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Price majorization and the inverse Lorenz function*

by

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Abstract

The paper presents an approach to the measurement of economic disparity in several commodities. We introduce a special view on the usual Lorenz curve and extend this view to the multivariate situation: Given a vector of shares of the total endowments in each commodity, the multivariate inverse Lorenz function (ILF) indicates the maximum percentage of the population by which these shares or less are held. Its graph is the Lorenz hypersurface. Many properties of the ILF are studied and the equivalence of the pointwise ordering of ILFs and the price Lorenz order is established. We also study similar notions for distributions of absolute endowments. Finally, several disparity indices are suggested that are consistent with these orderings.

Keywords: Multivariate Lorenz order, directional majorization, price Lorenz order, generalized Lorenz function, multivariate disparity indices.

AMS 1991 **subject classification:** Primary 62H05; secondary 52A20, 62P20.

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1 Introduction

Since the beginning of this century, economists have been interested in the quantitative description and statistical estimation of economic disparity. The classical Lorenz curve (Lorenz 1905) depicts the disparity of an attribute, usually income, in a given population. It is the basic notion for modelling and measuring economic inequality with respect to a single attribute of wellbeing.

But economic inequality does not arise from the distribution of income alone. Other attributes of affluence and well-being appear to be of similar interest in economic analysis. Households vary in income and assets, individuals in earning and education, countries in per capita income and mineral resources, etc. In modern theories of social choice the specific distributional inequality of attributes like these is considered; see Tobin (1970), Sen (1970, 1973), Kolm (1977). If inequality in two or more attributes is treated simultaneously, we face the problem of modelling and measuring multidimensional economic disparity. Special multidimensional indices and orderings have been proposed by Atkinson and Bourgouignon (1982, 1989), Kolm (1977), Maasoumi (1986), Marshall and Olkin (1979), Tsui (1995), and others.

Here, to describe the disparity of d attributes, we introduce a d-variate function, which we name the inverse Lorenz function (ILF). Two distributions are compared by the pointwise ordering of their ILFs. Our basic idea is briefly outlined as follows. We introduce a special view on the usual Lorenz curve and extend this view to the multivariate situation: The usual Lorenz curve can also be seen as the graph of the *inverse Lorenz function*: Given a portion of the total income, the inverse Lorenz function indicates the maximum percentage of the population that receives this portion (Section 2). In the multivariate case, we consider a vector of shares of the total endowment in each commodity and determine the largest percentage of the population by which this vector of shares or less is held. This percentage, depending on the vector of shares, is the d–variate inverse Lorenz function; its graph will be called the *Lorenz hypersurface* (Section 3).

The following example exhibits the ILF in two attributes.

Example 1. Let $(1000, 0), (0, 80), (0, 20)$ be the (wealth, income) vectors of three households; that is, the first household has wealth 1000 and income 0, etc. For income (wealth) alone, the usual univariate Lorenz curve plots the percentage $L(t)$ of total income (wealth) which is received by the percentage t of lower income (wealth) households; see Figure 1. Figure 2 shows the Lorenz surface of the bivariate distribution.

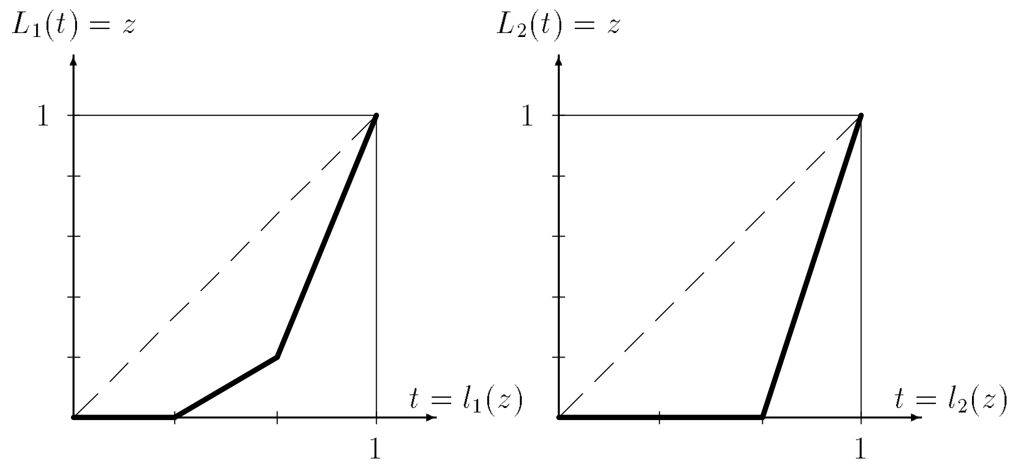

Figure 1. The Lorenz functions, $L_1(t)$ and $L_2(t)$, and inverse Lorenz functions, $l_1(z)$ and $l_2(z)$, of the univariate empirical distributions having support $\{0, 80, 20\}$ and $\{1000, 0, 0\}$.

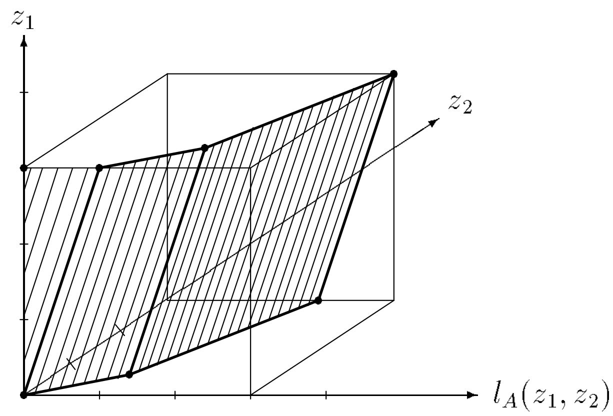

Figure 2. The Lorenz hypersurface of a bivariate empirical distribution (Example 1). The hypersurface (= graph of the bivariate ILF) is the hatched surface "spanned" by $(0,0,0)$, $(0,1,0)$, $(\frac{1}{3},1,0)$, $(\frac{2}{3},0,1)$, $(1,1,1)$, $(\frac{1}{3},0,\frac{1}{5})$ and $(\frac{2}{3},1,\frac{1}{5})$.

The Lorenz order between univariate distributions is the pointwise ordering of their Lorenz functions or, equivalently, of their ILFs. The pointwise ordering of univariate ILFs is equivalent to the majorization of vectors of shares.

The pointwise ordering of ILFs is one possibility of extending the univariate Lorenz order to a multivariate setting. Another economic meaningful notion is the *price*

Lorenz order (Section 4). Given a nonnegative price vector, any multivariate distribution of endowments yields a univariate distribution of expenditures. Consider the shares of expenditures in the population. A distribution of endowments majorizes another one in the price Lorenz order if, whatever the prices, the vector of expenditure shares of the first distribution, in the usual sense, majorizes that of the second. It comes out (Theorem 4.1) that the price Lorenz order is equivalent to the pointwise ordering of ILFs.

Like the usual univariate Lorenz order, the price Lorenz order deals with relative endowments or shares. To compare distributions of absolute endowments, we introduce similar orderings (Section 5).

Price majorization (usual majorization of expenditure vectors, whatever the prices) has been introduced by several authors under different names. Kolm (1977) terms it income more equal with nonnegative prices. The same is named second degree stochastic dominance by Muliere and Scarsini (1989), and majorization by positive linear combinations by Joe and Verducci (1992). Arnold (1987) gives an interpretation of it in terms of n individuals with money in d different currencies.

If also negative "prices" are included in the definition, price majorization turns to *directional majorization* (Marshall and Olkin (1979), Bhandari (1988)). Directional majorization implies price majorization, but not vice versa, as can be seen from simple examples. Directional majorization has been used by Kolm (1977) in an economic context. Koshevoy (1995) characterized it by means of the Lorenz zonotope, Koshevoy and Mosler (1996) characterized its generalization, directional order, by means of Lorenz zonoids.

Section 2 gives an account of vector majorization and the Lorenz order of univariate probability distributions. In Section 3 the inverse Lorenz function of a d-variate probability distribution is defined, its main properties are derived, and examples are given. For an empirical distribution in $\mathbb{R}^d_+$, the ILF is piecewise linear; the points which define its graph are determined in Proposition 3.2. Section 4 investigates the ordering of ILFs and its equivalence to the price Lorenz order, while the orderings for absolute endowments are treated in Section 5. Section 6 contrasts price majorization with the previous multicommodity orderings of Atkinson and Burgouignon (1982) and other multivariate majorizations. Section presents disparity indices that are consistent with the price Lorenz order, and Section 8 concludes the paper. An Appendix collects all proofs.

Some notation: $\mathbb{R}^d$ ($\mathbb{R}^d_+$, $\mathbb{R}^d_-$, $\mathbb{R}^d_{++}$) is the d–dimensional Euclidean space of (nonnegative, nonpositive, strictly positive) row vectors. $\mathbf{0}$ is the origin, $\mathbf{1} = (1, \ldots, 1)$. The prime denotes the transpose. $\mathbb{R}^{n \times d}$ is the space of $n \times d$ matrices. $\mathcal{F}^d$ is the class of probability distributions (or probability distribution functions) on $\mathbb{R}^d_+$, and let $\mathcal{F}_1^d$ the subclass of those $F \in \mathcal{F}^d$ that have a finite positive expectation vector, denoted $\mu_F = (\mu_1, \ldots, \mu_d)$, with all $\mu_j > 0$. Absolute continuity of a distribution refers to the Lebesgue measure.

2 Univariate majorization and Lorenz order

We start with a short discussion of univariate majorization and the Lorenz order of univariate probability distributions.

Consider a population of n persons among which income is distributed. An income profile $A = (a_1, \ldots, a_n)' \in \mathbb{R}^{n\times 1}$ is a vector that assigns to the i-th person his or her income $a_i \geq 0$. $a_{(1)} \leq a_{(2)} \leq \ldots \leq a_{(n)}$ are the ordered incomes. Given two income profiles $A = (a_1, \ldots a_n)'$ and $B = (b_1, \ldots b_n)'$, B is majorized by A if both profiles have the same total income and the poorest person in A receives no more than the poorest one in B, the poorest and the second poorest persons in A together receive no more than the poorest and the second poorest one in B do, and so on.

Formally speaking, an income profile A *majorizes* B, $A \succ B$, if one of the following equivalent conditions is fulfilled (Marshall and Olkin (1979)).

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \text{ and } \sum_{i=1}^{k} a_{(i)} \leq \sum_{i=1}^{k} b_{(i)} \text{ for } k = 1, \ldots n-1, \tag{1}$$

$$\sum_{i=1}^{n} g(b_i) \geq \sum_{i=1}^{n} g(a_i) \text{ for all } g : \mathbb{R} \to \mathbb{R} \text{ that are convex,} \tag{2}$$

$$B = TA \text{ for some bistochastic matrix } T. \tag{3}$$

(1) means that the Lorenz curve of A lies below that of B. Convexity of g in (2) can be interpreted in a framework of decision under risk: A subject, in evaluating alternative social states, considers himself to occupy each position in the population with equal probability and orders the states according to their expected value of individual disutility g. Then, convexity of g is tantamount to risk aversion of the subject. (3) says that B is an *average* of A.

The univariate Lorenz function and order have also been defined for probability distributions in place of distribution vectors (see, e.g., Gastwirth 1971), which is useful for many applications. Let us recall the definitions for general univariate probability distributions.

Definition 2.1 (Lorenz function, curve, and order) *Let $F \in \mathcal{F}_1^1$. The function $L_F : [0,1] \to [0,1]$,*

$$L_F(t) = \frac{\int_0^t F^{-1}(s)ds}{\int_0^1 F^{-1}(s)ds}, \quad 0 \le t \le 1, \tag{4}$$

is the Lorenz function *of F, where $F^{-1}(t) = \inf\{x \,|\, F(x) \ge t\}$ is the left continuous quantile function of F. The graph of $L_F(t)$, $\{(t, L_F(t)) | t \in [0,1]\}$, is the* Lorenz curve *of F. The* Lorenz order *on $\mathcal{F}_1^1$ is defined by*

$$F \succ_L G \quad \text{if} \quad L_F(t) \le L_G(t) \ \text{for all}\ t \in [0,1]. \tag{5}$$

For a general distribution $F \in \mathcal{F}_1^1$ with expectation $\mu > 0$, define

$$H_F(z) = \{h : \mathbb{R} \to [0,1] \,|\, \int_0^\infty \frac{x}{\mu} h(x) dF(x) \le z\},$$

$$l_F(z) = \max_{h \in H_F(z)} \int_0^\infty h(x) dF(x), \quad z \in [0,1]. \tag{6}$$

$l_F(z)$ is the maximal fraction of the population which receives the fraction z of the total income. From the definition follows immediately that l_F is concave, $l_F(1) = 1$, and

$$l_F(0) = F(0) = \sup\{t \,|\, L_F(t) = 0\}. \tag{7}$$

Lemma 1 (Koshevoy and Mosler 1996) *The restriction of L_F to $[l_F(0), 1]$ is strictly increasing, and l_F is its inverse function.*

Lemma 1 is the reason to name l_F the inverse Lorenz function. By the lemma and (7), the usual Lorenz order is the same as the pointwise ordering of inverse Lorenz functions. We have

$$F \succ_L G \quad \Longleftrightarrow \quad l_F(t) \ge l_G(t) \ \text{for all}\ t \in [0,1].$$

In the next section we extend the inverse Lorenz function to the multivariate case.

3 The multivariate inverse Lorenz function

Let $F \in \mathcal{F}_1^d$ be a probability distribution on $\mathbb{R}_+^d$, and $\mu = (\mu_1, \ldots \mu_d)$ its expectation vector. Then there exists a probability space $(\Omega, \mathcal{S}, P)$ and an $\mathcal{S}$-measurable map $X : \Omega \to \mathbb{R}_+^d$, a random vector, such that F is the distribution function of X, i.e.

$F(z)$ equals the probability of the set $\{\omega \in \Omega : X(\omega) \leq z\}$. Let us interpret the space Ω as a space of economic agents. An element $S \in \mathcal{S}$, an event, is seen as a coalition of agents; the probability $P(S)$ equals the relative size of the coalition S with respect to the size of the total space Ω. A value of the random vector, $X(\omega)$, is the endowment of a person $\omega \in \Omega$. $F(z)$ equals the relative size of the coalition whose agents have endowments less or equal to $z \in \mathbb{R}^d_+$. In such a model, a comparison of two distributions is equivalent to a comparison of distributions of endowments X and Y in two societies $(\Omega, \mathcal{S}, P)$ and $(\Omega', \mathcal{S}', P')$.

A probability distribution that gives equal mass to n, not necessarily different, points in $\mathbb{R}^d_+$, say $a_1, \ldots, a_n$, is an *n-point empirical distribution* and corresponds to the model $\Omega = \{1, \ldots, n\}$, $\mathcal{S} = 2^\Omega$, $P(S) = |S|/n$, $X(i) = a_i$. An $n \times d$ matrix $A = (a'_1, \ldots, a'_n)'$ describes the distribution of d commodities in the population. Its row a_i is the endowment vector of the i-th person, and its column a^k is the distribution of the k-th commodity. The total commodity vector equals $\sum_{i=1}^n a_i = (\sum_i a_i^1, \ldots, \sum_i a_i^d)$ and is positive in each component. Every such A is called a *distribution matrix.* Let $\overline{A} = (\bar{a}_i^k)$ be the matrix of shares, $\bar{a}_i^k = a_i^k / \sum_i a_i^k$. F_A denotes the n-point empirical distribution on the rows of A.

For $x \in \mathbb{R}^d_+$ we notate

$$\frac{x}{\mu} = \left(\frac{x_1}{\mu_1}, \ldots, \frac{x_d}{\mu_d} \right).$$

I.e., if X is a random vector distributed with F then $\widetilde{X} = \frac{X}{\mu}$ is the "normalization" of X having expectation $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^d_+$.

Definition 3.1 (Inverse Lorenz function) *Let a function* $l_F : [0,1]^d \to [0,1]$ *be defined by the rule*

$$l_F(z) = \max \int_{\mathbb{R}^d_+} h(x) dF(x), \tag{8}$$

where the maximum extends over all measurable functions $h : \mathbb{R}^d_+ \to [0,1]$ *for which*

$$\int_{\mathbb{R}^d_+} h(x) \frac{x}{\mu} dF(x) \leq z.$$

l_F *is said to be the* inverse Lorenz function (ILF) *of* F. *Its graph is the* Lorenz hypersurface.

The economic interpretation of the ILF is as follows: $l_F(z)$ equals the maximum percentage of people who hold a vector of shares less or equal $z = (z_1, \ldots, z_d)$. Indeed, to every person ω of the population Ω the vector $X(\omega)$ of endowments in d commodities is assigned. Then person ω holds the vector $\widetilde{X}(\omega)$ of shares of the mean endowment. A given measurable function $h : \mathbb{R}_+^d \to [0,1]$ may be considered as a *selection* of some part of the population: Of all those persons who have endowment vector x the percentage $h(x)$ is selected. Thus $\int_{\mathbb{R}_+^d} h(x)\frac{x}{\mu}dF(x)$ amounts to the total portion vector held by this subpopulation and $\int_{\mathbb{R}_+^d} h(x)dF(x)$ is the size of the subpopulation selected by h. Therefore, for a given vector $z \in [0,1]^d$, $\max \int h(x)dF(x)$, where the maximum is taken over all measurable h such that $\int h(x)\frac{x}{\mu}dF(x) \leq z$, is the maximum size of a part of population that holds a total portion vector z or less.

The next proposition collects basic properties of the ILF which are immediate from the definition.

Proposition 3.1 *l_F is strictly increasing, concave and continuous;*
$l_F(\mathbf{1}) = 1$; $l_F(\mathbf{0}) = F(\mathbf{0})$.

The case of an empirical distribution deserves particular attention. For an empirical distribution F_A with distribution matrix A, the definition of the inverse Lorenz function reads as follows.

Definition 3.2 *Let A be a distribution matrix. The function $l_A : [0,1]^d \to [0,1]$,*

$$l_A(z) = \max\{\frac{1}{n}\sum_{i=1}^{n}\theta_i \mid \sum_{i=1}^{n}\theta_i\bar{a}_i \leq z, 0 \leq \theta_i \leq 1\} \tag{9}$$

is the inverse Lorenz function (ILF) *of A.*

The Lorenz hypersurface of F_A is a piecewise linear hypersurface passing through a subset of

$$\left\{\left(\frac{|I|}{n}, \sum_{i\in I}\bar{a}_i^J, \mathbf{1}^{-J}\right) \mid I \subset \{1,\ldots,n\}, \emptyset \neq J \subset D = \{1,\ldots,d\}\right\}.$$

The notation $(\frac{|I|}{n}, \sum_{i\in I}\bar{a}_i^J, \mathbf{1}^{-J})$ means that the first coordinate (indexed 0) of this point is the cardinality of I over n and the remaining coordinates are $\sum_{i\in I}\bar{a}_i^j$ if $j \in J$ and 1 if $j \notin J$. If $\mathbb{R}^J$ denotes the coordinate subspace belonging to J, a^J is the projection of the vector $a \in \mathbb{R}^d$ on $\mathbb{R}^J$.

The following proposition characterizes the extreme points of the Lorenz hypersurface. Based on this proposition, the surface can be constructed by an algorithm.

Proposition 3.2 *Let A be a distribution matrix. Then a point $(\frac{|I|}{n}, \sum_{i \in I} \bar{a}_i^J, \mathbf{1}^{-J})$ is an extreme point of the Lorenz hypersurface if and only if the vector $\theta = (\mathbf{1}^J, \mathbf{0}^{-J})$ gives a solution to the following maximization problem*

$$\max\Big\{\frac{1}{n}\sum_{i=1}^{n}\theta_i \mid \sum_{i=1}^{n}\theta_i \bar{a}_i^J \le \sum_{i \in I}\bar{a}_i^J,\ \theta_i \in [0,1]\Big\}. \tag{10}$$

Obviously, in the case $d = 1$, such points have the form $(\frac{j}{n}, \sum_{i=1}^{j} \bar{a}_{(i)})$, $j = 0, \ldots, n$. We continue with additional properties of the ILF.

Proposition 3.3 *Let a sequence $\{F_n\}$ converge weakly to F. Then l_{F_n} converges uniformly to l_F.*

As a corollary to this proposition we get a sort of law of large numbers. Namely let $\hat{F}_n$ be the empirical distribution function of a random sample $X_1, \ldots . X_n$ from F. Then, because of Proposition 3.3 and the Glivenko–Cantelli theorem, the sample ILF converges uniformly to the true ILF with probability one,

$$P(\lim_{n\to\infty}\sup_z |l_{\hat{F}_n}(z) - l_F(z)| = 0) = 1. \tag{11}$$

Equation (11) can be regarded as a law of large numbers for random sets. Let

$$\mathcal{F}_2^d = \{F \in \mathcal{F}_1^d \,|\, \text{the function } x \mapsto e^{p'x} \text{ is } F\text{-integrable for any } p \in \mathbb{R}^d\}.$$

An important property is the uniqueness result (Koshevoy 1998, 1999):

Proposition 3.4 *Among all distributions $F \in \mathcal{F}_2^d$, the inverse Lorenz function l_F defines F uniquely, up to scale.*

Any probability distribution that has compact support, in particular any empirical distribution, lies in $\mathcal{F}_2^d$.

The following proposition exhibits the relation between the ILF of a probability distribution and the ILF of its marginal distributions.

Proposition 3.5 *Let F be a distribution in $\mathcal{F}_1^d$, and F^J its marginal with respect to coordinates in $J \subset \{1, \ldots, d\}$. Then*

$$l_F(z^J, \mathbf{1}^{-J}) = l_{F^J}(z^J). \tag{12}$$

Example 2 (Egalitarian distribution) As a special case, consider the *egalitarian distribution* E_c that gives unit mass to some point c in $\mathbb{R}^d_+$, $c \neq \mathbf{0}$. This means that all agents have the same endowment vector, $c \in \mathbb{R}^d_+$. We get $l_{E_c}(z) = \min_{i=1,\dots d} z_i$. Figure 3 shows the graph of l_{E_c} when $d = 2$.

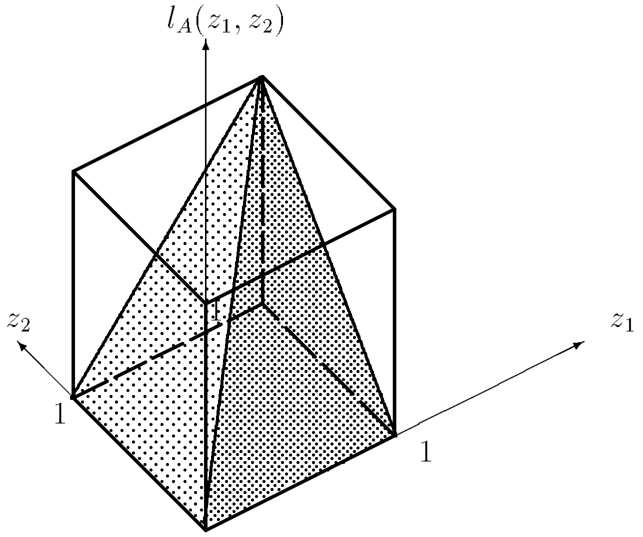

Figure 3. The Lorenz hypersurface of an egalitarian distribution when $d = 2$.

Example 3 (Similarly ordered distribution, $d = 2$). A distribution function F that satisfies $F(x_1, x_2) = \min\{F^{\{1\}}(x_1), F^{\{2\}}(x_2)\}$ is said to be *similarly ordered.* Here $F^{\{1\}}$ and $F^{\{2\}}$ are the marginals. In such a case there exist random variables X_1, X_2 with joint distribution F such that X_2 is an increasing function of X_1, viz. $X_2 = (F^{\{2\}})^{-1}(F^{\{1\}}(X_1))$. This means that endowments of agents can be ordered with respect to the coordinatewise order $\geq$ in $\mathbb{R}^d$. Then l_F is of the form:

$$l_F(z) = \begin{cases} l_{F^{\{2\}}}(z_2), & \text{if } \int_0^t (F^{\{1\}})^{-1}(s)ds \geq \int_0^t (F^{\{2\}})^{-1}(s)ds, \\ l_{F^{\{1\}}}(z_1), & \text{if } \int_0^t (F^{\{1\}})^{-1}(s)ds \leq \int_0^t (F^{\{2\}})^{-1}(s)ds, \end{cases}$$

where $l_{F^{\{1\}}}$ and $l_{F^{\{2\}}}$ are the inverse Lorenz functions of the marginals.

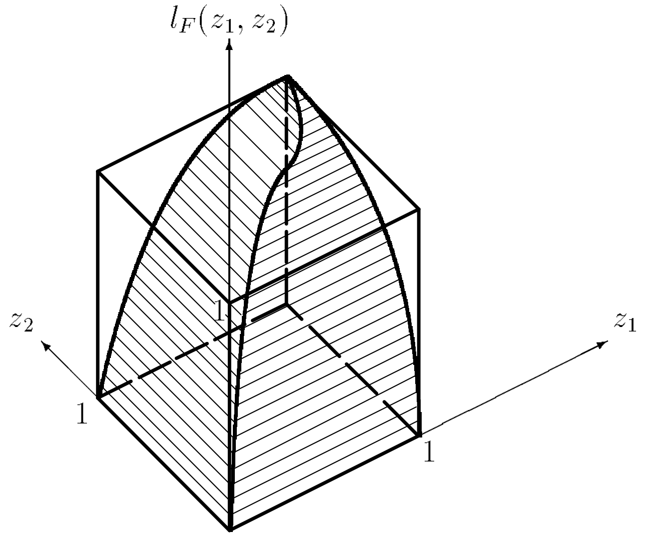

Figure 4. The Lorenz hypersurface of a similarly ordered distribution when $d = 2$.

4 The price Lorenz order

Now we turn to a multivariate extension of the Lorenz order, the *price Lorenz order* for d-variate distributions in $\mathcal{F}_1^d$.

For a given distribution F on $\mathbb{R}_+^d$ and for some $p \in \mathbb{R}_+^d$ and $t \in \mathbb{R}$, let $F_p(t)$ be the probability of the set

$$\{x \in \mathbb{R}_+^d \,|\, p'\frac{x}{\mu} \le t\}.$$

If X is a random vector in $\mathbb{R}_+^d$ distributed by F, the function $t \mapsto F_p(t)$ is the distribution function of the random variable $p'\widetilde{X}$ in $\mathbb{R}$. If p is interpreted as a vector of prices, F_p is the distribution of relative expenditures, i.e. expenditure shares, in the population under these prices.

Definition 4.1 (Price Lorenz order) *Of two distributions F and G in $\mathcal{F}_1^d$, F is said to be larger than G in the* price Lorenz order*, $F \succ_{PL} G$, if*

$$F_p \succ_L G_p \quad \textit{for every } p \in \mathbb{R}_+^d.$$

If X and Y are random vectors with distributions F and G, the Lorenz order $F \succ_{PL} G$ means that, for all prices $p \ge 0$, the distribution of $p'\widetilde{X}$ is larger than that of $p'\widetilde{Y}$ in the univariate Lorenz order.

The following theorem is the main result of this section.

Theorem 4.1 *For F and $G \in \mathcal{F}_1^d$, $F \succ_{PL} G$ holds if and only if*

$$l_F(z) \geq l_G(z) \quad \textit{for all} \;\; z \in [0,1]^d.$$

The theorem says that the price Lorenz order is equivalent to the pointwise ordering of ILFs. Note that the $\succ_{PL}$ is a relation on $\mathcal{F}_1^d$ that is reflexive and transitive, but not antisymmetric. On a factor space the relation is also antisymmetric, hence a partial order. This factor space is constructed from $\mathcal{F}_1^d$ with the equivalence relation $\sim$, $F \sim H$ if there exists a vector $(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}_{++}^d$ such that $F(x_1, \ldots, x_d) = H(\lambda_1 x_1, \ldots, \lambda_d x_d)$ for all $(x_1, \ldots, x_d) \in \mathbb{R}_+^d$. In fact, transitivity and reflexivity are obvious, while antisymmetry follows from Proposition 3.4.

In view of Proposition 3.2, Theorem 4.1 shows that price majorization is a multivariate extension of the system of inequalities $\sum_{i=1}^k \bar{a}_{(i)} \leq \sum_{i=1}^k \bar{b}_{(i)}$, $k = 1, \ldots, n-1$. Due to these propositions, we have:

Proposition 4.1 *For two given distribution matrices A and B, $F_A \succ_{PL} F_B$ is characterized by the following condition:*
If x is an extreme point of the Lorenz hypersurface of B, $x = (\frac{|I|}{n}, \sum_{i \in I} \bar{b}_i^J, \mathbf{1}^{-J})$, then

$$\max\{\sum_{i=1}^n \theta_i\} \geq |I|,$$

where the maximum is taken over all $0 \leq \theta_i \leq 1$ for which $\sum_{i=1}^n \theta_i \bar{a}_i^J \leq \sum_{i \in I} \bar{b}_i^J$.

Several properties of the univariate Lorenz order carry over immediately to the price Lorenz order. First there is a minimal distribution:

$$F \succ_{PL} E_c \quad \text{for all} \quad F \in \mathcal{F}_1^d, \;\; c \in \mathbb{R}_{++}^d,$$

where E_c is the egalitarian distribution at some c. Second, as the univariate Lorenz order is continuous, we get:

Proposition 4.2 *Let $F \succ_{PL} G_n$ hold for the sequence $\{G_n\}$ that* converges weakly *to G. Then $F \succ_{PL} G$.*

Next we state that $\succ_{PL}$ is a convex order in the following sense.

Proposition 4.3 *Let F_1, F_2, G be distribution functions such that $G \succ_{PL} F_1$ and $G \succ_{PL} F_2$ hold. Then $G \succ_{PL} \alpha F_1 + (1-\alpha) F_2$ for any $\alpha \in [0,1]$.*

With stochastically independent attributes, directional majorization is equivalent to usual Lorenz order of all univariate marginals (Koshevoy and Mosler (1996)). As directional majorization implies price Lorenz order, we get:

Proposition 4.4 *Let* $F, G \in \mathcal{F}$, $F(x) = \Pi_{k=1}^d F^{\{k\}}(x_k)$, $G(x) = \Pi_{k=1}^d G^{\{k\}}(x_k)$. *Then*

$$F \succ_{PL} G \quad \text{if and only if} \quad F^{\{k\}} \succ_L G^{\{k\}}, \ k = 1, \ldots, d.$$

The following proposition shows that in the case of similarly ordered distribution an analogous property holds.

Proposition 4.5 *Let* $F(x) = \min_k F^{\{k\}}(x_k)$ *and* $G(x) = \min_k G^{\{k\}}(x_k)$ *where* $F^{\{k\}}$ *and* $G^{\{k\}}$ *are the marginals. Then* $F \succ_{PL} G$ *if and only if* $F^{\{k\}} \succ_L G^{\{k\}}$, $k = 1, \ldots, d$.

5 Majorizations of absolute endowments

So far we have compared distributions of shares, or relative endowments. Distributions of absolute endowments can be treated in the same way.

Several authors, among them Shorrocks (1983) and Moyes (1987), have investigated Lorenz orders between univariate distributions of absolute endownments. For a probability distribution F on $\mathbb{R}_+$, the *generalized Lorenz function* is given by

$$\widehat{L}_F(t) = \int_0^t F^{-1}(s)ds, \quad 0 \le t \le 1.$$

Hence $\widehat{L}_F(0) = 0$, $\widehat{L}_F(1) = \mu_F$, and $\widehat{L}_F(t) = \mu_F L_F(t)$ if μ_F is finite. The *dual generalized Lorenz function* of F is

$$\widehat{DL}_F(t) = \mu_F - \widehat{L}_F(1-t), \quad 0 \le t \le 1.$$

For two univariate distributions F and G the pointwise ordering of these functions is considered,

$$F \succ^w G \quad \text{if} \ \widehat{L}_F(t) \le \widehat{L}_G(t) \ \text{ for all } t,$$

$$F \succ_w G \quad \text{if} \ \widehat{DL}_F(t) \ge \widehat{DL}_G(t) \ \text{ for all } t.$$

$F \succ^w G$ is called *supermajorization*, and $F \succ_w G$ is called *submajorization* of F over G. These relations among univariate distributions are also named weak majorizations; see Marshall and Olkin (1979). Obviously $F \succ^w G$ implies that $\mu_F \le \mu_G$, while $F \succ_w G$ implies that $\mu_F \ge \mu_G$. If both $F \succ^w G$ and $F \succ_w G$ hold then $F \succ_{PL} G$ and $\mu_F = \mu_G$. On the other hand, if $\mu_F = \mu_G$, submajorization as well as supermajorization coincide with the usual Lorenz order.

To define multivariate versions of submajorization and supermajorization, we need a notation similar to the above F_p. For a given d-variate distribution F in $\mathcal{F}^d$, $p \in \mathrm{IR}_+^d$ and $t \in \mathrm{IR}$, let $\widehat{F}_p(t)$ be the F-probability of the set

$$\{x \in \mathrm{IR}_+^d \,|\, p'x \leq t\}.$$

That means, if the commodity vector X is distributed with F then the expenditure $p'X$ is distributed with $\widehat{F}_p$.

Definition 5.1 (Price supermajorization, price submajorization) *For F and $G \in \mathcal{F}^d$, define*

$$F \succ_P^w G \quad \textit{if} \;\; \widehat{F}_p \succ^w \widehat{G}_p \;\; \textit{for all} \;\; p \in \mathrm{IR}_+^d,$$

$$F \succ_{wP} G \quad \textit{if} \;\; \widehat{F}_p \succ_w \widehat{G}_p \;\; \textit{for all} \;\; p \in \mathrm{IR}_+^d.$$

$\succ_P^w$ *is called* price supermajorization*,* $\succ_{wP}$ price submajorization*. Both* $\succ_P^w$ *and* $\succ_{wP}$ *are called* weak price majorizations*.*

We define the price majorization of multivariate probability distributions as follows.

Definition 5.2 (Price majorization) *For F and $G \in \mathcal{F}^d$, define*

$$F \succ_P G \quad \textit{if} \quad F \succ_{PL} G \;\; \textit{and} \;\; \mu_F = \mu_G.$$

Corollary 5.1 *If $\mu_F = \mu_G$ then*

$$F \succ_P^w G \quad \Leftrightarrow \quad F \succ_{wP} G \quad \Leftrightarrow \quad F \succ_P G.$$

Price majorization, as defined in Definition 5.2, is related to these two weak majorizations and can be characterized by the $\widehat{F}_p$ and $\widehat{G}_p$ as well.

Corollary 5.2 (i) $F \succ_P G \quad \Leftrightarrow \quad \mu_F = \mu_G$ *and* $\widehat{F}_p \succ_L \widehat{G}_p$ *for all* $p \in \mathrm{IR}_+^d$.
(ii) $F \succ_P^w G$ *and* $F \succ_{wP} G \quad \Leftrightarrow \quad F \succ_P G$.

Both corollaries are obvious from the Definition 5.1. For the weak majorizations we have another characterization which involves a generalized version of the ILF and its dual.

Definition 5.3 (Inverse generalized Lorenz function and its dual) (i) *Let*

$$\hat{l}_F(z) = \max \int_{\mathbb{R}_+^d} h(x) dF(x), \tag{13}$$

where the maximum is taken over all measurable $h : \mathbb{R}_+^d \to [0,1]$ *with*

$$\int_{\mathbb{R}_+^d} h(x) x dF(x) \leq z.$$

The function $\hat{l}_F : \mathbb{R}_+^d \to [0,1]$ *is called the* inverse generalized Lorenz function (IGLF) *of* F.
(ii) *The function* $\widehat{Dl}_F : \mathbb{R}_+^d \to [0,1]$ *is called the* dual inverse generalized Lorenz function (DIGLF) *of* F,

$$\widehat{Dl}_F(z) = \min_h \int_{\mathbb{R}_+^d} h(x) dF(x), \quad z \in \mathbb{R}_+^d, \tag{14}$$

where the minimum is taken over all measurable functions $h : \mathbb{R}_+^d \to [0,1]$ *for which*

$$\int_{\mathbb{R}_+^d} h(x) x dF(x) \geq z.$$

Note that, for an empirical distribution F_A with a matrix A, (13) becomes the function $\hat{l}_A : \mathbb{R}_+^d \to [0,1]$,

$$\hat{l}_A(z) = \max\{\frac{1}{n}\sum_{i=1}^n \theta_i \mid \frac{1}{n}\sum_{i=1}^n \theta_i a_i \leq z, 0 \leq \theta_i \leq 1\}. \tag{15}$$

Hildenbrand (1981) has used a function of the form (13) for an aggregate production function.

Theorem 5.1 (i) $F \succ_P^w G$ *if and only if*

$$\hat{l}_F(z) \geq \hat{l}_G(z) \quad \text{for all } z \in [\mathbf{0}, \mu_F],$$

(ii) $F \succ_{wP} G$ *if and only if*

$$\widehat{Dl}_F(z) \leq \widehat{Dl}_G(z) \quad \text{for all } z \in [\mathbf{0}, \mu_G].$$

As finite expectations do not necessarily exist for F and G, the d-variate intervals $[\mathbf{0}, \mu_F]$ and $[\mathbf{0}, \mu_G]$ may extend over the whole $\mathbb{R}_+^d$.

Table 1 summarizes the definitions of the orderings studied so far and the relations between them.

price supermajorization	$F \succ_P^w G$: $\forall p \geq 0 \ \hat{F}_p \succ^w \hat{G}_p$	$\Leftrightarrow$	ordering of IGLFs
	$\Uparrow$		$\Uparrow$
price majorization	$F \succ_P G$: $\mu_F = \mu_G$ and $\forall p \geq 0 \ F_p \succ G_p$	$\Leftrightarrow$	$F \succ_P^w G$ and $F \succ_{wP} G$
	$\Downarrow$		$\Downarrow$
price submajorization	$F \succ_{wP} G$: $\forall p \geq 0 \ \hat{F}_p \succ_w \hat{G}_p$	$\Leftrightarrow$	ordering of DIGLFs

Table 1. Price majorization and weak price majorizations and their characterization by inverse generalized Lorenz functions (IGLFs) and their duals (DIGLFs).

6 Relations to previous multicommodity orderings

Atkinson and Bourgouignon (1982) suggested another method of ranking distributions of commodity vectors: their first ordering ranks $F \succ_1 G$ if $F(x) \leq G(x)$ for every $x \in \mathbb{R}^d$, and their second ordering ranks $F \succ_2 G$ if $\int_{z \leq x} F(z)dz \leq \int_{z \leq x} G(z)dz$ for every $x \in \mathbb{R}^d$. The orderings are known in the literature as lower orthant ordering and lower orthant concave ordering, respectively; see, e.g., Shaked and Shanthikumar (1994).

Obviously, in dimension one, the two Atkinson-Bourgouignon orderings amount to usual first and second degree stochastic dominance. For $d \geq 1$, the first ordering implies the second one, and both can be characterized by inequalities on expected utilities. In particular, (see, for example, Scarsini (1985)), $F \succ_1 G$ if and only if $\int_{\mathbb{R}^d} u(x)dF(x) \geq \int_{\mathbb{R}^d} u(x)dG(x)$ for any utility function u from the class

$$\mathcal{U}_1 = \{u : \mathbb{R}^d \to \mathbb{R} \, : \, (-1)^{|J|} \frac{\partial^J u}{\partial x_J} \leq 0 \ \text{ for any } J \subset \{1, \ldots, d\}\}.$$

Here, the vector x_J includes the components x_j of x for $j \in J$, and $\frac{\partial^J u}{\partial x_J}$ is the partial derivative with respect to x_J.

The first Atkinson-Bourgouignon ordering, in dimension $d = 2$, implies the ordering of IGLFs or, equivalently, the weak price majorization. This can be seen from the characterizations of the rankings via expected utilities. A class of utility functions that characterizes the weak price majorization is

$$\mathcal{U}_{wP} = \{u : u(x) = v(\langle p, x \rangle) \text{ with an increasing convex real function } v \text{ and some } p \in \mathbb{R}_+^d\}.$$

When $d = 2$, we get $\mathcal{U}_1 = \{u : \mathbb{R}^2 \to \mathbb{R} : \frac{\partial u}{\partial x_1} \geq 0, \frac{\partial u}{\partial x_2} \geq 0, \frac{\partial^2 u}{\partial x_1 \partial x_2} \leq 0\}$, and the inclusion $\mathcal{U}_{wP} \subset \mathcal{U}_1$ holds. Then, $F \succ_1 G$ implies $F \succ_{wP} G$. For $d > 2$ the first Atkinson-Bourgouignon ordering does not imply the weak price majorization.

The second Atkinson-Bourgouignon ordering, in dimension $d = 1$, equals price supermajorization. But for $d \geq 2$ it neither implies nor is implied by weak price majorization.

Many other majorizations have been used in the analysis of economic disparity; see the survey by Mosler (1994). E.g., Kolm (1977) and Russell and Seo (1978) employ the multivariate dilation order, which is generated by expected utility inequalities for all concave utility functions. The dilation order implies the directional majorization and the latter implies the price majorization.

In dimension one, majorization and the Lorenz order are closely connected with Pigou-Dalton transfers. A *multivariate Pigou-Dalton transfer* is a transfer of commodity vectors among two agents such that the resulting two endowments are in the convex hull of the previous.

The price majorization relates to Pigou-Dalton transfers as follows. Consider a finite number, n, of agents. Since price majorization is equivalent to usual univariate majorization of expenditures, whatever the prices, it is tantamount saying that for every given price there exists a series of usual Pigou-Dalton transfers of expenditures. Each usual Pigou-Dalton transfer of expenditures corresponds to a multivariate transfer of endowments. In other words, price majorization means, that for every given price, there is a series of multivariate Pigou-Dalton transfers of endowments.

A distribution matrix A is *chain-majorized* by another distribution matrix B if A is obtained by a finite number of multivariate Pigou-Dalton transfers from B. Chain-majorization realizes Pigou-Dalton transfers of expenditures by the same series of multivariate Pigou-Dalton transfers of endowments for every price. Chain-majorization implies matrix majorization (= multivariate dilation), but for $n \geq 3$ and $d \geq 2$ the reverse is not true; see Marshall and Olkin (1979, p. 431). Thus the ordering induced by uniform multivariate Pigou-Dalton transfers is stronger than dilation, directional majorization and price majorization.

7 Multivariate disparity indices

The disparity of distributions may be measured by partial orderings, as presented above, or by real valued indices. An important question is which multivariate indices are consistent with the price Lorenz order, the price majorization, and the weak price majorizations, that is, which indices are increasing with respect to these orderings.

Kolm (1977, Theorem 5) suggests a class of indices of the form $I(A) = W(u(a_1), \ldots, u(a_n))$, where u is a convex function on $\mathbb{R}^d$, a utility function, and $W : \mathbb{R}^n \to \mathbb{R}$ is an index consistent with usual supermajorization ($\succeq^w$), and shows that such an index I is consistent with the dilation order. Tsui (1995) investigates similar indices.

We propose the following class of indices, which are consistent with the price Lorenz order. Let I be a univariate index that is decreasing with the univariate Lorenz order, e.g., the Gini index. Define an "average index" by

$$\mathcal{I}(F) = \int_{p \in \Delta} I(F_p) dp, \tag{16}$$

where $\Delta = \{p \in \mathbb{R}^d_+ \mid \sum_j p_j = 1\}$ denotes the unit simplex and integration is done with respect to some measure on Δ. A measure on the price simplex can reflect some information about possible prices. For example, if there is no information the measure is Lebesgue measure, or if some price vector p_0 is known for sure then the measure equals δ_{p_0}. Then $\mathcal{I}$ obviously decreases with the price Lorenz order, i.e. $F \succeq_L G$ implies $I(F) \leq I(G)$.

A different, particularly visual index is given by

$$R(F) = \frac{d+1}{d} \quad \times \quad \text{volume between the graphs of } l_F \text{ and } l_{E_c}. \tag{17}$$

$R(F)$ can be considered as a multivariate generalization of the usual Gini index. $R(F)$ is normed, $0 \leq R(F) < 1$, and obviously increasing with the price Lorenz order.

Koshevoy and Mosler (1997) investigate another multivariate generalization of the Gini index, the so called distance Gini index R_D,

$$R_D(F) = \frac{1}{2d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \parallel \frac{x}{\mu} - \frac{y}{\mu} \parallel dF(x)\, dF(y).$$

It can be shown that the distance Gini index R_D is consistent with directional majorization but not with the price Lorenz order.

8 Conclusions

A multivariate extension of the Lorenz curve, the inverse Lorenz function of a d-variate probability distribution, has been defined and studied. The pointwise ordering of inverse Lorenz functions induces an ordering of multivariate disparity, namely the price Lorenz order.

The approach to disparity measurement via the inverse Lorenz function and its graph, the Lorenz hypersurface, gives many new insights into the price majorization order. Proposition 4.1 provides an efficient algorithm to check for price Lorenz order and hence for price majorization among two given distributions. Many other properties of the orders have been clarified, among them their antisymmetry, continuity and convexity.

While the price Lorenz order compares distributions of share vectors, or relative endowments, related notions have been introduced for distributions of absolute endowments. Two cases of weak majorizations have been obtained where the total endowment vectors are ordered in one or the other direction.

The univariate Lorenz curve has three principal properties which are the reason for its almost universal use in the measurement of economic inequality. We discuss these properties and how they carry over to the multivariate notions of Lorenz hypersurface.

- The pointwise ordering by Lorenz functions is the Lorenz order.
- The Lorenz curve is visual: it indicates the degree of disparity as the bow is bent.
- Up to a scale parameter, the Lorenz curve fully describes the underlying distribution. No information is lost when we look at the Lorenz curve instead of the cumulative distribution function or the density.

All these properties are shared by the multivariate inverse Lorenz function.

- The pointwise ordering of inverse Lorenz functions is equivalent to the price Lorenz order.
- The Lorenz hypersurface is visual in the same way as the Lorenz curve is: increasing deviation from the egalitarian hypersurface indicates more disparity.
- The Lorenz hypersurface determines the underlying distribution matrix uniquely, up to a vector of scaling constants.

Appendix

Let us recall the definition of the Lorenz zonoid (Koshevoy and Mosler (1996)).

Definition A.1 (Lorenz zonoid). *For $F \in \mathcal{F}^d$ consider the set*

$$LZ(F) = \{z \in \mathbb{R}^{d+1} \quad | \quad z = (z_0, z_1, \dots z_d) = \zeta_F(h),\ h : \mathbb{R}^d_+ \to [0,1] \ \text{ measurable}\}$$

where

$$\zeta_F(h) = \left(\int_{\mathbb{R}^d_+} h(x) dF(x), \int_{\mathbb{R}^d_+} h(x) \frac{x}{\mu} dF(x) \right).$$

We call $LZ(F)$ the Lorenz zonoid.

A relation between the Lorenz hypersurface and the Lorenz zonoid is established by the following proposition.

Proposition A.1 *The graph of l_F is the intersection of the unit cube $[0,1]^{d+1}$ with the relative boundary of the set $LZ(F) + (\mathbb{R}_- \times \mathbb{R}^d_+)$.*

An equivalent form of this proposition is: The hypograph of l_F coincides with $[0,1]^{d+1} \cap \{LZ(F) + (\mathbb{R}_- \times \mathbb{R}^d_+)\}$.

Proof of Proposition A.1. A point (z_0, z) belongs to the graph of l_F iff $z_0 = \int_{\mathbb{R}^d_+} h(x) dF(x)$ where h is a solution of the following optimization problem

$$\int_{\mathbb{R}^d_+} h(x) dF(x) \quad \to \quad \max_h \tag{18}$$

$$\int_{\mathbb{R}^d_+} \frac{x}{\mu} h(x) dF(x) \quad \leq \quad z, \tag{19}$$

$$0 \leq h(x) \quad \leq \quad 1. \tag{20}$$

Let $p \in \mathbb{R}^d_+$ be a vector of Lagrange multipliers for (19). Then (18) to (20) are equivalent to the task

$$h(x)(-1 + p' \frac{x}{\mu}) \to \min_{0 \leq h(x) \leq 1}, \tag{21}$$

with complementary slackness conditions

$$p' \left(\int_{\mathbb{R}^d_+} \frac{x}{\mu} h(x) dF(x) - z \right) = 0.$$

From (21) follows that a solution of (18) is the function of the form

$$h(x) = \begin{cases} 1, & \text{if } p' \frac{x}{\mu} \leq 1 \\ 0, & \text{if } p' \frac{x}{\mu} > 1. \end{cases}$$

The point $(\int_{\mathbb{R}^d_+} h(x)dF(x), \int_{\mathbb{R}^d_+} \frac{x}{\mu}h(x)dF(x))$ with such a function $h(x)$ is an extreme point of $LZ(F)$ (Koshevoy and Mosler (1996)). With respect to the complementary slackness conditions, we get $(z_0, z) \in \partial(LZ(F) + \mathbb{R}_- \times \mathbb{R}^d_+)$. Q.E.D.

Propositions 3.2, 3.3 and 3.4 are concluded from this proposition as follows.

Proof of Proposition 3.2. For an empirical distribution F_A, the Lorenz zonoid is a convex polytope of the form

$$\sum_{i=1}^{n}\Big[\mathbf{0}, (\frac{1}{n}, \bar{a}_i)\Big],$$

see Koshevoy and Mosler (1998). Its extreme points are from the set of points of the form $(\frac{|I|}{n}, \sum_{i\in I}\bar{a}_i)$, $I \subset \{1,\dots,n\}$. Due to Proposition A.1, extreme points of l_A belong to the set of points of the form $(\frac{|I|}{n}, \sum_{i\in I}\bar{a}_i^J, \mathbf{1}^{-J})$. According to (9), a point of such a form is an extreme point of l_A if and only if it gives a solution to (8) with $z = (\sum_{i\in I}\bar{a}_i^J, \mathbf{1}^{-J})$. Q.E.D.

Proposition A.1 allows to rewrite the ordering of ILFs in the form

$$l_F \geq l_G \quad \text{iff} \quad \{LZ(F) + \mathbb{R}_- \times \mathbb{R}^d_+\} \supset \{LZ(G) + \mathbb{R}_- \times \mathbb{R}^d_+\}. \tag{22}$$

The latter inclusion can be written in terms of support functions.

Recall that the *support function* $\phi_K : \mathbb{R}^k \to \mathbb{R}$ of a convex set $K \subset \mathbb{R}^k$ is defined by

$$\phi_K(p) = \max\{p'x \mid x \in K\}, \quad p \in \mathbb{R}^k.$$

The function ϕ_K is continuous and convex. If $0 \in K$, then $\phi_K \geq 0$. There holds $K \subset R$ if and only if $\phi_K \leq \phi_R$. For $S + \{\mathbb{R}_- \times \mathbb{R}^d_+\}$, which is a cone, the support function in directions other than $\mathbb{R}_+ \times \mathbb{R}^d_-$ equals infinity. For these and other properties of support functions see Eggleston (1958).

Lemma 2 *For $F \in \mathcal{F}^d$, let ϕ_F denote the support function of $LZ(F)$. There holds*

$$\phi_F(p_0, p) = \int_{p'\frac{x}{\mu}+p_0\geq 0}(p'\frac{x}{\mu} + p_0)dF(x).$$

Proof. Koshevoy and Mosler (1996, Proposition 4.1).

Proof of Proposition 3.3. Because of Proposition A.1 and Lemma 2, $l_{F_n} \to l_F$ uniformly iff, for any $p \in \mathbb{R}^d_-$ and $p_0 \in \mathbb{R}_+$,

$$\phi_{F_n}(p_0, p) = \int_{p'\frac{x}{\mu}+p_0\geq 0}(p'\frac{x}{\mu} + p_0)dF_n(x) \to \int_{p'\frac{x}{\mu}+p_0\geq 0}(p'\frac{x}{\mu} + p_0)dF(x) = \phi_F(p_0, p).$$

Given $p \in \mathbb{R}^d_-$, the function $\max\{p'\frac{x}{\mu} + p_0, 0\}$, $x \in \mathbb{R}^d_+$, is a bounded function. Hence, weak* convergence $F_n \to_w F$ implies convergence of the support functions, $\phi_{F_n}(p_0, p) \to \phi_F(p_0, p)$, $p \in \mathbb{R}^d_-$, $p_0 \in \mathbb{R}_+$, and from this follows the uniform convergence $l_{F_n} \to l_F$. Q.E.D.

Proof of Proposition 3.5. This is an immediate consequence of the definition of the inverse Lorenz function. Q.E.D.

Proof of Theorem 4.1. According to (22), we have $l_F \geq l_G$ iff $\{LZ(F) + \mathbb{R}_- \times \mathbb{R}^d_+\} \supset \{LZ(G) + \mathbb{R}_- \times \mathbb{R}^d_+\}$. The latter inclusion is equivalent to

$$\phi_F(p_0, p) \geq \phi_G(p_0, p) \quad \text{for every } p_0 \in \mathbb{R}_+, p \in \mathbb{R}^d_-. \tag{23}$$

With respect to Lemma 2, (23) is equivalent to

$$\int_{p'\frac{x}{\mu_F}+p_0 \geq 0} \left(p'\frac{x}{\mu_F} + p_0\right) dF(x) \geq \int_{p'\frac{x}{\mu_G}+p_0 \geq 0} \left(p'\frac{x}{\mu_G} + p_0\right) dG(x) \tag{24}$$

for all $p_0 \in \mathbb{R}_+$, $p \in \mathbb{R}^d_-$.

Rewrite the Inequality (24) in the following form:

$$\int_{p_0 \geq t} (p_0 - t) dF_p(-t) \geq \int_{p_0 \geq t} (p_0 - t) dG_p(-t) \tag{25}$$

for all $p_0 \in \mathbb{R}_+$, $\quad p \in \mathbb{R}^d_-$. Due to homogeneity, $F_p(-t) = F_{-p}(t)$ holds.
According to the definition of the support function ϕ, we have $\phi_{F_p}(p_0, -1) = \int_{-t+p_0 \geq 0} (-t + p_0) dF_p(t)$. By this and Lemma 2, Inequality (25) is equivalent to that the ILF of G_p is below the ILF of F_p, $p \in \mathbb{R}^d_+$. Q.E.D.

Proof of Proposition 4.1. According to Theorem 4.1, $A \succ_{PL} B$ iff $l_A \geq l_B$. From the convexity of the subgraph of the Lorenz inverse function follows that the inequality $l_A \geq l_B$ holds iff $l_A(x) \geq l_B(x)$ holds for every extreme point x to the Lorenz hypersurface of F_B. In view of this, Proposition 3.2 yields the proof. Q.E.D.

Proof of Proposition 4.2. Because of (22) and properties of the support function, we have $F \succ_{PL} G_n$ iff, for any $p \in \mathbb{R}^d_-$, $p_0 \in \mathbb{R}_+$, $\phi_F(p_0, p) \geq \phi_{G_n}(p_0, p)$. Weak* convergence $G_n \to_w G$ (see the proof of Proposition 3.3) implies $\phi_{G_n}(p_0, p) \to_{n \to \infty} \phi_G(p_0, p)$, $p \in \mathbb{R}^d_-$, $p_0 \in \mathbb{R}_+$. Therefore, for any $p \in \mathbb{R}^d_-$, $p_0 \in \mathbb{R}_+$, $\phi_F(p_0, p) \geq \phi_G(p_0, p)$ holds, which is equivalent to $F \succ_{PL} G$. Q.E.D.

Proof of Proposition 4.3. Let us show that there holds

$$\begin{aligned} LZ(\alpha F_1 + (1-\alpha)F_2) + \mathbb{R}_- \times \mathbb{R}^d_+ &= \alpha(LZ(F_1) + \mathbb{R}_- \times \mathbb{R}^d_+) \\ &+ (1-\alpha)(LZ(F_2) + \mathbb{R}_- \times \mathbb{R}^d_+). \end{aligned} \tag{26}$$

Really, in view of Lemma 2, the support function to the set $LZ(\alpha F_1 + (1-\alpha)F_2) + \mathbb{R}_- \times \mathbb{R}_+^d$ in direction $p_0 \in \mathbb{R}_+$ and $p \in \mathbb{R}_-^d$ has the form

$$\int_{p'\frac{x}{\mu}+p_0\geq 0} \left(p'\frac{x}{\mu} + p_0\right) d(\alpha F_1(x) + (1-\alpha)F_2(x)).$$

The latter is equal to the sum of $\alpha \int_{p'\frac{x}{\mu}+p_0\geq 0}(p'\frac{x}{\mu} + p_0)dF_1(x)$ and $(1-\alpha)\int_{p'\frac{x}{\mu}+p_0\geq 0}(p'\frac{x}{\mu} + p_0)dF_2(x)$. That means that the support function to the set $LZ(\alpha F_1 + (1-\alpha)F_2) + \mathbb{R}_- \times \mathbb{R}_+^d$ is equal to the sum of the support functions to the sets $\alpha(LZ(F_1) + \mathbb{R}_- \times \mathbb{R}_+^d)$ and $(1-\alpha)(LZ(F_2) + \mathbb{R}_- \times \mathbb{R}_+^d)$. Therefore, (26) holds. According to Theorem 4.1 and Proposition A.1, there holds $G \succ_{PL} F_s$ for $s = 1$ and $s = 2$ iff $\{LZ(G) + \mathbb{R}_- \times \mathbb{R}_+^d\} \supset \{LZ(F_s) + \mathbb{R}_- \times \mathbb{R}_+^d\}$, $s = 1, 2$. By (26) and the monotonicity of the Minkowski sum,

$$\{LZ(G) + \mathbb{R}_- \times \mathbb{R}_+^d\} \supset \{LZ(\alpha F_1 + (1-\alpha)F_2) + \mathbb{R}_- \times \mathbb{R}_+^d\}.$$

Q.E.D.

Proof of Proposition 4.5. Let $F(x) = \min_{k=1,\ldots,d} F^{\{k\}}(x_k)$. Then there exist random variables $X_1, \ldots, X_d$ such that F is their distribution function and $X_k = (F^{\{k\}})^{-1}(F^{\{1\}}(X_1))$, $k = 2, \ldots, d$.

Show that

$$l_F(z) = \min_k l_{F^{\{k\}}}(z_k). \tag{27}$$

To see (27), consider first an empirical distribution, $F = F_A$. Then the matrix A is similarly ordered: For every i, $j = 1, \ldots, n$, either $a_i \leq a_j$ or $a_i \geq a_j$ holds in the usual componentwise order. In view of Proposition 3.2, the points $(\frac{k}{n}, \sum_1^k a_{(i)})$ belong to l_A. According to Koshevoy and Mosler (1996, Theorem 4.2),

$$pr_k(LZ(F) + \mathbb{R}_- \times \mathbb{R}_+^d) = LZ(F_k) + \mathbb{R}_- \times \mathbb{R}_+, \; k = 1, \ldots, d. \tag{28}$$

Hence the following inequality holds,

$$l_F(z) \leq \min_k l_{F^{\{k\}}}(z_k). \tag{29}$$

Therefore, $l_A(z) = \min_k l_{a^k}(z_k)$ in the empirical distribution case and, by Proposition 3.3, (27) is in general true. Now, from (29) follows that $G \prec_L \min_k G^{\{k\}}$. So, $F^{\{k\}} \succ_L G^{\{k\}}$, $k = 1, \ldots, d$, implies $\min_k F^{\{k\}} \succ_{PL} \min_k G^{\{k\}}$ and, hence, $\min_k F^{\{k\}} \succ_{PL} G$. If $F \succ_{PL} G$ then $F^{\{k\}} \succ_L G^{\{k\}}$, $k = 1, \ldots, d$, holds due to (28). Q.E.D.

Proof of Theorem 5.1. (i). In the case of absolute endowments, by analogy with Proposition A.1, we have $\widehat{l}_F \geq \widehat{l}_G$ iff $\{\widehat{Z}(F) + \mathbb{R}_- \times \mathbb{R}_+^d\} \supset \{\widehat{Z}(G) + \mathbb{R}_- \times \mathbb{R}_+^d\}$, where

$\widehat{Z}(F)$ is the lift zonoid of F (Koshevoy and Mosler (1998)). The latter inclusion is equivalent to

$$\int_{p'x+p_0\geq 0}(p'x+p_0)dF(x) \geq \int_{p'x+p_0\geq 0}(p'x+p_0)dG(x) \tag{30}$$

for all $p_0 \in \mathbb{R}_+$, $p \in \mathbb{R}^d_-$.

So, there holds (25) with absolute $\widehat{F}_p$ and $\widehat{G}_p$ in place of F_p and G_p. That yields the proof.

(ii). There holds $\widehat{Dl}_F \leq \widehat{Dl}_G$ iff $\{\widehat{Z}(F) + \mathbb{R}_+ \times \mathbb{R}^d_-\} \subset \{\widehat{Z}(G) + \mathbb{R}_+ \times \mathbb{R}^d_-\}$. The latter inclusion is equivalent to

$$\int_{p'x+p_0\geq 0}(p'x+p_0)dF(x) \leq \int_{p'x+p_0\geq 0}(p'x+p_0)dG(x) \tag{31}$$

for all $p_0 \in \mathbb{R}_-$, $p \in \mathbb{R}^d_+$. Rewrite the Inequality (31) as follows:

$$\int_{-p_0\leq t}(-p_0+t)d\widehat{F}_p(t) \leq \int_{-p_0\leq t}(-p_0+t)d\widehat{G}_p(t) \tag{32}$$

for all $p_0 \in \mathbb{R}_-$, $p \in \mathbb{R}^d_+$. Therefore, Inequality (32) is equivalent to that the DIGLF of $\widehat{F}_p$ is below the DIGLF of $\widehat{G}_p$, $p \in \mathbb{R}^d_+$. Q.E.D.

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