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Entrepreneurship and regional economic growth: Towards a general theory of start-ups

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Abstract

Start-ups of new firms are important for economic growth. However, start-up rates differ significantly between countries and within regions of the same country. A large empirical literature studies the reasons for this and attempts to identify the regional determinants of start-ups. In contrast, there is a much smaller theoretical literature that attempts the formal modelling of the start-up process within a region. In this paper, we attempt to contribute to this small literature by introducing a general theoretical model of the entrepreneurial start-up process. The model links start-ups to economic growth and can be applied to understand growth in a regional context. We derive five propositions that fit the stylized facts from the empirical literature: (i) growth in the regional economy is driven by an expansion in the number of start-up firms that supply intermediate goods and services; (ii) improvements in human capital will enhance the rate of start-ups; (iii) improvements in the relative rates of return to entrepreneurs and business conditions will raise start-up rates; (iv) an increase in regional financial concentration will reduce the start-up rate in a region and; (v) increased agglomeration/urbanization in a region has an a priori ambiguous effect on start-up rates.

Keywords: start-ups, entrepreneurship, frictions, economic growth

JEL classification: M13, O12, O40, L26
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1 Introduction

Entrepreneurship,\(^1\) the ‘process of starting and continuing to expand new businesses’ (Hart 2003: 5) is important for economic growth and development (Naudé 2008; Baumol et al. 2007; Schramm 2004).\(^2\) Substantial resources are devoted in developed as well as developing countries on raising the rate of entrepreneurship (Blanchflower and Oswald 1998: 27). The rate of entrepreneurship can be measured statically as the self-employment rate or dynamically as the start-up rate of business firms (Wennekers et al. 2002: 29). The start-up rate is increasingly seen as important, if not more so, as the growth of existing firms. They are more likely to grow (Lingelbach et al. 2005; Johnson et al. 2000), to create new jobs (Audretsch et al. 2006: 25; McMillan and Woodruff 2002: 166), and to promote new and flexible organizational forms (Kim et al. 2006: 20).

In many transition countries, where there were no significant private sectors to start out with, new firms often strengthened reforms by improving economic conditions (McMillan and Woodruff 2002: 153).

It is therefore understandable that many countries and regions devote increasing resources towards raising the rate of start-ups. It is however the case—and here the empirical evidence is fairly robust—that start-up rates of new firms differ significantly between countries and even between different regions of a country (Reynolds et al. 1994; Tödtling and Wanzenböck 2003; Stam 2006; Naudé et al. 2008). Given the relationship between entrepreneurship and employment creation (Fonseca et al. 2001) this difference in regional start-up rates can be an important reason for regional disparities (Naudé et al. 2008). A large empirical literature attempts to identify the determinants of start-ups, most of this focusing on the regional context. In contrast, there is a much smaller theoretical literature which attempts the formal modelling of the start-up process. Existing models aim to explain various aspects or obstacles of start-ups such as start-up costs (Fonseca et al. 2001), firm growth and size dynamics (e.g., Lucas 1978; Jovanovic 1982), credit constraints (e.g., Banerjee and Newman 1993; Evans and Jovanovoc 1989), and exit decisions (e.g., Bosma et al. 2005).

A general theory of entrepreneurial start-ups, and the effect that these might have on regional economic growth, is however still lacking. In this paper, we attempt to fill this gap, by formulating a theoretical model of the entrepreneurial start-up process. The model is a dynamic, non-linear model in contrast to many studies which have used linear models to empirically predict new firm births or introduce borrowing constraints in a homogeneous linear form (Reynolds 1993: 19; Mesnard and Ravallion 2005: 3). A further novel aspect is that we link start-ups to regional output growth in our model—an aspect that is paid scant attention in the theoretical literature. We derive a number of propositions and testable hypotheses from our model.

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\(^1\) There are many different views on the nature of entrepreneurship. Wennekers and Thurik (1999: 30) identify thirteen distinct roles of an entrepreneur. More commonly, however, and for purposes of this paper, entrepreneurship is seen as involving either/or risk-taking, innovation, arbitrage and co-ordination of production factors in the start-up of a business. (Fogel et al. 2006: 542).

\(^2\) According to Iyigun and Rodrik (2004: 1-2) ‘a key obstacle to growth in low-income environments is an inadequate level of entrepreneurship in non-traditional activities’.
The paper is structured as follows. Section 2 contains a brief overview of the relevant literature. Section 3 presents our theoretical model and Section 4 contains some concluding remarks.

2 The determinants of start-ups

In this section we provide a brief review over the literature on the regional determinants of entrepreneurial start-ups. The purpose is to identify the main determinants of start-ups. In the next section we build on this literature by formalizing start-ups in a general endogenous growth model.

At the outset it should be noted that most empirical studies on the determinants of start-up rates make use of regional data within a particular country, because comparisons between countries are often difficult due to non-comparable data and definitions (Reynolds 1993: 11). One can distinguish between studies that attempt to identify macro level constraints or determinants, those focusing on industry level and firm level, and those that attempt to test the predictions from theoretical models on the relationship between wealth, inequality, credit market constraints, and entrepreneurship.

On a macro level, evidence tends to suggest that macroeconomic conditions such as the state of employment, GDP and inflation, and interest rates matter. Highfield and Smiley (1987) empirically analysed the determinants of start-ups in the USA over the period 1948 to 1984, using rates of incorporation as an indicator of start-ups. They found that on a macroeconomic level, five factors were significant in affecting start-up rates: real GNP growth, expenditure on equipment (investment), the unemployment rate, real interest rates and inflation. Wang (2006) finds that in Taiwan production costs, capital costs, and unemployment are significant determinants of start-up rates. Bosma et al. (2005) using data for the Netherlands over 1960 to 1999 find that unemployment is statistically significantly associated with higher start-up rates. Macro level conditions will apply to all regions in a country, but will have different spatial implications to the extent that economic activity agglomerates, and to the extent that industry level composition and dynamics will differ across regions.

Reynolds (1992) found that industry level characteristics such as industry profits and entry barriers affect start-ups rates. Reynolds (1993) and Reynolds and Storey (1993) identified six broad determinants of entrepreneurial start-ups in a region, namely demand (measured by the size of the population and size of the economy), urbanization (agglomeration), unemployment, personal household wealth, specialization (industry level differentiation), and government spending on infrastructure, education, and health.

On an industry level, studies on Belgium, Germany, Korea, Norway, Portugal, and the UK have found a ‘mixed pattern’ of determinants/frictions. Table 1 below, taken from Reynolds (1992: 278), summarizes the significant frictions to start-ups found in these five country studies.
Table 1: Determinants of start-ups: empirical evidence from five countries

<table>
<thead>
<tr>
<th>Start-up frictions</th>
<th>Number significantly positive</th>
<th>Number significantly negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected profits</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Industry growth</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Product differentiation (marketing)</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Scale economies</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Industry size</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Capital requirements (finance)</td>
<td>–</td>
<td>4</td>
</tr>
</tbody>
</table>


Table 1 shows that generally, profits, industry growth, and industry size can be expected to be positively related to the rate of start-ups, and the capital requirements, the need for product differentiation can be expected to be negatively related to start-up rates. Scale economies seem to have an a priori ambiguous effect, although, as pointed out by Reynolds (1992: 282), start-up rates tend to be higher in urban areas than elsewhere, suggesting that agglomeration may matter for start-up rates. Agglomeration may have negative effects on start-up rates once an industry level perspective is taken, since incumbents in an industry might reduce what has been termed ‘market room’ or market potential. Evidence from the Belgian retail industry suggests that start-ups in this industry are affected by this market room (Carree and Dejardin 2007).

Firm level/individual level studies generally point to the importance of human capital, which has been confirmed as important for start-up rates in a number of theoretical models discussed in Section 2.1. Kim et al. (2006: 9) recognize that educational achievement may act as a signal for quality of human capital as it provides specific skills which will be needed in the new venture, it raises the status of the recipient, and it acts as a sorting mechanism. These benefits of human capital are important as mechanisms for reducing or overcoming frictions related to finance (educated entrepreneurs may find it easier to obtain credit), marketing (skilled entrepreneurs can better identify their market niche), network forming (educated entrepreneurs may have higher social standing and so better networks) (see also Backes-Gellner and Werner 2007). Indeed, evidence has been found to substantiate the view of ‘entrepreneurial ability’ in the Evans and Jovanovic (1989)-type models as overcoming frictions due to imperfect capital markets. This reflects the view that ‘one cannot invoke capital market imperfections to explain deficiencies of entrepreneurship... for part of the role of an entrepreneur is precisely to overcome such factor-market imperfections’ (Leff 1979: 47–48).

Related to human capital, but less easy to quantify or measure, is the effect of cultural factors and the entrepreneurs’ background. A number of studies have found that children whose parents were entrepreneurs are more likely to become entrepreneurs themselves (Giannetti and Simonov 2004; Dunn and Holtz-Eakin 2000; Davidsson and Honig 2003). However, for other cultural factors the evidence may be mixed. Research based on the Panel Study of Entrepreneurial Dynamics (PSID) in the USA found that cultural background was not a significant determinant of entrepreneurial entry, but that education and experience was (Kim et al. 2006: 18).
In the literature, the notion of start-up costs feature importantly (e.g., Mesnard and Ravallion 2005; Fonseca et al. 2001; 2007). These costs are seen as general costs such as legal compliance, red-tape, etc. to be incurred in formalising a business. Fonseca et al. (2007) use data from nine countries to find support for the view that start-up cost weakens the relationship between wealth and start-up rates.

Other determinants of start-up rates that have been identified on a firm level in the literature include tax rates. Reynolds (1992: 282) report that earlier USA studies have found that only property tax rates had a significantly negative effect on start-ups, with income tax being insignificant as a determinant or friction in the start-up process. More recently however Gentry and Hubbard (2000) found that progressive marginal tax rates discourage start-up rates and that the size of this effect will depend on the gap between taxes on wages and taxes on profits (assuming that there are no better opportunities for tax evasion in self-employment than in wage employment).

Finally, a large literature focuses on the effects of wealth and financial constraints of entrepreneurial start-ups, stemming from the work of Evans and Jovanovic (1989) and Banerjee and Newman (1993). In their models, individual wealth is a significant predictor of the probability to start a firm, and this is interpreted as due to credit market constraints.

While capital requirements are significant in the earlier studies reported by Reynolds (1992) and summarized in Table 1, the overall empirical evidence for the significance of financial constraints on start-ups have been mixed and on the negative side (Kim et al. 2006: 7; Munshi 2007; Paulson et al. 2006). Furthermore, Lloyd-Ellis and Bernhardt (2000: 160) point out that the frequent complaints by entrepreneurs captured in surveys about lack of access to finance ‘often masks technical and managerial inadequacies’. In fact, some authors doubt whether financial constraints are that binding on start-ups, and see it to be more important in the expansion of existing businesses rather than in the creation of new firms (Estrin et al. 2006: 702). Kim et al. (2006: 7) have also noted that start-up requirements, even in developed countries, may be quite low—so that they can be met through own savings, credit from family members and informal borrowing. In the words of Hurst and Lusardi (2004: 321) ‘even if some households that want to start small businesses are currently constrained in their borrowing, such constraints are not empirically important in deterring the majority of small business formation in the United States’.

Second, Banerjee and Newman (1993) suggest increased start-up rates (self-employment) as economic development levels improves, due to the fact that individuals now have on average greater access to financial resources. Empirical evidence from the GEM survey found however that there is a U-shaped relationship between start-up rates and level of economic development as measured through per capita income (Wennekers et al. 2005). Thus, in poor countries, start-up rates are high, and when a country develops, the start-up rate falls as more opportunities for (safer) wage-employment are created (Iyigun and Owen 1999: 454). After a certain threshold is reached, start-up rates start to increase again. The threshold that is relevant here refers to structural changes in the economy: much of the increases in start-ups in richer countries have been due to the development of the services sector, as well as due to changes in the structure of demand, as consumers demand more individualized goods where economies of scale do not apply (Wennekers et al. 2005; Bosma et al. 2005). This suggests that macro level factors
such as the state of wage employment and the structure of the regional economy are important determinants of the start-up rate.

In the next section, we present a general model of entrepreneurial start-ups, within an endogenous growth framework. We attempt to incorporate the insights from the literature surveyed in this section, and show how regional policies may affect the rate of start-ups.

3 The model

Most of the studies into start-ups as surveyed in the previous section have focused on particular determinants or obstacles to start-ups. They have also, despite the fact that most empirical studies of determinants have made use of regional data, not explicitly considered the regional (spatial) dimensions of the decision to start up a firm. In the model that we set out in this section, we attempt to address these shortcomings by constructing a general model that explains the most salient determinants of start-ups, and by incorporating a regional (spatial) dimension. The latter is based on the notion that when different regions (space) of a country are introduced into a model, it allows for space to permit regional monopolies of products and services to come into existence, wherein entrepreneurs can fulfil the functions of creating (innovating) new goods and services and imitating goods and services from other regions (arbitrage/disequilibrium). Thus our entrepreneur acts according to both the Schumpeterian and Kirznerian notions of entrepreneurship.

3.1 Aggregate production process in the region

In our model entrepreneurs start up small firms which we assume supply intermediate goods to a final goods sector. Intermediate goods include activities such as services, maintenance work, craft and trade, transport and logistics, commerce activities, and engineering. These small firm activities are local, and offered to the local producer of the final product. In treating start-ups as providers of intermediate goods, our model is related to that of Ciccone and Matsuyama (1996: 34) who show that if a particular (regional) economy produces a limited range of intermediate goods, the final (consumer) goods sector will be characterized as ‘primitive’ in production methods and will have little demand for sophisticated, new inputs. This will lead to lower incentives for potential entrepreneurs to start-up new firms. The economy can get stuck in such an underdevelopment trap with primitive production in its (small) modern sector. They also point out that there might, in such an ‘underdevelopment trap’ be a case for assistance to new start-ups since these can provide both pecuniary and technological externalities if they start producing new intermediate goods—which will induce final good producers to demand more of these (which will improve the incentives for other entrepreneurs to start up firms due to greater demand and the example provided in the application of new technology). In their model, as in ours, start-ups face positive start-up costs which include the costs of developing a product variation and all other cost of bringing a new good to the market.

Thus in a well-defined economic region a number $N$ of these small businesses exist, each producing a certain type of differentiated intermediate goods $x_i$. The aggregate
final product $Y$ of the region is produced with local human capital $H$ and $N$ local intermediate inputs $x_{ij}$ supplied by $N$, the small, local firms. We also assume here human capital is immobile between regions.

The production function\(^3\) for the representative local final product producing firm $i$ is

$$Y_i = AH_i^{1-\alpha} \sum_{j=1}^{N} (x_{ij})^\alpha \quad (1)$$

where $A$ represents the degree of urbanization. It represents the positive external effects from localization and urbanization economies on the firm's total factor productivity. Growth is driven by an expansion in $N$, denoting the number of small firms in the market and hence the number of different intermediate goods available. Suppose that all private firms have the same size, $x_{ij} = x_i$ (which is true in equilibrium) then the quantity of output is given by $Y_i = AH_i^{1-\alpha} Nx_i^{\alpha}$ which is a positive function on $N$. In order to focus on determinants of start-ups the final output sector can be modelled rather simply. Specifically, we propose a continuum of final good producing firms supplying to a competitive final goods market in the closest urban centre available to the firms in the specified region. The market price for the final good is given at price $P$.

Firms producing the final good maximize profits according to the profit function $\Pi^f_i = PY_i - \nu H_i - \sum p_j x_{ij}$ with $p_j$ denoting the price of intermediate inputs $x_{ij}$ and $\nu$ denoting the human capital wage rate. For simplicity we omit the representative firms index $i$ and assume the market for human capital to be competitive. Using the first order conditions we can derive the demand for intermediate inputs, namely

$$x_j = H \left( \frac{PA\alpha}{P_j} \right)^{\frac{1}{1-\alpha}} \quad (2)$$

and determine the wage rate of regional human capital, namely

$$\nu = (1-\alpha) \left( \frac{PY}{H} \right) \quad (3)$$

### 3.2 Start-up of firms

A start-up firm will offer a new intermediate product variation to the final output sector. Each product variation has certain properties that make the variation unique compared to other already existing variations. In order to get the new product to the market, the start-up firm has to invest in start-up costs. Start-up costs are a barrier to entry to the

\(^3\) This specification of a production function originates from Ethier (1982). Similarly, Romer (1987) used this specification to model technological change and growth, driven by newly invented variations of productive inputs.
market and by overcoming these, the entrepreneur opens up a new market segment. Start-up costs include all costs such as initial capital endowment information and organization and management costs, administrative costs, costs of acquiring and developing a business idea and business plan suitable to obtain finance from a financial intermediation.

In this model we assume that start-up costs $\chi$ are determined by the degree of urbanization $A$ and the density of product variations already in the market indicated by the density of competitors of the SME $\delta = N/Y$. With an increasing density of competitors, it becomes more difficult and hence costly to set up a new variety. For the urbanization parameter $A$ both, a negative as well as a positive effect can be possible. An increasing degree of urbanization will increase the potential ability of the final product sector to absorb additional varieties. However, more urbanization also may lead to increasing congestion. Hence at this stage, we leave it open if one or the other effect is dominant

$$\chi = \chi(\delta, A) = \varepsilon(A)\delta$$

In addition to start-up costs there are permanent costs to operate the business. These costs are denoted by $c_x$. These can also include the opportunity costs of starting a firm and becoming self-employed. Due to these start-up costs, once a small firm is set up it will remain the only supplier for the specific good and remain a monopolist for the specific product variation.

Since there is no immediate income, start-up costs must be credit financed at the loan rate $r_l$. We assume that external financing is the only viable option. To simplify, we assume a firm that revolves loans infinitely and services interest only (i.e., Ponzi finance is excluded). Denoting the deposit rate $r_d$, the present value of set-up costs ($V_s$) including finance is

$$V_s = \chi(\delta, A) \frac{r_l}{r_d}$$

(4)

3.3 Activities of start-ups

Due to barriers of entry and costs of challenging an existing market, every new firm can act as a local monopoly. Each period’s profits are determined by the price of the product variation $p_j$ and the periodic costs $c_x$. These periodic operating costs per $x$ are assumed to be the only production costs per unit output of the intermediate good. These costs may also include opportunity costs of being an entrepreneur or a satisfying entrepreneurial net income which is instantaneously consumed. Hence net periodic profits are $\pi^*_j = p_j - c_{sj}$ and the expected net present value of such a monopoly $V_m$ is

$$EV_m(t) = (1 - \vartheta)\int_0^\infty (p_j - c_{sj})x_je^{-\tau_\vartheta(u)(u-t)}du$$

(5)
where \( \vartheta \) represents the given individual probability of business failure and \((1 - \vartheta)\) is an indicator for market opportunities and hence the probability of success. Monopoly profits are maximized by the optimal choice of the intermediate good price \( p_j \)\(^4\)

\[
p_j = \frac{c_{ij}}{\alpha} \tag{6}
\]

As a result, the net present value of a new firm is

\[
EV_m(t) = \frac{\left(1 - \vartheta\right)(1 - \alpha)}{r_d} H \left( \frac{PA\alpha^{1+\alpha}}{(c_x)^{\alpha}} \right)^{\frac{1}{1-\alpha}} \tag{7}
\]

As long as there is no steady state equilibrium, start-up entrepreneurs realize a net rent. However, in a steady state equilibrium the net present value of the new firm will just cover total set-up costs which are \( V_s = V_m \). Thus periodic monopoly rents are eventually fully distributed as income of the entrepreneur and under competition used to finance start-up costs. We can extend this to take into account non-pecuniary benefits of entrepreneurship but for the sake of simplicity leave this for future elaboration.

With respect to financial markets, start-up activities from the private small business sector lead to a perfectly elastic loan demand\(^5\)

\[
r_i = \frac{(1 - \vartheta)(1 - \alpha)}{\varepsilon(A)} H^2 \left( \frac{\alpha^{1+\alpha}}{p^{1-\alpha}c_x^{1-\alpha}} \right) \frac{1+3\alpha}{A^{1-\alpha}} \tag{8}
\]

### 3.4 Households

The description of households linked to the final output sector is standard. The representative household maximizes a constant relative risk aversion (CRRA) utility function

\[
U = \int_0^\infty \frac{C_t^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt
\]

with \( \rho \) denoting the time preference and \( 1/\theta \) indicating the intertemporal elasticity of substitution. As the aggregate income flow consists of interest income from deposits \( Dr_d \) and rental income \( \nu H \), the budget constraint is given by

\[\text{4 See Appendix 1a.}\]

\[\text{5 This is a result of the equality of the firm’s net present value (7) and total set-up costs (4), see Appendix 1b.}\]
This income can be consumed or saved on deposits. Optimization results in the familiar Ramsey rule

\[ \gamma_c = \frac{r_d - \rho}{\theta} \]  

(9)

where \( \gamma_c \) is the growth rate of consumption.

### 3.5 Financial intermediation

Financial intermediation occurs when an economic agent transfers surplus funds (savings) from households and firms to borrowing households and firms in the form of loans. In the formal market economy the function of a financial intermediary is performed by banks. Following the contribution of Stiglitz and Weiss (1981) it is widely accepted that informational asymmetries and agency problems can result in newer, smaller firms finding it difficult to access sufficient external finance, i.e., being credit rationed (Bonnet et al. 2005: 2).

The problem of small firms being credit rationed can be more severe in remote regions of a country where sometimes no more than one or two operating banking groups exist, and where monitoring costs can be prohibitively large. Also, proximity to banks in lagging regions is often limited. As a consequence access to formal bank borrowing may be an obstacle to the start-up of small businesses, with entrepreneurs having to consider other, more informal, options to obtain start-up funding (Mason and Harrison 2002).

However, a weakness of informal finance is that the informal financial sector tends to have a highly concentrated structure (Montiel et al. 1993: 1). A region with a high concentration market power by financial intermediaries (formal and informal) will face high credit rationing of start-ups. Hence, this region will have low start-up rates, low investment, and overall lower economic growth than regions with better access (less concentration, more collateral). Over time this differential access to financial intermediation can lead to widening income inequalities between different regions (Orford et al. 2003: 17).

In order to model concentration of market power in the financial market we assume a historical given number of banks \( B \). Each bank \( b \) offers deposits \( D_b \) to households and loans \( K_b \) to potential start-up firms. Banks have symmetric monitoring costs \( c \). The profit function of bank \( b \) is given as

\[ \pi_b = (1 - \vartheta) r_l K_b - r_p(D) D_b - c_b K_b - \bar{c} \]  

(10)

where \( r_p(D) \) is the deposit demand function and \( D \) denotes total deposits in the region, and \( \vartheta \) is the expected default rate of the loans given to SMEs. As the bank modelled as
a pure intermediary the balance sheet of the bank can be written as $K_b = D_b$ when the
bank maximizes profits subject to the balance sheet constraint. Then, the first order
condition (FOC) is

$$\frac{\partial \pi}{\partial D_b} = (1 - \theta) r_i - r_d(D) + \frac{dr_d}{dD} \frac{\partial D}{\partial D_b} D_b - c_b = 0$$

Rearranging the FOC and using the definition of total deposits ($D = BD_b$) and the
definition of the elasticity of the deposit demand function

$$\eta = -\frac{dD r_d}{dr_d D} = \text{const.}$$

we obtain an optimal deposit rate for banks offered to the public

$$r_d = \left( (1 - \theta) r_i - c_b \right) \left( 1 + \frac{1}{B \eta} \right) \quad (11)$$

The solution of the banks’ optimization problem gives a region’s loan-deposit rate
spread. As can be seen from Equation (11) a region’s spread is determined by two
factors, namely the costs of monitoring ($c$) and the concentration of banks measured by
the index $(1 + 1/B \eta)$. A lower number of banks in a region will increase the concentration
of financial intermediaries and widen the interest spread.

In theory we are free to go one step further by looking at full market equilibrium. We
would endogenously determine the number of banks and apply the zero profit condition
to the banking sector. This would take away the pressure for market entry by additional
banks. But, as we would like to focus on the number of banks rather than on some
additional factors which might determine this number and the interest spread in the
market, we end our analysis of the financial intermediation at this stage.

3.6 Solving the model

So far we have three central equations to solve for. First, we have a fully elastic loan
demand function (see Equation 8)

$$r_i = \frac{(1 - \theta)(1 - \alpha)}{\varepsilon(A)} H^2 P^{1 - \alpha} C_{1 - \alpha} \alpha \frac{-2 \alpha}{3 - \alpha} \frac{1 + 3 \alpha}{3 - \alpha} A^{1 - \alpha}. \quad \text{(Loan demand curve)}$$

Second, we have the Ramsey Rule for intertemporal optimal decisions (see Equation 9).

$$\gamma_c = \frac{r_d - \rho}{\theta} \quad \text{(Ramsey Rule)}$$

Third, from the financial sector characterized by concentration (defined as a small
number of banks) we obtain the loan-deposit rate spread (see Equation 11) for the
supply of loans by banks. Hence, we obtain the loan supply function
Proposition 1: The loan demand function (8), the loan supply function of the banking sector (11), and the Ramsey Rule (9) determine the steady state growth rate of the SME sector \( \gamma_N \) as well as the regional growth process in general \( \gamma_Y \) (14):\(^6\)

\[
\gamma_Y = \gamma_N = \frac{1}{\theta} \left[ \left( 1 - \vartheta \right)^2 \left( 1 - \alpha \right) \frac{\alpha}{\varepsilon(A)} c_1^{1+\alpha} A^{1-\alpha} - c_h \right] \left( 1 + \frac{1}{B\eta} \right) - \rho
\]

The resulting equilibrium for the particular region can be graphically illustrated as in Figure 1. Equations (8) and (11) represent the economy in the loan market and with (9), and the steady state assumption we can determine the steady state growth rate for the number of start-ups in the region, as well as the growth rate of the region in general (see Proposition 1).

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\(^6\) For a proof see Appendix 1c.
4 Implications

The model constructed in Section 3 allows us to investigate the effects of various frictions on the start-up rate of firms. In the sub sections that follow we will illustrate the impact of profits, education, and agglomeration on the start-up of firms. We will use Figure 1b to illustrate these effects. Both in endogenous growth theory and the new economic geography these factors are important determinants of regional growth, inequality, and divergence, and as we proceed we will justify the focus on these factors as determinants of entrepreneurial start-ups.

4.1 Education

Proposition 2: An increase in human capital in the region will raise the rate of start-ups:

\[ \frac{d\gamma_N}{dH} > 0 \]

The importance of education for start-ups builds on the spill-over models following Lucas (1988). It suggests that entrepreneurial start-up rates will be higher in localities that have a relatively highly educated population and an economic environment favourable to the accumulation of knowledge.

The education level in a region or country (e.g., as measured by the proportion of the population with a particular qualification) has a clear effect on the equilibrium growth rate of start-ups in our model, as is illustrated in Figure 1b. Increasing human capital will increase marginal productivity of intermediates in the final product sector. Hence the demand for intermediates increases, and the Net Present Value of each product variation potentially increases. As a result, small firms are able to pay a higher loan rate—this results in the loan demand function shifting upwards in Figure 1b. With increasing deposit rates in credit market equilibrium, the supply of financial resources will increase and hence also the start-up rate of new businesses.

4.2 Profits and business environment

Proposition 3: Any factor that improves the profits of start-up firms will raise the start-up rate:

\[ \frac{d\gamma_N}{d\pi_j} > 0 \]

An important assumption in the model is that start-up firms are local monopolies for product variations of intermediate goods in their region. For these firms the present value of the sum of positive periodic profits \[ \pi_j^+ = (p_j - c_y)\kappa_j \] is maximized. If these periodic profits increase due to, for instance, decreasing periodic costs \( c_y \), start-ups are able to pay higher loan rates. In this case, the loan demand curve in Figure 1b shifts

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7 For a proof see Appendix 2.
8 For a proof see Appendix 3.
upwards. With a higher deposit rate start-ups have better access to financial sources and the growth of the number of start-ups will accelerate.

Proposition 4: An improvement in the business environment and hence an increasing chance of a successful survival of the start-up firm \((1 - \vartheta)\) will raise the start-up rate:

\[
\frac{d\gamma_N}{d\vartheta} < 0 \tag{9}
\]

As far as the business environment is concerned, our model captures these by the start-up success rate \((1 - \vartheta)\). The value of \((1 - \vartheta)\) may differ for a variety of reasons—such as uncertainty, political instability, including a deterioration in the quality of entrepreneurs. Opportunities for start-ups to supply intermediaries are particularly positive if the industry sector is in good shape. However, in this model, the failure rate is exogenous. With a decreasing rate of failure, the start-up firms would be able to finance a higher loan rate and the loan demand curve in Figure 1b shifts upwards. Further, from the viewpoint of the banks a decreasing default rate would increase the deposit rate the banks are able to pay and hence let the loan supply curve turn right. The banks are able to offer a higher deposit rate for each loan rate. The total effect is an increase in the growth rate of start-ups.

4.3 Agglomeration, competition, and congestion

For the ‘once-off’ start-up costs \(\chi\) we assume that these costs are determined by the degree of urbanization \(A\) and the density of start-up (small firm) competitors \(\delta = N / Y\) and hence product variations already in the market. With a higher density of competitors, it becomes more difficult and hence costly to set up a new variety (e.g., more needs to be spent on finding an additional niche).

Proposition 5: The effect of urbanization and agglomeration is ambiguous as increasing congestion costs \(-\eta_{e,A}\) may overcompensate positive externalities from agglomeration effects.\(^{10}\)

Thus, for the urbanization parameter \(A\) both a negative as well as a positive effect can be possible. An increasing degree of urbanization will increase the potential ability of the final product sector to absorb additional varieties. Increasing urbanization also may lead to increasing congestion. Hence at this stage, we leave it open if one or the other effect is dominant.

The inclusion of urbanization and agglomeration is based on explanations of entrepreneurship that emphasize market size, spill-overs, and imperfect competition. Henderson (2000: 2) points out that a high degree of urban concentration is essential for a country to kick-start industrial development. In the endogenous growth literature, urban agglomerations are important for entrepreneurial activity because they provide these dynamic information externalities that are important for innovation (as in Romer

\[^{9}\text{For a proof see Appendix 4.}\]

\[^{10}\text{For a proof see Appendix 5.}\]
1987 and Lucas 1988). A large strand of the literature deals with the benefits of agglomeration for the birth and growth of new firms (see e.g., Tödtling and Wanzenböck 2003: 352–3). However, there are other factors, often ignored in the literature, which can work in the opposite direction. With increasing economic size and density congestion costs will increase. Congestion costs include higher levels of competition and barriers to entry and can also reflect a failure of small start-up firms to benefit from knowledge spill-overs due to agglomeration (Acs and Varga 2004). Tödtling and Wanzenböck (2003: 353) identify another channel through which increasing agglomeration can have a discouraging effect on start-ups, namely through the establishment of ‘lock-in’ situations due to industrial specialization. They point to the dominance of sectors with high entry barriers and the absence of specialized business services needed for new firm start-ups.

In our model both, benefits and disadvantages of urbanization and agglomeration via increasing congestion costs, are covered. An increase in $A$ is beneficial as long as positive externalities overcompensate potentially increasing congestion costs. In this case the loan demand curve shifts upward with an increasing $A$. Increasing congestion costs indicated by $\eta_\epsilon \eta_\eta$ are covered by the parameter $\epsilon(A)$ which potentially raises start-up costs with an increase in the agglomeration parameter $A$. If congestions costs become high enough the loan demand curve in Figure 1b shifts downwards.

Which effect will dominate may largely be an empirical question, and differ between localities. Naudé et al. (2008) find evidence of congestion effects in the metropolitan areas of South Africa.

4.4 Concentration of market power in financial markets

Proposition 6: An increase in regional financial concentration will reduce the start-up rate in that region:11 $$\frac{d\gamma_N}{dB} < 0$$

We illustrate the effect of access to finance (and bank concentration) with the help of Figure 1b. In the figure concentration of market power is indicated by the number of banks $B$ operating in the local financial market. This reflects bank concentration and monopoly power, as well as the physical proximity of entrepreneurs to banks. With a decreasing number of banks (increased physical proximity) and an increasing local market power the loan supply curve shifts upwards in Figure 1b. The equilibrium spreads and loan rate increases and hence the start-up rate declines due to more expensive loans. It can be noted that a similar effect will occur, if for some reason, a bank’s monitoring costs increase—which could result if the number of potential entrepreneurs which can be serviced by each branch increases. Increasing financial concentration and lower proximity to a bank would therefore be a friction that reduces the start-up rate of firms.

11 For a proof see Appendix 6.
5 Concluding remarks

Start-up rates differ significantly between countries and within regions of the same country. A large empirical literature studies the reasons for this and attempts to identify the regional determinants of start-ups, and the obstacles or impediments to start-ups across different regions. However, the theoretical literature which attempts the formal modelling of the start-up process within a region and its impact on regional growth is much smaller. In this paper, we try to contribute to this small literature by introducing a general endogenous growth model of the entrepreneurial start-up process. The model links start-up frictions to economic growth and can be applied to understand growth in a regional context.

Our model has two crucial ‘foundational’ assumptions. We assume that when spatial considerations come into play, alert entrepreneurs can identify region specific opportunities and by utilizing these become local monopolists. In doing this they can imitate technological innovations from other regions, or innovate. We also assume that these entrepreneurs are especially active in supplying intermediate goods and services to final good producers. This distinction has been shown in the literature to be important to link entrepreneurs with qualitative and structural economic change in a region. It allows us to relate increases in the number of start-ups (which correspond to greater diversity in intermediate goods) to increases in economic growth.

As in basic models of entrepreneurial occupational choice, access to finance plays an important role in our model. We generalize our model to include other constraints by linking these to the availability of finance. Thus for instance our model is able to show, as in Fonseca et al. (2007), that higher start-up costs reduce start-up rates through a capital market effect. As the start-up costs need to be financed, the access to external finance becomes important. Even if the financial intermediation is called sometimes ‘bank’ in this model, it is easy to think of other lenders (even the Mafia) to the start-up entrepreneur. The important issue addressed is that there is market concentration and hence inefficiency in the financial sector. Further, we derived propositions that education, the relative opportunity costs of being an entrepreneur, and the business environment are positively related to start-ups. We also show that agglomeration (urbanization) is ambiguously related to start-ups, and that regional financial concentration will have an adverse impact on start-ups. In this latter regard, we are one of a few models concerned about the impact of financial sector concentration on regional economic development.

We set out to provide a basic general theoretical model that could account for the start-up process in a manner that allows for regional distinctions. Further extensions could include linking it to the structural economic transformation experienced by economies as they develop (so that start-ups not only influence the economic growth rate, but also the structural composition and rate of urbanization in the economy), by allowing for growth in firm sizes, and to model the process of firm exit.
References


Appendix

Appendix 1a: Maximize the expected present value of the cash flow, of the new firm including probability of success \((1 - \vartheta)\) and failure \(\vartheta\):

\[
EV_m(t) = (1 - \vartheta) \int_0^{\infty} (p_j - c_x)x_j e^{-r_d(v;x)(v-t)} dv + \vartheta 0,
\]

\[
= \frac{(1 - \vartheta)}{r_d} (p_j - c_x) H_i \left( \frac{Aa}{p_j} \right)^{\frac{1}{1-\alpha}}
\]

\[
\max_{p_j} : EV_m(t) \to \frac{dEV_m}{dp_j} = 0
\]

\[
\frac{dEV_m}{dp_j} = \frac{(1 - \vartheta)}{r_d} H_i \left( \frac{Aa}{p_j} \right)^{\frac{1}{1-\alpha}} - \frac{1}{1 - \alpha} \left( \frac{1 - \vartheta}{r_d} (p_j - c_x) H_i \left( \frac{Aa}{p_j} \right)^{\frac{1}{1-\alpha}} \right)
\]

\[
= \frac{(1 - \vartheta)}{r_d} H_i \left( \frac{Aa}{p_j} \right)^{\frac{1}{1-\alpha}} \left[ 1 - \frac{(p_j - c_x)}{1 - \alpha} \frac{1}{p_j} \right] = 0
\]

\[
p_j = \frac{c_x}{\alpha}
\]

Determining the resulting periodic profit: \(\pi_j\) and profit reactions:

\[
\frac{d\pi_j}{dc_x} \quad \pi_j = (p_j - c_x)x_j \quad \text{with} \quad c_x = \bar{c}_x
\]

\[
= \left( \frac{1}{\alpha} - 1 \right) c_x H_i \left( \frac{Aa^2}{c_x} \right)^{\frac{1}{1-\alpha}}
\]

\[
= (1 - \alpha)HA \frac{1}{1-\alpha} \alpha^{\frac{\alpha}{1-\alpha}} c_x^{\frac{-\alpha}{1-\alpha}}
\]

\[
\frac{d\pi_j}{dc_x} = -\alpha HA \frac{1}{1-\alpha} \alpha^{\frac{\alpha}{1-\alpha}} c_x^{\frac{-\alpha}{1-\alpha}}
\]

Determining the expected maximum present value of the new firm:

\[
EV_m = \frac{1 - \vartheta}{r_d} (p_j - c_x) H_i \left( \frac{Aa}{p_j} \right)^{\frac{1}{1-\alpha}}
\]

\[
= \frac{1 - \vartheta}{r_d} c_x \left( \frac{1}{\alpha} - 1 \right) H_i \left( \frac{Aa^2}{c_x} \right)^{\frac{1}{1-\alpha}}
\]
\[
= \frac{1 - \theta}{r_d} c_{x}^{1/\alpha} (1 - \alpha) a^{-1} H(a^2) \frac{1}{1 - \alpha} A \frac{1}{1 - \alpha} c_{x}^{\frac{1}{1 - \alpha}} a^{\frac{1}{1 - \alpha}}
\]

\[
= \frac{1 - \theta}{r_d} c_{x}^{\frac{1}{1 - \alpha}} (1 - \alpha) H A \frac{1}{1 - \alpha} a^{-1/\alpha} a^{\frac{2}{1 - \alpha}}
\]

\[
= \frac{1 - \theta}{r_d} (1 - \alpha) H \left( \frac{A a^{1 + \alpha}}{c_{x}^{\alpha}} \right) \frac{1}{1 - \alpha}
\]

**Appendix 1b**: Solving for the loan demand rate: \( r_l \)

\[
\chi(\delta, A)\frac{r_l}{r_d} = \frac{(1 - \theta)}{r_d} (1 - \alpha) H \left( \frac{A a^{1 + \alpha}}{c_{x}^{\alpha}} \right) \frac{1}{1 - \alpha}
\]

\[
\varepsilon(A) \left( \frac{X}{H} \right)^{-\alpha} (HA)^{-1} \frac{r_l}{r_d} = \frac{(1 - \theta)}{r_d} (1 - \alpha) H \left( \frac{A a^{1 + \alpha}}{c_{x}^{\alpha}} \right) \frac{1}{1 - \alpha}
\]

\[
\varepsilon(A) P \frac{\alpha}{c_{x}} A^{\frac{1}{1 - \alpha}} \frac{a}{1 - \alpha} r_l = (1 - \theta)(1 - \alpha) H^2 \left( \frac{A a^{1 + \alpha}}{c_{x}^{\alpha}} \right) \frac{1}{1 - \alpha}
\]

\[
\varepsilon(A) P \frac{\alpha}{c_{x}} A^{\frac{1}{1 - \alpha}} \frac{a}{1 - \alpha} r_l = (1 - \theta)(1 - \alpha) H^2 c_{x}^{\frac{a}{1 - \alpha}} A^{\frac{2}{1 - \alpha}}
\]

\[
r_l = \frac{(1 - \theta)(1 - \alpha) H^2 P \frac{\alpha}{c_{x}} A^{\frac{1}{1 - \alpha}} \frac{a}{1 - \alpha} A^{\frac{2}{1 - \alpha}}}{\varepsilon(A)}
\]

**Appendix 1c**: Solving for the steady state growth rate: \( \gamma_N \)

**Proof of Proposition 1**:

\[
r_d = \left( (1 - \theta) r_l - c_b \right) \left( 1 + \frac{1}{B \eta} \right), \quad \gamma_c = \frac{r_d - \rho}{\theta}
\]

\[
= \left( (1 - \theta) \left( (1 - \theta) (1 - \alpha) \frac{\varepsilon(A)}{H^2 P \frac{\alpha}{c_{x}} A^{\frac{1}{1 - \alpha}}} - c_b \right) \left( 1 + \frac{1}{B \eta} \right) \right)
\]

\[
\gamma_N = \gamma_c = \frac{\frac{(1 - \theta) (1 - \alpha) \frac{\varepsilon(A)}{H^2 P \frac{\alpha}{c_{x}} A^{\frac{1}{1 - \alpha}} - c_b} \left( 1 + \frac{1}{B \eta} \right) - \rho}{\theta}}
\]

\[
\gamma_N = \frac{1}{\theta} \left[ \left( (1 - \theta) (1 - \alpha) \frac{\varepsilon(A)}{H^2 P \frac{\alpha}{c_{x}} A^{\frac{1}{1 - \alpha}} - c_b} \left( 1 + \frac{1}{B \eta} \right) - \rho \right) \right]
\]

\[
Y = A H^{1 - a} X^a \rightarrow \dot{Y} = A H^{1 - a} x^{\alpha} \dot{X} \rightarrow \frac{\dot{Y}}{Y} = \gamma_Y = \gamma_N
\]
Appendix 2: Show positive relation between $\gamma_N$ and potential human capital $H$, $\frac{dy_N}{dH}$

Proof of Proposition 2:

\[
\gamma_N = \frac{1}{\theta} \left[ \left( \frac{1-\theta}{\theta} \right)^2 \left( 1 - \alpha \right) H^2 P^{2\theta} \frac{x}{c_x} \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} - \frac{c_b}{1 + \frac{1}{B\eta}} \right] \left( 1 + \frac{1}{B\eta} \right) - \rho
\]

\[
\frac{d\gamma_N}{dH} = \frac{2}{\theta} \left( \frac{1-\theta}{\theta} \right)^2 \left( 1 - \alpha \right) H^2 P^{2\theta} \frac{x}{c_x} \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \left( 1 + \frac{1}{B\eta} \right) > 0
\]

Appendix 3: show a positive relation between $\gamma_N$ and potential periodic monopoly profits $\pi_j$, if e.g. costs of running the business will decrease $c_x$.

Proof of proposition 3:

\[
\gamma_N = \frac{1}{\theta} \left[ \left( \frac{1-\theta}{\theta} \right) \left( 1 - \alpha \right) H^2 P^{2\theta} \frac{x}{c_x} \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} - \frac{c_b}{1 + \frac{1}{B\eta}} \right] \left( 1 + \frac{1}{B\eta} \right) - \rho
\]

\[
d\gamma_N = -\frac{1}{\theta} \left( 1 - \alpha \right) \frac{2}{\theta} \left( \frac{1-\theta}{\theta} \right) H^2 P^{2\theta} \frac{x}{c_x} \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \left( 1 + \frac{1}{B\eta} \right) \frac{dc_x}{dc_N} = -\frac{d\pi_j}{\pi_j} \frac{d\pi_j}{aHA \frac{1}{\alpha} \frac{1}{1-\alpha} c_x \frac{1}{1-\alpha}}
\]

\[
\gamma_N = \frac{2}{\theta} \left( \frac{1-\theta}{\theta} \right) \left( 1 - \alpha \right) H^2 P^{2\theta} \frac{x}{c_x} \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \left( 1 + \frac{1}{B\eta} \right) \frac{d\pi_j}{\pi_j} \frac{d\pi_j}{aHA \frac{1}{\alpha} \frac{1}{1-\alpha} c_x \frac{1}{1-\alpha}}
\]

\[
\frac{d\gamma_N}{d\pi_j} = \frac{2}{\theta} \left( \frac{1-\theta}{\theta} \right) \left( 1 - \alpha \right) H^2 P^{2\theta} \frac{x}{c_x} \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \left( 1 + \frac{1}{B\eta} \right) > 0
\]

Appendix 4: show a negative relation between $\gamma_N$ and business environment indicated by the risk of failure $\vartheta$.

Proof of proposition 4:

\[
\gamma_N = \frac{1}{\theta} \left[ \left( \frac{1-\theta}{\theta} \right)^2 \left( 1 - \alpha \right) H^2 P^{2\theta} \frac{x}{c_x} \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} - \frac{c_b}{1 + \frac{1}{B\eta}} \right] \left( 1 + \frac{1}{B\eta} \right) - \rho
\]

\[
\frac{d\gamma_N}{d\vartheta} = -\frac{2}{\theta} \left( \frac{1-\theta}{\theta} \right)^2 \left( 1 - \alpha \right) H^2 P^{2\theta} \frac{x}{c_x} \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \left( 1 + \frac{1}{B\eta} \right) < 0
\]
Appendix 5: Show an ambiguous relation between $\gamma N$ and degree of urbanization and agglomeration $A$ : \[ \frac{d\gamma_N}{dA} \]

Proof of proposition 5:

\[
\gamma_N = \frac{1}{\theta} \left[ \left(1 - \beta\right)^2 \left(1 - \alpha\right) \frac{H^2 \gamma^a \alpha^{\frac{a}{2+a}} A^{\frac{1}{2+a}} - c_b}{\varepsilon(A)} \left(1 + \frac{1}{B\beta}\right) - \rho \right]
\]

\[
d\gamma_N = \frac{1}{\theta} \left(1 + \frac{1}{B\beta}\right) \left[ \frac{\left(1 - \beta\right)^2 \left(1 - \alpha\right)}{\varepsilon(A)} \left[ H^2 \gamma^a \alpha^{\frac{a}{2+a}} A^{\frac{1}{2+a}} \right] \frac{2}{1 - \alpha} A^{\frac{2}{1-a}} \right] \frac{d\alpha}{dA}
\]

\[
= \frac{1}{\theta} \left(1 + \frac{1}{B\beta}\right) \left[ (1 - \beta) \gamma \frac{2}{1 - \alpha} - (1 - \beta) \gamma \right] \frac{d\epsilon}{dA} \frac{d\alpha}{dA}
\]

\[
\frac{d\gamma_N}{dA} = \frac{1}{\theta A} \left(1 + \frac{1}{B\beta}\right) (1 - \beta) \gamma \left[ \frac{2}{1 - \alpha} - \eta_{\epsilon,\alpha} \right]
\]

Appendix 6: Show an ambiguous relation between $\gamma N$ and degree of urbanization and agglomeration $A$ : \[ \frac{d\gamma_N}{dB} \]

Proof of proposition 6:

\[
\gamma_N = \frac{1}{\theta} \left[ \left(1 - \beta\right)^2 \left(1 - \alpha\right) \frac{H^2 \gamma^a \alpha^{\frac{a}{2+a}} A^{\frac{1}{2+a}} - c_b}{\varepsilon(A)} \left(1 + \frac{1}{B\beta}\right) - \rho \right]
\]

\[
\frac{d\gamma_N}{dB} = -\frac{1}{\theta B^2 \beta} \left[ \left(1 - \beta\right)^2 \left(1 - \alpha\right) \frac{H^2 \gamma^a \alpha^{\frac{a}{2+a}} A^{\frac{1}{2+a}} - c_b}{\varepsilon(A)} \right]
\]