Working Paper
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Memorandum, No. 2011,02

Provided in Cooperation with:
Department of Economics, University of Oslo

Suggested Citation: Framstad, Nils Chr. (2011) : Portfolio separation properties of the skew-elliptical distributions, Memorandum, No. 2011,02, University of Oslo, Department of Economics, Oslo

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MEMORANDUM

No 02/2011

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PORTFOLIO SEPARATION PROPERTIES OF THE SKEW-ELLIPtical DISTRIBUTIONS

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Version: February 1, 2011.

Abstract. The two fund separation property of the elliptical distributions is extended to the skew-elliptical and by adding a number of funds equalling the rank of the skewness matrix. Some elements of the generalization to singular extended skew-elliptical distributions are covered.

Key words and phrases: Portfolio separation, mutual fund theorem, stochastic dominance, singular extended skew-elliptical distributions.

MSC (2000): 91B28, 60E05, 49K45.
JEL classification: G11, C61, D81, D53.

0 Introduction

The concept of portfolio separation, a.k.a. the mutual fund theorem, should be well known. Since Tobin [12], numerous works have generalized the result in terms of the preferences which admit separation (like Cass and Stiglitz [2] or even as recently as Schachermayer et al. [11], using a modern approach), or in terms of distributions (Ross [10]). The concept of risk measures falls somewhat in between, see e.g. this author [6] and independently, De Giorgi et al. [7].

This note extends the results of Owen and Rabinovitch [9] and Chamberlain [3], who point out that the elliptical (also frequently referred to as «elliptically contoured») distributions admit two fund separation. It will turn out that a similar result holds for the skew-elliptical class (Branco and Dey [1] and Díaz-García and González-Farías [4]), at the expense of requiring an additional number of funds corresponding to the rank of the skewness matrix. The latter introduce the wider singular extended skew-elliptical (SESE) class, and one of these generalizations will be covered herein. We shall restrict ourselves to the single-period discrete time case. Using this author’s refinement [5] of the approach given by Khanna and Kulldorff [8], there will be a continuous-time analogue if the probability law is infinitely divisible (hence the discrete-time setup is more general in terms of probability distributions).

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1 The result

Consider a single period investment in a numéraire (enumerated with a zero) returning $Y_0$ per monetary unit invested, and another $p$ investment opportunities with returns vector $Y_0 \mathbf{1} + \mathbf{u} + \mathbf{Y}$, so that the return with investments $\mathbf{u}$ in the $p$ opportunities and $w - \mathbf{u}^\dagger \mathbf{1}$ (where $w$ is initial wealth) in the numéraire, will be

$$X = wY_0 + \mathbf{u}^\dagger (\mathbf{\mu} + \mathbf{Y}),$$

(1)

(where the $\dagger$ superscript denotes transposition). The market will be assumed free of arbitrage opportunities and of redundant investment opportunities (having removed the latter from the market).

The probability distribution of $\mathbf{\mu} + \mathbf{Y}$ will be considered conditional on $Y_0$ – therefore, we can (and will) without loss of generality assume $Y_0 = 0$ (or, for that matter, a risk-free return). $\mathbf{\mu}$ will be a location parameter, enabling us to assume location at zero in the representation to follow – note however, that we do not assume finite moments of any order.

Recall that an elliptical (a.k.a. elliptically contoured) random variable $Z$, has characteristic function of the form $e^{-i\mathbf{\theta}^\dagger \mathbf{\delta}} \psi(\mathbf{\theta}^\dagger \mathbf{M} \mathbf{\theta})$, where the matrix $\mathbf{M}$ is positive definite. The underlying spherical distribution (i.e. $\mathbf{M}^{-1/2}(\mathbf{Z} - \mathbf{\delta})$) can then be written as a mixture $\mathbf{RS}$ of a positive radial variable $R$, and $\mathbf{S}$ which is independent and uniform on the sphere. A singular elliptical distribution in the sense of [4], is obtained by relaxing the requirement to positive semidefinite $\mathbf{M}$. Therein, it is assumed that $R$ is absolutely continuous, but an approximation argument will allow for general $R$.

This paper does only to a limited extent use singular properties covered by [4], but will utilize their multivariate generalization of the case treated in [1]. Following their notation, one takes as starting point a singular elliptical vector $\mathbf{E} = (\mathbf{E}_1^\dagger, \mathbf{E}_2^\dagger)^\dagger$ located at $\mathbf{\delta} = \mathbf{0}$ and with associated matrix $\mathbf{M} = \left( \begin{array}{cc} \Sigma & 0 \\ 0 & \Delta \end{array} \right)$, and where the marginals $\mathbf{E}_1$ and $\mathbf{E}_2$ ($p$-vector and $q$-vector, respectively) have associated positive semidefinite matrices $\Sigma \in \mathbb{R}^{p \times p}$ and $\Delta \in \mathbb{R}^{q \times q}$ – observe that each $\mathbf{E}_i$ is allowed intra-dependent components. Now for arbitrary non-random $\mathbf{\mu} \in \mathbb{R}^p$, $\nu \in \mathbb{R}^q$, $\mathbf{D} \in \mathbb{R}^{q \times p}$, then

$$[\mathbf{\mu} + \mathbf{E}_1 | \mathbf{D} \mathbf{E}_1 + \mathbf{E}_2 - \nu \geq 0]$$

has the singular vector-variate skew-elliptical distribution. In [4], this is parametrized as $\text{SESE}^r_q(p, k_1, \mu, \Sigma, k, \mathbf{D}, \nu, \Delta, h^{(p)})$ where $r, k$ and $k_1$ are the ranks of $\Sigma$, $\Delta$ and $\Delta + \mathbf{D} \Sigma \mathbf{D}^\dagger$, respectively, and $h^{(p)}$ denotes the density generating function with respect to some appropriate Hausdorff measure (which is not unique – however, the results won’t depend on the choice). We remark that integrability assumptions are not needed, despite the literature’s common use of terms like e.g. covariance matrix.

We shall assume $\mathbf{\mu} + \mathbf{Y}$ to have such a distribution. Then $\mathbf{Y}$ belongs to the same class, except with location $\mathbf{\mu}$ replaced by null. In order to ensure absence of arbitrage and of redundant investment opportunities, we shall assume $\Sigma$ positive definite (so that in particular, $r = p$); the only «singular» property left then is a possible rank-deficiency of $\Delta$. We can adapt the following special case from [4, Theorem 5.1]:

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LEMMA. Suppose that $Y$ is absolutely continuous and distributed 

$$Y \sim \text{SESE}^{(p)}(q, k_1, 0, \Sigma, k, D, \nu, \Delta, h_p^{(p)}),$$

(2)

where $\Sigma$ is positive definite and $h_p^{(p)}$ is the density generating function with respect to $p$-dimensional Lebesgue measure. Then, for any non-random non-null $p$-vector $u$:

$$u^\dagger Y \sim \text{SESE}^{(1)}(q, k_1, 0, u^\dagger \Sigma u, \text{rank}(\Delta_u), D_u, \nu, \Delta_u, h)$$

(3)

where $h = h_1^{(1)}$ is a univariate density-generating function, and

$$D_u = \frac{1}{u^\dagger \Sigma u} D \Sigma u, \quad \Delta_u = \Delta + D \Sigma D_\dagger - D_u (u^\dagger \Sigma u) D_\dagger u.$$  

(4)

Recalling that non-absolutely continuous components in the underlying radial distribution can be recovered by approximation, we then have the following:

THEOREM. Assume the market (1) with the returns distributed according to (2), where $\Sigma$ is positive definite. Suppose the agents rank portfolios according to first-order stochastic dominance of the return. Then we have $2 + \text{rank}(D)$ fund separation. Furthermore, under the additional constraint of $u^\dagger 1 = w$ (i.e. the absence of opportunity to invest in the («safe») numéraire), we have $1 + \text{rank}(D^\dagger, 1^\dagger \Sigma^{-1})$ fund separation.

Proof. We observe from (4) that the distribution (3) depends on $u$ only through $\sqrt{u^\dagger \Sigma u} \in \mathbb{R}_+$ and $D \Sigma u \in \mathbb{R}^q$. For given values $Q > 0$ and $Qq \in \mathbb{R}^q$ of these, the agent will

$$\max_u \mu^\dagger u \quad \text{subject to} \quad u^\dagger \Sigma u = Q^2, \quad D \Sigma u = Qq$$

or equivalently, putting $v = \Sigma u$, $a = \mu \Sigma^{-1}$

$$\max_v a^\dagger v \quad \text{subject to} \quad v^\dagger \Sigma^{-1} v = Q^2, \quad Dv = Qq,$$

where for the case without safe investment opportunity, augment with the additional constraint $1^\dagger u = (1^\dagger \Sigma^{-1}) v = w$. Now the constraints $Dv = Qq$ form $\text{rank}(D)$ linear equations in $v$. Rewriting these constraints – including $1^\dagger \Sigma^{-1} v = w$ if appropriate – into $\tilde{D}v = \tilde{q}$ where $\tilde{D}$ has full rank, the proof is now a standard procedure: The associated Lagrangian becomes

$$a^\dagger v - \lambda^\dagger \tilde{D}v - \Lambda v^\dagger \Sigma^{-1} v,$$

which is stationary when $a - \lambda^\dagger \tilde{D} = 2\Lambda \Sigma^{-1} v = 2\Lambda u$. To complete the proof, we merely need to address degeneracies: First, if the constraint qualification fails (where the ellipsoid $v^\dagger \Sigma^{-1} v = Q^2$ is tangent to one of the hyperplanes), the solution is obtained as a limiting case, and spanned by the rows of $\tilde{D}$. Finally, the case $\Lambda = 0$ is only possible when $a$ is spanned by the rows of $\tilde{D}$, and the one fund saved this way will be replaced by an additional orthogonal vector in order to achieve the desired dispersion $Q^2$ (since no risk aversion is assumed).

Observe that the result reduces to three-fund separation for the setup of Branco and Dey [1] (who restrict their analysis to $D$ being a vector), and that by putting $D = 0$ we recover the Owen and Rabinovich [9] two-fund separation property as a corollary.
References


