

# Pricing and Hedging of Oil Futures - A Unifying Approach -

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## **Abstract**

We develop and empirically test a continuous time equilibrium model for the pricing of oil futures. The model provides a link between no-arbitrage models and expectation oriented models. It highlights the role of sufficient inventories for oil futures pricing and for the explanation of backwardation and contango situations. In an empirical study the hedging performance of our model is compared with five other one- and two-factor pricing models. The hedging problem considered is related to Metallgesellschaft's strategy to hedge long-term forward commitments with short-term futures. The results show that the downside risk distribution of our inventory based model stochastically dominates those of the other models.

JEL Classification: G13, Q40

## Introduction

In the mid eighties highly liquid spot markets for crude oil superseded the integrated contract system of the major oil companies. As prices in the spot market tend to be highly volatile, risk management became an increasingly important issue in the oil business. This is reflected in the success of oil futures contracts at the New York Mercantile Exchange (NYMEX) and the International Petroleum Exchange (IPE). For example, in 1998 more than 30 million crude oil futures contracts were traded on NYMEX, which represents a volume of 30 billion barrels, more than the worldwide oil production.

An effective use of futures contracts in risk management requires an understanding of the factors determining futures prices and of the price sensitivities with respect to these underlying risk factors. In particular, the change of oil futures markets from backwardation<sup>1</sup> into contango and vice versa should be carefully modeled. It is well known that one-factor cost-of-carry models with fixed parameters are unable to explain both a positive and a negative basis.

In the recent literature on commodity futures pricing<sup>2</sup> mainly two approaches have been followed. The first one is based on the notion of a “convenience yield”, defined as the benefit which accrues to the owner of the commodity but not to the owner of the futures contract (Brennan (1991)). For example, this benefit would result from the right of the owner of the physical commodity to use it for production purposes whenever necessary.<sup>3</sup> The convenience yield, net of storage costs, has been modeled in different ways. Brennan and Schwartz (1985) take the convenience yield as a constant fraction of the spot price. Gibson and Schwartz (1990) and Brennan (1991) introduce a mean-reverting Gaussian convenience yield. Schwartz (1997), Model 3, and Hilliard and Reis (1998) extend the stochastic convenience yield model by adding stochastic interest rates and jumps in the spot price process, respectively. Despite their different complexity all these models share the following feature: Oil futures prices are determined by the *current* oil price and the costs and benefits of *storing* oil.

The second approach to valuing oil futures was put forward in the recent literature by Ross (1997) and Schwartz (1997), Model 1, and extended by Schwartz and Smith (1997). This approach builds

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<sup>1</sup> Backwardation is usually defined as a downward-sloping term-structure of futures prices. When the futures curve is upward-sloping, the market is said to be in contango.

<sup>2</sup> The literature on commodity futures is enormous and dates back at least to Keynes (1930). Here, we focus on models which are designed for direct application in pricing and hedging.

<sup>3</sup> The theory of storage relates the benefits of holding inventories to the level of inventories. It dates back to Working (1949), Telser (1958) and Brennan (1958).

on the idea that a replication of futures contracts by storing or short selling the physical commodity is made impracticable or impossible due to market frictions. Thus, futures prices cannot be deduced from *current* spot prices and the costs and benefits of *storage*. Instead they are determined by the *expected* spot price at maturity of the contract. As a consequence, the drift rate of the spot price process becomes crucially important for valuation. In particular, prices of long-term contracts will strongly depend on whether the spot price process is mean-reverting or not. When the spot price process is mean-reverting either backwardation or contango can result from models of this class depending on the current spot price level.

Both valuation approaches take extreme views of the structure of the spot market. In the first approach, potential arbitrageurs trade without transaction costs and can build long or short positions in the physical commodity without limits. In the second approach, any link between current spot prices and futures prices is broken due to market frictions.

The contribution of our paper to the literature is twofold. On the theoretical side we develop an equilibrium model with a representative investor which provides a connection between these two approaches and identifies the determinants of oil futures prices in different market situations. While the model presented is simple enough for practical use in pricing and hedging, it shows that costs and benefits of storage, the current spot price, the characteristics of the spot price process, the level of inventories and risk premia all play a role in oil futures pricing. On the empirical side we test the hedging performance of our model against five alternatives which represent the two valuation approaches.

Our model has a number of interesting theoretical implications. First, one key insight is that the characteristics of futures prices change fundamentally as soon as no discretionary<sup>4</sup> inventories are available. If the spot price of oil is low and inventories are high, the market price of oil risk is positive and futures prices coincide with those of simple cost-of-carry models. If, on the other hand, spot prices are high and inventories are zero, the market price of oil risk becomes zero and futures prices equal the spot prices expected for the expiration dates. Second, both backwardation and contango situations can be explained endogenously without using a convenience yield variable. As expected, backwardation occurs for high oil prices and contango for low oil prices. Third, the oil price sensitivity of futures prices strongly depends on the oil price level and the level of discretionary inventories. If the spot oil price is high, the sensitivity of the futures price is low, if the oil price is low, the sensitivity equals the discount

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<sup>4</sup> As Routledge et al. (2000), we use the term discretionary inventories for inventories which are not directly committed to production.

factor. This result has important consequences for hedging long-term forward commitments with short-term futures.

Schöbel (1992) developed an inventory based continuous time equilibrium model for commodity futures pricing which shares some important features of our model. Using the concept of a state dependent convenience yield, his model's state space captures two regimes: a complete market region, where the spot instrument is available for arbitrage as long as the convenience yield is nil and an incomplete market region, where arbitrage is impossible due to a relative scarceness of the spot, whenever the convenience yield becomes positive. While in Schöbel (1992) the model is driven by an exogenous level of inventories, which results in a mean-reverting spot price process, we determine inventories endogenously and model the spot price process directly. Because we use a short sale restriction for the inventories held by the representative investor, we do not require to introduce a convenience yield.

Our model also has qualitatively similar implications as the discrete time equilibrium model of Routledge et al. (2000). However, the underlying economic mechanisms are quite different. In Routledge et al. (2000) risk neutral agents determine futures prices according to the expected spot price at the maturity of the contract. The agents' storage decisions become important, as they influence the endogenously determined spot price process. In our approach risk averse traders can engage in spot and futures positions. The level of inventories held affects the oil price sensitivity of the traders' wealth and in turn changes the risk premium demanded for oil futures. One advantage of our approach is that it fits nicely into the framework of standard models and can be easily extended to a multi-factor setting. It provides a rich modeling framework which allows a consistent integration of different stochastic factors determining the costs and benefits of storage as well as the oil price dynamics.

The empirical performance of our model is assessed by applying it to the problem of hedging long-term forward commitments with short-term futures, a problem which has recently received much attention.<sup>5</sup> We compare the performance of the dynamic hedging strategy resulting from our model with the performance of five other strategies based on one- and two-factor models from the literature. Three of these competing models belong to the first, two models to the second valuation approach. In the basic hedge problem we consider a forward contract with ten years to maturity, which is hedged by successively rolling over short-term futures. For each hedging strategy we determine the probability distribution of the hedged position's terminal value using a bootstrap methodology. Under ideal conditions this value

should be zero with probability one. The most important empirical result is that for each of the five hedging strategies the distribution of terminal losses is first order stochastically dominated by the strategy derived from our equilibrium model. A stability analysis supplements the empirical study. This analysis quantifies the errors introduced by our bootstrap approach, the parameter sensitivity of the hedging strategies and the impact of an “extremal event” like the Gulf Crisis. The main result is basically stable as our equilibrium model still dominates most of the other models in the loss region.

In a recent study, Neuberger (1999) also analyzes the performance of different strategies to hedge long-term exposures with short-term futures. The main difference to our investigation is that we consider, as the studies of Brennan and Crew (1997) and Ross (1997), hedging strategies based on no-arbitrage or equilibrium models which imply that the long-term commitment can be hedged perfectly. Instead, Neuberger assumes a linear relationship between the prices of currently traded futures and the price at which new futures are expected to open. He imposes no additional restrictions on the stochastic development of futures prices and does not assume that the long-term contract can be replicated by a sequence of short-term futures. Both approaches complement each other and have their specific advantages and disadvantages. The remaining part of the paper is organized as follows: Section I develops the basic one-factor version of the model. Section II illustrates the model characteristics and reports some results of a comparative static analysis. In Section III some possible extensions, including the introduction of a convenience yield and a stochastic mean-reversion level of the spot price process are discussed. Section IV shortly summarizes the hedging strategies derived from the six competing models. Section V includes a detailed empirical analysis of the hedging performance and Section VI concludes.

## I. Model Setup

In this section we develop our basic continuous time partial equilibrium model of oil futures prices. The interest rate and the dynamics of the oil price are taken as exogenous, the dynamics of futures prices are endogenous. The individual investor is assumed to have time-additive preferences of the form

$$E_0 \left[ \int_0^{\hat{t}} e^{-\rho t} \ln(C(t)) dt \right], \quad (1)$$

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<sup>5</sup> This attention was mainly attracted by the case of the Metallgesellschaft AG. The main contributions to the debate surrounding this case are collected in Culp and Miller (1999).

where  $E_0[.]$  denotes the expectations operator conditional on the information in  $t = 0$ ,  $C(t)$  represents time  $t$  consumption flow and  $\hat{T}$  is the planning horizon of the investor. The investor receives only capital income and can choose between immediate consumption and three investment alternatives. First, there is the possibility to buy and store oil, paying a constant rate  $K$  of storage costs per barrel, and sell oil out of inventories. Second, long or short positions in oil futures can be taken and third, long or short positions in an asset earning a riskless rate of  $r$  are possible. For the moment,  $r$  is assumed to be exogenous and constant.<sup>6</sup> The investor can be interpreted as a trader who is active in both the spot and the futures markets for oil.

The oil price  $S$  is given exogenously and its logarithm  $\ln S$  is driven by an Ornstein-Uhlenbeck process with positive mean-reversion parameter  $\gamma$ , stationary mean  $\Theta - (\sigma^2 / 2\gamma)$  and positive volatility  $\sigma$ . This results in the following price process

$$dS = \gamma(\Theta - \ln S)S dt + \sigma S dz, \quad (2)$$

which is well supported empirically<sup>7</sup> and ensures that prices will always be positive. The futures price  $F$  is assumed to be at most a function of the investor's wealth  $W$ , the oil price  $S$  and  $T - t$ , the time to expiration of the contract. It is further assumed that the futures price follows a diffusion process

$$dF = \beta_F(S, t)dt + \sigma_F(S, t)dz, \quad (3)$$

with drift  $\beta_F$  and diffusion coefficient  $\sigma_F$ , and that the futures contract is continuously marked to market. Hence,  $dF$  equals the linearized gains and losses of one contract long held over the time period  $dt$ .

The investor is a price taker and trades in continuous markets. There are no transaction costs and no limits on the storage capacity.

The investor's decision problem is to maximize (1) subject to the budget constraint

$$dW = [aW(\gamma(\Theta - \ln S) - K/S) + (1-a)Wr + n\beta_F - C]dt + [aW\sigma + n\sigma_F]dz, \quad (4)$$

<sup>6</sup> This assumption is made to leave the model structure as simple as possible. Extensions of the model are discussed in Section III.

<sup>7</sup> The stationarity of oil prices has been documented in the literature, e.g. in Bessembinder et al. (1995). Pilipovic (1997), p. 74 ff. compares several models of the oil price dynamic. An Ornstein-Uhlenbeck process of the log oil price provides the best explanation of the statistical properties of observed prices.

where  $a$  is the proportion of wealth – after consumption – held in inventories of oil and  $n$  is the number of futures contracts taken. Equation (4) reflects that wealth can either be consumed or invested in oil or in the riskless asset. Long or short positions in futures can be initially taken and rebalanced without any cash consequences. The investor's choice variables are  $C \geq 0$ ,  $a$  and  $n$ .

The dynamic optimization problem is easily solved by standard methods.<sup>8</sup> In the case of a logarithmic utility function an analytical solution for the derived utility of wealth function exists and the first order conditions for  $a$  and  $n$  are

$$\gamma(\Theta - \ln S) - K/S - r - a\sigma^2 - n\sigma\sigma_F / W = 0 , \quad (5)$$

$$\beta_F - a\sigma\sigma_F - n\sigma_F^2 / W = 0 . \quad (6)$$

In equilibrium we have to consider that futures contracts are in zero net supply and that the aggregated discretionary inventories must be non-negative. Therefore, if the investor is taken to be representative for the futures market, in equilibrium  $n$  must be equal to zero. Moreover, the representative investor cannot short positions in discretionary inventories. Intuitively speaking, whenever a short position in oil is attractive for the representative investor, no discretionary inventories are available in the market and thus no oil can be borrowed to execute a short sale.

Using the conditions  $n^* = 0$  and  $a \geq 0$  together with equation (5), we receive the following proportion  $a^*$  of wealth optimally invested in oil:

$$a^*(S) = \max \left[ \frac{\gamma(\Theta - \ln S) - K/S - r}{\sigma^2}, 0 \right]. \quad (7)$$

$a^*$  has a unique maximum at  $\hat{S} = K / \gamma$ . In the following we assume that  $a^*(\hat{S})$  is positive, i.e.  $\gamma(\Theta - \ln(K / \gamma)) - \gamma - r > 0$ . Then it can be shown that there exist critical positive oil prices  $\tilde{S}$  and  $\bar{S}$  such that the discretionary inventory is zero for  $S < \tilde{S}$  and  $S > \bar{S}$ .

Equation (7) states that the representative investor stores oil whenever an instantaneous positive return, net of storage and financing costs, is expected. Otherwise storage is zero. The optimal storage  $a^*$  increases with the expected return received from holding inventories and decreases with the risk in terms of  $\sigma^2$ .

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<sup>8</sup> Compare Merton (1971) or Ingersoll (1987), Chapter 13.

Assuming that the futures price in equilibrium is a sufficiently smooth function of the oil price and time, from (6) and Ito's lemma for  $\beta_F$  and  $\sigma_F$  the fundamental partial differential equation

$$\frac{\partial F}{\partial S}(\gamma(\Theta - \ln S) - \sigma\lambda(S))S + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} \sigma^2 S^2 = 0 \quad (8)$$

is obtained, where

$$\lambda(S) = a^*(S)\sigma = \max\left[\frac{\gamma(\Theta - \ln S) - K/S - r}{\sigma}, 0\right] \quad (9)$$

is the market price of oil risk<sup>9</sup>. For our analysis of the futures price in Section II it is important to note that the market price of risk is positive for low oil prices and zero for high oil prices. The fundamental valuation equation (8) has to be solved for the futures price subject to the terminal condition  $F_T(T, S) = S(T)$ . No analytical solution of (8) is known, but numerical solutions based on finite difference methods or Monte Carlo methods can be obtained easily.

## II. Model Analysis

We start the analysis of futures prices resulting from (8) with two special cases. These allow us to illustrate the economic intuition behind the model and its relationship to other models. First, assume that the current oil price  $S(t)$  is very high compared to the critical oil price  $\bar{S}$  and that  $P(S(\tau) \leq \bar{S}; t \leq \tau \leq T)$  and thus  $P(a^*(S(\tau)) > 0; t \leq \tau \leq T)$  are negligible. When no inventories are held, the covariance of changes in the oil price with changes in the optimally invested wealth is zero. For the logarithmic utility function this implies that the risk premium is zero. With  $\lambda = 0$ , equation (8) together with the terminal condition can be solved analytically and we receive the following expression for the futures price:

$$F_T(t, S) = \exp\left[e^{-\gamma(T-t)} \ln S(t) + \left(\Theta - \frac{\sigma^2}{2\gamma}\right)(1 - e^{-\gamma(T-t)}) + \frac{\sigma^2}{4\gamma}(1 - e^{-2\gamma(T-t)})\right]. \quad (10)$$

This pricing formula is identical to the one resulting from Model 1 of Schwartz (1997). It equals the expected spot price at maturity of the contract, given the current spot price and the oil price dynamics in (2). Thus, in situations where the spot price is very high and no



inventories are likely to be held over the life of the contract, the futures price is exclusively driven by expectations about the spot price dynamics.

Second, assume that the current oil price  $S(t)$  is so low that  $P(S(\tau) \leq \bar{S}; t \leq \tau \leq T)$  is close to one.<sup>10</sup> Then, with probability close to one, the storage of oil has a positive expected instantaneous return throughout the life of the contract and both the inventories  $a^*(S(\tau))$  and the market price of oil risk  $\lambda(S(\tau))$ ,  $t \leq \tau \leq T$ , will be strictly positive. Thus, the fundamental valuation equation (8) becomes

$$\frac{\partial F}{\partial S}(K + rS) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} \sigma^2 S^2 = 0, \quad (11)$$

with the following solution for the futures price:

$$F_T(t, S) = e^{r(T-t)} S(t) + \frac{K}{r} [e^{r(T-t)} - 1]. \quad (12)$$

The price in (12) is identical to the one obtained from a simple cost-of-carry model, i.e. it depends only on the current spot price and the costs of storage. The intuition behind this result is as follows: When the investor's optimal strategy would always lead to positive inventories during the life of the futures contract, any deviation from (12) can be exploited by standard arbitrage strategies. This is obvious if the futures price would exceed (12). If the futures price were lower than in (12), the investor could sell some oil from inventories, take a long position in the futures and buy the oil back at the expiration date. This would provide some additional, riskless income compared to the original strategy  $a^*$ , i.e. the original strategy cannot be optimal.<sup>11</sup>

Looking from another perspective, equation (12) results from a specific risk adjustment. If the market price of oil risk  $\lambda = a^* \sigma$  is positive, it is proportional to the fraction of the investor's

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<sup>9</sup> The valuation equation (13) can be considered as a special case of the fundamental valuation equation for derivatives in Cox et al. (1985a).

<sup>10</sup> For reasonable parameter values, as empirically determined in Section V,  $\tilde{S}$  becomes 0.41 \$/barrel. Even if the current oil price were as low as 5 \$/barrel, the probability for an oil price below  $\tilde{S}$  in one year were less than  $10^{-100}$ . Therefore, the event  $\{S(\tau) \leq \tilde{S}; t \leq \tau \leq T\}$  is neglected in the following argumentation.

<sup>11</sup> When there is a positive probability for a stock out during the live of the futures contract, selling oil from the optimal inventory and buying it back later via a futures contract is no longer a utility increasing strategy as there are two effects. First, there is an additional gain resulting from selling current stocks and buying them back later. Second, it is necessary to adjust the original strategy in the

wealth held in inventories. The more oil the representative investor stores, the higher is the instantaneous covariance between changes in aggregate wealth and changes in the oil futures price. As the investor is risk averse, changes in aggregate wealth will be hedged with short positions in the futures in order to smooth the consumption stream. Thus, when inventories are high, short positions in futures become relatively more attractive, which drives futures prices downwards.

The special cases (10) and (12) are also useful reference cases for the following comparative static analysis of the futures price. Our basic parameter scenario is defined by the following values:<sup>12</sup> The futures contract has six months to maturity,  $\Theta = \ln(20.5)$ <sup>13</sup>,  $\gamma = 2.5$ ,  $\sigma = 0.35$ ,  $K = 4$  and  $r = 0.05$ . For these parameter values the two critical oil prices  $\tilde{S}$  and  $\bar{S}$  are  $\tilde{S} = 0.41$  \$/barrel and  $\bar{S} = 18.42$  \$/barrel, i.e. the discretionary inventories  $a^*(S)$  are positive if  $0.41\$ < S < 18.42\$$ , and zero otherwise.

First, we consider a variation of the current spot price  $S(t)$ . Both pricing equations (10) and (12) provide upper bounds for the futures price resulting from our simple equilibrium model. For relatively high current spot prices  $S(t)$ , futures prices converge to the prices as if oil could not be stored. For low current spot prices they are close to those given by the simple cost-of-carry model. Figure 1 illustrates this behavior of the futures price. The futures prices of the equilibrium model have been obtained by a Monte Carlo method based on the antithetic variable technique and a control variable taken from the Schwartz model.

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(Insert Figure 1)

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As Figure 1 shows, the futures price obtained from the equilibrium model always increases with the spot price. However, the oil price sensitivity is quite different for high and low spot prices. This will be important for delta hedging strategies based on the model, as fairly different hedge-ratios may result in situations of high and low oil prices.

Next consider the consequences of a parameter variation.  $K$  and  $r$  determine the costs of storing oil. When  $K$  and  $r$  grow, less oil will be stored and the market price of oil risk decreases. Thus, futures prices tend to increase and move towards those of the Schwartz

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case of a stock out, implying a loss in utility. As both effects have to be taken into account (12) provides only an upper bound for the futures price.

<sup>12</sup> These parameter values are in accordance with the empirical results of Section V.

<sup>13</sup> For  $\Theta = \ln(20.5)$  the stationary mean of the log oil price process equals  $\ln(20)$ .

model<sup>14</sup>, in which storage of oil is not possible. The futures price is also an increasing function of the mean level parameter  $\Theta$  of the log oil price process. An increasing mean-reversion parameter  $\gamma$  can move futures prices upwards or downwards, depending on whether the current spot price is below or above its average level. Finally, an increase in  $\sigma$  has two effects. First, as is seen from (10), futures prices can increase with  $\sigma$ . The reason for this unexpected reaction is that the logarithmic spot price is assumed to follow an Ornstein-Uhlenbeck process and therefore,  $\sigma$  increases the drift of the oil price in (2). Second, a higher volatility affects the futures price negatively as it increases the current risk premium  $\lambda\sigma S$  and the expected risk premia over the life of the futures contract.

Another important issue refers to the endogenous term-structure of futures prices. Figure 2 presents three different term structures for the same parameters as used in Figure 1 and maturities of up to twelve months. The only difference lies in the value of the current oil price, which is 15 \$/barrel, 19 \$/barrel and 25 \$/barrel respectively. For the low oil price, the term structure is upward-sloping, for the medium one it is slightly humped and for the high one it is downward-sloping. Thus, for low oil prices there is a contango and for high ones a backwardation situation. As in the purely expectation-based models the concept of a convenience yield is not needed to explain backwardation. Backwardation simply occurs when the oil price expected for the expiration date is declining with the maturity of the futures contracts. By the mean-reversion property of log oil prices this is the case if the current oil price is relatively low so that no discretionary inventories are held and backwardation cannot be exploited by arbitrage trading.

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(Insert Figure 2)

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Another interesting point to notice is that the term structure does not react symmetrically to changes in the difference between the current spot price and its mean level. When the spot price is below its mean level, the short end of the term structure becomes almost linear, reflecting the similarity to model (12). For high spot prices, there can be a considerable curvature at the short end of the term structure, which results from the similarity to model (10).

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<sup>14</sup> With  $K \rightarrow \infty$  oil will become a non-storable good. In this case futures prices are as in (10).

### III. Model Extensions

The one-factor model discussed in Section II has been kept simple in order to highlight its main features. In this section we discuss possible extensions of this model for two reasons. First, we want to demonstrate that the model can be easily extended to cover more general market settings. Second, and more importantly, we want to show how some models presented in the recent literature can be nested in our approach.

Consider first a possible convenience yield from storing oil. If the owner of inventories uses oil for production, a disruption in the production process due to a stock out becomes less likely with higher inventories, i.e. the storage of oil has a benefit even if the oil price is not expected to rise. This typical argument for a convenience yield cannot be directly used for our model as the investor cannot be classified as an end user of oil, a speculator, an arbitrageur or a hedger. But the investor could lend oil to a producer who receives a convenience yield from holding non-discretionary inventories.<sup>15</sup> Thus, by means of repo transactions in oil the investor would have a share in this convenience yield.

A simple way to model a convenience yield has been proposed by Brennan and Schwartz (1985), where the convenience yield rate, net of storage costs, equals a constant proportion  $y$  of the spot price. The Brennan-Schwartz model leads to the following futures price:

$$F_T(t, S) = e^{(r-y)(T-t)} S(t). \quad (13)$$

Replacing the storage costs  $K$  by the net convenience yield  $y \cdot S$ , our approach results in the same fundamental valuation equation as given in (8), except that  $\lambda$  is substituted by  $\bar{\lambda}$ , with

$$\bar{\lambda} = \max \left[ \frac{\gamma(\Theta - \ln S) - r + y}{\sigma}, 0 \right]. \quad (14)$$

If the current oil price  $S(t)$  is low, so that the probability of a zero stock is negligible, the solution of (8) with the convenience yield dependent market price of risk equals (13). The introduction of a convenience yield makes the storage of oil more attractive. With a positive net convenience yield, more inventories will be held than in the basic model and higher risk premia for long positions in oil or futures will be demanded. This results in lower futures prices.

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<sup>15</sup> In our continuous time approach, the amount of oil lent has to be adjusted continuously in order to insure that the optimal consumption and investment strategy is not affected.

In the next step a stochastic net convenience yield rate and stochastic interest rates are introduced. Assume the following processes for the convenience yield rate and the short-term interest rate:

$$dy = \mu_y(y, t)dt + \sigma_y(y, t)dz_y, \quad (15)$$

$$dr = \mu_r(r, t)dt + \sigma_r(r, t)dz_r, \quad (16)$$

$$dz_y dz_y = \rho_y dt, \quad dz_r dz_r = \rho_r dt \quad \text{and} \quad dz_y dz_r = \rho_{yr} dt.$$

$\mu_y, \sigma_y, \mu_r$  and  $\sigma_r$  are unspecified Lipschitz- and growth constrained functions in time and  $y$  or  $r$ , respectively.  $z_y$  and  $z_r$  are Wiener processes and  $\rho_y, \rho_r$  and  $\rho_{yr}$  constant correlation parameters.

With the oil price process from (2) and the processes in (15) and (16) we derive as in Section I the following fundamental valuation equation for futures prices:

$$\begin{aligned} & \frac{\partial F}{\partial S}(\gamma(\Theta - \ln S) - \bar{\lambda}\sigma)S + \frac{\partial F}{\partial y}(\mu_y(y, t) - \lambda_y\sigma_y(y, t)) + \frac{\partial F}{\partial r}(\mu_r(r, t) - \lambda_r\sigma_r(r, t)) \\ & + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}\sigma_y^2(y, t) + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma_r^2(r, t) \\ & + \frac{\partial^2 F}{\partial S \partial y}\sigma S \sigma_y(y, t)\rho_y + \frac{\partial^2 F}{\partial S \partial r}\sigma S \sigma_r(r, t)\rho_r + \frac{\partial^2 F}{\partial y \partial r}\sigma_y(y, t)\sigma_r(r, t)\rho_{yr} = 0 \end{aligned} \quad (17)$$

and terminal condition  $F_T(T, S) = S(T)$ . The market price of convenience yield risk  $\lambda_y = \bar{\lambda}\rho_y$  and the market price of interest rate risk  $\lambda_r = \bar{\lambda}\rho_r$  are endogenously determined. Both  $\lambda_y$  and  $\lambda_r$  are proportional to the market price of oil risk  $\bar{\lambda} = \max[(\gamma(\Theta - \ln S) - r + y)/\sigma, 0]$ , with  $\rho_y$  and  $\rho_r$  as factors of proportionality. In the special case that the innovations in oil prices are uncorrelated with innovations in interest rates and in the convenience yield, the market prices of interest rate risk and convenience yield risk are zero. In equilibrium the sign and magnitude of a risk premium in the drift of futures prices depends on the instantaneous correlation between the changes in the risk factor and the changes in the representative investor's wealth. If drift- and diffusion coefficients in (15) and (16) are defined appropriately the close relationship between our model and some models presented in the literature becomes evident. If  $y$  is assumed to follow an Ornstein-Uhlenbeck process and  $r$  is non-stochastic, futures prices similar to those of the Gibson and Schwartz (1990) model will result when the current

oil price is sufficiently low. When both  $y$  and  $r$  follow an Ornstein-Uhlenbeck process and oil prices are low, the model prices are similar to those in the three-factor model of Schwartz (1997). Schwartz (1997) assumes constant market prices of convenience yield risk and interest rate risk, however, whereas in our model these market prices of risk change in general with the oil price level.

A second line of possible model extensions does not focus on the costs and benefits of storage but on the oil price dynamics. The mean level parameter  $\Theta$  of the log oil price process as well as the volatility  $\sigma$  can be modeled as stochastic factors. Assume the following stochastic processes for these factors:

$$d\Theta = \mu_{\Theta}(\Theta, t)dt + \sigma_{\Theta}(\Theta, t)dz_{\Theta}, \quad (18)$$

$$d\sigma = \mu_{\sigma}(\sigma, t)dt + v(\sigma, t)dz_{\sigma}, \quad (19)$$

$$dz_{\Theta}dz_{\Theta} = \rho_{\Theta}dt, \quad dz_{\sigma}dz_{\sigma} = \rho_{\sigma}dt \quad \text{and} \quad dz_{\Theta}dz_{\sigma} = \rho_{\Theta\sigma}dt.$$

Here,  $\mu_{\Theta}, \sigma_{\Theta}, \mu_{\sigma}, v$  are suitably defined functions of time and  $\Theta$  or  $\sigma$ , respectively,  $z_{\Theta}, z_{\sigma}$  are Wiener processes and  $\rho_{\Theta}, \rho_{\sigma}, \rho_{\Theta\sigma}$  constant correlation parameters. The fundamental valuation equation for the corresponding three-factor model has the following form:

$$\begin{aligned} & \frac{\partial F}{\partial S}(\gamma(\Theta - \ln S) - \lambda\sigma)S + \frac{\partial F}{\partial \Theta}(\mu_{\Theta}(\Theta, t) - \lambda_{\Theta}\sigma_{\Theta}(\Theta, t)) + \frac{\partial F}{\partial \sigma}(\mu_{\sigma}(\sigma, t) - \lambda_{\sigma}v(\sigma, t)) \\ & + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 F}{\partial \Theta^2}\sigma_{\Theta}^2(\Theta, t) + \frac{1}{2}\frac{\partial^2 F}{\partial \sigma^2}v^2(\sigma, t) \\ & + \frac{\partial^2 F}{\partial S \partial \Theta}\sigma S \sigma_{\Theta}(\Theta, t)\rho_{\Theta} + \frac{\partial^2 F}{\partial S \partial \sigma}\sigma S v(\sigma, t)\rho_{\sigma} + \frac{\partial^2 F}{\partial \Theta \partial \sigma}\sigma_{\Theta}(\Theta, t)v(\sigma, t)\rho_{\Theta\sigma} = 0 \end{aligned} \quad (20)$$

Again, the market price of  $\Theta$ -risk and the market price of volatility risk have the structure  $\lambda_{\Theta} = \lambda \rho_{\Theta}$  and  $\lambda_{\sigma} = \lambda \rho_{\sigma}$ , where  $\lambda = \max\left[\frac{\gamma(\Theta - \ln S) - K/S - r}{\sigma}, 0\right]$  denotes the endogenous market price of oil risk. This market price depends on the amount of discretionary inventories as in Section I.

Assuming a Brownian motion with drift for  $\Theta$ , a constant volatility of the log oil price process and discretionary inventories of zero with probability one, the two-factor model of Schwartz and Smith (1997) is obtained with  $\lambda = \lambda_{\Theta} = 0$ . With a constant  $\Theta$  and a mean-reverting

square root process for  $\sigma$ , a stochastic volatility extension of the one-factor model of Schwartz (1997) results under the assumption that oil will never be stored.

#### **IV. Risk-Minimal Hedging Strategies**

In order to evaluate the empirical performance of our model, we consider the problem of hedging long-term forward commitments with short-term futures contracts. This “Metallgesellschaft problem” has received much attention in the literature.<sup>16</sup> There is still a controversy on the appropriate hedging strategy and whether a continuation of the implemented hedge strategy would have resulted in lower losses than the closing of all contracts.

We compare the hedge results of our model with those of five other models proposed in the literature. We include three one-factor and two two-factor models which serve as representatives of the no-arbitrage-based and the expectations-based approaches discussed in the introduction. The simple cost-of-carry model (12) and the Brennan/Schwartz model (13) are one-factor-no-arbitrage models. In the first model storage of oil results in costs only whereas in the second the benefits of storage are modeled by a constant convenience yield rate. Schwartz (1997), Model 1, is used as a one-factor model which neither allows storage nor sale of oil. The model of Schwartz and Smith (1997) is used as representative of two-factor models in which oil is assumed to be a non-traded asset and Gibson and Schwartz’s (1990) model with a stochastic convenience yield represents the class of two-factor-no-arbitrage approaches.

Within these models the long-term forward commitment can be hedged perfectly for every planning horizon up to the maturity of this contract by a dynamically rebalanced portfolio of futures which are rolled over at their maturity dates.

For the one-factor models a single futures is sufficient to achieve a riskless hedge whereas for the two-factor models two futures contracts with different maturities are needed. The forward commitment is perfectly hedged if at every point of time the sensitivity of the discounted forward price equals the sensitivity of the futures portfolio with respect to each of the stochastic factors. For the one-factor models the appropriate number of futures contracts  $h_t^1$  to hedge one forward contract is determined by equation (21), where  $T_1$  and  $T$  are the expiration dates of the current futures in this hedge and the forward, respectively.<sup>17</sup>

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<sup>16</sup> See the collection of papers in Culp and Miller (1999).

<sup>17</sup> We do not have to distinguish between forward prices and futures prices here, as none of the models analyzed assumes stochastic interest rates. See Cox et al. (1981).

$$h_t^1 \frac{\partial F_{T_1}}{\partial S} = e^{-r(T-t)} \frac{\partial F_T}{\partial S}. \quad (21)$$

Substituting the prices from (10), (12) and (13) into (21) provides the following values for the hedge-ratio  $h_t^1$ :

$$h_t^1 = e^{-r(T_1-t)}, \quad \text{for the cost-of-carry model,} \quad (22)$$

$$h_t^1 = \frac{e^{-r(T-t)} F_T}{F_{T_1}}, \quad \text{for the Brennan/Schwartz model,} \quad (23)$$

$$h_t^1 = e^{-\gamma(T-T_1)} \frac{e^{-r(T-t)} F_T}{F_{T_1}}, \quad \text{for the Schwartz model.} \quad (24)$$

The cost-of-carry model implies a hedge-ratio close to one, which corresponds to the strategy followed by the Metallgesellschaft. This hedge-ratio is not equal to one because of the continuous mark-to-market of the futures contracts which requires to tail the hedge over the relatively short period until the short-term futures expires. Note that there is no long-term tailing-the-hedge effect as a forward commitment and not a spot position in oil is hedged. The hedge-ratio of the Brennan/Schwartz model is smaller than one when the discounted forward price lies below the futures price. This will either be the case when the market is in backwardation or the effect of discounting dominates. The latter is likely for forwards with several years to expiration. Hedge-ratios resulting from the Schwartz model are even smaller. They are identical to those in (23) except for a multiplicative factor smaller than one. This factor decreases with the time to maturity of the forward and with the mean-reversion parameter  $\gamma$  of the spot price process.

For the equilibrium model hedge-ratios have to be obtained numerically. Figure 3 shows the number of one-month futures needed to hedge a six-months forward for varying spot prices. The model parameters are identical to those in Figures 1 and 2. For comparison reasons, hedge-ratios of the cost-of-carry model and the one-factor model of Schwartz (1997) are included in the figure.

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(Insert Figure 3)

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As Figure 3 shows, the relations between the equilibrium model, the cost-of-carry model (12) and Model 1 of Schwartz (1997) also translates into the hedging strategies. For low spot



prices, the hedge-ratio of the equilibrium model is close to one. For high oil prices it approaches the hedge-ratio of the Schwartz-model, which is considerably lower. More importantly, Figure 3 clearly shows the strong spot price dependence of a hedging strategy based on the equilibrium model.

For the two-factor model of Gibson and Schwartz (1990) futures prices take the following form:

$$F_T(t, y, S) = S(t) \cdot \exp\left[-y(t)\left[1 - e^{-\alpha(T-t)}\right] / \alpha + A(\cdot)\right], \quad (25)$$

where  $\alpha > 0$  is the mean reversion parameter of the convenience yield dynamics,  $y(t)$  is the current convenience yield rate and  $A(\cdot)$  is for fixed parameters a function of time only. The appropriate numbers  $h_t^1$  and  $h_t^2$  of short-term futures contracts to hedge a long-term forward can be deduced from the following system of equations:

$$h_t^1 \frac{\partial F_{T_1}}{\partial S} + h_t^2 \frac{\partial F_{T_2}}{\partial S} = e^{-r(T-t)} \frac{\partial F_T}{\partial S}, \quad (26)$$

$$h_t^1 \frac{\partial F_{T_1}}{\partial y} + h_t^2 \frac{\partial F_{T_2}}{\partial y} = e^{-r(T-t)} \frac{\partial F_T}{\partial y}.$$

Substituting the prices according to (25) into (26), the following hedging positions in the futures contracts with expiration dates  $T_1$  and  $T_2$  ( $T_1 < T_2 \leq T$ ) result.

$$h_t^1 = \left[1 - \frac{1 - e^{-\alpha(T-T_1)}}{1 - e^{-\alpha(T_2-T_1)}}\right] \frac{e^{-r(T-t)} F_T}{F_{T_1}}, \quad (27)$$

$$h_t^2 = \left[\frac{1 - e^{-\alpha(T-T_1)}}{1 - e^{-\alpha(T_2-T_1)}}\right] \frac{e^{-r(T-t)} F_T}{F_{T_2}}.$$

Note that the futures contract with maturity  $T_1$  is always shorted if  $T_2 < T$ , whereas always long positions are held in the second futures. The net position  $h_t^1 + h_t^2$  is close to (23), the number of futures contracts taken in the corresponding one-factor-model.

In the model of Schwartz and Smith (1997) the mean level parameter  $\Theta$  of the log oil price process is stochastic and follows a Brownian motion with drift.<sup>18</sup> The corresponding futures prices are given by (28), where  $B(\cdot)$  is for fixed parameters a function of time only.

$$F_T(t, S, \Theta) = \exp \left[ e^{-\gamma(T-t)} \ln(S(t)) + \left( \Theta(t) - \frac{\sigma^2}{2\gamma} \right) (1 - e^{-\gamma(T-t)}) + B(\cdot) \right]. \quad (28)$$

The hedge portfolio for the Schwartz/Smith model results again from (26), using the price function (28) and replacing the convenience yield rate in (26) by  $\Theta$ . Explicitly the numbers of futures contracts are

$$h_t^1 = \left[ 1 - \frac{1 - e^{-\gamma(T-T_1)}}{1 - e^{-\gamma(T_2-T_1)}} \right] \frac{e^{-r(T-t)} F_T}{F_{T_1}},$$

$$h_t^2 = \left[ \frac{1 - e^{-\gamma(T-T_1)}}{1 - e^{-\gamma(T_2-T_1)}} \right] \frac{e^{-r(T-t)} F_T}{F_{T_2}}. \quad (29)$$

A comparison of (29) and (27) shows that the hedge positions have the same structure. This is due to the formal equivalence of the two models.<sup>19</sup> The only difference is that the mean-reversion parameter  $\alpha$  of the convenience yield in (27) is replaced by the mean reversion parameter  $\gamma$  of the oil price in (29). However, the economic basis of the two models is very different. This difference will carry over to the empirical implementation of the two models and to the size of the hedging portfolios.

Yet other approaches to hedge long-term forwards could be considered. Neuberger (1999) derives a risk-minimal strategy which uses a hedge portfolio consisting of multiple short-term futures contracts with different times to expiration. This strategy turns out to be very robust and performs well empirically. Purely data driven hedging strategies are proposed by Edwards and Canter (1995) and Pirrong (1997). As the main focus of our study is to compare and empirically test different no-arbitrage and equilibrium valuation models by means of their hedging performance, these alternative approaches are not pursued any further.

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<sup>18</sup> In Schwartz and Smith (1997) the log oil price is the sum of two unobservable stochastic factors. Here, we chose an equivalent formulation with the oil price and  $\Theta$  as stochastic factors.

<sup>19</sup> See Schwartz and Smith (1997), Section 4.

## **V. Empirical Comparison of Hedging Strategies**

### **A. Methodology**

It is the forward commitments' time to maturity of up to ten years which makes the hedging problem of the Metallgesellschaft AG both an interesting and difficult one. As short-term futures have to be rolled over many times, hedging strategies can be strongly exposed to basis risk. A long hedge horizon also complicates the empirical evaluation of different hedging strategies as the available time series of data do not allow to generate a sufficient number of independent hedge results for quantifying both the expected return and the risk of a strategy. Here, we follow a bootstrap methodology similar to Ross (1997) and Bollen and Whaley (1998) to simulate hedge portfolios. The structure of our empirical study is depicted in Figure 4.

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(Insert Figure 4)

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In a first step, the data model is specified which captures the main features of historical spot and futures prices for oil. This model allows us to simulate different oil price scenarios. The second step comprises the calibration of the valuation models to the current term-structure of futures prices. In the third step 20.000 time series of spot and futures prices are simulated for the hedge period of ten years. For each time series the dynamic hedge strategy of each of the models and the hedge results are determined. In the final step the performance of the models is assessed.

### **B. Data**

The futures data set available for the empirical study consists of daily settlement prices of the NYMEX crude oil futures contract over the period 1 July 1986 – 25 November 1996. The contract is settled by physical delivery of 1,000 barrels of West Texas Intermediate (WTI) crude in Cushing, Oklahoma. Up to 1989 the longest maturity contracts were 12 months. Currently, contracts for the next 30 consecutive months are traded. However, trading activity concentrates on the shortest maturity contracts. Trading terminates on the third business day prior to the 25<sup>th</sup> calendar day of the month preceding the delivery month.

Daily spot prices for WTI crude in Cushing were provided by Platt's, the leading oil price information service. These prices also cover the period from 1 July 1986 – 25 November 1996.

As we change the hedge positions once a month when the short-term futures are rolled over, monthly observations are sufficient for the analysis. Thus, for each month of the data period we selected the spot price and the futures prices for maturities of up to twelve months on the third business day prior to the 25<sup>th</sup> calendar day.<sup>20</sup> This provides us with a total of 13 time series with 125 observations each.

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(Insert Figure 5)

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Figure 5 shows the time series of the oil price and the six month futures basis, defined as the spot price minus the futures price for delivery in six months. The oil price exhibits a considerable variability with an annualized return volatility of 33%. The period of the Gulf Crisis between July 1990 and February 1991 can be clearly identified, where the oil price reached a level of up to 35 \$/barrel. As the visual impression suggests, spot price and basis are positively correlated with a correlation coefficient of 0.73.

### **C. Data Model**

To generate plausible price scenarios, it is important to capture the main features of spot and futures prices. This leads to the following conditions:

- (i) Prices should always be positive.
- (ii) Futures prices should be closely tied to the spot price to avoid unrealistic deviations.
- (iii) There should be a positive correlation between the oil price level and the basis.

The simplest way to fulfill the first condition is to model and simulate logarithmic prices. A link between futures prices and spot prices is established when futures prices are generated indirectly via a model of the basis. Moreover, if such a model uses the relative basis, i.e. the basis divided by the oil price, the absolute deviation between the oil price and the futures price is likely to be smaller for low oil prices than for high oil prices. The third condition can be fulfilled by allowing for cross sectional correlation between the innovation terms of the stochastic processes describing the log spot price and the relative basis in the data model.

Our data model builds on the log spot price and the relative basis for futures with up to twelve months to maturity. The spot price, the one-month basis and the two-month basis are needed to generate the prices of the one-month futures and the two-month futures, which will be used

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<sup>20</sup> In the period before 1989 contracts with maturities of more than nine months were occasionally not traded. In these cases the missing futures price was replaced with the price for the longest maturity



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(Insert Table 1)

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The Ljung-Box tests give no indication of autocorrelation in the residuals of the data model. Moreover, there is little evidence for ARCH effects in the residuals. For none of the equations are ARCH effects significant on a 1% significance level, only for some of the equations on a 5% level. In summary, if there is any time series dependence at all in the residuals, it is only weak. This allows us to simulate price paths by standard bootstrapping methods. Whole residual vectors  $[u_{0t}, \dots, u_{12t}]$  are drawn to maintain the strong cross sectional correlation between the residuals of different equations at the same point of time.

The simulation procedure starts with the observed values for the log spot price and relative bases on 22 July 1992. This day was selected for two reasons. First, the Metallgesellschaft (MGRM) was already active in the futures market. However, the volume was small so that no price effect has to be expected. Second, at that time the term structure of futures prices was in a typical backwardation situation with a spot price of 21.80 \$/barrel and futures prices for delivery in one, six and twelve months of 21.55 \$/barrel, 21.09 \$/barrel and 20.48 \$/barrel respectively. A residual vector is drawn and the simulated log spot price and basis for the next month are calculated according to the data model. Starting from these new values, the subsequent residual vector is drawn etc.. In this way paths for a time horizon of up to ten years are generated. Corresponding paths for the futures prices are easily obtained from the simulated values of the log spot price and the relative bases. Finally, the simulation of the ten-year price paths is replicated 20.000 times. This provides us with a sufficient number of scenarios to investigate the risk and return of the different hedging strategies.

#### **D. Implementation of Hedging Strategies**

In order to implement the hedging strategies as described in Section IV, we first have to specify which futures contracts to use as hedge instruments. Given the higher liquidity of the very short-term contracts we chose to use the futures with one month to maturity for the one-factor strategies and the futures with one and two months to maturity when two hedging instruments are required.

For the calculation of the hedge-ratios  $h_t = e^{-r(t_1-t)}$  derived from the cost-of-carry model, only the time to expiration  $t_1 - t$  of the short-term futures and the interest rate  $r$ , which is set equal to 5 % p.a., are required. As in our empirical study hedge positions are rolled over once a

month at the last trading day of the expiring contract and are not rebalanced inbetween, we obtain constant hedge-ratios  $h_t = e^{-0.05/12} \approx 0.9958$ . For all other strategies prices of the one-month futures, the two-month futures and the long-term forward as well as certain model parameters are needed to calculate hedge-ratios. For the model of Brennan and Schwartz (1985) information on prices and the interest rate is sufficient. The hedging strategies based on the models of Schwartz (1997), Schwartz and Smith (1997) and Gibson and Schwartz (1990) demand the additional knowledge of either the mean-reversion parameter  $\gamma$  or  $\alpha$ . As for the equilibrium model futures prices are not available in closed form, hedge-ratios have to be obtained numerically, which requires the current spot price, the parameters  $\gamma$ ,  $\Theta$  and  $\sigma$ , the storage costs  $K$  and the interest rate.

Price information entering the hedge-ratios in (23), (24), (27) and (29) is obtained from the data model. Simulated futures prices for contracts of up to twelve months to maturity are directly available. The forward price for delivery in ten years is set according to the pricing rule of the Metallgesellschaft as the average of the one- to twelve-month futures prices plus 2.1 \$. Forward prices for intermediate maturities are calculated by linear interpolation of the one-year futures price and the ten-year forward price. As time elapses over the hedge horizon, these forward prices are needed to calculate the hedge-ratios according to the Brennan/Schwartz, Schwartz, Gibson/Schwartz and Schwartz/Smith strategies.

The parameters of the Ornstein-Uhlenbeck process underlying the oil price dynamics in the one-factor model of Schwartz (1997) and the equilibrium model were estimated from the time series of logarithmic spot prices by the method of maximum likelihood. Monthly observations form 1 July 1986 to 22 July 1992 were used, thus the estimates utilize only information which was available at the beginning of our hedging period. The resulting parameter estimates for the log oil prices are  $\hat{\gamma} = 2.71$ ,  $\hat{\Theta} = 3.02$  and  $\hat{\sigma} = 0.36$ , with standard errors of 0.71, 0.05 and 0.03, respectively. The storage costs  $K$  per barrel of oil were set equal to 4 \$.<sup>24</sup>

The one-factor models are in general not flexible enough to fit even the current one-month futures and ten-year forward prices correctly. This deficiency causes a problem for the comparison between the hedging strategies. Under ideal model conditions and time continuous hedging the corresponding strategies would all result in riskless hedged positions. This provides us with a natural reference point for the empirical comparison of different models. However, if the one-month futures or the ten-year forward are not correctly priced initially,

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<sup>24</sup> Ross (1997), p. 388, estimates the storage costs per barrel and year to be 20% of the current spot price. For a spot price of 20 \$/barrel this would be 4 \$ per year.

some hedging error is introduced even under ideal conditions and no common reference point exists. To resolve this problem we modified the one-factor models slightly and introduced calendar time dependent parameters. Time dependent storage costs  $K$  were used in the cost-of-carry model, a time dependent convenience yield rate  $y$  in the Brennan/Schwartz model and a time dependent  $\Theta$  in the Schwartz and in the equilibrium model. These parameters can be chosen to fit the initial term structure of futures prices correctly. The procedure is identical with the fitting of spot rate models to the current term structure of interest rates as suggested by Cox et al. (1985b) and implemented by Hull and White (1990, 1993). Interestingly, it turns out that the time dependence of the parameters does not change the hedge-ratio formulas (22), (23) and (24), however.

The two-factor model of Gibson and Schwartz (1990) is flexible enough to fit three futures prices and the price of the ten-year forward exactly for an appropriate choice of the identified model parameters.<sup>25</sup> One of these identified parameters is the mean-reversion parameter  $\alpha$  of the convenience yield rate process. We calibrated the model parameters to the prices of the one-month, two-month, six-month and ten-year futures contracts prevailing on 22 July 1992, the starting point of the hedging periods. This leads to a value for  $\alpha$  of 5.62. Of course, there exist more sophisticated methods to estimate  $\alpha$ .<sup>26</sup> As our focus lies on the hedging strategies, however, we prefer to analyze the sensitivity of the hedging results for a wide range of parameter values instead of relying on one parameter estimate only. This sensitivity analysis will be presented in Subsection F.

As Schwartz and Smith (1997) have shown, their model is structurally equivalent to the one of Gibson and Schwartz (1990). Thus, it is impossible to differentiate between the two mean-reversion parameters  $\alpha$  and  $\gamma$  if they are estimated implicitly, despite the two parameters have a different economic meaning. Whereas the implicitly estimated value of 5.62 for the convenience yield process is in line with estimates reported in the literature<sup>27</sup>, it is unrealistically high for the mean-reversion parameter of the log oil price process. Therefore, we proxy the  $\gamma$ -parameter of the Schwartz/Smith model by its estimated value from the time-series data, i.e. we use a value of 2.71. With  $\gamma$  set equal to 2.71 the remaining free model

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<sup>25</sup> Here, we interpret the unobserved state variable  $y(t)$  as another free parameter.

<sup>26</sup> See Schwartz (1997), p. 932 f.

<sup>27</sup> Compare the values in Gibson and Schwartz (1990), p. 966, Table II and Brennan and Crew (1997), p. 177, Figure 4.



parameters were then chosen such that the prices of the one- and two-month futures and the ten-year forward are fit correctly.

Given the specifications discussed above, Figure 6 shows how the hedge-positions  $h_t^1$  for the one-factor models change over time when the hedging period is initially ten years. The figure assumes a term structure of futures prices which is flat at 20\$/barrel and does not change over the hedging period.

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(Insert Figure 6)

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Considerable differences between the strategies are visible. While the cost-of-carry model leads to a constant hedge-ratio of one, the Brennan/Schwartz model starts with a hedge-ratio of about 0.6, which gradually increases towards a value of one at the end of the hedging period. The hedging positions taken according to the Schwartz model are negligible most of the time. This is due to the significant mean-reversion of the oil price. Only in the last year of the hedging period are considerable positions in the one-month futures taken. The strategy resulting from the equilibrium model is similar to that of the Schwartz model, though slightly more futures contracts are held in the last two years of the hedging period. We want to stress, however, that the hedge-ratios of the equilibrium model strongly depend on the current oil price, which is kept constant here. As Figure 3 shows, in a contango situation considerably higher hedge-ratios could result.

Figure 7 shows how the hedge-positions resulting from the Gibson/Schwartz model and the Schwartz/Smith model change over time. As in Figure 6, the term structure of futures prices is assumed to be flat at 20\$/barrel over the whole hedging period. This figure confirms the result that the investor always holds a long position in the two-month futures and a short position in the one-month futures. The net hedge position, i.e. the sum of  $h_t^1$  and  $h_t^2$  is identical for both models and equals the hedge-ratio of the Brennan/Schwartz model shown in Figure 6. Due to the different values for  $\alpha$  and  $\gamma$ , the net hedge position is spread differently between the two hedging instruments, however. In general, the absolute values of  $h_t^1$  and  $h_t^2$  decrease if  $\alpha$  or  $\gamma$  increases.

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(Insert Figure 7)

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## E. Hedging Results

Given the specifications discussed in the previous subsections, the hedge strategies can now be evaluated for the 20.000 simulated price paths. We hedged a forward commitment to deliver one barrel of oil in ten years<sup>28</sup> and calculated the \$-value of the hedged position at the end of the hedge horizon. Futures contracts were rolled over at their last trading day into contracts with the same initial maturity. During a month the hedge portfolio was not rebalanced. This introduces a discretization error for all but the cost-of-carry and the Brennan/Schwartz strategies.<sup>29</sup> Intermediate payments from the futures are either invested or financed assuming a fixed interest rate of 5% p.a.. Summary statistics for the distribution of the hedge results, based on the 20.000 observations, are shown in Table 2. The results refer to one barrel of oil with an initial ten-year forward price of 23.12 \$ resulting from the pricing rule of the Metallgesellschaft. If the models would work perfectly, the hedge results should have a mean and a standard deviation of zero. For comparison reasons, results for the unhedged forward commitment are added in the last column of the table.

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(Insert Table 2)

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The table provides a number of interesting insights. First, all hedging strategies lead to gains on average. These are highest for the cost-of-carry model, with a mean of 36.31 \$ or 167 % of the spot price on 22 July 1992. A comparison with the results for the unhedged position shows that only 3.32 \$ of these gains can be attributed to the forward commitment.<sup>30</sup> The remaining gains have been earned by rolling over the short-term futures contracts. Such a strategy is expected to be profitable when the market is predominantly in backwardation and the oil price has no downward trend. This was the case for the data period 1986 to 1996.

Second, the almost 1:1 hedge resulting from the cost-of-carry model leads not only to the highest mean but - with one exception, the Schwartz/Smith model - also to the highest risk of all strategies. This holds irrespective of whether risk is expressed by the standard deviation or by some measure of downside risk, such as the loss probability or the mean loss. These results

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<sup>28</sup> The Metallgesellschaft had to hedge a whole portfolio of forward commitments with different maturities. Here, we concentrate on a single component in order to demonstrate the effects more clearly.

<sup>29</sup> The impact of these discretization errors on the hedging results is analyzed in Subsection F for the exemplary case of the Gibson/Schwartz model.

<sup>30</sup> There is a gain on the unhedged position on average, because the ten-year forward price at the beginning of the hedging period lies above the mean level of the oil price.

support the view that the strategy of the Metallgesellschaft was a speculative one, highly profitable on average, but highly risky also. If Metallgesellschaft would have continued its strategy at the end of November 1993 level of about 200 million barrels, in the worst case she would have suffered a loss of 10 billion \$ or about eight times the loss she had to cover until end of December 1993. The 1% quantile of the hedged positions distribution equals -17.42 \$ per barrel and corresponds to a total loss of 3.5 billion \$. The average loss of -0.62 \$ per barrel results in a loss of 124 million \$ or about 10% of the loss at the end of December 1993.

Third, if the average value and the standard deviation of the hedged position's value are considered, the six models fall into two groups. The first group consists of Schwartz's model and our equilibrium model with means and standard deviations of about the same size as the unhedged position. The second group contains the remaining four models. Surprisingly both two-factor models belong to the second group and are outperformed with respect to each of the three risk measures by the models of the first group and partly by the unhedged position. These results can be explained by two observations. First, the hedge strategies related with the models of the second group have consistently higher net positions in futures than those of the first group. The roll over of these futures contracts exposes the hedged position to a considerable basis risk. Second, due to the mean-reversion of the spot price, price fluctuations will average out over the long hedge period of ten years. This explains the relatively low risk of the unhedged position and of the models of the first group, as these models lead to smaller futures positions.

Fourth, the hedge strategy derived from our equilibrium model first order stochastically dominates both, the strategy based on the Schwartz model and the unhedged position. This is demonstrated by Figure 8, which shows the distribution function of the terminal value resulting from the unhedged forward commitment and the two hedge strategies.

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(Insert Figure 8)

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Especially interesting is the behavior of the different strategies in the loss region of the distribution function. Of course the first order stochastic dominance relations between the equilibrium based strategy, the Schwartz strategy and the unhedged position shown in Figure 8 still hold. Interestingly, the equilibrium strategy dominates also the other four strategies if one considers only the loss region of the value distribution. This strong result is shown in Figure 9.

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(Insert Figure 9)

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## **F. Stability Analysis**

Overall, none of the model based hedging strategies provides fully satisfactory results for the ten-year hedge horizon in comparison with the unhedged position. This may be for different reasons:

- (i) One potential problem arises from the use of a data model to simulate representative price scenarios. It is possible that the valuation models explain the futures prices well but the data structure is not appropriately captured by the data model. Therefore, the hedging quality appears to be too low.
- (ii) The hedge portfolios are adjusted only once a month. This introduces discretization errors for some models compared to a theoretically correct continuous adjustment of the hedge portfolio.
- (iii) The hedge-ratios depend on unknown model parameters. If these parameters are estimated with error the hedging performance is reduced.
- (iv) Finally, the valuation models may not appropriately explain the observed futures prices and their dynamic development. In this respect one can also imagine that the models work well in normal periods but collapse in extreme situations like the Gulf Crisis.

These four possible explanations are now discussed in turn. The errors introduced by the data model and the monthly adjustments of the hedging strategies can be isolated and quantified when the whole analysis is replicated with model consistent data. Therefore, we replaced the historical futures prices by a "new set of futures data" calculated according to the Gibson/Schwartz model from the historical spot prices and the calibrated model parameters. This "new data set" was then used together with the historical spot prices to reestimate the data model. Finally, the whole simulation study was repeated for a ten-year hedge horizon. The resulting hedging errors of the Gibson/Schwartz strategy give some indication on the joint effect of an inappropriate data model and the discrete hedge adjustments. It turns out that the standard error of the hedge results lies below 0.2, which is extremely small compared to the value of 11.81 given in Table 2. This shows that only a small proportion of the hedging errors can be attributed to a misspecified data model and to the non-continuous rebalancing of the hedging positions.

The effect of an imprecise estimation of the parameters  $\gamma$  and  $\alpha$  can be quantified by a sensitivity analysis. As far as the  $\gamma$  parameter in the Schwartz model is concerned, the study already covers the extreme cases of the parameter range. For  $\gamma \rightarrow 0$  the hedging strategy converges to that of the Brennan/Schwartz model and for very large values of  $\gamma$  one essentially obtains an unhedged position. If  $\gamma$  decreases gradually, starting from a value of 2.71, the general tendency in the results of Table 2 is confirmed: Decreasing values of  $\gamma$  result in increasing hedge-ratios, means and standard deviations of the hedge results.<sup>31</sup>

Within the group of the two-factor models the results for the structurally equivalent Gibson/Schwartz and Schwartz/Smith models, as shown in Table 2, must be a consequence of the different values used for the two mean-reversion parameters  $\alpha$  and  $\gamma$ . Table 3 provides hedging results for a wide range of different parameter values. The smallest value of 1.49 was the  $\gamma$ -estimate obtained by Schwartz and Smith (1997) by means of the Kalman filter technique.<sup>32</sup> The maximum value of 15 is close to the  $\alpha$ -estimate of Gibson and Schwartz (1990).

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(Insert Table 3)

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Again, lower parameter values than the implicitly estimated mean-reversion parameter of the convenience yield ( $\alpha = 5.62$ ) reduces the hedge performance with respect to all three risk measures. The minimum of the risk measures is attained for parameter values of about 9.

For this “optimal” parameter value, which cannot be extracted from the data, the loss probability and the mean loss improves considerably, but the volatility of the position value is still higher than for the unhedged position.

In order to investigate the influence of the Gulf Crisis on the hedging performance of different strategies, we excluded the period between July 1990 and February 1991 from the data set, estimated the data model with the remaining observations and repeated the hedging study. As the time of the Gulf Crisis was a period of highly volatile spot prices and bases, we expected lower standard deviations and lower mean values of the hedging results.

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(Insert Table 4)

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<sup>31</sup> Detailed results are available from the authors.

<sup>32</sup> See Schwartz and Smith (1997), p. 22, Table II.

Table 4 provides the hedging results. As expected, all strategies show a lower mean value compared to the results of Table 2. The differences are considerable for the cost-of-carry strategy and the Brennan/Schwartz strategy, which take relatively large positions in the one-month futures. However, the risk of the strategies does not change as expected. For the one-factor models, especially the cost-of-carry model and the Brennan/Schwartz model, the standard deviations of the hedge results even increase. This result is driven by two effects. First, the exclusion of the extreme observations during the Gulf War reduces the volatility of the basis and the risk of the roll-over hedging strategies. Second, when the period between July 1990 and February 1991 is not used for the estimation of the data model, the mean-reversion of the relative basis becomes smaller, which leads to an increase in the volatility of the basis. It turns out that the second effect compensates the first one. For the cost-of-carry model and the Brennan/Schwartz model it even dominates. Overall, the “extremal event” of the Gulf Crisis cannot explain the high risk of some of the analyzed hedging strategies. The general characteristics of the different strategies remain unaffected by the exclusion of the period of the Gulf Crisis from the data set.

## **VI. Summary and Conclusion**

In this paper we develop a continuous time partial equilibrium model for the pricing of oil futures. This model provides a link between two important valuation approaches and offers a unifying framework to nest all kinds of models for futures prices: storage-based explanations, models using the concept of a convenience yield and pure expectation-based models in which oil is a non-traded asset. Therefore, the model is able to identify the roles played by different stochastic factors which drive the costs and benefits of storage as well as the oil price dynamics.

Our model has a number of interesting implications. First, the characteristics of futures prices depend crucially on the amount of discretionary inventories available. If the spot price of oil is low and inventories are high, the market price of oil risk becomes positive and futures prices coincide with those of simple cost-of-carry models. If, on the other hand, spot prices are high and inventories are zero, the market price of oil risk is zero and futures prices are equal to the spot prices expected for the expiration dates. Second, both backwardation and contango situations can be explained endogenously without the conception of a convenience yield. Backwardation occurs for high oil prices and contango for low oil prices. Third, the oil price sensitivity of futures prices strongly depends on the spot price level and the related level of

discretionary inventories. If the spot oil price is high, the sensitivity is low. For a low oil price the sensitivity equals the discount factor. This result has important implications for delta hedging strategies derived from the model.

In the empirical part of the paper the hedge performance of our model is compared with five other one- and two-factor pricing models. We investigate the implied dynamic strategies to hedge long-term forward commitments with short-term futures. The empirical results show that for a long-term hedge horizon the downside risk distribution of the strategy derived from our model stochastically dominates those of the other models.

Even though the hedge results of the basic one-factor version of our model are encouraging, the potential of the proposed modeling framework has to be further explored. First, the stability analysis of the hedge performance of our model should be generalized by varying the hedge horizon and by considering interest rate risk. Second, it needs to be analyzed which possible risk factors are most important and what kind of specification works best for which kind of application. For example, it is unlikely that a one-factor model is flexible enough to yield a high explanatory power in a pricing study. Third, another interesting and empirically testable implication of our model concerns the relation between current futures prices and future spot prices. Theoretically, in our model futures prices are downward biased predictors of future spot prices in general, but the magnitude of this bias strongly depends on the current oil price and the level of inventories. The bias will be highest when the current oil price is low since this implies a high market price of oil risk.

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Table 1: Estimation results of the data model, monthly observations between July 1986 and November 1996

Equation	Parameter			$R^2$	Ljung-Box-Test (12 Lags)	LR-Test for ARCH (12 Lags)
	$\hat{a}$ (St. dev.)	$\hat{b}$ (St. dev.)	$\hat{c}$ (St. dev.)		Test Statistic (P-Value)	Test Statistic (P-Value)
(30)	0,542 (0,150)	0,942 (0,088)	-0,124 (0,086)	0,71	16,93 (0,152)	23,86 (0,021)
(31)	0,001 (0,002)	0,464 (0,046)	-	0,46	17,43 (0,134)	9,38 (0,670)
(32)	0,002 (0,003)	0,636 (0,047)	-	0,61	18,49 (0,101)	10,52 (0,570)
(33)	0,003 (0,003)	0,709 (0,048)	-	0,65	17,31 (0,138)	11,98 (0,447)
(34)	0,004 (0,004)	0,745 (0,048)	-	0,66	16,20 (0,182)	13,94 (0,305)
(35)	0,005 (0,004)	0,768 (0,048)	-	0,67	16,57 (0,166)	16,71 (0,161)
(36)	0,007 (0,004)	0,944 (0,085)	-0,186 (0,081)	0,69	11,48 (0,488)	15,34 (0,223)
(37)	0,007 (0,004)	0,960 (0,086)	-0,194 (0,083)	0,70	10,98 (0,531)	17,33 (0,137)
(38)	0,008 (0,005)	0,974 (0,086)	-0,202 (0,084)	0,70	10,71 (0,554)	19,70 (0,073)
(39)	0,009 (0,005)	0,986 (0,086)	-0,211 (0,085)	0,71	10,41 (0,580)	22,35 (0,034)
(40)	0,009 (0,005)	0,993 (0,087)	-0,212 (0,085)	0,71	10,60 (0,564)	24,75 (0,016)
(41)	0,009 (0,005)	0,994 (0,087)	-0,212 (0,087)	0,71	10,85 (0,541)	24,28 (0,018)
(42)	0,009 (0,005)	1,014 (0,089)	-0,213 (0,090)	0,71	11,65 (0,474)	24,13 (0,020)

Table 2: Hedging results for a forward commitment to deliver one barrel of oil in ten years

	Hedging Strategies						
	Cost-of-Carry	Brennan/Schwartz	Schwartz (Model 1) $\gamma = 2.71$	Equilibrium	Gibson/Schwartz $\alpha = 5.62$	Schwartz/Smith $\gamma = 2.71$	No Hedge
Mean (in \$)	36.31	29.68	4.42	5.98	22.66	16.90	3.32
St. dev. (in \$)	25.53	18.94	1.92	2.50	11.81	28.06	3.28
Minimum (in \$)	-50.74	-37.00	-4.07	-3.05	-18.77	-92.16	-17.35
25% Quantile (in \$)	18.37	16.47	3.16	4.29	14.59	-1.79	1.55
50% Quantile (in \$)	34.91	28.97	4.45	5.87	22.41	17.00	3.72
75% Quantile (in \$)	52.87	42.08	5.71	7.59	30.53	35.91	5.58
Maximum (in \$)	169.64	120.10	11.91	16.11	77.73	152.58	12.42
Loss Probability	6.61%	5.06%	1.28%	0.53%	2.56%	27.00%	13.99%
Mean Loss (in \$)	-0.62	-0.35	-0.01	-0.01	-0.10	-4.76	-0.36

**Table 3: Hedging results for the Gibson/Schwartz and the Schwartz/Smith models for different values of the parameters  $\alpha$  and  $\gamma$ . The forward commitment is to deliver one barrel of oil in ten years**

	Hedge Results					
	Gibson/Schwartz; Schwartz/Smith					
	$\alpha = \gamma = 1.49$	$\alpha = \gamma = 2.71$	$\alpha = \gamma = 5.62$	$\alpha = \gamma = 9$	$\alpha = \gamma = 12$	$\alpha = \gamma = 15$
Mean (in \$)	8.16	16.90	22.66	24.68	25.50	25.97
St. dev. (in \$)	56.97	28.06	11.81	9.50	9.81	10.32
Minimum (in \$)	-246.93	-92.16	-18.77	-13.45	-13.54	-13.54
25% Quantile (in \$)	-29.36	-1.79	14.59	18.15	18.75	18.88
50% Quantile (in \$)	9.10	17.00	22.41	24.38	25.23	25.64
75% Quantile (in \$)	46.93	35.91	30.53	30.91	31.87	32.64
Maximum (in \$)	284.89	152.58	77.73	65.32	68.78	71.58
Loss Probability	43.45%	27.00%	2.56%	0.24%	0.25%	0.34%
Mean Loss (in \$)	-18.89	-4.76	-0.10	-0.01	-0.01	-0.01

Table 4: Hedging results for a forward commitment to deliver one barrel of oil in ten years.  
 The period of the Golf Crisis was excluded form the data set used to specify the data model

	Hedging Strategies						
	Cost-of-Carry	Brennan/Schwartz	Schwartz (Model 1) $\gamma = 2.71$	Equilibrium	Gibson/Schwartz $\alpha = 5.62$	Schwartz/Smith $\gamma = 2.71$	No Hedge
Mean (in \$)	16,53	13,21	4.35	5.09	14.96	16.35	3,92
St. dev. (in \$)	29,74	22,66	1.92	2.86	10.03	27.82	2,53
Minimum (in \$)	-84,49	-69,09	-3.61	-5.27	-23.38	-100.65	-9,42
25% Quantile (in \$)	-4,14	-2,38	3.07	3.15	8.19	-1.80	2,35
50% Quantile (in \$)	14.89	12,47	4.35	5.03	15.09	17.04	4,06
75% Quantile (in \$)	35,42	28,07	5.62	6.95	21.78	35.21	5,67
Maximum (in \$)	154,94	110,66	12.74	18.42	55.36	123.90	12,10
Loss Probability	29,92%	28,54%	1.25%	3.31%	6.92%	27.15%	6,52%
Mean Loss (in \$)	-5,06	-3,79	-0.01	-0.04	-0.31	-4.89	-0,10

Figure 1: Future prices of a contract with six months to maturity as functions of the current spot price  $S(t)$ . The results are based on the parameter values  $\Theta = \ln(20.5)$ ,  $\gamma = 2.5$ ,  $\sigma = 0.35$ ,  $K = 4$ ,  $r = 0.05$  and  $T - t = 6$  months

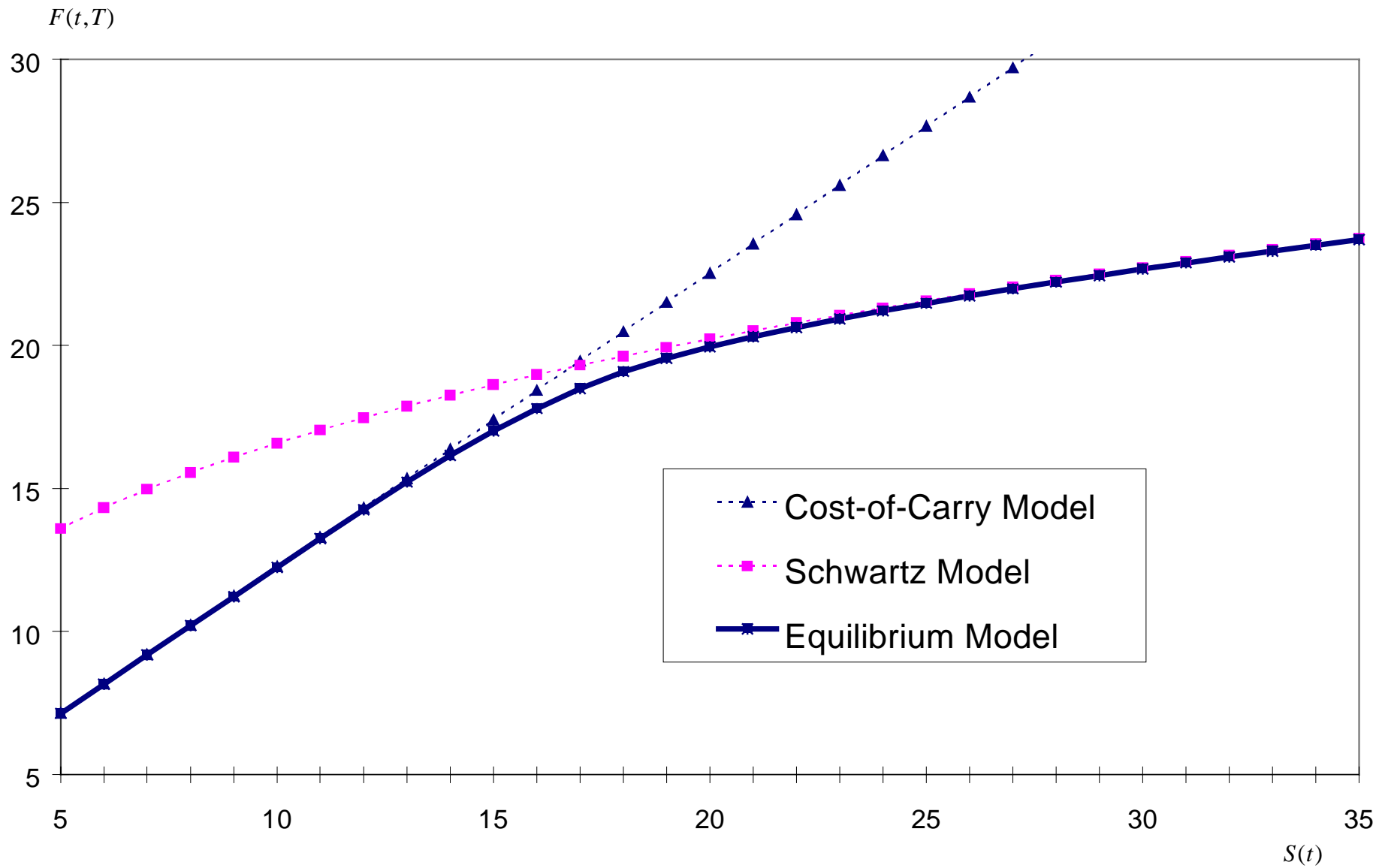


Figure 2: Different term structures of future prices arising from the model. The results are based on the parameter values  $\Theta = \ln(20.5)$ ,  $\gamma = 2.5$ ,  $\sigma = 0.35$ ,  $K = 4$  and  $r = 0.05$

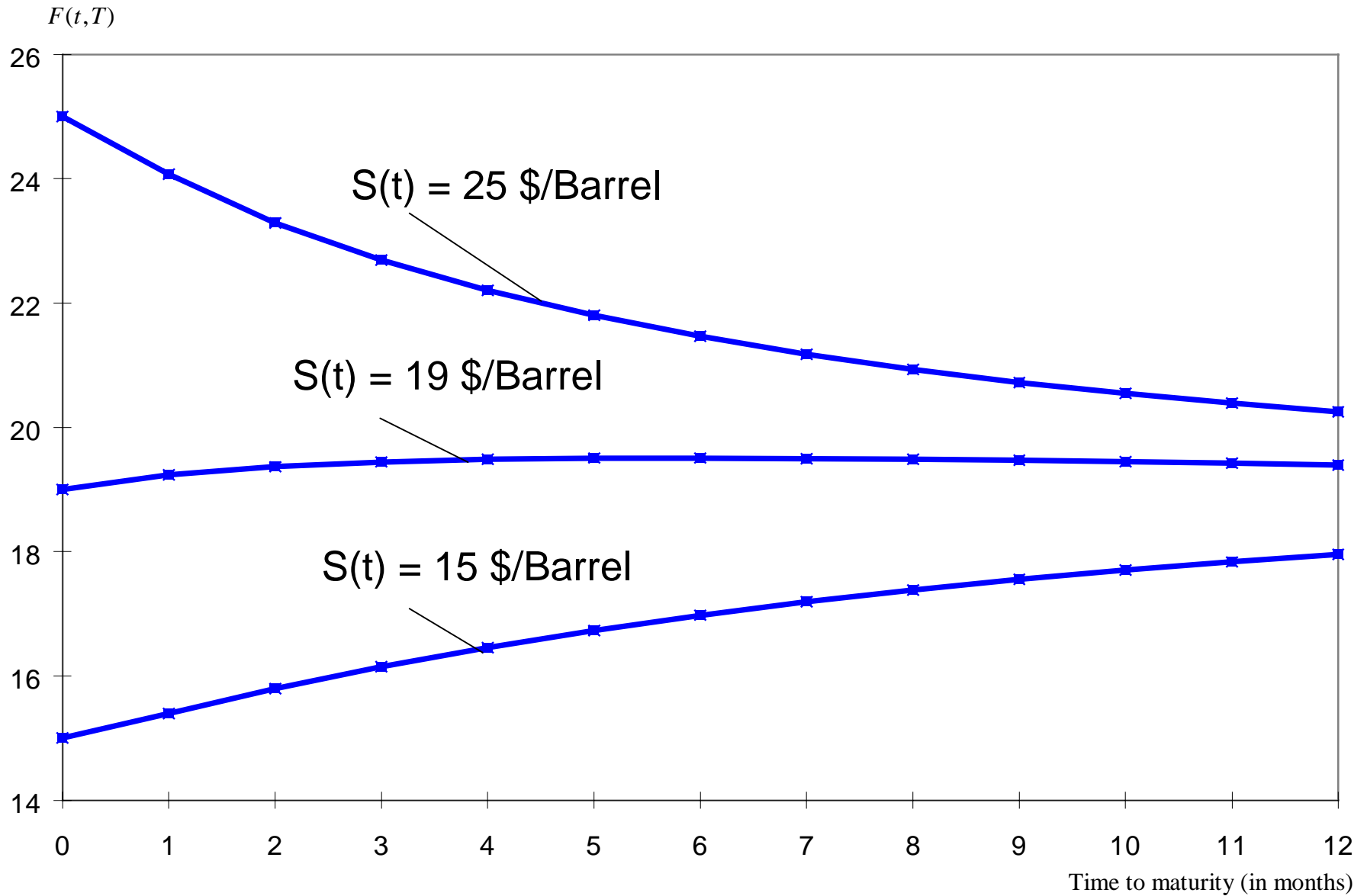


Figure 3: Numbers of one-month futures needed to hedge a six-month forward as a function of the current spot price  $S(t)$ . The results are based on the parameter values  $\Theta = \ln(20.5)$ ,  $\gamma = 2.5$ ,  $\sigma = 0.35$ ,  $K = 4$  and  $r = 0.05$

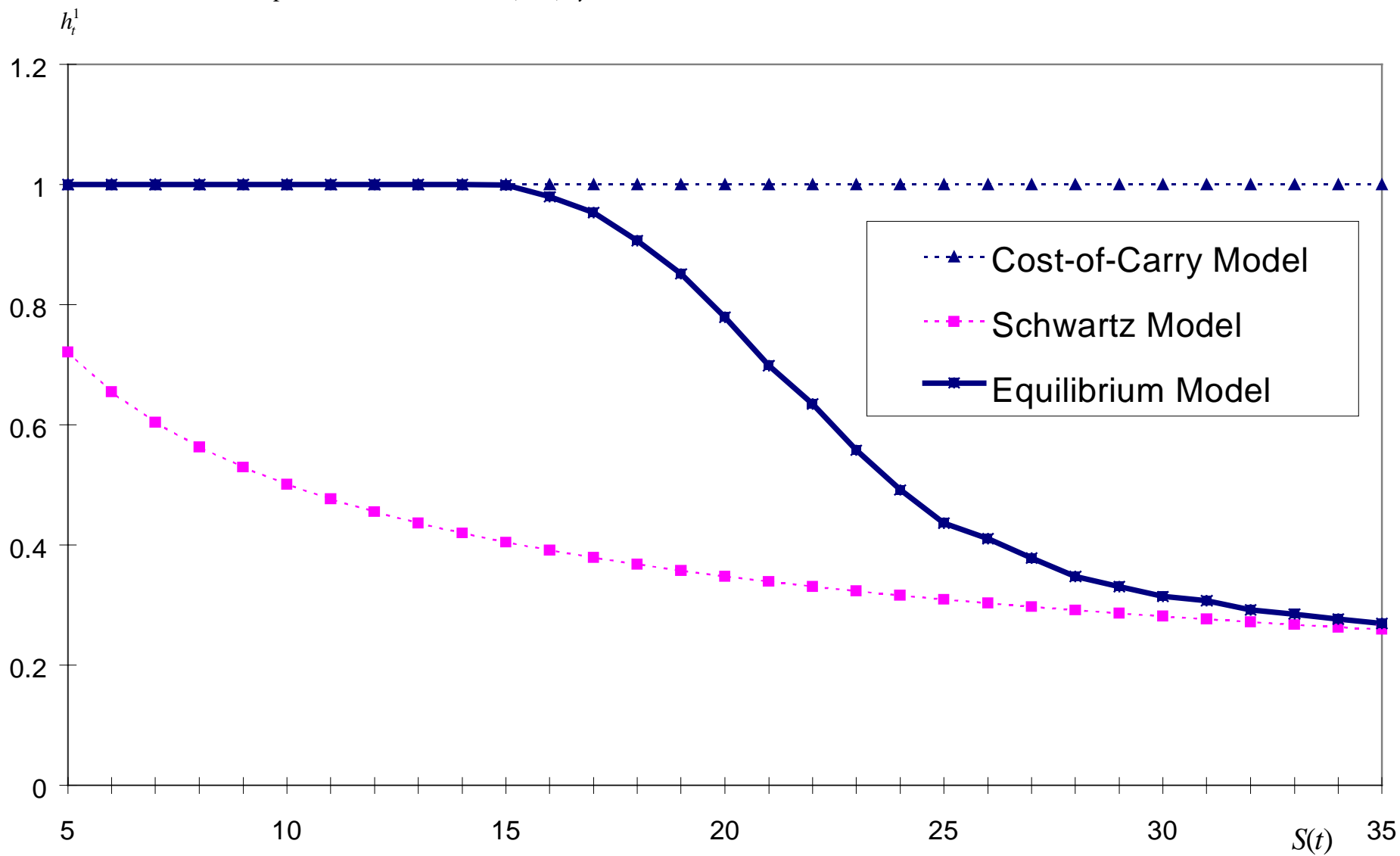




Figure 4: Structure of the empirical study

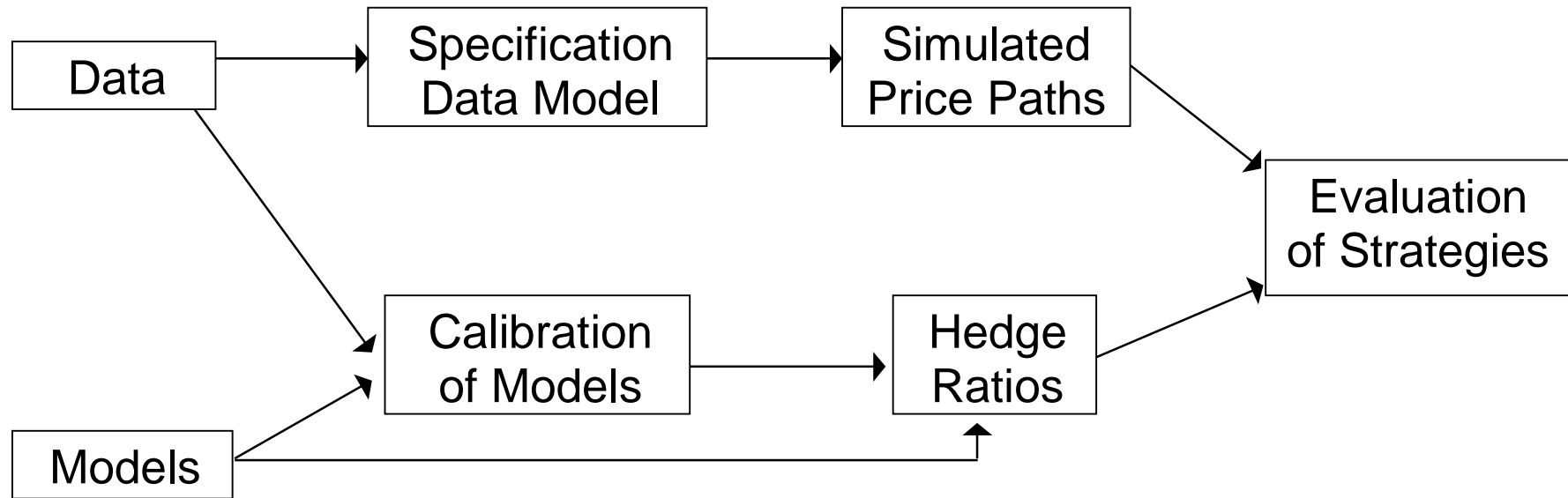


Figure 5: Price of West Texas Intermediate Crude Oil (WTI) and six-month basis (spot – six-month future)

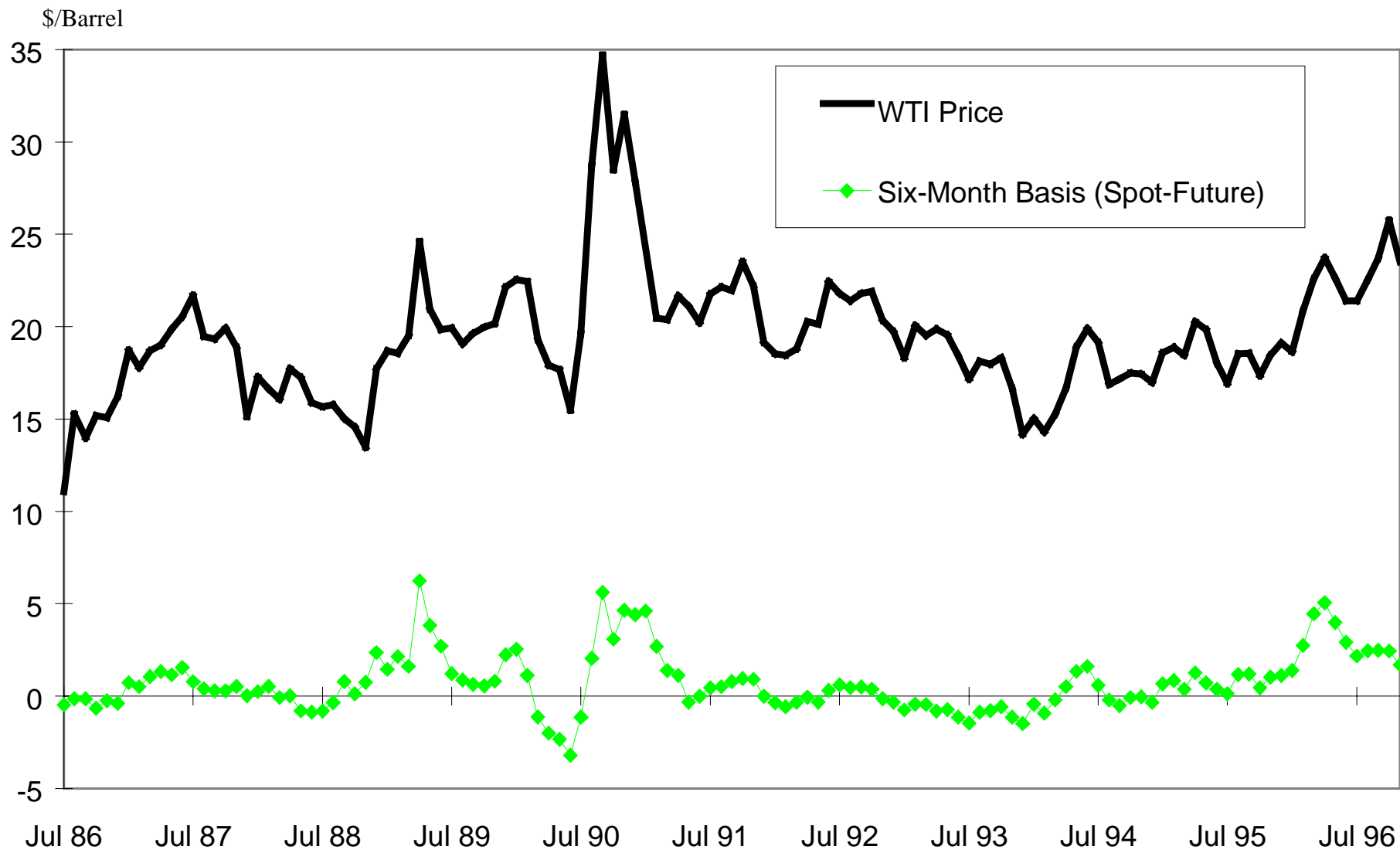


Figure 6: Hedge-ratios  $h_t^1$  for the one-factor models over time. The hedge horizon is ten years and parameters used are  $\gamma = 2.71$ ,  $\sigma = 0.36$ ,  $K = 4$  and  $r = 0.05$ . The mean level of the log oil price  $\Theta$  was chosen as a time dependent parameter to fit the initial term structure

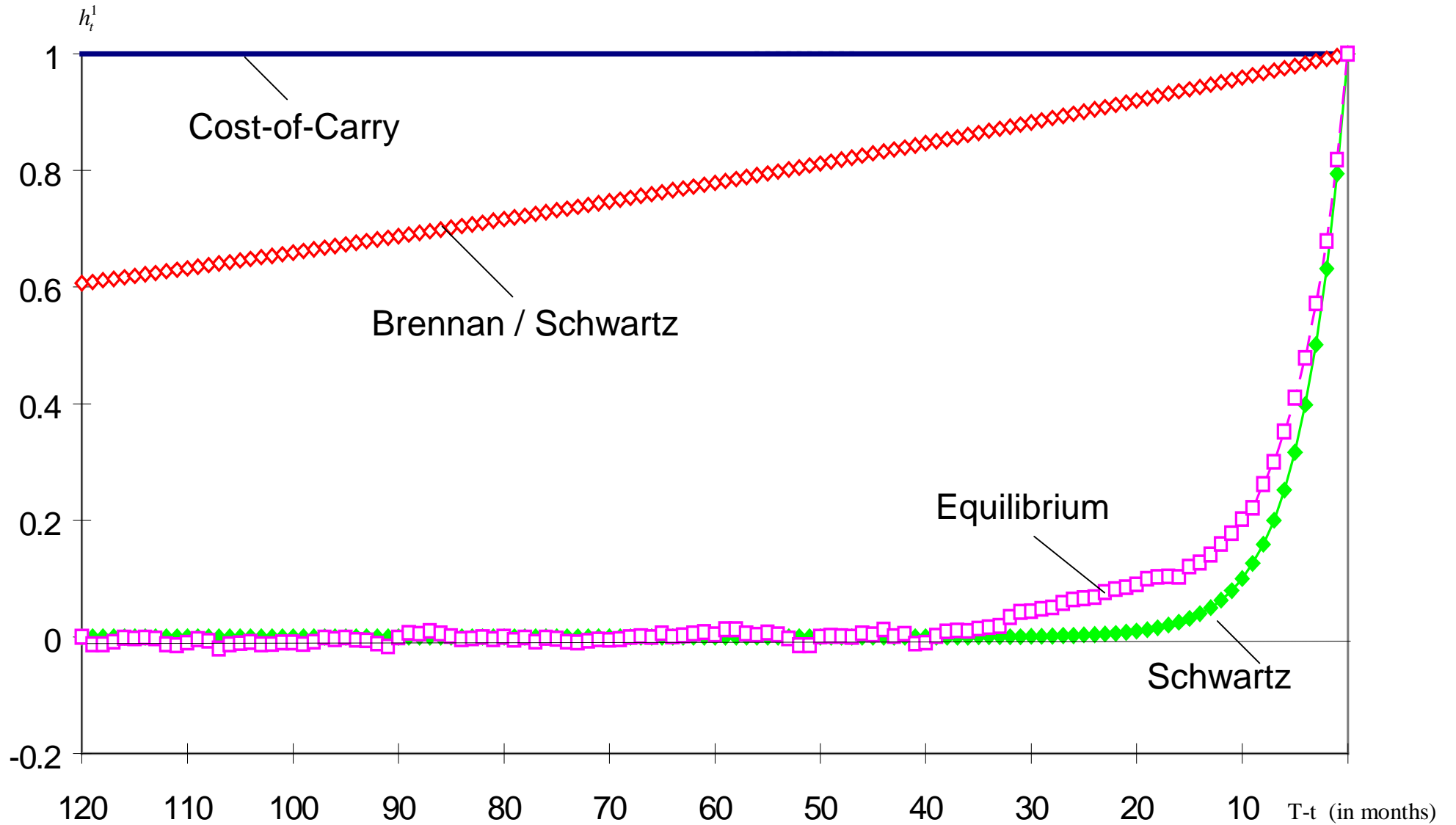


Figure 7: Hedge positions  $h_t^1$  and  $h_t^2$  for the two-factor models over time. The hedge horizon is ten years. For the Gibson/Schwartz model a parameter value of  $\alpha = 5.62$  is used, for the Schwartz/Smith model a value of  $\gamma = 2.71$

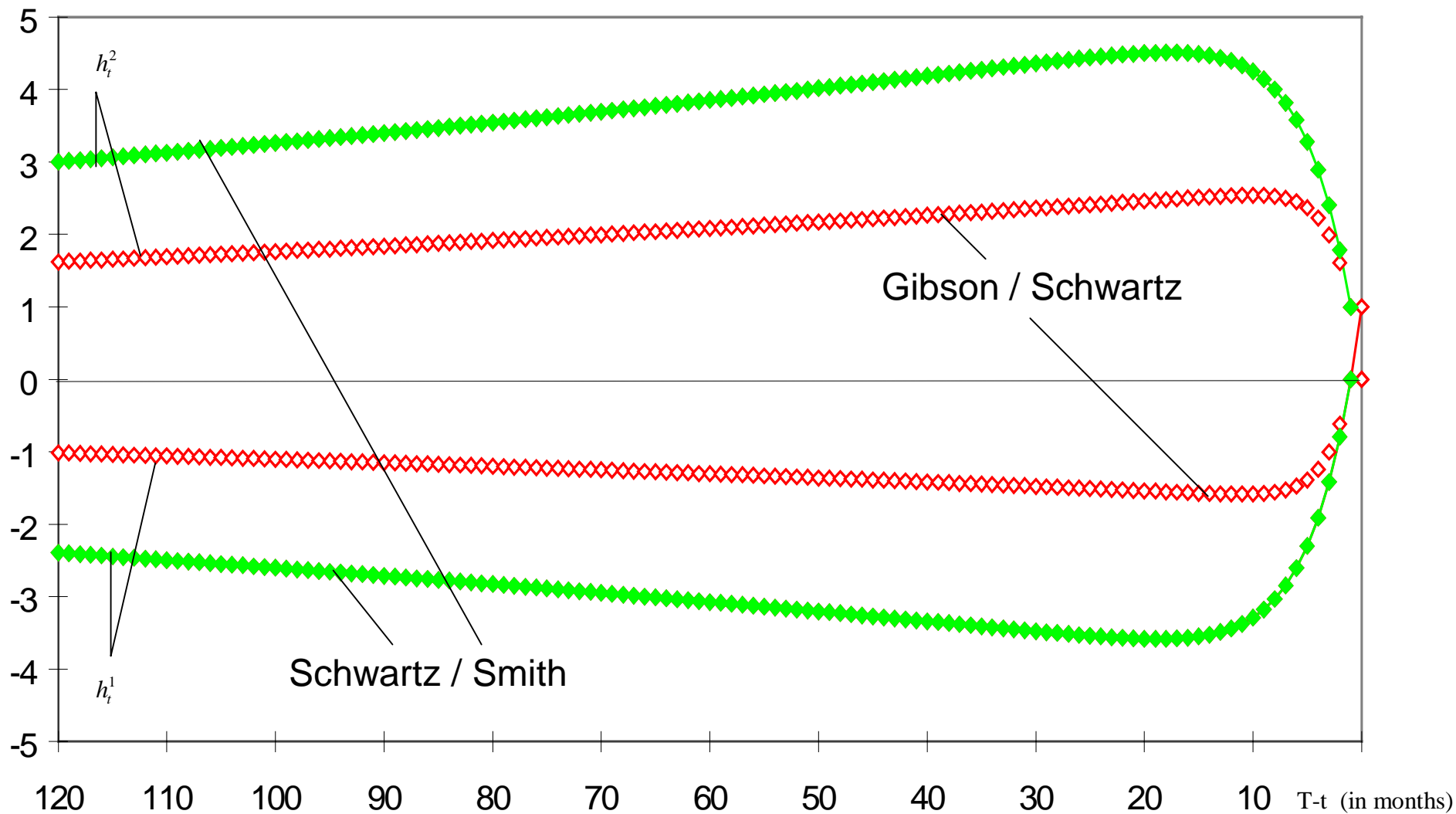


Figure 8: Cumulative distribution function of the hedge results at the end of the ten-year hedge horizon for the equilibrium model, the Schwartz model and an unhedged ten-year forward commitment to deliver one barrel of oil

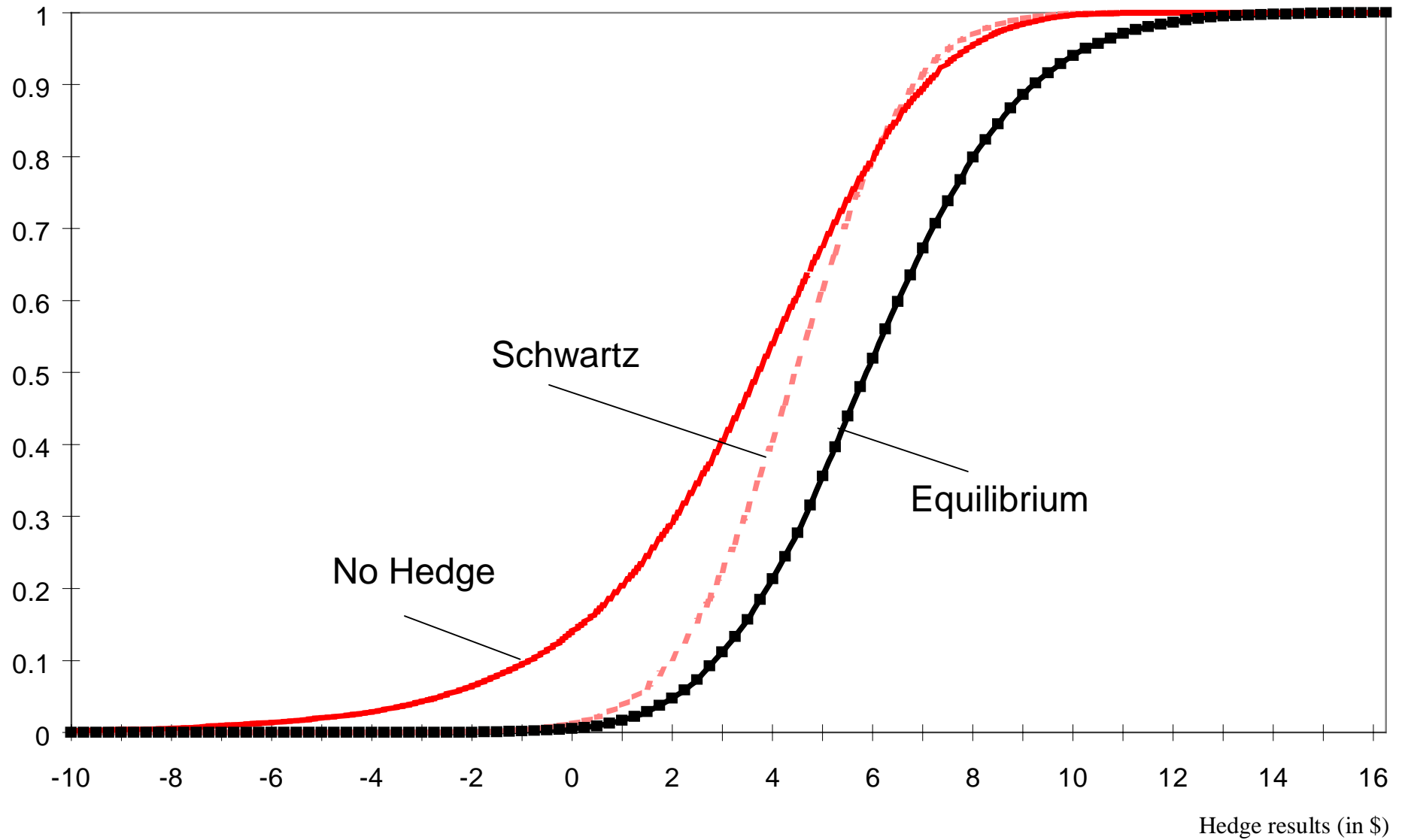


Figure 9: Cumulative distribution function of the hedge results at the end of the ten-year horizon, restricted to losses. Results are given for the equilibrium model, the cost-of-carry model, the Brennan/Schwartz model, the Gibson/Schwartz model and the Schwartz/Smith model

