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MEMORANDUM

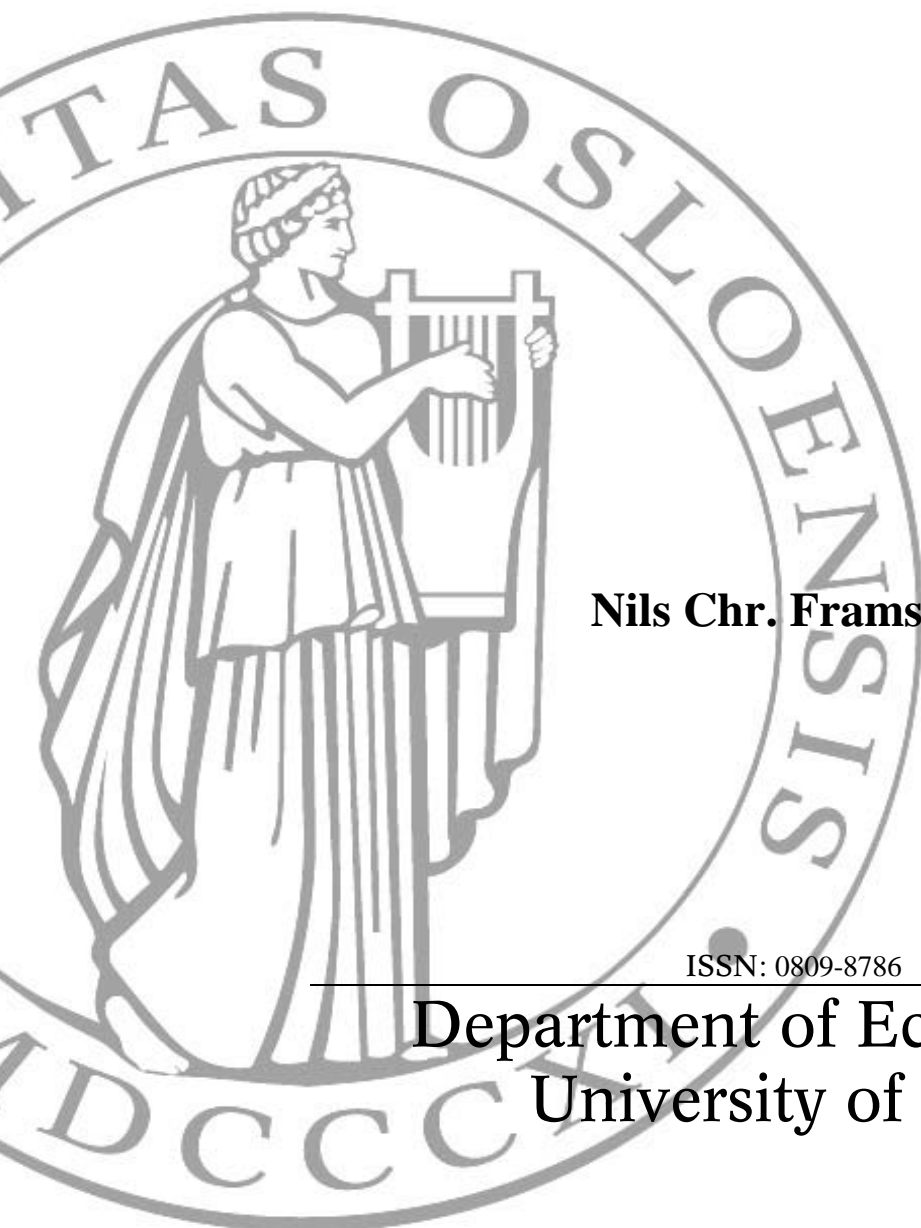
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PORTFOLIO SEPARATION PROPERTIES OF THE SKEW-ELLIPTICAL DISTRIBUTIONS

NILS CHR. FRAMSTAD^{*†}

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Abstract. The two fund separation property of the elliptical distributions is extended to the skew-elliptical and by adding a number of funds equalling the rank of the skewness matrix. Some elements of the generalization to *singular* extended skew-elliptical distributions are covered.

Key words and phrases: Portfolio separation, mutual fund theorem, stochastic dominance, singular extended skew-elliptical distributions.

MSC (2000): 91B28, 60E05, 49K45.

JEL classification: G11, C61, D81, D53.

0 Introduction

The concept of portfolio separation, a.k.a. the mutual fund theorem, should be well known. Since Tobin [12], numerous works have generalized the result in terms of the preferences which admit separation (like Cass and Stiglitz [2] or even as recently as Schachermayer et al. [11], using a modern approach), or in terms of distributions (Ross [10]). The concept of risk measures falls somewhat in between, see e.g. this author [6] and independently, De Giorgi et al. [7].

This note extends the results of Owen and Rabinovitch [9] and Chamberlain [3], who point out that the elliptical (also frequently referred to as «elliptically contoured») distributions admit two fund separation. It will turn out that a similar result holds for the skew-elliptical class (Branco and Dey [1] and Díaz-García and González-Farías [4]), at the expense of requiring an additional number of funds corresponding to the rank of the skewness matrix. The latter introduce the wider singular extended skew-elliptical (SESE) class, and one of these generalizations will be covered herein. We shall restrict ourselves to the single-period discrete time case. Using this author's refinement [5] of the approach given by Khanna and Kulldorff [8], there will be a continuous-time analogue if the probability law is infinitely divisible (hence the discrete-time setup is more general in terms of probability distributions).

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1 The result

Consider a single period investment in a numéraire (enumerated with a zero) returning Y_0 per monetary unit invested, and another p investment opportunities with returns vector $Y_0\mathbf{1} + \boldsymbol{\mu} + \mathbf{Y}$, so that the return with investments \mathbf{u} in the p opportunities and $w - \mathbf{u}^\dagger\mathbf{1}$ (where w is initial wealth) in the numéraire, will be

$$X = wY_0 + \mathbf{u}^\dagger(\boldsymbol{\mu} + \mathbf{Y}), \quad (1)$$

(where the « \dagger » superscript denotes transposition). The market will be assumed free of arbitrage opportunities and of redundant investment opportunities (having removed the latter from the market).

The probability distribution of $\boldsymbol{\mu} + \mathbf{Y}$ will be considered conditional on Y_0 – therefore, we can (and will) without loss of generality assume $Y_0 = 0$ (or, for that matter, a risk-free return). $\boldsymbol{\mu}$ will be a location parameter, enabling us to assume location at zero in the representation to follow – note however, that we do not assume finite moments of any order.

Recall that an elliptical (a.k.a. elliptically contoured) random variable \mathbf{Z} , has characteristic function of the form $e^{-i\boldsymbol{\theta}^\dagger\boldsymbol{\delta}}\psi(\boldsymbol{\theta}^\dagger\mathbf{M}\boldsymbol{\theta})$, where the matrix \mathbf{M} is positive definite. The underlying spherical distribution (i.e. $\mathbf{M}^{-1/2}(\mathbf{Z} - \boldsymbol{\delta})$) can then be written as a mixture $R\mathbf{S}$ of a positive radial variable R , and \mathbf{S} which is independent and uniform on the sphere. A *singular elliptical distribution* in the sense of [4], is obtained by relaxing the requirement to positive semidefinite \mathbf{M} . Therein, it is assumed that R is absolutely continuous, but an approximation argument will allow for general R .

This paper does only to a limited extent use singular properties covered by [4], but will utilize their multivariate generalization of the case treated in [1]. Following their notation, one takes as starting point a singular elliptical vector $\mathbf{E} = (\mathbf{E}_1^\dagger, \mathbf{E}_2^\dagger)^\dagger$ located at $\boldsymbol{\delta} = \mathbf{0}$ and with associated matrix $\mathbf{M} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta} \end{pmatrix}$, and where the marginals \mathbf{E}_1 and \mathbf{E}_2 (p -vector and q -vector, respectively) have associated positive semidefinite matrices $\boldsymbol{\Sigma} \in \mathbf{R}^{p \times p}$ and $\boldsymbol{\Delta} \in \mathbf{R}^{q \times q}$ – observe that each \mathbf{E}_i is allowed intra-dependent components. Now for arbitrary non-random $\boldsymbol{\mu} \in \mathbf{R}^p$, $\boldsymbol{\nu} \in \mathbf{R}^q$, $\mathbf{D} \in \mathbf{R}^{q \times p}$, then

$$[\boldsymbol{\mu} + \mathbf{E}_1 \mid \mathbf{D}\mathbf{E}_1 + \mathbf{E}_2 - \boldsymbol{\nu} \geq \mathbf{0}] \quad (\text{component-wise inequality, i.e. positive orthant})$$

has the *singular vector-variate skew-elliptical distribution*. In [4], this is parametrized as $\text{SESE}_r^{(p)}(q, k_1, \boldsymbol{\mu}, \boldsymbol{\Sigma}, k, \mathbf{D}, \boldsymbol{\nu}, \boldsymbol{\Delta}, h_r^{(p)})$ where r , k and k_1 are the ranks of $\boldsymbol{\Sigma}$, $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta} + \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\dagger$, respectively, and $h_r^{(p)}$ denotes the density generating function with respect to some appropriate Hausdorff measure (which is not unique – however, the results won't depend on the choice). We remark that integrability assumptions are not needed, despite the literature's common use of terms like e.g. covariance matrix.

We shall assume $\boldsymbol{\mu} + \mathbf{Y}$ to have such a distribution. Then \mathbf{Y} belongs to the same class, except with location $\boldsymbol{\mu}$ replaced by null. In order to ensure absence of arbitrage and of redundant investment opportunities, we shall assume $\boldsymbol{\Sigma}$ positive definite (so that in particular, $r = p$); the only «singular» property left then is a possible rank-deficiency of $\boldsymbol{\Delta}$. We can adapt the following special case from [4, Theorem 5.1]:

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LEMMA. Suppose that \mathbf{Y} is absolutely continuous and distributed

$$\mathbf{Y} \sim \text{SESE}_p^{(p)}(q, k_1, \mathbf{0}, \boldsymbol{\Sigma}, k, \mathbf{D}, \boldsymbol{\nu}, \boldsymbol{\Delta}, h_p^{(p)}), \quad (2)$$

where $\boldsymbol{\Sigma}$ is positive definite and $h_p^{(p)}$ is the density generating function with respect to p -dimensional Lebesgue measure. Then, for any non-random non-null p -vector \mathbf{u} :

$$\mathbf{u}^\dagger \mathbf{Y} \sim \text{SESE}_1^{(1)}(q, k_1, \mathbf{0}, \mathbf{u}^\dagger \boldsymbol{\Sigma} \mathbf{u}, \text{rank}(\boldsymbol{\Delta}_\mathbf{u}), \mathbf{D}_\mathbf{u}, \boldsymbol{\nu}, \boldsymbol{\Delta}_\mathbf{u}, h) \quad (3)$$

where $h = h_1^{(1)}$ is a univariate density-generating function, and

$$\mathbf{D}_\mathbf{u} = \frac{1}{\mathbf{u}^\dagger \boldsymbol{\Sigma} \mathbf{u}} \mathbf{D} \boldsymbol{\Sigma} \mathbf{u}, \quad \boldsymbol{\Delta}_\mathbf{u} = \boldsymbol{\Delta} + \mathbf{D} \boldsymbol{\Sigma} \mathbf{D}^\dagger - \mathbf{D}_\mathbf{u} (\mathbf{u}^\dagger \boldsymbol{\Sigma} \mathbf{u}) \mathbf{D}_\mathbf{u}^\dagger. \quad (4)$$

Recalling that non-absolutely continuous components in the underlying radial distribution can be recovered by approximation, we then have the following:

THEOREM. Assume the market (1) with the returns distributed according to (2), where $\boldsymbol{\Sigma}$ is positive definite. Suppose the agents rank portfolios according to first-order stochastic dominance of the return. Then we have $2 + \text{rank}(\mathbf{D})$ fund separation. Furthermore, under the additional constraint of $\mathbf{u}^\dagger \mathbf{1} = w$ (i.e. the absence of opportunity to invest in the («safe» numéraire), we have $1 + \text{rank}(\mathbf{D}^\dagger, \mathbf{1}^\dagger \boldsymbol{\Sigma}^{-1})$ fund separation.

Proof. We observe from (4) that the distribution (3) depends on \mathbf{u} only through $\sqrt{\mathbf{u}^\dagger \boldsymbol{\Sigma} \mathbf{u}} \in \mathbf{R}_+$ and $\mathbf{D} \boldsymbol{\Sigma} \mathbf{u} \in \mathbf{R}^q$. For given values $Q > 0$ and $Q\mathbf{q} \in \mathbf{R}^q$ of these, the agent will

$$\max_{\mathbf{u}} \boldsymbol{\mu}^\dagger \mathbf{u} \quad \text{subject to} \quad \mathbf{u}^\dagger \boldsymbol{\Sigma} \mathbf{u} = Q^2, \quad \mathbf{D} \boldsymbol{\Sigma} \mathbf{u} = Q\mathbf{q}$$

or equivalently, putting $\mathbf{v} = \boldsymbol{\Sigma} \mathbf{u}$, $\mathbf{a} = \boldsymbol{\mu} \boldsymbol{\Sigma}^{-1}$

$$\max_{\mathbf{v}} \mathbf{a}^\dagger \mathbf{v} \quad \text{subject to} \quad \mathbf{v}^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{v} = Q^2, \quad \mathbf{D} \mathbf{v} = Q\mathbf{q},$$

where for the case without safe investment opportunity, augment with the additional constraint $\mathbf{1}^\dagger \mathbf{u} = (\mathbf{1}^\dagger \boldsymbol{\Sigma}^{-1}) \mathbf{v} = w$. Now the constraints $\mathbf{D} \mathbf{v} = Q\mathbf{q}$ form $\text{rank}(\mathbf{D})$ linear equations in \mathbf{v} . Rewriting these constraints – including $\mathbf{1}^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{v} = w$ if appropriate – into $\check{\mathbf{D}} \mathbf{v} = \check{\mathbf{q}}$ where $\check{\mathbf{D}}$ has full rank, the proof is now a standard procedure: The associated Lagrangian becomes

$$\mathbf{a}^\dagger \mathbf{v} - \boldsymbol{\lambda}^\dagger \check{\mathbf{D}} \mathbf{v} - \Lambda \mathbf{v}^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{v},$$

which is stationary when $\mathbf{a} - \boldsymbol{\lambda}^\dagger \check{\mathbf{D}} = 2\Lambda \boldsymbol{\Sigma}^{-1} \mathbf{v} = 2\Lambda \mathbf{u}$. To complete the proof, we merely need to address degeneracies: First, if the constraint qualification fails (where the ellipsoid $\mathbf{v}^\dagger \boldsymbol{\Sigma}^{-1} \mathbf{v} = Q^2$ is tangent to one of the hyperplanes), the solution is obtained as a limiting case, and spanned by the rows of $\check{\mathbf{D}}$. Finally, the case $\Lambda = 0$ is only possible when \mathbf{a} is spanned by the rows of $\check{\mathbf{D}}$, and the one fund saved this way will be replaced by an additional orthogonal vector in order to achieve the desired dispersion Q^2 (since no risk aversion is assumed). \square

Observe that the result reduces to three-fund separation for the setup of Branco and Dey [1] (who restrict their analysis to \mathbf{D} being a vector), and that by putting $\mathbf{D} = \mathbf{0}$ we recover the Owen and Rabinovich [9] two-fund separation property as a corollary.

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