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Under-connected and over-connected networks

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Under-connected and Over-connected Networks

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Abstract

Since the seminal contribution of Jackson & Wolinsky 1996 [A Strategic Model of Social and Economic Networks, JET 71, 44-74] it has been widely acknowledged that the formation of social networks exhibits a general conflict between individual strategic behavior and collective outcome. What has not been studied systematically are the sources of inefficiency. We approach this gap by analyzing the role of positive and negative externalities of link formation. We find general results that relate situations of positive externalities with stable networks that cannot be "too dense" in a well-defined sense, while situations with negative externalities, tend to induce "too dense" networks.

Keywords: Networks, Network Formation, Connections, Game Theory, Externalities, Spillovers, Stability, Efficiency

JEL-Classification: D85, C72, L14

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1 Introduction

The importance of social and economic networks has been widely recognized in economics, as well as in other social sciences. Applications include personal contacts (e.g. Granovetter, 1974), scientific collaborations (e.g. Newman, 2004), trade between countries (e.g. Goyal and Joshi, 2006b), embeddedness of companies (e.g. Uzzi, 1996), and even marriages of ancient trading families (Padgett and Ansell, 1993).

Given the prevalence of network structure in many economic situations, it seems natural to ask how networks change, when agents alter the network structure in order to pursue their goals. It was a major contribution of the economics literature to propose such models based on game theoretic concepts. The seminal contribution of Jackson and Wolinsky (1996) has shown a central problem in strategic network formation: there is a tension between stability and efficiency, meaning that individual interest can be at odds with societal welfare. Since then, there was a flourishing literature on specific situations of strategic network formation of which two small surveys can be found in Jackson (2004) and Goyal and Joshi (2006a). The various network formation games provide micro-based models and analyze which networks are stable (and which are efficient).

What has not been explicitly studied are the *sources of inefficiency*. In particular the question is, *how* do stable networks generally differ from efficient networks? And, *why* does individual interest not always lead to efficient outcomes?

We approach these questions by analyzing the role of externalities (or spillovers) of link formation. Simply put, positive externalities define situations where agents can profit (at least do not suffer) from others who form a relationship; while negative externalities mean that they do not benefit from that. We argue that both types of externalities are not very unnatural. Network formation games where direct and indirect connections are the source of benefits represent examples for positive externalities. On the other hand, in a context of competition or rival goods, negative externalities occur.

Our main result for positive externalities is that there is no stable network that can be socially improved by the severance of links. This result is not dependent on the particular shape of the utility functions or the degree of homogeneity. We provide some examples taken from the literate - among them is the connections model (see Jackson and Wolinsky, 1996) - and illustrate the implications of our result. For negative externalities the tension between stability and efficiency is just the other way around: In various models of network formation, we observe that efficient networks get altered by individuals adding links, a process that leads to stable networks that are "too dense" from a societal point of view. In the context of transfers, we show that no stable network can be socially improved by the addition of links. Without the assumption of transfers, additional insights are won by restricting attention to a large class of network formation models, where the utility function only depends on the number of links all the players have. Among them are the co-author model, firstly introduced in Jackson and Wolinsky (1996), and the model of patent races by Goyal and Joshi (2006a).

This paper is organized as follows: the next section introduces the model. The implications of positive externalities on the tension of stability and efficiency are shown in section 3. Section 4 addresses negative externalities. Section 5 concludes.

2 Model and Definitions

Let $N = \{1, ..., n\}$ be a (finite, fixed) set of agents/players, with $n \geq 3$. A network/graph g is a set of unordered pairs, $\{i, j\}$ with $i \neq j \in N$, that represents the bilateral connections in a non-directed graph. Thus, $ij := \{i, j\} \in g$ means that player i and player j are linked under g. Let g^N be the set of all subsets of N of size two and G be the set of all possible graphs, $G = \{g : g \subseteq g^N\}$.
By $N_i(g)$ we denote the neighbors of player i in network g, $N_i(g) := \{j \in N \mid ij \in g\}$. Similarly, $L_i(g)$ denotes the set of player i's links, $L_i(g) := \{ij \in g\}$. We define $d_i(g) :=\mid L_i(g) \mid=\mid N_i(g) \mid$ called player i's degree.

For each player $i \in N$ a utility function $u_i : G \to \mathbb{R}$ expresses his preferences over the set of possible graphs. $u = (u_1, ...u_n)$ denotes the profile of utility functions. Decisions typically do not depend on absolute utility, but on changes in utility. Let $mu_i(g, l)$ be the marginal utility of player i, when the set of links l are deleted from network g, that is $mu_i(g, l) := u_i(g) - u_i(g \setminus l)$ for $(l \subseteq g)$. Equivalently, we denote $mu_i(g \cup l, l) := u_i(g\cup) - u_i(g)$ as the marginal utility of adding the set of links l to network g.

Each (exogenously given) u defines a situation that is the basis for a strategic network formation game. Different modeling approaches yielded different notions of stability. We employ three of the most common stability notions.[1] The first notion is based on a cooperative framework and was introduced by Jackson and Wolinsky (1996).

Definition 1. *A network g is pairwise stable (PS), if no link will be cut by a single player, and no two players want to form a link:*

(i) $\forall ij \in g, \quad u_i(g) \geq u_i(g \backslash ij)$ *and* $u_j(g) \geq u_j(g \backslash ij)$ *and*

(ii) $\forall ij \notin g, \quad u_i(g \cup ij) > u_i(g) \Rightarrow u_j(g \cup ij) < u_j(g)$.

This well-known definition captures the idea that links can be severed by any involved player, whereas the formation of a link requires the consent of both players. Pairwise stability is a basic notion that can be refined in multiple ways (e.g. unilateral stability, Buskens and Van de Rijt, 2005; strong stability, Jackson and Van de Nouweland, 2005; or bilateral stability, Goyal and Vega-Redondo, 2007).[2] One of the refinements builds on Myerson's link formation game and is called pairwise Nash stability or pairwise equilibrium Goyal and Joshi (2006a).

Definition 2. *A network g is pairwise Nash stable (PNS), if there exists a Nash equilibrium in the corresponding link formation game that supports this network and no link will be added by two players. This boils down to the following conditions:*

(i) $\forall i \in N, \quad \nexists l \subseteq L_i(g) : u_i(g \setminus l) > u_i(g)$ *and*

(ii) $\forall ij \notin g, \quad u_i(g \cup ij) > u_i(g) \Rightarrow u_j(g \cup ij) < u_j(g)$.

[1] A game theoretic foundation and a comparison of the three notions can be found in Bloch and Jackson (2006).

[2] Therefore, all results that exclude networks from being pairwise stable would also hold for the other notions, i.e. if pairwise stable networks fail to be efficient, then this tension cannot be fixed with a stronger stability notion.

The third notion of stability is based on the idea of transfers and can be found in Bloch and Jackson (2007).

Definition 3. *A network g is pairwise stable with transfers (PS^t), if there does not exist any pair of players that can jointly benefit by adding, respectively cutting, their link:*

(i) $\forall ij \in g, \quad u_i(g) + u_j(g) \geq u_i(g \setminus ij) + u_j(g \setminus ij)$ *and*

(ii) $\forall ij \notin g, \quad u_i(g) + u_j(g) \geq u_i(g \cup ij) + u_j(g \cup ij)$.

We denote by $[PS(u)], [PNS(u)], [PS^T(u)]$ the sets of stable networks for a utility profile u. While it holds that $[PNS(u)] \subseteq [PS(u)]$, the relation to pairwise stability with transfers is not that simple (see Bloch and Jackson, 2006).

While stability tries to answer which networks emerge based on individual preferences, efficiency addresses the evaluation of networks from a societal point of view. To formally capture efficiency, we use a welfare function $w : G \to \mathbb{R}$ that typically (but not necessarily) is only dependent on the vector of utility, given a network g. A welfare function w satisfies monotonicity if $u_i(g) \geq u_i(g') \quad \forall i \in N \quad \Longrightarrow \quad w(g) \geq w(g')$. A special case is the additive/utilitarian welfare function: $w^u(g) = \sum_{i \in N} u_i(g)$.

Definition 4. *A network g^* is called efficient with respect to the welfare function w, if it is a welfare maximizing network, that is $w(g^*) \geq w(g) \quad \forall g \in G$.*

In many network formation games we observe a general tension between stability and efficiency.[3] Individual interest often conflicts with social welfare. In the following we want to ask under what conditions this tension is observed. Specifically, we ask whether networks are "locally" efficient in a sense that neither links can be added nor severed to increase overall welfare, and if not, we want to know how overall welfare can be improved. We use the following two definitions in order to describe non-efficient networks.

Definition 5. *A network g is called* over-connected *(with respect to the welfare function w), if $\exists g' \subset g$ such that $w(g') > w(g)$.*

Definition 6. *A network g is called* under-connected *(with respect to the welfare function w), if $\exists g' \supset g$ such that $w(g') > w(g)$.*

A network is over-connected, if it is "too dense" in the sense that overall welfare can be improved by cutting links. Similarly, under-connected networks are "not dense enough". Efficient networks are neither over-connected nor under-connected, while all supernetworks of efficient networks are either over-connected or efficient and all subnetworks are either under-connected or efficient. Note that for any given w, a network can satisfy both, one, or none of these two properties. To shed some light into the tension between stability and efficiency we will ask whether and under what conditions stable networks are over-connected respectively under-connected. From the perspective of a social planer, this gives some insights whether to subsidize or to tax the formation of links in order to arrive in a socially preferred outcome.

[3] See Jackson and Wolinsky (1996) for a general statement.

3 Positive Externalities

Externalities describe the spillovers of link formation: Positive externalities simply capture that two players forming a link cannot mean harm for others.

Definition 7. *A profile of utility functions u satisfies positive externalities, if $\forall g \in G, \forall ij \notin g, \forall k \in N \setminus \{i,j\}$ it holds that*

$$u_k(g \cup ij) \geq u_k(g).$$

Being required for any network, any link and any player, this property seems quite restrictive. However, we argue that there are many such contexts, and we can easily find examples in the literature on strategic network formation that satisfy this property. Among them are "Provision of a pure public good" (Goyal and Joshi, 2006a), "Market sharing agreements" (Goyal and Joshi, 2006a), and the "Connections model" (Jackson and Wolinsky, 1996), which we discuss below. In case of a utility function that is additive separable into costs and benefits (where costs only depend on the own links), positive externalities are implied by a simple monotonicity property of the benefit function. Since players who pay for certain links have to share their overall benefits with other players, individual incentives to establish links can be lower than their collective value. The subsequent results follow this intuition.

Theorem 1. *If a profile of utility functions u satisfies positive externalities, then no pairwise Nash stable network is over-connected with respect to any monotonic welfare function w, that is $\forall g \in [PNS(u)]$ it holds that $\nexists g' \subset g : w(g') > w(g)$.*

All proofs can be found in the appendix. To prove this result, we show that any player is worse off in a subnetwork g' of a PNS network g.[4] Because of PNS a player cannot prefer a network $\tilde{g}(\subset g)$ that has only been reduced by some of his own links. Furthermore, positive externalities imply that he cannot prefer a subnetwork $g' \subset \tilde{g}$.

Necessary for the above Theorem is the property of pairwise Nash stable networks that no player can benefit by unilaterally severing a set of own links. For pairwise stability, however, each link is considered one by one. Thus, in order to derive a similar result for pairwise stability, we need an additional assumption.

Definition 8. *A profile of utility functions u is convex in own current links, if $\forall i \in N, \forall g \in G$, and $\forall l \subseteq L_i(g)$ it holds that $mu_i(g,l) \geq \sum_{ij \in l} mu_i(g, ij)$.*

Since convexity in own links is sufficient for $[PNS(u)] = [PS(u)]$,[5] the next corollary is immediately implied.

Corollary 1. *If u satisfies positive externalities and convexity in own links, then no $g \in [PS(u)]$ is over-connected (with respect to any monotonic welfare function).*

[4] The result also implies that no subnetwork of a PNS network is Pareto better.

[5] See Calvó-Armengol and Ilkiliç (2007).

While PS requires that no agent improves his utility by cutting a link, pairwise stability with transfers is a bit weaker in this respect (because the other player involved can compensate him for keeping the link). To establish the corresponding result for pairwise stability with transfers, we restrict attention to the utilitarian welfare function.

Theorem 2. *If u satisfies positive externalities and convexity in own links, then no $g \in [PS^t(u)]$ is over-connected with respect to the utilitarian welfare function.*

The results excluding over-connectedness have trivial implications for the complete and empty network. As any network is a subnetwork of the complete network, it follows that (a) if the complete network is stable, then it must also be efficient. Since any network is a supernetwork of the empty network it follows that, (b) if the empty network is uniquely efficient, then no other network can be stable. More insights can be won, when studying specific examples.

The Connections Model Revisited

The connections model was introduced in Jackson and Wolinsky (1996). It models the flow of resources (like information or support) via shortest paths in a network. Let $d_{ij}(g)$ denote the distance of players i and j in network g (which is defined to be ∞ for unconnected pairs), then the utility of each player can be written as

$$u_i^{CO}(g) = w_{ii} + \sum_{j \neq i} \delta^{d_{ij}(g)} w_{ij} - \sum_{j: ij \in g} c_{ij}, \quad with \;\; \delta \in (0,1). \tag{1}$$

It is easy to see that the connections model satisfies positive externalities. If ij forms in some network g, then the utility of a player $k \neq \{i,j\}$ either does not change or increases as some of k's distances are shortened, because $d_{km}(g \cup ij) \leq d_{km}(g)$ for all ij and m. Moreover, Calvó-Armengol and Ilkiliç (2007) show that $u^{CO}(\cdot)$ satisfies convexity in own current links. Consequently (by Theorem 1, Corollary 1, and Theorem 2), no pairwise (Nash) stable network can be over-connected w.r.t any monotonic welfare function and no pairwise stable network with transfers can be over-connected w.r.t. the utilitarian welfare function. While stable networks depend on the dyadic specifications of value and costs (w_{ij}, c_{ij}), the results excluding over-connectedness imply that the welfare of a stable network can never be improved by severing links.

There are more specific results for the connections model in its symmetric version, setting $w_{ij} = 1$, $c_{ij} = c$ ($\forall i \neq j$) and considering the utilitarian welfare function w^u only. This has been studied in Jackson and Wolinsky (1996), Jackson (2003), Hummon (2000), and Buechel (2008) among others. Jackson and Wolinsky (1996, Prop. 1 and Prop. 2) show that for low costs ($c < \delta - \delta^2$) the complete network is efficient (and uniquely pairwise stable); for medium costs ($\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$) the star network is efficient; while for very high costs ($c > \delta + \frac{n-2}{2}\delta^2$) the empty network is efficient. Their famous statement of inefficiency in the connections model is the following: "For $\delta < c$, any pairwise stable

network which is non-empty is such that each player has at least two links and thus is inefficient."[6]

What does our result excluding over-connectedness add to their discussion of inefficiency? First, there is the above mentioned trivial implication for the empty network: Since any network is a supernetwork of the empty network, it follows that if the empty network is uniquely efficient, then no other network can be stable. Thus, the statement of inefficiency is restricted to $\delta < c < \delta + \frac{n-2}{2}\delta^2$. Second, the result on over-connectedness adds a new point of view on the flavor of inefficiency. This can be illustrated in the following example, which is also taken from Jackson and Wolinsky (1996, Ex. 1).

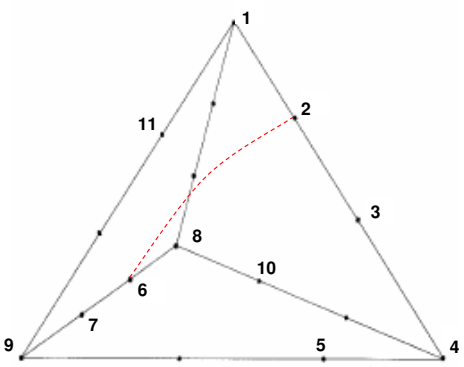

Figure 1: Example of an inefficient network ("Tetrahedron").

Example 1. *The network in fig. 1, called "Tetrahedron", is stable for costs $c > \delta$, where the star network is uniquely efficient.*[7] *The tetrahedron is "too dense" in the sense that it has* 18 *links, while the efficient network has* 15*. Accordingly, Jackson and Wolinsky (1996, p. 51) label it as "over-connected". However by the Corollary 1, it is not over-connected according to the definition used in this paper. This means that the welfare of the tetrahedron cannot be improved by leaving out some of its links. Moreover, we claim that the tetrahedron is under-connected for the parameters such that it is pairwise stable. In the appendix we show that the addition of a link between the players "2" and "6" would strictly improve utilitarian welfare. The same point as in the Tetrahedron can be illustrated in a circle graph of $n \geq 7$: both networks are under-connected for any costs for which they are pairwise stable.*

The example illustrates two different viewpoints on inefficiency (in the connections model). From the viewpoint of a social planer that can unrestrictedly manipulate a given network, some stable networks are "too dense" in the sense that less links are needed to form the efficient one. From the viewpoint of a social planer who is restricted to either foster or hinder the formation of links (e.g. by taxes or subsidies), many stable networks in the connections model are "not dense enough" (under-connected), while none is "too dense" (over-connected).

[6] Jackson and Wolinsky (1996), p. 51.

[7] More precisely, g^{Tetra} is pairwise stable iff $\delta - \delta^5 + \delta^2 - \delta^4 + \delta^2 - \delta^5 + 2(\delta^3 - \delta^4) \leq c \leq \delta - \delta^8 + \delta^2 - \delta^7 + \delta^3 - \delta^6 + 2(\delta^4 - \delta^5)$.

Market Sharing Agreements

Besides the connections model, it is easy to find further examples for positive externalities (and convexity in own links). Among them is the model of "market sharing agreements" described in Goyal and Joshi (2006a). In this model, there are n firms and n markets, where each firm has one home market and can be active in all other markets, too. Before starting a Cournot competition in each market, bilateral agreements can be made to stay out of each other's home market.

The reduction of competitors in the own market might be profitable. However, all remaining competitors in the market benefit from these activities without paying for it. That is why the utility function of this example exhibits positive externalities. In addition, it satisfies convexity in own current links, such that all results above (Theorem 1, Corollary 1, and Theorem 2) apply. Consequently, the positive externalities lead to rather too few agreements with respect to a monotonic welfare function. Note that such a function only covers firms' utility, but not consumers'. Let us have a closer look on one more example for positive externalities.

Provision of a Pure Public Good

The model "provision of a pure public good" is also taken from Goyal and Joshi (2006a) who extended a model of Bloch (1997). n players choose an output level x_i (second stage), which is valuable for everybody $\tilde{\pi}_i(x) = \sum_{i \in N} x_i$. Collaboration (knowledge sharing) between any two players is costly, but can reduce the marginal costs of producing the output (first stage).[8] Assuming that any player chooses his output quantity optimally, the utility of a player i is:

$$u_i^{PG}(g) = \frac{1}{2}(d_i(g)+1)^2 + \sum_{j \in N \setminus i} (d_j(g)+1)^2 - c d_i(g),$$

where the first term is the difference of own output and production costs, the second term is the output of all other agents, and the last term is the costs of collaboration.

Not surprisingly, the network formation situation of the first stage satisfies positive externalities, because other agents' cooperations lower their costs, increase their optimal output and, hence, is beneficial to all. To see this, observe that the addition of foreign links increases the middle term of the utility function. Note that in this example the externalities are strict in the sense that the addition of any link in any network increases the utility of all agents that are not involved. Moreover, given these specific functional forms, u satisfies convexity in own links (as also noted by Goyal and Joshi, 2006a). Thus,

[8] Agent i's cost of producing the output is $f_i(x_i, g) = \frac{1}{2}(\frac{x_i}{d_i(g)+1})^2$. Fixing the number of collaborators $d_i(g)$, the utility maximizing output quantity of an agent i can be derived by $\max_{x_i \in \mathbb{R}_+} x_i + \sum_{j \in N \setminus i} x_j - \frac{1}{2}(\frac{x_i}{d_i(g)+1})^2 =: F(x)$. This yields $F'(x) = 0 \iff x_i^* = (d_i(g)+1)^2$. Then, plugging in the optimal output ($F(x^*)$) for any agent into the objective function and subtracting the linking costs yields the utility of one agent.

the three results (Theorem 1, Corollary 1 and Theorem 2) imply again that no stable network can be over-connected.

Consider very low costs $c \leq \frac{3}{2}n^2 - \frac{3}{2}(n-1)^2 =: lb$ such that the complete network is stable. By the results (Theorem 1, Corollary 1 and Theorem 2) above, the complete network must also be efficient for these costs. In fact, since the externalities are strict, there exists an $\epsilon > 0$ such that g^N is efficient for $c \leq lb + \epsilon$. The tension can be illustrated for $lb < c < lb + \epsilon$. In this cost range the complete network would still be efficient. However, the stable networks are not complete.

The model can be interpreted as a doubled public goods problem. In the second stage there is the classic public goods problem, where individual output x_i is chosen "too low" (from a collective perspective). This problem persists, but in addition (in the first stage) players tend to choose "too few" links reducing the cost of provision, such that the outcome is even worse. In the same manner any network formation situation with positive externalities can be interpreted as a public goods problem. Utility maximizing agents simply do not internalize the positive effects that establishing a bilateral link means for other agents.

4 Negative Externalities

Negative externalities in network formation occur, when adding links is not beneficial for the players not involved. Formally, we speak of weakly negative externalities (further denoted as negative externalities), if the following holds:

Definition 9. *A profile of utility functions u satisfies negative externalities, if $\forall g \in G, \forall ij \notin g, \forall k \in N \setminus \{i,j\}$ it holds that*

$$u_k(g \cup ij) \leq u_k(g).$$

When considering negative externalities in the economics literature, equilibrium analysis usually shows that individuals do rather "too much" (pollute, etc.) than being socially optimal. In that sense it is intuitive to think about "too dense" networks as being stable. For stability only individual incentives are considered and not the overall welfare. Network formation games with negative externalities are thus expected to be over-connected, and not under-connected. The intuition is clear, however, this section will be more complex to analyze. It is the structure of pairwise stability and related concepts that makes it more difficult to show that a network is not under-connected, than to show that a network is not over-connected. The reason for this is that in the first case supernetworks are considered and links can be cut by only one player, whereas building a link requires the consent of both involved players. In general, even when analyzing utility functions with negative externalities, it can well be that a pairwise stable network is under-connected with respect, e.g. to the utilitarian welfare function. To see this, suppose that one player i gains a lot from a link with player j, but the link is simply not formed, because j looses a little bit. Assuming that all others do not loose a lot either, the addition of the link ij could produce higher welfare.

A simple way out of this issue is using the stability concept "Pairwise Stability with transfers" [PS^t]. This concept helps ensure that any single link that is not in a network

g, which is pairwise stable with transfers, cannot be welfare improving. To ensure that not a set of links can be welfare improving, we introduce for our main result on negative externalities the concept of concavity in own new links:

Definition 10. *A profile of utility functions u is concave in own new links, if for all $i \in N$, for all $g \in G$ and for all links l such that $l \subseteq L_i(g^N)$, and $l \cap g = \emptyset$ the following holds:*

$$mu_i(g \cup l, l) \geq \sum_{ij \in l} mu_i(g \cup ij, ij).$$

Utility functions satisfying this property imply that networks, which are pairwise stable with transfers are not under-connected with respect to the utilitarian welfare function:

Theorem 3. *Suppose a profile of utility functions u satisfies negative externalities and concavity in own new links, then no network $g \in G$, which is pairwise stable with transfers, is under-connected with respect to the utilitarian welfare function.*

Pairwise stability with transfers differs from pairwise stability significantly. In general, neither $[PS(u)] \subseteq [PS^t(u)]$ nor $[PS(u)] \supseteq [PS^t(u)]$. The concept of pairwise stability with transfers rather stems from a game, in which players can pay transfers to make others willing to build a link. Formally, pairwise stability with transfers has been introduced in Bloch and Jackson (2007), who slightly changed the following notion of "Pairwise Stability allowing for side payments" (Jackson and Wolinsky, 1996):

Definition 11. *A network g is pairwise stable with side payments (PS^s), if*

(i) $\forall ij \in g, \quad u_i(g) \geq u_i(g \backslash ij)$ *and* $u_j(g) \geq u_j(g \backslash ij)$ *and*

(ii) $\forall ij \notin g, \quad u_i(g) + u_j(g) \geq u_i(g \cup ij) + u_j(g \cup ij)$.

From this definition it follows immediately that $[PS^s(u)] = [PS^t(u)] \cap [PS(u)]$.[9] Thus, all results for pairwise stability with transfers carry over to pairwise stability allowing for transfers.

The assumptions on the utility function, namely negative externalities and concavity in own links, appear in a lot of networks formation models, some of which will be analyzed subsequently. These however can be relaxed, by requiring them to hold only for smaller set of networks $\tilde{G} \subseteq G$, which has to fulfill the convexity notion that $\forall g \in \tilde{G}$ it holds that $[g, g^N] \subseteq \tilde{G}$, where $[g, g^N]$ is defined as the set of all networks containing g, $[g, g^N] := \{g' \in G | g' \supseteq g\}$.

Remark 1. *Suppose u satisfies negative externalities and concavity in own new links on a domain $\tilde{G} \subseteq G$, such that for all $g \in \tilde{G}$ it holds that $[g, g^N] \subseteq \tilde{G}$. Then no $g \in \tilde{G}$, which is pairwise stable with transfers is under-connected with respect to the utilitarian welfare function.*

[9] Pairwise stability with transfers is a stronger concept for links not in g, whereas pairwise stability is stronger for links in g. Pairwise stability allowing for side payments captures both strong conditions.

The remark is shown in the appendix. It becomes very handy when analyzing network formation games that fulfill the properties of Theorem 3 only on a certain domain $\tilde{G}$. We will see examples of network formation games such that the utility function is concave in own new links only for connected networks and it is straightforward to see that for the set of connected networks $\bar{G}$ it holds that $g \in \bar{G} \Rightarrow [g, g^N] \subseteq \bar{G}$.[10]

Using pairwise stability with transfers is a way of bounding the gain of any single player from any additional link. Without this notion or requiring "large externalities" we cannot exclude that an addition of a single link produces higher utilitarian welfare. Since this notion of welfare is standard in network formation models, we do not want to derive properties that do not hold for the utilitarian welfare function. Furthermore, a detailed discussion of large externalities is beyond the scope of this paper. It is obvious, that by requiring externalities to be large enough, one can make any pairwise stable network over-connected and not under-connected. Thus, when looking for welfare implications of stable networks in games with negative externalities, we can rather ask whether a network is not Pareto under-connected, that is whether there does not exists a Pareto better supernetwork.

Definition 12. *A network $g \in G$ is Pareto under-connected, if there exists a network $g' \supset g$, such that $u_i(g') \geq u_i(g)$ for all $i \in N$, and there exists a $j \in N$ such that $u_j(g') > u_j(g)$.*

To exclude Pareto under-connectedness, the following definition of transitivity is needed:

Definition 13. *A utility function satisfies transitivity of negative (positive) marginal utility in new links for $g \in G$, if for all ij, $jk \notin g$ the following holds:*

$$mu_i(g|ij) < (>)0, \quad mu_j(g|jk) < (>)0 \quad \Rightarrow \quad mu_i(g|ik) < (>)0.$$

This definition captures the idea of transitivity: if player A does not want to connect to B and B does not want to connect to C, then A does not want to connect to C. The following Theorem presents the result that no pairwise stable network can be Pareto under-connected.

Theorem 4. *Suppose a profile of utility functions satisfies negative externalities, and concavity in own new links. If g is pairwise stable and u satisfies transitivity of negative marginal utility for g, then g cannot be Pareto under-connected.*

Without any other restrictions on the utility function, except concavity in own new links, it is necessary to exclude the case that a set of links l can be added such that each involved individual has at least two links in l, and one link's gain compensates for the other's loss, while externalities are small enough. This is exactly what transitivity in negative marginal utility excludes. The property of transitivity, however, is only needed locally, that is it is only needed for pairwise stable networks. It could thus also be seen as a stability refinement.

[10] The set of connected networks is formally defined as $\bar{G} := \{g \in G \mid \quad \forall i, j \in N, \quad d_{ij}(g) < \infty\}$.

Degree Dependent Utility Functions

A lot of applications and examples of network formation games with negative externalities can be found under what we call "degree dependent utility functions". For instance, Goyal and Joshi (2006a) present network formation models, where the utility functions depend only on the number of own links, the number of links of neighbors and on the number of links of non-neighbors. They analyze two different models:

- Playing the field:
$$u_i^{PF}(g) = \Phi\big(d_i(g), D(g_{-i})\big) - d_i(g)c. \tag{2}$$
- Local Spillovers:
$$u_i^{LS}(g) = \Psi_1(d_i(g)) + \sum_{j \in N_i} \Psi_2(d_j(g)) + \sum_{k \notin N_i} \Psi_3(d_k(g)). \tag{3}$$

Here g_{-i} represents the network, obtained by deleting player i and all his links and $D(g) = \sum_{i \in N} d_i(g)$. The first utility function, playing the field, presented by Goyal and Joshi (2006a) features a very specific structure in that each utility does not depend on network positions, but rather on the number of own and the sum of all other players' links. In contrast, the local spillover utility function includes a little bit more of the network structure. Here, neighborhood distribution of links matters. In the playing the field case, players do not care to whom to connect, they are not able to distinguish between different players, whereas in the local spillover case, players are able to distinguish between players and might have preferences of whom to connect to. For the following we will generalize both cases and try to shed some light on the tension between stability and efficiency when considering the following utility function:

Definition 14. *A profile of utility functions is called homogeneous degree dependent utility function, if for all $i \in N$ there exist function $f : \mathbb{R}^n \to \mathbb{R}$ such that:*

$$u_i^{DD}(g) := f\left(d_i(g), \big(d_j(g)\big)_{j \in N_i(g)}, \big(d_k(g)\big)_{k \notin N_i(g)}\right). \tag{4}$$

This utility function captures both the local spillovers function and the playing the field function, as well as other examples. The distinction between neighbors and non-neighbors is only made for illustration, since in a lot of examples those degrees are treated differently. Abusing notation, we will also write $u_i^{DD}\big(d_i, (d_j)_{j \in N_i}, (d_k)_{k \notin N_i}\big)$ instead of $u_i^{DD}(g)$.

For the analysis of both playing the field and general degree dependent utility functions, we focus on the case of negative externalities, since the tension between efficiency and pairwise stability, respectively pairwise Nash stability in case of positive externalities seems to be sufficiently covered in section 3.

In the playing the field case, negative externalities imply that u^{PF} is decreasing in its second argument, i.e. $u^{PF}(l, k{+}1) - u^{PF}(l, k) \leq 0$ for all $l = \{0, ..., n{-}1\}$, $k = \{0, ..., n{-}2\}$. General degree dependent utility functions satisfy negative externalities, if for all $i \in N$,

for all $d_i \in \{0, ..., n-1\}$, and for all $d_{-i}, \tilde{d}_{-i} \in \{0, ..., n-1\}^{n-1}$ such that $\tilde{d}_k \geq d_k$ for all $k \in N \setminus \{i\}$ the following holds:

$$u_i^{DD}(d_i, d_{-i}) \geq u_i^{DD}(d_i, \tilde{d}_{-i}).$$

For the special cases of degree dependent utility functions, the results of section 4 carry over and some assumptions are automatically satisfied. For playing the field utility functions, it becomes immediately clear from (2), that u^{PF} satisfies our notion of transitivity for all $g \in G$, because either a player wants a link to any remaining player or to none. Thus the corollary follows:

Corollary 2. *Suppose that the profile of utility functions of a network formation game is given by (2), and u^{PF} satisfies negative externalities and concavity in own new links. Then no pairwise stable network g is Pareto under-connected.*

Due to the special structure and homogeneity of the utility functions given by (2) and (3) it is often observed that regular networks are pairwise stable, i.e. which feature equal degree distributions.[11] Since the utility functions given by (4) are homogeneous, regular pairwise stable networks are also pairwise stable with transfers. This is true, since no single player wants to add or cut a link in regular pairwise stable networks. Furthermore, the star is a common observed stable network. In the star, $n-1$ players share equal degree and equal degree of others. The center cannot add any links. Thus, same considerations hold for the star. These observations lead to the following result:

Corollary 3. *Suppose that the profile of utility functions is given by (4) and, furthermore, satisfies negative externalities and concavity in own new links. Then, no regular pairwise stable network is under-connected. Moreover, if the star is pairwise stable, then it is not under-connected.*

The corollary also holds for playing the field and local spillover utility functions, since degree based utility functions satisfying (4) contain these two cases.

Applications of degree dependent utility functions can be found plenty. Among those are the provision of a public good and the market sharing agreements presented in section 3, but also there are several examples for negative externalities, of which we present some below.

The Co-author Model

The co-author model has been introduced by Jackson and Wolinsky (1996) in their seminal paper and describes the utility of joint work. The nodes of the network are interpreted as researchers, who spend time writing papers. A link between two researchers i and j represents a collaboration between both researchers. The amount of time a researcher

[11] See Goyal and Joshi (2006a) for a detailed analysis of stable networks in playing the field and local spillover games.

spends on a project is inversely related to the number of projects he is involved in. The payoff function is given by:

$$u_i^{CA}(g) = \sum_{j \in N_i(g)} \left(\frac{1}{d_i(g)} + \frac{1}{d_j(g)} + \frac{1}{d_i(g) d_j(g)} \right) = 1 + \left(1 + \frac{1}{d_i(g)} \right) \sum_{k \in N_i(g)} \frac{1}{d_j(g)},$$

and $u_i^{CA}(g) = 0$, if $d_i = 0$. The utility depends only on own degree and neighbors degree and thus is degree dependent. Obviously, the functional form satisfies negative externalities as the utility of players decrease, when neighbors are adding links, i.e. increasing their degree. From Jackson and Wolinsky (1996) we get:

Proposition 1. (Jackson and Wolinsky (1996)) *In this co-author model, if n is even, then any efficient network consists of $n/2$ separate pairs. Any pairwise stable network can be partitioned into fully intra connected components, each of which has a different number of members (if m is the number of members of one such component and k is the next largest size, then $m > k^2$).*

We can add to this proposition that none of the stable networks contains a singleton component, since each player is better off connecting to some player than to none, and each player i wants a link to a player j, for whom $d_j \leq d_i$. Moreover, it needs to be remarked that *any* network consisting of $n/2$ separate pairs is efficient, if n is even. For n being odd, any network that consists of $(n-2)/2$ separate pairs and three players, which are connected by 2 links, is efficient. Thus, no pairwise stable network is efficient and moreover a pairwise stable network such that each component is of even size contains an efficient network, implying that these pairwise stable networks are over-connected. Generally, we can show that any pairwise stable network is over-connected.

Proposition 2. *In the co-author model if $n \geq 3$, then any pairwise stable network is over-connected.*

The proof is straightforward and uses the proposition of Jackson and Wolinsky (1996) component-wise (for any completely connected component of the pairwise stable networks). It is welfare better for any component of at least size three to be connected like one of the efficient networks. Thus any component of any pairwise stable network contains a welfare better subcomponent, implying that any pairwise stable network contains a welfare better subnetwork.

Patent Races

Goyal and Joshi (2006a) derive this model as a variation of the classical patent race model.[12] In addition to the classical model, firms can join R&D collaborations to accelerate research. The first firm to develop the new product is awarded a patent. The random time $\tau(l_i(g))$ at which the innovation happens is given by

$$Pr(\{\tau(d_i(g)) \leq t\}) = 1 - \exp(-d_i(g)t).$$

[12]See Dasgupta and Stiglitz (1980) among others.

Assuming risk neutrality, payoff of 1 in case of receiving the patent and 0 else, and a discount factor ρ, the firm i get the following expected payoff:

$$\begin{aligned} u_i^{PR}\big(d_i(g), D(g_{-i})\big) &= E_t[exp(-\rho t)Pr(\tau(d_i(g)) = t)\prod_{j\neq i} Pr(\tau(d_j(g)) > t)] - d_i(g)c \\ &= \frac{d_i(g)}{\rho + D(g)} - d_i(g)c \quad = \quad \frac{d_i(g)}{\rho + 2d_i(g) + D(g_{-i})} - d_i(g)c. \end{aligned}$$

This model is thus a playing the field utility function. Moreover, it satisfies negative externalities, since links of other firms reduce the probability to innovate firstly. Also, since u_i^{PR} is a concave function of $d_i(g)$, it is concave in own new links. From Theorem 3 we can thus conclude that no pairwise stable network with transfers is under-connected. In fact, it is straightforward to calculate the efficient networks, since the utilitarian welfare is given by:

$$w^{PR}(g) = \sum_{i\in N} u_i^{PR}(g) = \sum_{i\in N}\left(\frac{d_i(g)}{\rho + D(g)} - d_i(g)c\right) = \frac{D(g)}{\rho + D(g)} - D(g)c.$$

In this case the utilitarian welfare only depends on the total number of links and thus any network that contains the optimal number of total links is efficient. The distribution of links and the structure of the network do not matter for efficiency. We can easily calculate that for $\frac{\rho}{(\rho+2(k+1))(\rho+2k)} < c < \frac{\rho}{(\rho+2k)(\rho+2(k-1))}$ any network which contains k links is efficient and no other networks are efficient.

It requires a little bit more to characterize stable networks. However, for this matter we can apply Theorem 3 in order to bound the total number of links.

Proposition 3. *Suppose that $\frac{\rho}{(\rho+2k+2)(\rho+2k)} < c$, then all networks g, which are pairwise stable with transfers, have to contain more than k total links, in other words $D(g) \geq 2k$.*

The example shows that Theorem 3 not only describes the tension between stability and efficiency, but it can also be applied to characterize the stable networks (resp. the efficient ones).

Free Trade Agreements

This model has been introduced by Goyal and Joshi (2006b). We analyze the most basic setup here. In this example there are n countries. In each country there is one firm producing a homogeneous good. The firm may sell the product in the domestic market as well as in foreign markets. If two countries do not have a free trade agreement (FTA) the importing country charges tariffs. Given a configuration of FTA's the firms then compete in each market by choosing quantities. We denote the quantity output of firm j in country i by Q_i^j. In each country $i \in N$, a firm faces an identical inverse demand given by $P_i = \alpha - Q_i$, where $Q_i = \sum_{j\in N} Q_i^j$, and $\alpha > 0$. All firms have a constant and identical marginal cost of production, $\alpha > \gamma > 0$. In the basic model, linear demand and identical tariffs are assumed. Forming an FTA lowers the tariff to 0. Assuming high tariffs

$T > \alpha$, firm i sells in country j if and only if there is a FTA between the two countries, i.e. $ij \in g$. Given firm i is active in market j, then its output is given by $Q_j^i = \frac{\alpha-\gamma}{d_j(g)+1}$. The utility function of country i is given by its "social welfare" derived as the sum of consumer surplus, firm's profits and tariff revenue, which can be simplified in our basic setup:

$$\begin{aligned} u_i^{FTA}(g) &= 1/2Q_i^2 + \big((P_i(g)-\gamma)Q_i^i(g) + \sum_{j\neq i}(P_j(g)-\gamma-T_j^i(g))Q_j^i(g)\big) + \sum_{j\neq i} T_i^j(g)Q_i^j(g) \\ &= 1/2\left(\frac{(\alpha-\gamma)(d_i(g)+1)}{d_i(g)+2}\right)^2 + \sum_{j\in(N_i(g)\cup\{i\})}\left(\frac{\alpha-\gamma}{d_j(g)+2}\right)^2. \end{aligned}$$

Again, this utility function is degree dependent. Moreover, it is straightforward to see that u^{FTA} satisfies negative externalities. To see the reasoning for negative externalities, suppose a free trade partner j of country i signs a free trade agreement with country k. Then firm k will enter market j and thus reduces the Cournot output of firm i, lowering the country's welfare function. If two non-trade partners of i sign a free trade agreement, then i's payoff remains unaffected. Straightforward calculations show that u_i^{FTA} satisfies concavity in own new links, whenever $d_i \geq 1$. From Goyal and Joshi (2006b) we get the following characterization of stability and efficiency:

Proposition 4. (Goyal and Joshi, 2006b) *The complete network is a stable trading network. Furthermore, a network such that one component has $n-1$ countries and is complete and the other component is a single country can be stable. However, the complete network is uniquely efficient.*

In this case, the network such that there is one component with $n-1$ countries and one unconnected country is under-connected. It can be shown, that this network is also pairwise stable with transfers, if n is sufficiently large. These networks however, do not fall in the restricted domain in Remark 1, since concavity in own new links only holds for u_i^{FTA} if $d_i \geq 1$. Thus Remark 1 only applies to all $g \in \tilde{G}$ such that $\tilde{G} := \{g \in G \mid d_i(g) \geq 1 \quad \forall i \in N\}$, and hence does not exclude the case of under-connected pairwise stable networks $g \notin \tilde{G}$.

5 Concluding Remarks

We have introduced the notion of over-connected and under-connected networks in order to contribute to a better understanding of the tension between stability and efficiency in situations of strategic network formation. A network that is not over-connected and not under-connected is locally efficient, which means that there is neither a subnetwork nor a supernetwork with higher welfare.

The basic argument is that positive spillovers/externalities lead to situations where agents are not willing to form a link, although it would have been collectively beneficial. Negative externalities have the opposite effect: agents form links not internalizing the harm it does for others. Without restriction to any specific network formation model, we have

shown the following: for positive externalities, no stable network can be over-connected. For negative externalities no stable network can be under-connected when some other conditions are met. The contribution of these results is two-fold. First, they shed light into the general tension between stability and efficiency (positive externalities tend to induce under-connected networks, while negative externalities tend to induce under-connected networks) giving a social planer a clear signal in which situations rather to impede and when to foster bilateral relationships. Secondly, the results can be used in specific network formation models to better characterize stable and efficient networks. We have presented this for a few examples, while there are many other models of strategic network formation that satisfy the required conditions. Furthermore, our results can be applied to very general, possibly heterogeneous utility functions, which have not been analyzed so far.

APPENDIX

Proof of Theorem 1. Let $g \in [PNS(u)]$. We show that for all $g' \subset g$ it holds that $u_i(g') \leq u_i(g)$ for all $i \in N$. Let $l := l(g, g') = g \setminus g'$ for some $g' \subset g$, and denote $l_i := l_i(g, g') = l \cap L_i(g)$ and $l_{-i} := l \setminus l_i$. Since $g \in PNS(u)$, we get that $u_i(g) \geq u_i(g \setminus l_i)$. Since u satisfies positive externalities, it holds for all $\tilde{g} := g \setminus l_i$ that $u_i(\tilde{g}) \geq u_i(\tilde{g} \setminus l_{-i})$ (because player i does not own a link in l_{-i}), i.e. $l_{-i} \cap L_i(g) = \emptyset$). Thus: $u_i(g) \geq u_i(g \setminus l_i) \geq u((g \setminus l_i) \setminus l_{-i}) = u(g')$. The same argument holds for all $i \in N$, implying that $w(g) \geq w(g')$ for any welfare function satisfying monotonicity. □

Proof of Corollary 1. Calvó-Armengol and Ilkiliç (2007) show that (1-)convexity in own links is sufficient for $[PS(u)] = [PNS(u)]$. Thus, Theorem 1 applies. □

Proof of Theorem 2. Let g be pairwise stable with transfers. We show that for all $g' \subset g$ it holds that $\sum_{i \in N} u_i(g') \leq \sum_{i \in N} u_i(g)$. Suppose that u satisfies positive externalities and concavity in own new links. For $g' \subset g$, let $l := l(g, g') := g \setminus g'$ and for each $i \in N$ let $l_i := l_i(g, g') := l \cap L_i(g')$ and $l_{-i} := l_{-i}(g, g') := l \setminus l_i$. Given these definitions we have to show that

$$\sum_{i \in N} u_i(g) - \sum_{i \in N} u_i(g') = \sum_{i \in N} mu_i(g, l) \geq 0. \tag{5}$$

Since u satisfies positive externalities, it holds for all $i \in N$ that:

$$u_i(g') \leq u_i(g' \cup l_{-i}). \tag{6}$$

Convexity in own current links implies for all $i \in N$:

$$mu_i(g, l_i) \geq \sum_{ij \in l_i} mu_i(g, ij). \tag{7}$$

Now, since g is pairwise stable with transfers (3), (6) and (7) imply:

$$\begin{aligned}\sum_{i\in N} mu_i(g,l) &\overset{(6)}{\geq} \sum_{i\in N} mu_i(g,l_i)\\ &\overset{(7)}{\geq} \sum_{i\in N}\sum_{j:ij\in l_i} mu_i(g,ij)\\ &\overset{(*)}{=} \sum_{ij\in l}[mu_i(g,ij)+mu_j(g,ij)] \overset{(3)}{\geq} 0,\end{aligned}$$

where the equality (*) holds, because for each link $ij \in l$ it holds that $ij \in l_k$ if and only if $k \in \{i,j\}$ and only links in l are considered.[13] □

Proposition 5. *In the symmetric connections model, g^{Tetra} is under-connected with respect to the utilitarian welfare function for any parameters δ and c, for which g^{Tetra} is pairwise stable.*

Proof. We have to show that if δ and c are such that $g^{Tetra} \in PS(u_{\delta,c})$, then $\exists g' \supset g^{Tetra}$ for which $w_{\delta,c}(g') > w_{\delta,c}(g^{Tetra})$. Specifically, we show that the condition

$$c \leq \delta - \delta^8 + \delta^2 - \delta^7 + \delta^3 - \delta^6 + 2(\delta^4 - \delta^5) := ub \tag{8}$$

is necessary for stability, but sufficient for $w_{\delta,c}(g^{Tetra} \cup 26) > w_{\delta,c}(g^{Tetra})$. The label of the players correspond to figure 1.

The first part was done in Jackson and Wolinsky (1996) already. Suppose that $c > ub$, then player 1 benefits from cutting 12 (because his change in benefits is just ub).

For the second part denote by $\beta_i := \sum_{j\neq i} \delta^{d_{ij}(g^{Tetra}\cup 26)} - \sum_{j\neq i} \delta^{d_{ij}(g^{Tetra})}$ the marginal benefits for player i and by $\Delta := \sum_{i\in N} \beta_i$ the sum of marginal benefits. This allows to write

$$w_{\delta,c}(g^{Tetra} \cup 26) > w_{\delta,c}(g^{Tetra}) \Longleftrightarrow \Delta > 2c. \tag{9}$$

It is straightforward to derive that

$$\begin{aligned}\beta_1 &= \beta_8 = \delta^2 - \delta^4 + \delta^3 - \delta^4\\ \beta_2 &= \beta_6 = \delta - \delta^5 + \delta^2 - \delta^4 + \delta^2 - \delta^5 + 2(\delta^3 - \delta^4)\\ \beta_3 &= \delta^2 - \delta^5 + \delta^3 - \delta^4 + \delta^3 - \delta^5\\ \beta_4 &= \beta_9 = \beta_{10} = \beta_{11} = \delta^3 - \delta^4\\ \beta_7 &= \delta^2 - \delta^5 + \delta^3 - \delta^4 + \delta^3 - \delta^5,\end{aligned}$$

and $\beta_i = 0$ for all other i.

[13] Consider a link $ij \in l$: the link appears only in l_i and l_j and thus $\sum_{k\neq i,j}\left(1_{\{ij\in l_k\}}[u_i(g\cup ij)-u_i(g)]\right) = 0$ and $\sum_{k\in N}\left(1_{\{ij\in l_k\}}[u_i(g\cup ij)-u_i(g)]\right) = u_i(g\cup ij)-u_i(g)+u_j(g\cup ij)-u_j(g)$. The equality (*) thus holds, because $\sum_{ij\in l}\sum_{k\in N}\left(1_{\{ij\in l_k\}}[u_i(g\cup ij)-u_i(g)]\right) = \sum_{k\in N}\sum_{ij\in l}\left(1_{\{ij\in l_k\}}[u_i(g\cup ij)-u_i(g)]\right) = \sum_{k\in N}\left(\sum_{j:kj\in l_k}[u_i(g\cup ij)-u_i(g)]\right)$.

This yields

$$\Delta = 2(\delta - \delta^5) + 4(\delta^2 - \delta^4) + 4(\delta^2 - \delta^5) + 12(\delta^3 - \delta^4) + 2(\delta^3 - \delta^5). \tag{10}$$

To show that $\Delta > 2c$ under the condition $c \leq ub$, it is sufficient to show that $\Delta > 2ub$ holds. Recall that,

$$2ub(g) = 2(\delta - \delta^8) + 2(\delta^2 - \delta^7) + 2(\delta^3 - \delta^6) + 4(\delta^4 - \delta^5). \tag{11}$$

Thus,

$$\Delta > 2ub \Longleftrightarrow 6\delta^2 + 12\delta^3 - 20\delta^4 - 4\delta^5 + 2\delta^6 + 2\delta^7 + 2\delta^8 > 0 \tag{12}$$

Numerically it can be checked that (12) holds for all $\delta \in (0, 1)$ (we used Maple). □

Proof of Theorem 3. Let g be pairwise stable with transfers. We show that for all $g' \supset g$ it holds that $\sum_{i \in N} u_i(g') \leq \sum_{i \in N} u_i(g)$. Suppose that u satisfies negative externalities and concavity in own new links. For $g' \supset g$, let $l := l(g, g') := g' \setminus g$ and for each $i \in N$ let $l_i := l_i(g, g') := l(g, g') \cap L_i(g')$ and $l_{-i} := l_{-i}(g, g') := l(g, g') \setminus l_i(g, g')$. Since u satisfies negative externalities, it holds for all $i \in N$ that:

$$u_i(g') \leq u_i(g' \setminus l_{-i}(g, g')). \tag{13}$$

Concavity in own new links implies for all $i \in N$:

$$u_i(g \cup l_i(g, g')) - u_i(g) \leq \sum_{j: ij \in l_i(g,g')} u_i(g \cup ij) - u_i(g). \tag{14}$$

Now, since g is pairwise stable with transfers, (13) and (14) imply:

$$\begin{aligned}
\sum_{i \in N} (u_i(g') - u_i(g)) &= \sum_{i \in N} (u_i(g \cup l_i \cup l_{-i}) - u_i(g)) \\
&\overset{(13)}{\leq} \sum_{i \in N} (u_i(g \cup l_i) - u_i(g)) \\
&\overset{(14)}{\leq} \sum_{i \in N} \Big(\sum_{j: ij \in l_i} [u_i(g \cup ij) - u_i(g)] \Big) \\
&\overset{(*)}{=} \sum_{ij \in l} u_i(g \cup ij) - u_i(g) + u_j(g \cup ij) - u_j(g) \overset{(3)}{\leq} 0,
\end{aligned}$$

where the equality (*) holds, because for each link $ij \in l$ it holds that $ij \in l_1(k)$ if and only if $k \in \{i, j\}$ and only links in l are considered. □

Proof of Remark 1. Let $g \in \tilde{G}$ be pairwise stable with transfers. Since for all $g' \supset g$ it holds that $g' \in \tilde{G}$, and u satisfies the assumptions of Theorem 3 on $\tilde{G}$, we can show equivalently to the proof of Theorem 3 that for all $\bar{g} \supset g$ it holds that $w^u(\bar{g}) \leq w^u(g)$. □

Proof of Theorem 4. We have to show that for all $g' \supset g$ the following holds:

$$\exists i \in N: \quad u_i(g') > 0 \Rightarrow \exists j \in N: \quad u_j(g') < u_j(g).$$

Similarly to the above proofs, for $g' \supset g$ let $l = g' \setminus g$, $l_i = l \cap L_i(g')$ and $l_{-i} = l \setminus l_i$. Let $N(l) := \{i \in N : l_i \neq \emptyset\}$ be the set of players who are involved in at least one link in l. Suppose that $u_k(g') = u_k(g)$ for all $k \notin N(l)$ (otherwise the result is immediately established, because of non-positive externalities). Similarly to the proof of Theorem 3, concavity in own links and negative externalities imply for all $i \in N$:

$$u_i(g') - u_i(g) \overset{(13)}{\leq} u_i(g \cup l_i) - u_i(g) \overset{(14)}{\leq} \sum_{j:ij \in l_i} u_i(g \cup ij) - u_i(g). \tag{15}$$

Thus, for g' to be Pareto preferred to g there has to exist an $i : u_i(g') > u_i(g)$ and thus because of (15) there has to exist $i_1 \in N$: $u_i(g \cup ii_1) > u_i(g)$. But then because of pairwise stability of g: $u_{i_1}(g \cup ii_1) < u_{i_1}(g)$. However, for i_1 to have $u_{i_1}(g') \geq u_{i_1}(g)$, there has to exist an i_2 such that $u_i(g \cup i_1 i_2) > u_i(g)$, since (15) holds and already $u_{i_1}(g \cup ii_1) < u_{i_1}(g)$. Continuing in this manner, for g' to be Pareto preferred to g, there has to exists sequence $(i_k)_{k=1,...,K}$ of pairwise distinct players $i_k \in N(l)$ such that $u_{i_k}(g \cup i_k i_{k+1}) > u_{i_k}(g)$. By pairwise stability of g, $mu_{i_k}(g \cup i_k i_{k+1}, i_k i_{k+1}) > 0$ implies $mu_{i_{k+1}}(g \cup i_{k+1} i_k, i_{k+1} i_k) < 0$ for all $k = 1, ..., K-1$. Since transitivity of negative marginal utility in new links holds, and $mu_{i_{k+1}}(g \cup i_{k+1} i_k) < 0$ for all $k = 1, ..., K-1$, we get that $mu_{i_k}(g \cup i_k i_j) < 0$ for all $j < k$, and thus $i_k \notin \{i, i_1, ..., i_{k-1}\}$, and hence $K \leq |N(l)|$. But since $N(l)$ is finite, we get for the last player i_K in the sequence:

$$u_{i_K}(g') - u_{i_K}(g) \overset{(13)}{\leq} u_{i_K}(g \cup l_{i_K}) - u_{i_K}(g) \overset{(14)}{\leq} \sum_{j:i_K j \in l_{i_K}} u_{i_K}(g \cup ij) - u_{i_K}(g) < 0,$$

since $u_{i_K}(g \cup ij) - u_{i_K}(g) < 0$ for all $j \in \{i, i_1, ..., i_{K-1}\}$. Thus, if there exists a $i \in N$ such that $u_i(g') > u_i(g)$, then there has to exist a player j such that $u_j(g') < u_j(g)$, completing the proof. □

Proof of Corollary 3. We show here the obvious, that a regular pairwise stable network is also pairwise stable with transfers, the remaining is implied by Theorem 3. If the utility function satisfies (4) and the network is regular, then all players receive the same utility, since all arguments of the utility functions are equal. Alike are the marginal utilities of any two involved players from forming a link. Since g is pairwise stable, we have for any $i, j \in N$: $mu_i(g \cup ij, ij) = mu_j(g \cup ij, ij) \leq 0$, and thus $mu_i(g \cup ij, ij) + mu_j(g \cup ij, ij) \leq 0$, implying pairwise stability with transfers. In the star for any two peripheral players same considerations hold. □

Proof of Proposition 2. All pairwise stable networks consist of completely connected components that can be ordered according to size such that each larger component of size m satisfies $m > n^2$, where n is the size of the smaller component. There cannot be singleton components. Note that this implies that there exists at least one component of size 3, if $n \geq 3$. Since any even sized network of $n/2$ separate pairs and any odd sized network of $(n-2)/2$ pairs and the remaining three players being connected by 2

links is strongly efficient, it is also component efficient for any component of size n and, hence, strictly welfare better than any completely connected component of at least size 3. For the exact calculations see Jackson and Wolinsky (1996). Hence any completely connected component of size 3 or larger contains a welfare better subcomponent, whereas a completely connected component of size 2 is component welfare maximizing, implying the result. □

Proof of Proposition 3. Let $c < \frac{\rho}{(\rho+2k+2)(\rho+2k)}$, then for the welfare maximizing number of links it holds that $1/2D^*(g) \geq k$. Since any network, which contains $1/2D^*(g)$ links is welfare maximizing, any network, which has less than $1/2D^*(g)$ links is under-connected. By Theorem 3 no pairwise stable network can be under-connected, since u^{PR} satisfies negative externalities and concavity in own new links. Thus, any network $g \in [PS^t]$ has to contain at least $1/2D^*(g) \geq k$ links. □

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