additional implementation specification required.

## 2.2 Models based on bid-ask-spread data

### 2.2.1 Add-on model with bid-ask-spread: Bangia et al. (1999)

Bangia, Diebold, Schuermann and Stroughair (1998, 1999) developed a simple liquidity adjustment of a VaR-measure based on bid-ask-spread. Liquidity cost is measured with the bid-ask-spread. To determine risk as the worst achievable transaction price, the worst bid-ask-spread is added to the worst mid-price. Bangia et al. define relative, liquidity-adjusted total risk as

$$L - VaR = 1 - \exp(z\sigma_r) + (\mu_S + \hat{z}_S\sigma_S) \quad (3)$$

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<sup>5</sup>To keep the sample as large as possible, we reduced the rolling window up to 20 days at the beginning of the sample, in order to also include the first two years into the results period. This discriminates models using skewness and kurtosis, but they nevertheless show superior performance as will be shown in section 3.

where  $\sigma_r$  is the variance of the continuous mid-price return over the appropriate horizon and  $\mu_S$  and  $\sigma_S$  are the mean and variance of the bid-ask-spread,  $z$  is the percentile of the normal distribution for the given confidence.<sup>6</sup>  $\hat{z}_S$  is the empirical percentile of the spread distribution in order to account for non-normality in spreads.<sup>7</sup>

The Bangia et al. (1999)-approach acknowledges that spreads can rise over time, especially in crises, but neglects that liquidity costs rise with order size beyond the bid-ask-spread. The latter underestimates liquidity risk. The add-on approach also assumes perfect liquidity-return correlation, which is not observed in reality.<sup>8</sup> Overall, the approach is simple and easy to implement, also because data is available in many markets.

### 2.2.2 Modified add-on model with bid-ask-spread: Ernst et al. (2008)

Ernst, Stange and Kaserer (2008) suggest a different way to account for future time variation of prices and spreads. While Bangia et al. assume a normal distribution for future prices and take the historical distribution for future spreads, Ernst et al. use non-normal distributions for prices and spreads, which account for skewness and kurtosis. The non-normal distribution is estimated with a Cornish-Fisher approximation. This alternative parametric specification defines relative, liquidity-adjusted total risk as

$$L - VaR = 1 - \exp(\mu_r + \tilde{z}_r \sigma_r) \times \left(1 - \frac{1}{2} (\mu_S + \tilde{z}_S \sigma_S)\right) \quad (4)$$

where  $\mu$  and  $\sigma$  are mean and variance of mid-price return and spread respectively.  $\tilde{z}$  is the non-normal-distribution percentile adjusted for skewness and kurtosis according to the Cornish-Fisher expansion

$$\tilde{z} = z + \frac{1}{6}(z^2 - 1) * \gamma + \frac{1}{24}(z^3 - 3z) * \kappa - \frac{1}{36}(2z^3 - 5z) * \gamma^2 \quad (5)$$

where  $z$  is the appropriate percentile of the normal distribution,  $\gamma$  is the skewness, and  $\kappa$  the excess-kurtosis of the respective distribution. Ernst et al. (2008) show, that this approach yields more precise risk forecasts than the original specification of Bangia et al. (1999).

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<sup>6</sup>The critique of Loebnitz (2006) that worst spread need to be deducted from worst, not from current mid-prices, performs worse than the original specification. Cp. table 8 on page 19 in the appendix.

<sup>7</sup>The empirical percentile is calculated as  $\hat{\alpha}_S = (\hat{S}_\alpha - \mu_S)/\sigma_S$ , where  $\hat{S}_\alpha$  is the percentile spread of the past 20-day historical distribution.

<sup>8</sup>Cp. analysis of Stange and Kaserer (2008c).



<b>Liquidity Coefficient <math>\theta</math></b> in Euro per million shares	<b>Index</b>				
	<b>DAX</b>	<b>MDAX</b>	<b>SDAX</b>	<b>TECDAX</b>	<b>All</b>
<b>Mean</b>	0.03	0.30	5.24	0.37	1.84
<b>Median</b>	0.01	0.05	0.17	0.04	0.03
<b>Max.</b>	0.23	12.50	1,777.00	24.40	1,777.00
<b>Min.</b>	-0.12	-14.30	-53.10	-3.95	-53.10
<b>Std. Dev.</b>	0.05	1.37	91.90	2.94	52.20
<b>Signif. fraction at 95 % confidence</b>	53%	36%	45%	54%	44%
<b>Signif. fraction at 99 % confidence</b>	44%	27%	37%	46%	36%

Table 1: Estimates of the liquidity measure  $\theta$

Table shows cross-sectional statistics of the estimated liquidity coefficient  $\theta$ . The All-column contains the average over all indices. Significant fraction shows percentage of stocks with statistically significant theta at confidence level of 95 % and 99 % respectively.

## 2.3 Models based on transactions or volume

### 2.3.1 Transaction regression model: Berkowitz (2000)

Berkowitz (2000a,b) determines liquidity price impact from a regression of past trades while controlling for other risk factors. From this point of view, future price is driven by risk factor changes and the liquidity impact of trading  $N_t$  number of shares as follows

$$P_{mid,t+1} - P_{mid,t} = C + \theta N_t + x_{t+1} + \epsilon_t \quad (6)$$

where  $\theta$  is the regression coefficient,  $x_{t+1}$  is the effect of risk factor changes on the mid-price,  $C$  is a constant and  $\epsilon_t$  the error term of the regression.  $\theta$  can be understood as absolute liquidity cost per share traded. Although the original model is constructed on the basis of transaction data, we have tried to tune it as best as possible for the use in daily risk forecasts. Therefore, we approximated the transaction price with  $P_{mid,t+1}$ .

As the author does not go into implementation detail, we choose to estimate market risk effects as

$$x_{t+1} = \beta \times r_{M,t} \times P_{mid,t} \quad (7)$$

where  $\beta = Cov(r, r_M) / \sigma_{r_{market}}$  is the beta factor for each individual stock return on the 160-stock, value-weighted market portfolio return  $r_M$  over the sample period.<sup>9</sup>

Table 1 presents the regression estimates of the liquidity measure  $\hat{\theta}$  for the sample period. The regression produces positive and negative estimates, which is slightly counter-intuitive as the liquidation of a position should always induce a price discount.  $\hat{\theta}$  also varies strongly as indicated by standard deviation, minimum and

<sup>9</sup>Although this proceeding leads to a conceptually doubtful overlap between estimation and forecast period, this overlap generates a bias in favor of the model. Nevertheless, even positively biased estimates for this model provide poor results as will become apparent in section 3.

maximum. In general, average liquidity costs per share are very small, in the order of one Euro per million shares. Only about half of the stocks have  $\hat{\theta}$ -values that are statistically significant different from zero. Therefore, we already doubt at this stage, that the liquidity measure implemented in daily data will produce accurate results.

We now calculate continuous, liquidity-adjusted net return as

$$r_{net,t}(q) = \ln \left( 1 + \left[ \beta \times r_{M,t} - \hat{\theta} \times \frac{N_t + n}{P_{mid,t}} \right] \right) \quad (8)$$

for each standard-volume number of shares  $n = q/P_{mid,t}$  to allow for later comparison with other liquidity risk models. The optimal trading strategy of the original model requires  $1/h$ 'th of the position to be liquidated each day of the  $h$ -day horizon. For our daily horizon, the full position will be liquidated at once. We then define relative, liquidity-adjusted total risk as

$$L - VaR(q) = 1 - \exp \left( \mu_{r_{net}(q)} + \hat{z} \sigma_{r_{net}(q)} \right) \quad (9)$$

where  $\mu_{r_{net}(q)}$  is the 20-day rolling net return mean and  $\sigma_{r_{net}(q)}$  is the EWMA-estimated net return variance.  $\hat{z}$  is the empirical percentile of the net return distribution.

While the liquidity measure of Berkowitz (2000a) seems to be quite noisy, the approach has the general advantage to be based on transaction data only, which makes it a valid alternative in markets where liquidity cost data are not available.

### 2.3.2 Volume-based price impact: Cosandey (2001)

Cosandey (2001) introduces another simple framework to estimate price impact from volume data. He assumes, that total value traded in the market is constant and is split over the number of traded shares  $N_t$ . If an additional position of  $n = q/P_{mid,t}$  is liquidated, total value will then be split over  $N_t + n$  shares. Thus, net return is calculated as

$$r_{net,t+1}(q) = \ln \left( r_{t+1} \times \frac{N_t}{N_t + n} \right) \quad (10)$$

This implicitly assumes zero liquidity elasticity and no future time-variation of liquidity modeled by  $N_t$ .<sup>10</sup>

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<sup>10</sup>Equation 10 is much simpler, but equivalent to the formula in the original paper.

Their relative, liquidity-adjusted VaR is then implemented similar to the Berkowitz (2000a) approach described above as

$$L - VaR(q) = 1 - \exp(\mu_{rnet(q)} + \hat{z}\sigma_{rnet(q)}) \quad (11)$$

where  $\mu_{rnet(q)}$  is the 20-day rolling net return mean and  $\sigma_{rnet(q)}$  is the EWMA-estimated net return variance.  $\hat{z}$  is the empirical percentile of the net return distribution.<sup>11</sup>

The framework of Cosandey (2001) is easy and straightforward to implement and also has the merit to be based on market volume data only, which are available for many assets. If the simplifying assumptions significantly distort reality remains to be seen.

## 2.4 Models based on weighted spread data

In this section, we present models, that account for the fact that liquidity cost increase with order size by using limit order book data. We use the liquidity cost measure 'weighted spread', which calculates the liquidity costs compared with the fair price when liquidating a position quantity  $q$  against the limit order book. Weighted spread  $WS$  can be calculated as follows

$$WS_t(q) := \frac{a_t(v) - b_t(v)}{P_{mid,t}} \quad (12)$$

$a_t(v)$  is the volume-weighted ask price of trading  $v$  shares calculated as  $a_t(v) = \sum_i a_{i,t} v_{i,t} / v$  with  $a_{i,t}$  being the ask-price and  $v_{i,t}$  the ask-volume of individual limit orders. An order of size  $q$  is executed against several limit orders until individual limit order sizes add-up to  $q$ , i.e.  $q/P_{mid} = v = \sum_i v_i$ .  $b_t(v)$  is defined analogously. Weighted spread - similar to the bid-ask-spread - is the cost of a round-trip for position  $q$ .<sup>12</sup> In the following, weighted spread is used in liquidity risk models suggested in the literature.

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<sup>11</sup>We deviate from the original simulation approach, because, in our view, the key feature of this approach is new liquidity measure. Using a parametrization keeps approaches as comparable as possible. We also work with smoother continuous rather than discrete returns.

<sup>12</sup>For more detail on this liquidity measure cp. Stange and Kaserer (2008a,c).

#### 2.4.1 Limit order model: Francois-Heude and van Wynendaele (2001)

Francois-Heude and Van Wynendaele (2001) propose to calculate relative, liquidity-adjusted total risk as

$$L - VaR(q) = 1 - \exp(-z\sigma_r) \left( 1 - \frac{\mu(q)_{WS}}{2} \right) + \frac{1}{2} (WS_t(q) - \mu(q)_{WS}) \quad (13)$$

where  $z$  is the normal percentile and  $\sigma_r$  the standard deviation of the mid-price return distribution.  $\mu(q)_{WS}$  is the average spread for a security for order quantity  $q$ , and  $WS_t(q)$  is the spread at time  $t$ . In the second term, the average weighted spread is subtracted from worst mid-prices. However, as average spread might be different from the actual spread in time  $t$ , a correction term for the difference is added as a third term. The correction term is calculated on current not on worst mid-prices which can lead to misestimation.

In the original paper, Francois-Heude and Van Wynendaele interpolate the liquidity cost function only from the best five limit-order-quotes made available by the Paris Stock Exchange. In favor of their approach, we use the liquidity cost function estimated as weighted spread from the whole limit order book as described at the beginning of this section.

While the add-on of the correction term is conceptually doubtful and time-variation of liquidity is neglected, Francois-Heude and Van Wynendaele (2001) provide an interesting venue to use limit order book information as liquidity measure.

#### 2.4.2 T-distributed net return model with weighted spread: Giot and Gramming (2005)

Giot and Grammig (2005) define a net return based on the weighted spread as

$$rnet_t(q) = r_t \times \left( 1 - \frac{WS_t(q)}{2} \right) \quad (14)$$

and then compute relative, liquidity-adjusted total risk as

$$L - VaR(q) = 1 - \exp(\mu_{rnet(q)} + z_{ST}\sigma_{rnet(q)}) \quad (15)$$

where  $z_{ST}$  is the chosen percentile of the student t-distribution.<sup>13</sup> In order to ensure comparability, we stay with the EWMA-modeling of volatility and do not replicate their approach accounting for conditional heteroskedasticity. Because we implement

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<sup>13</sup>We take percentiles from the student distribution with 19 degrees of freedom due to the 20-day rolling window.

their approach to daily instead of intraday data, we ignore their adjustment for diurnal variation in weighted spread.

The approach by Giot and Grammig (2005) is a simple parametric alternative, but the validity of the t-distribution assumption remains to be tested.

#### 2.4.3 Empirical net-return model with weighted spread: Stange and Kaserer (2008)

Stange and Kaserer (2008c) calculate liquidity risk with weighted spread based on empirical percentiles as

$$L - VaR(q) = 1 - \exp(\mu_{rnet(q)} + \hat{z}(q) \times \sigma_{rnet(q)}) \quad (16)$$

where  $\hat{z}$  denotes the empirical percentile of the net return distribution. This circumvents the assumption of t-distributed net returns.

#### 2.4.4 Modified risk models with weighted spread

**a. Modified add-on model with weighted spread** In an analogous application of Ernst et al. (2008), it is also possible to use Cornish-Fisher approximated percentiles of the return and spread distribution separately and calculate risk as

$$L - VaR(q) = 1 - \exp(\mu_r + \tilde{z}_r \sigma_r) \times \left( 1 - \frac{1}{2} (\mu_{WS(q)} + \tilde{z}_{WS(q)} \sigma_{WS(q)}) \right) \quad (17)$$

where  $\tilde{z}$  denote the percentiles estimated with the Cornish-Fisher approximation (5) of the respective distribution. While assuming perfect correlation between mid-price returns and liquidity, this parametrization allows to account for liquidity risk as add-on. Also, forecasting of two parameters separately might prove to be more precise.

**b. Modified net-return model with weighted spread** In another analogous application of Ernst et al. (2008), it is also possible to use Cornish-Fisher approximated percentiles of the net return distribution and calculate risk as

$$L - VaR(q) = 1 - \exp(\mu_{rnet(q)} + \tilde{z}(q) \times \sigma_{rnet(q)}) \quad (18)$$

where  $\tilde{z}$  is the percentile estimated with the Cornish-Fisher approximation (5). This alternative parametrization does not rely on the assumption of t-distributed net returns or perfect return-liquidity correlation.

Market statistics	II/2002	2003	2004	2005	2006	2007	Total period <sup>a</sup>
<b>Average continuous period return<sup>b</sup></b>							
DAX	-52%	24%	6%	27%	20%	22%	13%
MDAX	23%	39%	15%	25%	25%	-1%	21%
SDAX	-36%	35%	11%	28%	29%	4%	16%
TECDAX	n/a	52%	3%	26%	24%	32%	27%
Total	-35%	24%	10%	26%	24%	11%	14%
<b>Average period return volatility (annualized)<sup>c</sup></b>							
DAX	64%	41%	22%	19%	23%	25%	29%
MDAX	54%	39%	28%	26%	30%	35%	34%
SDAX	65%	47%	35%	31%	36%	38%	40%
TECDAX	n/a	54%	42%	31%	38%	44%	42%
Total	60%	44%	32%	27%	32%	36%	37%
<b>Average daily transaction volume in thsd. Euro</b>							
DAX	93,500	94,399	98,037	119,563	165,833	250,835	140,985
MDAX	1,384	2,297	4,035	6,242	11,034	18,243	7,735
SDAX	36	160	237	514	958	2,129	730
TECDAX	n/a	1,813	2,345	2,308	4,769	7,946	3,836
Total	20,431	19,543	20,268	25,206	35,797	54,891	30,167

Table 2: Market conditions during sample period

Table shows the for our investigation most interesting market statistics over the sample period. All values show per-stock averages. a. annualized; b. Including dividend returns due to price series being adjusted for corporate capital actions; c. Annualized volatility with  $\sqrt{250}$ ; All values equal-weighted.

## 3 Empirical comparison

### 3.1 Description of data

For our empirical analysis we used daily data for the 160 stocks in the major German indices DAX, MDAX, SDAX and TecDAX, for the period 7/2002 to 12/2007. Price, volume and spread data were taken from Thomson Financial Datastream. Weighted spread data were obtained directly from Deutsche Börse AG. Deutsche Börse AG calculates weighted spread under the name of Xetra Liquidity Measure (XLM) as daily averages of the limit order book for each stock in the index.

Table 2 summarizes market conditions during the sample period. Except for the downturn in the second half of 2002, markets were bullish over the sample period. Due to the market decline at the beginning of the sample period, average period return is 14 %. Volatility exhibits a reversed pattern compared to returns. Daily transaction volume significantly increased during the sample period.

Table 3 shows descriptive statistics for the liquidity cost measures bid-ask-spread and weighted spread. Across all order sizes and indices, liquidity costs are 2.16 %. DAX shows the lowest liquidity costs, followed by MDAX, TECDAX and SDAX.

Liquidity cost in % of mid-price		Order size (in thsd. Euro)															
		Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
DAX	Mean	0.17%	n/a	0.12%	0.14%	n/a	0.19%	n/a	0.32%	0.57%	n/a	1.14%	1.97%	2.35%	2.73%	3.07%	1.21%
	Median	0.10%	n/a	0.09%	0.10%	n/a	0.13%	n/a	0.19%	0.30%	n/a	0.50%	0.86%	1.13%	1.39%	1.65%	0.36%
	Std. Dev.	0.23%	n/a	0.11%	0.14%	n/a	0.23%	n/a	0.52%	1.09%	n/a	3.19%	4.25%	4.26%	4.47%	4.81%	3.14%
	Obs.	41,469	0	42,050	42,050	0	42,050	0	42,050	42,046	0	42,003	41,099	39,340	37,633	35,734	406,055
MDAX	Mean	0.58%	0.57%	0.72%	0.91%	1.10%	1.28%	1.66%	2.27%	3.22%	3.85%	4.29%	n/a	n/a	n/a	n/a	1.81%
	Median	0.34%	0.31%	0.38%	0.49%	0.60%	0.70%	0.90%	1.28%	2.07%	2.67%	3.14%	n/a	n/a	n/a	n/a	0.83%
	Std. Dev.	0.74%	1.12%	1.70%	1.84%	2.08%	2.21%	2.75%	3.21%	3.88%	4.10%	4.29%	n/a	n/a	n/a	n/a	3.02%
	Obs.	72,201	73,758	73,579	72,940	72,265	71,637	70,309	67,443	59,688	52,416	45,735	0	0	0	0	659,770
SDAX	Mean	1.29%	1.63%	1.93%	2.52%	3.06%	3.52%	4.08%	4.94%	6.42%	7.81%	9.12%	n/a	n/a	n/a	n/a	3.39%
	Median	0.83%	1.05%	1.25%	1.61%	1.95%	2.25%	2.79%	3.66%	5.20%	6.43%	7.53%	n/a	n/a	n/a	n/a	2.06%
	Std. Dev.	1.77%	2.65%	2.52%	3.33%	4.19%	4.99%	5.21%	5.41%	5.49%	6.22%	8.26%	n/a	n/a	n/a	n/a	4.66%
	Obs.	68,950	68,097	63,514	60,019	56,838	53,900	48,316	38,876	22,700	14,020	8,878	0	0	0	0	435,158
TECDAX	Mean	0.51%	0.62%	0.82%	1.15%	1.51%	1.88%	2.50%	3.41%	4.81%	5.77%	6.49%	n/a	n/a	n/a	n/a	2.45%
	Median	0.39%	0.49%	0.62%	0.83%	1.03%	1.22%	1.63%	2.37%	3.69%	4.59%	5.29%	n/a	n/a	n/a	n/a	1.27%
	Std. Dev.	0.44%	0.47%	0.70%	1.17%	1.78%	2.43%	2.96%	3.52%	4.05%	4.67%	5.49%	n/a	n/a	n/a	n/a	3.33%
	Obs.	36,320	36,497	36,497	36,490	36,439	36,313	35,663	33,369	26,481	20,369	15,885	0	0	0	0	314,003
All	Mean	0.71%	n/a	0.98%	1.25%	n/a	1.75%	n/a	2.60%	3.24%	n/a	3.81%	n/a	n/a	n/a	n/a	2.16%
	Median	0.39%	n/a	0.51%	0.65%	n/a	0.89%	n/a	1.45%	1.96%	n/a	2.32%	n/a	n/a	n/a	n/a	1.03%
	Std. Dev.	1.17%	n/a	1.84%	2.30%	n/a	3.28%	n/a	3.87%	4.20%	n/a	5.19%	n/a	n/a	n/a	n/a	3.64%
	Obs.	218,940	0	215,640	211,499	0	203,900	0	181,738	150,915	0	112,501	0	0	0	0	1,814,986

Table 3: Descriptive statistics of liquidity cost measures

Table shows cross sectional statistics of liquidity costs for a round-trip of a specific size during the sample period. The Min.-column shows statistics for the bid-ask-spread, whereas the remaining columns show the weighted spread measure for standardized order sizes. All column contains the average over standardized order sizes excluding the Min.-column

Naturally, liquidity costs rise with rising order size, mounting up to 9.12 % on average for the largest order size of the most illiquid index SDAX.

### 3.2 Backtesting framework

We test the validity of risk forecasts for each model by comparing predicted risk with actual returns. Actual daily returns when liquidating a position of quantity  $q$  are calculated under the assumption that the position has to be immediately liquidated against the limit order book

$$rnet_t(q) = r_t + \ln \left( 1 - \frac{1}{2} W S_t(q) \right) \quad (19)$$

That this assumption is valid in a large range of risk-related situations has been shown in Stange and Kaserer (2008c).

We calculate a L-VaR for all models at  $1 - \alpha = 99\%$  confidence. If the L-VaR model correctly predicts risk, actual return should exceed VaR in only 1 % of all cases. We denote the number of actual exceedances with  $N$  and the number of days in the sample with  $T$ ; the actual exceedance frequency follows as  $N/T$ . If the actual exceedance frequency,  $N/T$ , deviates from the predicted exceedance frequency,  $\alpha$ , on a statistically significant basis is determined with the Kupiec (1995)-statistic.

Kupiec (1995) defines a likelihood ratio (LR) test statistic as

$$LR_{uc} = -2\ln [(1 - \alpha)^{T-N}\alpha^N] + 2\ln [(1 - N/T)^{T-N}(N/T)^N] \quad (20)$$

which is chi-squared distributed with one degree of freedom under the null hypothesis that  $\alpha = N/T$ . Taking a confidence interval of 95 %, the null hypothesis, that predicted and actual exceedance frequencies equal, would be rejected for  $LR_{uc} < 3.84$ .<sup>14</sup> The test statistic will reject an L-VaR model if the actual exceedance frequency is significantly below 1 % (model overestimates risk) or significantly above 1 % (model underestimates risk).

For each model, we calculated the percentage of stocks, where the predicted loss frequency did not deviate from the realized loss frequency on a statistically significant basis. For these stocks, the model correctly predicts risk. The percentage of stocks with correct risk estimation is called acceptance rate. If acceptance rates are averaged over all order sizes, we excluded bid-ask-spread rates to avoid double counting.<sup>15</sup> For stocks, where the deviation between predicted and real loss rates was significant, we determined if the violation occurred because risk was overestimated (fewer actual losses than predicted) or underestimated (more actual losses than predicted). These respective stock fractions were also determined.

When comparing models, we used a common sample. Our large period of 5.5 years, i.e. 1.423 days, allows for very robust results of the Kupiec-statistic.

### 3.3 Backtesting results and comparison

#### 3.3.1 Overall model ranking

Figure 1 shows the overall ranking of the tested liquidity risk models.<sup>16</sup> Liquidity risk models are ranked by the overall average percentage of stocks, for which risk was correctly estimated according to the Kupiec-statistic. In general, models based on the larger data set, limit order data, show superior performance with an acceptance rate of above 70 %. Best performing with 74 % is the Cornish-Fisher modified add-on approach with weighted spread by Ernst et al. (2008) (2.4.4a) and the empirical net return model based on weighted spread by Stange and Kaserer (2008c) (2.4.3). This is closely followed by modified weighted spread net return (2.4.4b) and the t-distributed net-return approach by Giot and Grammig (2005) (2.4.2) with a 71 % acceptance rate.

<sup>14</sup>The choice of the confidence region for the test statistic is independent of the confidence level selected for the L-VaR-calculation.

<sup>15</sup>As bid-ask-spreads are reported for non-standardized order sizes only, there is potential overlap with weighted spread of small sizes.

<sup>16</sup>Detailed individual model statistics can be found in the appendix.



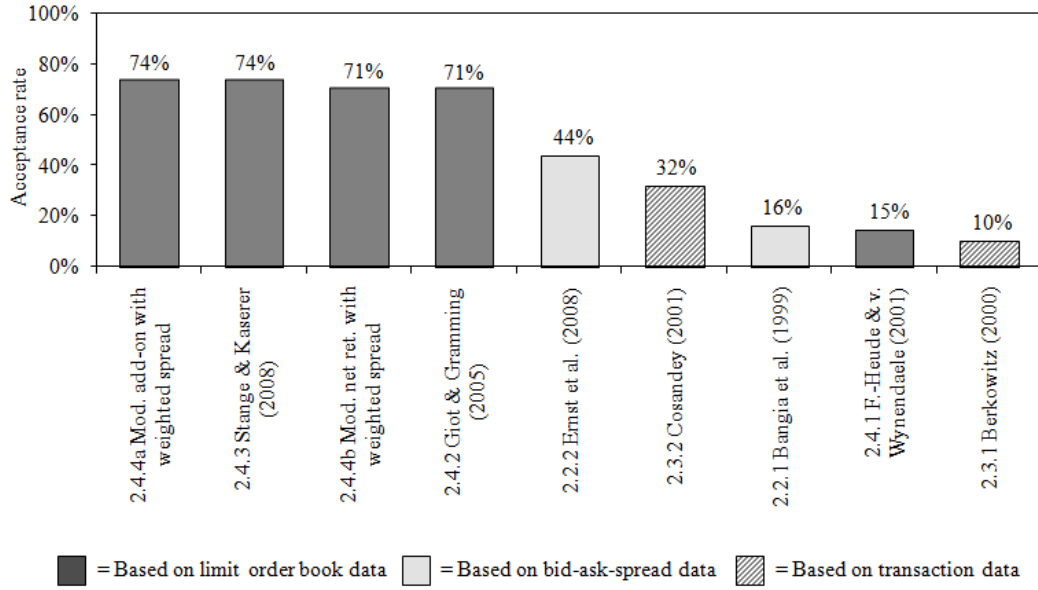


Figure 1: Ranking of liquidity risk models by overall acceptance rate

Figure shows overall acceptance rate averaged over all stocks and order sizes for each model. Acceptance rate is the percentage of stocks with statistically significant precise risk estimation according to Kupiec (1995).

We would have expected the net-return Cornish-Fisher approach (2.4.4b) to be higher ranked than if return and liquidity percentiles are separately estimated (2.4.4a), because correlation between return and liquidity are correctly accounted for. We hypothesize that forecasting of return and liquidity costs are more precise because the dynamics of both components are modeled separately. This compensates for the neglect of correlation. The t-distribution approach by Giot and Grammig (2005) (2.4.2) seems to only partially account for the non-normality. The limit-order-book approach by Francois-Heude and Van Wynendaele (2001) (2.4.1) is far behind on the second last place. We believe this is caused by the conceptual weakness of this model as described above.

Although the model of Ernst et al. (2008) (2.2.2) does not account for the price impact via weighted spread data, it surpasses with 44 % acceptance rate the Cosandey (2001) with 32 % acceptance. Bangia et al. (1999) (2.2.1) is with 16 % overall acceptance better than Francois-Heude and Van Wynendaele (2001) (2.4.1) and Berkowitz (2000a) (2.3.1). Our implementation attempt of the latter in daily data does not provide satisfactory results.

Acceptance rate	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
2.4.4a Mod. add-on with weighted spread	81%	80%	76%	77%	82%	79%	79%	75%	66%	63%	60%	78%	63%	58%	56%	74%
2.4.3 Stange & Kaserer (2008)	78%	73%	76%	79%	77%	77%	82%	77%	71%	71%	75%	50%	42%	41%	34%	74%
2.4.4b Mod. net ret. with weighted spread	77%	73%	72%	72%	75%	72%	77%	71%	67%	62%	65%	83%	67%	53%	52%	71%
2.4.2 Giot & Gramming (2005)	62%	59%	58%	57%	63%	65%	76%	81%	85%	88%	84%	92%	86%	82%	88%	71%
2.2.2 Ernst et al. (2008)	79%	82%	77%	65%	55%	51%	35%	29%	19%	11%	13%	15%	10%	7%	4%	44%
2.3.2 Cosandey (2001)	n/a	47%	39%	35%	22%	25%	21%	28%	33%	43%	44%	9%	6%	6%	6%	32%
2.2.1 Bangia et al. (1999)	39%	44%	31%	18%	16%	13%	11%	8%	7%	6%	5%	0%	0%	0%	0%	16%
2.4.1 F.-Heude & v. Wynendaele (2001)	3%	15%	15%	13%	16%	12%	14%	14%	18%	21%	20%	8%	9%	6%	12%	15%
2.3.1 Berkowitz (2000)	n/a	7%	9%	10%	8%	10%	8%	12%	10%	5%	14%	17%	14%	17%	19%	10%

Table 4: Acceptance rate of liquidity risk models by order size

Table shows acceptance rate by order size averaged over all stocks for each model. Acceptance rate is the percentage of stocks with statistically significant precise risk estimation according to Kupiec (1995).

### 3.3.2 The impact of order size on model performance

The overall rank calculated as order-size average is influenced by the selection of size classes included. We therefore also calculated averages by individual order sizes. Table 4 shows the acceptance rate of the tested liquidity risk models by order size. The modified add-on model with weighted spread (2.4.4a) performs best in small to medium order sizes, while the best performing model in larger order sizes is Giot and Grammig (2005) (2.4.2). The t-distribution seems to capture liquidity risk in larger order sizes very efficiently. The relatively low performance of the modified risk models with weighted spread (2.4.4a/b) in larger sizes is probably due to rising skewness and kurtosis for weighted spread in larger sizes caused by single outliers which leads to imprecise Cornish-Fisher estimates.<sup>17</sup> Also, the assumption of perfect correlation leads to an overestimation of risk which has a significant impact in larger order sizes.<sup>18</sup> Our hypothesis that the lower performance of the modified net return model with weighted spread (2.4.4b) is driven by the low forecastability of net return dynamics is substantiated. In lower order sizes performance is more acceptable, because dynamics are mainly driven by mid-price return and liquidity is neglectable. In larger order sizes performance drops as liquidity dynamics are lost in the compounding of the net return.

Models based on bid-ask-spreads (2.2.1 and 2.2.2) - not accounting for order size - show expectedly declining performance with rising order size, while Ernst et al. (2008) (2.2.2) consistently dominates. Cosandey (2001) (2.3.2) shows a quite good performance for medium sizes, but very low at large order sizes. The assumption of linear price impact probably distorts results at order size extremes.

The discussion shows that overall ranking results remain valid with one exception. The rank of the top-performing limit order models is not fixed and - depending on the

<sup>17</sup>Cp. Jaeger (2004), p.16. and Zangari (1996), p.10.

<sup>18</sup>Cp. Stange and Kaserer (2008a).

Overestimated L-VaR in % of total stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
2.4.4a Mod. add-on with weighted spread	11%	2%	5%	6%	8%	8%	11%	16%	19%	23%	21%	14%	29%	39%	35%	12%
2.4.3 Stange & Kaserer (2008)	3%	1%	3%	5%	8%	8%	8%	16%	23%	23%	23%	50%	55%	56%	63%	14%
2.4.4b Mod. net ret. with weighted spread	3%	3%	5%	6%	9%	9%	12%	16%	19%	23%	17%	6%	24%	41%	48%	12%
2.4.2 Giot & Gramming (2005)	0%	0%	0%	2%	1%	1%	2%	2%	3%	5%	5%	3%	6%	15%	12%	2%
2.2.2 Ernst et al. (2008)	8%	4%	3%	2%	1%	1%	1%	0%	0%	0%	0%	0%	0%	0%	0%	0%
2.3.2 Cosandey (2001)	n/a	50%	59%	65%	77%	73%	78%	72%	67%	57%	56%	91%	91%	94%	94%	67%
2.2.1 Bangia et al. (1999)	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
2.4.1 F.-Heude & v. Wynendaele (2001)	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
2.3.1 Berkowitz (2000)	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Underestimated L-VaR in % of total stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
2.4.4a Mod. add-on with weighted spread	9%	18%	19%	17%	11%	13%	9%	9%	14%	14%	20%	8%	9%	3%	9%	14%
2.4.3 Stange & Kaserer (2008)	20%	26%	21%	16%	16%	15%	10%	7%	6%	6%	2%	0%	3%	3%	3%	13%
2.4.4b Mod. net ret. with weighted spread	19%	23%	23%	22%	16%	19%	10%	12%	14%	15%	17%	11%	9%	6%	0%	17%
2.4.2 Giot & Gramming (2005)	38%	41%	41%	41%	35%	33%	23%	17%	12%	7%	11%	6%	9%	3%	0%	26%
2.2.2 Ernst et al. (2008)	13%	14%	21%	33%	44%	48%	65%	70%	81%	89%	87%	85%	90%	93%	96%	55%
2.3.2 Cosandey (2001)	n/a	3%	2%	0%	1%	1%	1%	0%	0%	0%	0%	0%	3%	0%	0%	1%
2.2.1 Bangia et al. (1999)	61%	56%	69%	82%	84%	88%	89%	92%	93%	94%	95%	100%	100%	100%	100%	84%
2.4.1 F.-Heude & v. Wynendaele (2001)	96%	85%	85%	87%	84%	88%	86%	86%	82%	79%	80%	92%	91%	94%	88%	85%
2.3.1 Berkowitz (2000)	n/a	93%	91%	90%	92%	90%	92%	88%	90%	95%	86%	83%	86%	83%	81%	90%

Table 5: Over- and underestimation rate by order size

Tables show over- and underestimation rate by order size averaged over all stocks for each model. Over- and underestimation rate is percentage of stocks with statistically significant deviating risk estimation according to Kupiec (1995).

order size in question, the modified add-on model (2.4.4a), the Stange and Kaserer (2008c) model (2.4.3) and the Giot and Grammig (2005) model (2.4.2) are probably all good choices in practice.

### 3.3.3 Type of misestimation

To allow a more detailed analysis of the reasons behind the individual model performance, table 5 shows the over- and underestimation rate of the tested liquidity risk models by order size. The first four limit order models (2.4.2 to 2.4.4) are quite balanced and show underestimation as well as overestimation. Stange and Kaserer (2008c) (2.4.3), for example, overestimates the risk of 14 % of the stocks and underestimates the risk for 13 % of the stocks. As mentioned earlier, the severe underestimation of bid-ask-spread models (2.2.x) in large order sizes is expected due to their design. The strong general underestimation of the Francois-Heude and Van Wynendaele (2001) can probably be traced to the neglect of time variation, because the crises increase of liquidity cost has not been incorporated. The failure of the implementation of the Berkowitz (2000a)-model in daily data, probably lies in the fact that we had to use mid-price instead of transaction prices, which seems to smooth and therefore underestimate any liquidity effects. Based on these arguments, these two models (2.4.1 and 2.3.1) should probably be ruled out for practical implementation.

Acceptance rate	Index				
	DAX	MDAX	SDAX	TECDAX	All
<b>2.4.4a Mod. add-on with weighted spread</b>	70%	72%	76%	74%	74%
<b>2.4.3 Stange &amp; Kaserer (2008)</b>	61%	69%	79%	83%	74%
<b>2.4.4b Mod. net ret. with weighted spread</b>	64%	70%	74%	68%	71%
<b>2.4.2 Giot &amp; Gramming (2005)</b>	70%	62%	75%	77%	71%
<b>2.2.2 Ernst et al. (2008)</b>	46%	49%	38%	44%	44%
<b>2.3.2 Cosandey (2001)</b>	29%	23%	36%	31%	32%
<b>2.2.1 Bangia et al. (1999)</b>	10%	16%	16%	16%	16%
<b>2.4.1 F.-Heude &amp; v. Wynendaele (2001)</b>	5%	9%	18%	19%	15%
<b>2.3.1 Berkowitz (2000)</b>	22%	8%	6%	4%	10%

Table 6: Acceptance rate of liquidity risk models by index

Table shows acceptance rate by index sub-sample averaged over all order sizes and all stocks for each model. Acceptance rate is the percentage of stocks with statistically significant precise risk estimation according to Kupiec (1995).

### 3.3.4 Robustness of model rank

As natural sub-samples, we used the four indices in our sample to check for the robustness of the model rank. Figure 6 shows the acceptance rate of the tested liquidity risk models by index. The performance in the least liquid SDAX where liquidity effects are largest, is of particular importance. The first four models based on limit order data (2.4.2 to 2.4.4) keep their superior performance, but switch ranks in some sub-samples. The modified add-on models (2.4.4a) delivers high acceptance rates more consistently than other models with acceptance rates never below 70 %. Therefore, the modified add-on model is recommendable when limit order book data are available.

Ernst et al. (2008) based on bid-ask-spread data (2.2.2) consistently outperforms all other non-limit order data models as well as Francois-Heude and Van Wynendaele (2001) (2.4.1). Our adaptation of Berkowitz (2000a) has particular low acceptance rates in the less liquid indices. Its performance is best in the DAX, where liquidity is of minor importance. Hence, it cannot be recommended for daily risk forecasts. Above results are therefore generally confirmed.

Although a shortening of the period length reduces the reliability of the Kupiec-statistic, we split the period into two sub-periods and calculated separate results for each sub-period as another robustness test. The acceptance rate by sub-period is presented in table 7. As the sub-period is significantly shorter than the full period, results of the Kupiec statistic are not directly comparable to the full period statistic. Levels can therefore not be compared across tables. We will look at the relative model rank only. In the first sub-period (II/2002 to I/2005), the model ranking is slightly different. The model of Stange and Kaserer (2008c) (2.4.3) and Giot and Grammig (2005) (2.4.2) now dominate the modified risk models. We

Acceptance rate	Sub-period	
	II/2002 - I/2005	II/2005 - II/2007
<b>2.4.4a Mod. add-on with weighted spread</b>	72%	79%
<b>2.4.3 Stange &amp; Kaserer (2008)</b>	79%	79%
<b>2.4.4b Mod. net ret. with weighted spread</b>	69%	73%
<b>2.4.2 Giot &amp; Gramming (2005)</b>	79%	74%
<b>2.2.2 Ernst et al. (2008)</b>	43%	60%
<b>2.3.2 Cosandey (2001)</b>	33%	28%
<b>2.2.1 Bangia et al. (1999)</b>	24%	23%
<b>2.4.1 F.-Heude &amp; v. Wynendaele (2001)</b>	26%	22%
<b>2.3.1 Berkowitz (2000)</b>	6%	11%

Table 7: Acceptance rate of liquidity risk models by sub-period

Table shows acceptance rate by sub-period averaged over all order sizes and all stocks for each model. Acceptance rate is the percentage of stocks with statistically significant precise risk estimation according to Kupiec (1995).

hypothesize that this effect is driven by inefficient skewness and kurtosis estimates which are themselves caused by outliers during the turbulent first sub-period. An improved estimation technique for higher moments might help improve results in this particular situation, a point which is, however, left to future research. The ranking of the second sub-period (II/2005 to II/2007) is preserved.

The first four limit-order models therefore switch ranks in some sub-periods but keep their superiority as a group. The remaining ranking is preserved.

## 4 Conclusion

In this paper, we have put a large selection of traceable liquidity risk models to the test in order to find out which is most suitable for daily risk estimation. We implemented Bangia et al. (1999), Berkowitz (2000a), Cosandey (2001), Francois-Heude and Van Wynendaele (2001), Giot and Grammig (2005), Stange and Kaserer (2008c) and Ernst et al. (2008) in a large sample of daily stock data over 5.5 years. We used a standard Kupiec (1995)-statistic to determine if models provide precise risk forecasts on a statistically significant basis.

We find, that available data is the main driver of the preciseness of risk forecasts. Models based on limit order data generally outperform models based on bid-ask-spread or volume data. The latter (Cosandey (2001) and Berkowitz (2000a)) are highly approximate and should only be used if no other data are available. If limit order book data available, an approach based on empirical or t-distributed net returns (Stange and Kaserer (2008c) or Giot and Grammig (2005)) as well as the modified add-on model adapted from Ernst et al. (2008) all show satisfactory results. Ernst

et al. (2008) with limit order data shows slightly more consistent outperformance. If only bid-ask-spread data can be obtained, the modified add-on model with bid-ask-spreads by Ernst et al. (2008) is recommendable. On the basis of volume data, Cosandey (2001) provides suitable results for daily forecasts.

Several issues are open to future research. An analogous empirical comparison in intraday data would complement our daily-data analysis. Implementation solutions for theoretical models would allow to include those in comparative, empirical analysis. Also models for less liquid markets still need to be developed and should then be tested accordingly. We also hypothesize that a model explicitly integrating a return-liquidity correlation estimate could further improve results. Overall, our paper provides indications, which of the tested models is most suitable for practical daily risk forecasts.

## 5 Appendix

Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
DAX	23%	n/a	26%	23%	n/a	20%	n/a	11%	6%	n/a	0%	0%	0%	0%	0%	10%
MDAX	41%	39%	28%	18%	18%	11%	11%	7%	4%	4%	2%	n/a	n/a	n/a	n/a	16%
SDAX	44%	42%	31%	13%	10%	8%	7%	5%	6%	7%	8%	n/a	n/a	n/a	n/a	16%
TECDAX	34%	44%	32%	14%	12%	10%	10%	6%	8%	6%	6%	n/a	n/a	n/a	n/a	16%
All	39%	44%	31%	18%	16%	13%	11%	8%	7%	6%	5%	0%	0%	0%	0%	16%

Table 8: Acceptance rate of Bangia et al. (1999)-approach (2.2.1) by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.

Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
DAX	23%	n/a	20%	20%	n/a	20%	n/a	11%	3%	n/a	0%	0%	0%	0%	0%	8%
MDAX	38%	39%	28%	18%	18%	10%	10%	7%	4%	4%	2%	n/a	n/a	n/a	n/a	16%
SDAX	40%	40%	28%	13%	10%	8%	6%	5%	6%	7%	8%	n/a	n/a	n/a	n/a	15%
TECDAX	26%	38%	32%	14%	12%	10%	8%	6%	8%	6%	6%	n/a	n/a	n/a	n/a	15%
All	35%	42%	29%	18%	16%	12%	10%	8%	6%	6%	5%	0%	0%	0%	0%	16%

Table 9: Acceptance rate of Bangia et al. (1999)-approach (2.2.1) with Loebnitz correction by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. L-VaR is calculated as  $L - VaR = 1 - \exp(\alpha\sigma_r) \times (1 - 1/2(\mu_S + \hat{\alpha}_S\sigma_S))$ . The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.

Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
DAX	76%	n/a	86%	81%	n/a	78%	n/a	72%	50%	n/a	26%	15%	10%	7%	4%	46%
MDAX	85%	85%	80%	72%	61%	52%	41%	32%	16%	10%	7%	n/a	n/a	n/a	n/a	49%
SDAX	77%	79%	68%	52%	39%	34%	21%	15%	12%	12%	13%	n/a	n/a	n/a	n/a	38%
TECDAX	76%	84%	82%	67%	55%	49%	33%	20%	12%	10%	10%	n/a	n/a	n/a	n/a	44%
All	79%	82%	77%	65%	55%	51%	35%	29%	19%	11%	13%	15%	10%	7%	4%	44%

Table 10: Acceptance rate of Ernst et al. (2008)-approach with bid-ask-spread (2.2.2) by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.

Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
<b>DAX</b>	n/a	n/a	24%	24%	n/a	24%	n/a	25%	27%	n/a	25%	17%	14%	17%	19%	22%
<b>MDAX</b>	n/a	10%	4%	5%	12%	6%	16%	6%	4%	9%	4%	n/a	n/a	n/a	n/a	8%
<b>SDAX</b>	n/a	7%	4%	7%	5%	5%	5%	7%	4%	4%	8%	n/a	n/a	n/a	n/a	6%
<b>TECDAX</b>	n/a	3%	3%	3%	3%	0%	0%	8%	8%	4%	16%	n/a	n/a	n/a	n/a	4%
<b>All</b>	n/a	7%	9%	10%	8%	10%	8%	12%	10%	5%	14%	17%	14%	17%	19%	10%

Table 11: Acceptance rate of Berkowitz (2000a)-approach (2.3.1) by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.

Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
<b>DAX</b>	n/a	n/a	77%	77%	n/a	60%	n/a	34%	14%	n/a	0%	9%	6%	6%	6%	29%
<b>MDAX</b>	n/a	46%	30%	28%	19%	16%	16%	14%	13%	18%	20%	n/a	n/a	n/a	n/a	23%
<b>SDAX</b>	n/a	39%	30%	25%	21%	19%	21%	31%	47%	63%	75%	n/a	n/a	n/a	n/a	36%
<b>TECDAX</b>	n/a	47%	33%	22%	20%	18%	22%	27%	35%	40%	50%	n/a	n/a	n/a	n/a	31%
<b>All</b>	n/a	47%	39%	35%	22%	25%	21%	28%	33%	43%	44%	9%	6%	6%	6%	32%

Table 12: Acceptance rate of Cosandey (2001)-approach (2.3.2) by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.

Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
<b>DAX</b>	3%	n/a	3%	3%	n/a	3%	n/a	3%	3%	n/a	3%	8%	9%	6%	12%	5%
<b>MDAX</b>	4%	11%	6%	6%	9%	7%	10%	12%	9%	12%	12%	n/a	n/a	n/a	n/a	9%
<b>SDAX</b>	2%	13%	19%	15%	14%	13%	15%	19%	21%	24%	32%	n/a	n/a	n/a	n/a	18%
<b>TECDAX</b>	6%	16%	16%	16%	20%	18%	12%	15%	29%	23%	24%	n/a	n/a	n/a	n/a	19%
<b>All</b>	3%	15%	15%	13%	16%	12%	14%	14%	18%	21%	20%	8%	9%	6%	12%	15%

Table 13: Acceptance rate of Francois-Heude and Van Wynendaele (2001)-approach (2.4.1) by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.



Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
DAX	86%	n/a	53%	53%	n/a	58%	n/a	61%	61%	n/a	72%	92%	86%	82%	88%	70%
MDAX	57%	41%	43%	45%	51%	55%	65%	74%	86%	88%	87%	n/a	n/a	n/a	n/a	62%
SDAX	57%	65%	65%	63%	67%	76%	81%	90%	89%	85%	87%	n/a	n/a	n/a	n/a	75%
TECDAX	64%	61%	67%	67%	76%	73%	80%	88%	91%	93%	85%	n/a	n/a	n/a	n/a	77%
All	62%	59%	58%	57%	63%	65%	76%	81%	85%	88%	84%	92%	86%	82%	88%	71%

Table 14: Acceptance rate of Giot and Grammig (2005)-approach with weighted spread (2.4.2) by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.

Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
DAX	86%	n/a	78%	78%	n/a	83%	n/a	78%	64%	n/a	57%	50%	42%	41%	34%	61%
MDAX	79%	69%	74%	77%	72%	68%	73%	63%	58%	64%	70%	n/a	n/a	n/a	n/a	69%
SDAX	76%	78%	78%	84%	77%	73%	82%	82%	80%	78%	83%	n/a	n/a	n/a	n/a	79%
TECDAX	74%	71%	80%	82%	90%	92%	90%	85%	77%	75%	90%	n/a	n/a	n/a	n/a	83%
All	78%	73%	76%	79%	77%	77%	82%	77%	71%	71%	75%	50%	42%	41%	34%	74%

Table 15: Acceptance rate of Stange and Kaserer (2008c)-approach with weighted spread (2.4.3a) by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.

Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
DAX	88%	n/a	64%	69%	n/a	69%	n/a	81%	78%	n/a	81%	78%	63%	58%	56%	70%
MDAX	84%	86%	80%	78%	84%	81%	78%	68%	54%	51%	49%	n/a	n/a	n/a	n/a	72%
SDAX	77%	84%	81%	78%	81%	79%	73%	75%	70%	67%	55%	n/a	n/a	n/a	n/a	76%
TECDAX	78%	63%	67%	73%	78%	80%	82%	77%	76%	73%	71%	n/a	n/a	n/a	n/a	74%
All	81%	80%	76%	77%	82%	79%	79%	75%	66%	63%	60%	78%	63%	58%	56%	74%

Table 16: Acceptance rate of modified add-on approach with weighted spread (2.4.3b) by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.

Acceptance rate in % of stocks	Order size (in thsd. Euro)															
	Min.	10	25	50	75	100	150	250	500	750	1000	2000	3000	4000	5000	All
<b>DAX</b>	85%	n/a	61%	61%	n/a	58%	n/a	67%	75%	n/a	74%	83%	67%	53%	42%	64%
<b>MDAX</b>	76%	78%	76%	73%	76%	70%	74%	68%	60%	59%	63%	n/a	n/a	n/a	n/a	70%
<b>SDAX</b>	77%	80%	79%	76%	75%	71%	76%	69%	74%	66%	58%	n/a	n/a	n/a	n/a	74%
<b>TECDAX</b>	72%	55%	61%	71%	73%	76%	82%	71%	62%	60%	71%	n/a	n/a	n/a	n/a	68%
<b>All</b>	77%	73%	72%	72%	75%	72%	77%	71%	67%	62%	65%	83%	67%	53%	52%	71%

Table 17: Acceptance rate of modified net return approach with weighted spread (2.4.3c) by index and order size

Table shows the fraction of stocks, where the L-VaR-model has been accepted by Kupiec-statistics. The min-column measures the acceptance rate for the minimum spread level, i.e. the bid-ask-spread. The All-column measures the average over all standardized order sizes, i.e. without the min-column.

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