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## Optimal resource extraction under stochastic terms of trade

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**Optimal Resource Extraction  
under Stochastic Terms of Trade**

by

Axel Behrens

September 1989

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# Optimal Resource Extraction under Stochastic

## Terms of Trade

This paper examines the optimal extraction of an exhaustible resource owned by a small open economy, when the terms of trade faced by this country follows an exogenous given time path that is subject to stochastic fluctuations. First, we will see that if the functions involved in the first order conditions are nonlinear, the optimal extraction path is changed due to uncertainty. A "certainty-equivalence" solution would not bring out this result. Second, the ability of this country to withhold production in times when it's not profitable to deplete the resource (due to stochastic fluctuations of the terms of trade), provides an incentive to slow down the rate of production under uncertainty.

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## 1. Introduction

This paper<sup>1</sup> examines the case where a small open economy owns a stock of exhaustible resources like Chromium, Copper etc. which are not consumed domestically but exported. The resource earnings are used to import consumption goods. We will discuss the optimal production of this resource, when the terms of trade (faced by the country) follows an exogenous given time path that is subject to some stochastic fluctuations. It will come about that uncertainty concerning future values of the terms of trade affects the extraction dynamics for the following reasons. First, if the functions involved in the first-order conditions are nonlinear, the expected future value of these functions differs from a world of certainty. Consequently uncertainty has an impact on the behaviour of resource-exporting-countries. Note that this change in behaviour cannot be captured with "certainty-equivalence"<sup>2</sup>. Second, if the costs of extraction exceed the gain from the resource use for some time due to stochastic fluctuations, the country can keep the resource in the ground but maintain the option of future extraction when depletion will become profitable again. So uncertainty creates an incentive to slow down production.

Note that we will neither be concerned with the determination of the terms of trade, nor with the reason for the stochastic fluctuation. We will not consider market equilibrium, since we make the small country assumption.

This paper follows the tradition of "cake-eating-problem" papers<sup>3</sup>. *Kemp (1976)*, *Loury (1978)*, *Gilbert (1979)*, *Deshmukh/Pliska (1980)*, *Arrow/Chang (1982)* and *Hartwick (1983)* examine resource exploration and exploitation when the level of resources or the success of exploration are unknown. *Long (1975)* deals with the case of resource extraction under the possibility of expropriation. Aspects of uncertainty in the demand for natural resources are discussed e.g. in *Lewis (1977)*, *Pindyck (1981)*, *Sengupta (1986)* and *Anderson/Anderson*

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<sup>2</sup> This is the case, if the random variables in a stochastic control system are replaced by their first moments and then the system is solved with methods of deterministic control theory.

<sup>3</sup> Cf. *Kemp (1976)*, "How to eat a cake of unknown size".

(1987). *Dasgupta/Heal (1974)*, *Hoel (1978)* and *Dasgupta/Stiglitz (1981a,b)* describe the effects of uncertainty about a possible development of a substitute for the resource. *Pindyck (1980)* and *Deshmukh/Pliska (1983)* present more general models that contain several types of uncertainty.

In the next section we present the basic assumptions for this problem. For simplicity the time index for the variables is omitted when there is no ambiguity. In section 3 we describe the optimal resource extraction under uncertainty. In comparison with the deterministic case the effects of incomplete information about the dynamics of the terms of trade on the optimal depletion path of the resource are worked out.

## 2. The Basic Assumptions

The country's initial resource stock is  $R_0$ . The change in the stock  $R$  is given by  $dR(t) = -q(t) dt$ , where  $q$  is the extraction rate. The total costs of the extraction  $TC$  depend on the extraction rate  $q(t)$  and the current resource level  $R(t)$ , so that<sup>4</sup>

$$(2.0) \quad TC = TC(q, R) \text{ with } TC_q > 0, TC_{qq} > 0, TC_{qqq} > 0, \text{ and } TC_R < 0.$$

Such a specification of the cost function might be reasonable in the case of resource depletion, because it gets more difficult to deplete additional resource units, and if a country owns a large stock of resources the effort to deplete resources might be smaller than in the case of a small stock.

In each period the extracted resources are sold on the world market at price  $p^f$ . There is no possibility of storage. The resource earnings are used to import consumption goods  $D$  at price  $p^D$ . Since we make the small-country assumption, the prices cannot be influenced by the home country, i.e. the terms of trade are exogenous. It is assumed that the country's budget constraint must hold in each period. That assumption is made to create a frame of reference and to distinguish this case from the one of a resource producing firm, as discus-

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<sup>4</sup> A subscript denotes the partial derivative of a function.

sed e.g. in *Pindyck (1981)*, because otherwise the "Fisher-Separation-Theorem" would make these two problems similar.

$$(2.1) \quad p^f q - TC(q, R) - p^D D = 0.$$

Dividing (2.1) by  $p^D$  and defining the terms of trade as  $p := p^f / p^D$  it follows that

$$(2.2) \quad D = pq - C(q, R),$$

where the total costs of extraction  $C(q, R)$  are now defined in units of the imported good  $D$ . The country is assumed to have a social welfare function  $W$  with the usual properties. The only source of welfare is the consumption of the imported good.

$$(2.3) \quad W = W(D),$$

From (2.2) and (2.3) we can write:

$$(2.4) \quad W = W\{D(p, q, R, t)\}.$$

The country is assumed to maximize its discounted social welfare function over time  $(0, T)$ . Future welfare is being discounted by the country's given time preference rate  $\delta$ .  $q(t)$  is the control variable and  $p(t)$  and  $R(t)$  are the state variables.

### 3. Extraction Dynamics under Uncertainty

To introduce uncertainty over future values of the terms of trade we assume that:

$$(3.1) \quad dp = bdt + \sigma dZ,$$

where  $Z$  stands for a Wiener process<sup>5</sup>. So  $dp$  is normally distributed with  $E(dp) = bdt$  and  $var(dp) = \sigma^2 dt$ . That implies that current  $p$  is known exactly, but the uncertainty about future terms of trade grows with time. Under uncertainty, the country will maximize its expected discounted social welfare over time  $(0, T)$ . The stochastic dynamic optimization problem is, with  $J(p, R, t)$  as the optimal value function:

$$(3.2) \quad J(\cdot) = \max_{\{q\}} E \left\{ \int_0^T W \{ D(p, q, R, s) \} e^{-\delta s} ds \right\}$$

$$(3.3) \quad \text{s.t.} \quad dp = bdt + \sigma dZ$$

$$(3.4) \quad dR = -qdt$$

$$(3.5) \quad \begin{array}{l} p(0) = p_0 \\ R(0) = R_0 \\ p, R, q > 0 \end{array} \quad \Bigg|$$

Expanding Equation (3.2) using Itô's lemma, taking the expectation, dividing by  $dt$  and letting  $dt \rightarrow 0$  gives the partial differential equation<sup>6</sup>:

$$(3.6) \Rightarrow \begin{aligned} 0 &= \max_{\{q\}} \{ e^{-\delta t} W \{ D(p, q, R, t) \} + (1/dt) EdJ \} \\ 0 &= J_t + \max_{\{q\}} \{ e^{-\delta t} W \{ D(p, q, R, t) \} \\ &\quad + bJ_p - qJ_R + 0.5J_{pp}\sigma^2 \} \end{aligned}$$

<sup>5</sup> Note that Eqn. (3.1) is the limiting form of the discrete time difference equation  $p(t+h) - p(t) = bh + \sigma e(t) \sqrt{h}$  as  $h \rightarrow 0$ , where  $e(t)$  is a serially uncorrelated normal random variable with zero mean and unit variance.

<sup>6</sup> Cf. Mangel (1985).



This is the desired "dynamic programming equation". Assume that the marginal value of a terms of trade improvement is decreasing ( $J_{pp} < 0$ ), uncertainty then has a negative impact on the optimal value function. From (3.6) we get the first order condition<sup>7</sup>

$$(3.7) \quad e^{-\delta t} W_q = e^{-\delta t} W'(D) \cdot (p - C_q) = J_R.$$

Eqn. (3.7) is the well-known result. The discounted marginal utility  $e^{-\delta t} W_q$  that could be obtained by selling an additional unit of the resource should equal the shadow price of the resource  $J_R$  at each moment<sup>8</sup>.

To determine the expected depletion path  $(1/dt) Edq$  we again make use of Itô's lemma and see that the expected extraction dynamics satisfy<sup>9</sup>:

$$(3.8) \quad \frac{1}{dt} E dq = \frac{1}{W_{qq}} \{ -W_R + \delta W_q + qW_{qR} - bW_{qp} - F(\bullet) \}$$

$$\text{with} \quad F(\bullet) = \sigma^2 (0.5 q_p^2 W_{qqq} + 0.5 W_{qpp} + q_p W_{qpq})$$

#### 4. The Effects of Uncertainty

Equation (3.8) deviates from the deterministic case by the factor  $F(\bullet)$ . Supposing that  $q_p > 0$ , we can state the following:

If  $W_q$  is a nonlinear function of  $p$  and  $q$ , uncertainty will change the expected extraction dynamics. If  $W_q$  is a convex (concave) function of  $p$  and  $q$ , i.e.  $W_{qqq}$ ,  $W_{qpp}$ ,  $W_{qpq} > 0$  ( $< 0$ ) extraction will be less rapid (more rapid) than it was in the case of certainty. And also in the case of a "certainty-equivalence" solution. It is not obvious, without looking

<sup>7</sup> It can be seen that the second-order conditions are satisfied and, therefore, the first-order conditions are sufficient.

<sup>8</sup> Note that if  $C_q$  exceeds  $p$  for some time due to the stochastic fluctuations the country keeps the resource in the ground.

<sup>9</sup> See Appendix.

at the specification of  $W(D)$  and  $C(q,R)$ , whether  $W_{qqq}$ ,  $W_{qpp}$ ,  $W_{qpq}$  are positive or negative. With  $D = pq - C(q,R)$  and  $W = W(D)$  it follows that

$$(a) \quad W_{qqq} = W'''(D) \cdot (p - C_q)^3 + W''(D) \cdot 3(p - C_q) \cdot (-C_{qq}) + W'(D) \cdot (-C_{qqq})$$

$$(b) \quad W_{qpq} = W'''(D) \cdot q \cdot (p - C_q)^2 + W''(D) \cdot (2p - 2C_q - qC_{qq})$$

$$(c) \quad W_{qpp} = W'''(D) \cdot q^2 \cdot (p - C_q) + 2W''(D) \cdot q$$

We are not able to determine the direction in which the (expected) optimal depletion path is changed due to uncertainty. Several effects can be distinguished:

- (i)  $W''' > 0$  means that expected marginal welfare for the future will be higher than it would be in the case of certainty. This leads to more conservative (less rapid) extraction.
- (ii)  $W'' < 0$  reflects diminishing marginal welfare, which works against the first effect.
- (iii) If  $C_{qqq} > 0$ , the expected marginal costs will be higher in the future. This implies an incentive to speed up production.
- (iv) Another point should be stressed. If  $C_q$  exceeds  $p$  for some time due to stochastic fluctuations Eqn. (3.8) no longer holds. The country will keep the resource in the ground and maintain the option of extracting it at some future time. This means that uncertainty concerning future values of the terms of trade creates an incentive to slow down production even if  $F$  equals zero.

The net effect is ambiguous.

## 5. Concluding Comments

What makes our analysis different from others is the fact that here third derivatives play an important role. That comes about, because if the functions involved in the first order conditions are nonlinear, it has an effect on the expected value of these functions, since the expectations operator  $E$  is a linear operator. This technical point, however, has very plausible economic reasons. It is clear that if the expected marginal gains or the expected

marginal costs from an action are changed due to uncertainty that will change optimal behaviour.

### Appendix

To determine the expected depletion path  $(1/dt) Edq$  we differentiate (3.6) with respect to  $R$ .

$$(A1) \quad e^{-\delta t} W_R + (1/dt) EdJ_R = 0.$$

"Itô's differential generator" is applied to both sides of (3.7)

$$(A2) \quad (1/dt) Ed(e^{-\delta t} W_q) = (1/dt) EdJ_R.$$

From (A1) and (A2) we get:

$$(A3) \quad (1/dt) Ed(e^{-\delta t} W_q) = -e^{-\delta t} W_R$$

$$(A4) \Rightarrow (-\delta) e^{-\delta t} W_q + e^{-\delta t} (1/dt) EdW_q = -e^{-\delta t} W_R$$

$$(A5) \Rightarrow (1/dt) EdW_q = \delta W_q - W_R$$

Now we need  $EdW_q$ . A Taylor expansion gives<sup>10</sup>:

$$(A6) \quad \begin{aligned} EdW_q = E \{ & W_{qq} dq + W_{qp} dp + W_{qR} dR + W_{qt} dt \\ & + 0.5 W_{qqq} (dq^2) + 0.5 W_{qpp} (dp^2) \\ & + 0.5 W_{qRR} (dR^2) + W_{qpq} dpdq + W_{qpr} dpdR \\ & + W_{qqR} dqdR + o(dt) \} \end{aligned}$$

<sup>10</sup> Where  $o(dt)$  is defined as  $\lim o(dt)/dt = 0$  as  $t \rightarrow 0$ .

Taking the expectation, dividing by  $dt$  and letting  $dt \rightarrow 0$  we get

$$(A7) \quad (1/dt) EdW_q = W_{qq} Edq + bW_{qp} - qW_{qR} + 0.5 q_p^2 \sigma^2 W_{qqq} \\ + 0.5 \sigma^2 W_{qpp} + q_p \sigma^2 W_{qpq}$$

Substituting (A7) into (A5), we get (3.8).

## References

- Anderson, P., T.M. Anderson (1987), The Extraction of Exhaustible Resources under Demand Uncertainty, *Resources and Energy* 9, 223-231.
- Arrow, K.J., S. Chang (1982), Optimal Pricing, Use, and Exploration of Uncertain Natural Resource Stocks, *Journal of Environmental Economics and Management* 9, 1-10.
- Dasgupta, P., G. Heal (1974), The Optimal Depletion of Exhaustible Resources, *Review of Economic Studies* 41, 3-28.
- Dasgupta, P., J.E. Stiglitz (1981a), Resource Depletion under Technological Uncertainty, *Econometrica* 49, 85-104.
- \_\_\_\_\_, \_\_\_\_\_ (1981b), Market Structure and Resource Extraction under Uncertainty, *Scandinavian Journal of Economics* 83, 318-333.
- Deshmukh, S.D., S.R. Pliska (1980), Optimal Consumption and Exploration of Nonrenewable Resources under Uncertainty, *Econometrica* 48, 177-200.
- \_\_\_\_\_, \_\_\_\_\_ (1983), Optimal Consumption of a Nonrenewable Resource with Stochastic Discoveries and a Random Environment, *Review of Economic Studies* 50, 543-554.
- Gilbert, R. (1979), Optimal Depletion of an Uncertain Stock, *Review of Economic Studies* 46, 47-57.
- Hartwick, J.M. (1983), Learning about and Exploiting Exhaustible Resource Deposits of Uncertain Size, *Canadian Journal of Economics* 16, 391-410.
- Hoel, M. (1978), Resource Extraction when a Future Substitute has an Uncertain Cost, *Review of Economic Studies* 45, 637-644.
- Kemp, M. (1976), "How to eat a cake of unknown size" in Kemp, M., Three Topics in the Theory of International Trade, Chap. 23, North Holland, American Elsevier.
- Lewis, T.R. (1977), Attitudes towards Risk and the Optimal Exploitation of an Exhaustible Resource, *Journal of Environmental Economics and Management* 4, 111-119.
- Long, N.V. (1975), Resource Extraction under the Uncertainty about Possible Nationalization, *Journal of Economic Theory* 10, 42-53.
- Loury, G.C. (1978), The Optimal Exploitation of an Unknown Reserve, *Review of Economic Studies* 45, 621-636.

Mangel, M. (1985), *Decision and Control in Uncertain Resource Systems*, Academic Press, New York.

Pindyck, R.S. (1980), Uncertainty and Exhaustible Resource Markets, *Journal of Political Economy* 88, 1203-1225.

\_\_\_\_\_ (1981), The Optimal Production of an Exhaustible Resource when Price is Exogenous and Stochastic, *Scandinavian Journal of Economics* 83, 279-288.

Sengupta, J.K (1986), *Stochastic Optimization and Economic Models*, Reidel, Dordrecht.