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Working Paper

On the stability of CSS under the replicator dynamic

Working Papers, No. 436

Provided in Cooperation with:
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Suggested Citation: Louge, Fernando (2010) : On the stability of CSS under the replicator dynamic, Working Papers, No. 436, Bielefeld University, Institute of Mathematical Economics (IMW), Bielefeld,
https://nbn-resolving.de/urn:nbn:de:hbz:361-17302

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Working Papers

Institute of Mathematical Economics

436

July 2010

On the Stability of CSS under the Replicator Dynamic

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http://www.wiwi.uni-bielefeld.de/~imw/Papers/showpaper.php?436
ISSN: 0931-6558

On The Stability of CSS under the Replicator Dynamic*

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July 9, 2010

Abstract

This paper considers a two-player game with a one-dimensional continuous strategy. We study the asymptotic stability of equilibria under the replicator dynamic when the support of the initial population is an interval. We find that, under strategic complementarities, Continuously Stable Strategy (CSS) have the desired convergence properties using an iterated dominance argument. For general games, however, CSS can be unstable even for populations that have a continuous support. We present a sufficient condition for convergence based on elimination of iteratively dominated strategies. This condition is more restrictive than CSS in general but equivalent in the case of strategic complementarities. Finally, we offer several economic applications of our results.

JEL Classification: C73

1 Introduction

In finite games, the static concept of an Evolutionarily Stable Strategy (ESS) captures asymptotic stability under many dynamics.[1] The literature of evolution in games with a continuum of strategies has been less successful in this regard. Eshel and Motro (1981) introduced an analogous concept to ESS: the Continuously Stable Strategy (CSS). This paper provides some sufficient conditions under which a CSS is asymptotically stable under the replicator dynamic.

It is well known that CSS is not sufficient for asymptotic stability in general. In fact, it has been shown that a stronger concept than CSS, the Neighborhood Invader Strategy (NIS), is necessary for convergence from *arbitrary* initial populations.[2]

*I am grateful to Frank Riedel and Bill Sandholm for valuable comments.

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[1]The first result in this direction was provided by Taylor and Jonker (1978). See Sandholm (2010) for a recent survey.

[2]See Eshel and Sansone (2003) and Cressman (2005). Apaloo (1997) provides an early treatment of NIS.

In this paper we show that positive convergence results can be obtained when the support of the population is a continuum. In particular, if the game exhibits strategic complementarities, the CSS requirement is sufficient for the evolutionarily stability of the equilibrium. For general games, however, CSS can be unstable even for populations that have a continuous support.

We introduce a closely related solution concept, the *Dominance Solvable Strategy* (DSS), that guarantees convergence under two conditions on the initial population: (i) the support is in a neighborhood of the DSS (2) the support is an interval. In general, DSS is a stronger condition than CSS, but they coincide in the case of strategic complementarities. Moreover, in the latter case, if a CSS/DSS is the unique equilibrium, then the convergence occurs from even from distributions that are far way from the equilibrium, provided they satisfy the continuous support.

A recurring property in evolutionary game theory is the one of *superiority*. Namely, a strategy has to perform against an invader better than the invader against itself. A CSS guarantees that locally superior strategies to an invader are in the direction of the equilibrium. However, the convergence fails since these strategies might not be in the support of the population. The related concept of NIS requires that the equilibrium itself is locally superior against all potential invaders.

This paper focuses on the following question: if the population contains all intermediate strategies between the invader and the equilibrium, is a CSS asymptotically stable? The question is particularly relevant since they are economic applications where the equilibrium satisfies the CSS condition but are not NIS, such as first price auctions, as pointed out in Louge and Riedel (2010).

Unfortunately, the answer to our question is negative in general games, although true in games with strategic complementarities. We present an example in which the unique equilibrium is a CSS (and a NIS) and it is unstable even for a population such that the support is a continuum.[3] The main problem is that, even though all locally superior strategies to an invader are in the direction of the equilibrium, the best response to the invader is not 'close' to the equilibrium. Our DSS concept, on the other hand, requires explicitly all best responses to the invading strategy to be closer to the equilibrium than the invader itself. Theorem 3 shows that this condition is sufficient for convergence.

We continue to provide some applications of our result to known economic models. First, we consider the evolutionary model of a first price auction of Louge and Riedel (2010). Since the unique Nash equilibrium is a CSS but not a NIS, our main theorem provides a convincing argument in favor of the stability of the Nash equilibrium. Second, show that the equilibrium in a contest model à la Tullock (1980) is stable under the replicator dynamic. Third, we present a Cournot duopoly. Since this model has been studied extensively in the literature, we will use it to show consistency between this

[3]Our example elaborates on one provided in Eshel and Sansone (2003).

paper and other positive results obtained in the literature. Finally, we consider Bertrand competition with differentiated products. If demand and cost functions are linear, the unique Nash equilibrium is always a CSS. Convergence is also obtained, despite the Nash equilibrium not always being a NIS.

We focus on the replicator dynamic since it is perhaps the most well-known dynamic in evolutionary game theory. Different models based on imitative behavior lead to this dynamic (see Schlag, 1998 and Sandholm, 2010). Our main argument is based on elimination of iterated dominated strategies. Similar results were obtained by Samuelson and Zhang (1992) for finite games, Cressman and Hofbauer (2005) for quadratic payoff functions and Heifetz, Shannon, and Spiegel (2007) in a model of evolution of preferences. Our solution concept tries to capture the conditions that allow iterated domination as established by Moulin (1984). The results from Moulin allow us also to conclude that convergence in the replicator dynamic implies convergence in the Cournot dynamic.

The paper is organized as follows. Section 2 introduces the model. In section 3 we define our notion of iterated elimination of dominated strategies for a continuum and show that they become extinct under the replicator dynamic. Section 4 introduces the concept of DSS and shows its relation with CSS. Section 5 contains our main result, namely, the convergence to DSS for general games and a more powerful result for strategic complementarities. Section 6 presents applications of our results. Section 7 gives further comparisons between our results and the ones known in the literature. Some concluding remarks are offered in section 8. All the proofs are collected in the Appendix.

2 The Model

We consider a two-player game $G = (S, \pi)$ played by a single population. The strategy space is $S = [\alpha, \beta] \subset \mathbb{R}$ and $\pi : S \times S \to \mathbb{R}$ is the (symmetric) payoff function, where $\pi(x, y)$ is the payoff to strategy x against an opponent that plays strategy y. Let $BR(y) = \operatorname{argmax}_{x \in S} \pi(x, y)$ be the best response to strategy y. For most of our results, we will need the following assumptions:

Assumption 1. The payoff function π is continuous.

Assumption 2. The payoff function π is strictly quasiconcave in the first argument. That is, for all $x_1 \neq x_2$, $y \in S$ and $\lambda \in (0, 1)$, $\pi(\lambda x_1 + (1 - \lambda)x_2, y) > \min\{\pi(x_1, y), \pi(x_2, y)\}$.

Many economic applications satisfy these assumptions naturally (see Section 6). Additionally, they make the model very tractable. First, since the strategy set is compact, Assumption 1 guarantees that $BR(y)$ is well defined and continuous at all y. Second, Assumption 2 implies that BR is a function and that $\pi(x, y)$ is single peaked: $\pi(x, y)$ is strictly increasing in the first argument if $x < BR(y)$ and strictly decreasing if $x > BR(y)$. Finally, they imply that there is at least one symmetric (pure) Nash equilibrium.

We will say that a game G exhibits strategic complementarities if $BR(y)$ is strictly increasing.

A population is a probability measure P on S. The set of populations is denoted by $\Delta(S)$. Let P and Q be two populations. The expected payoff of P against Q is given by

$$\Pi(P,Q) = \iint_S \pi(x,y)P(dx)Q(dy) \tag{1}$$

The continuity of π and the compactness of S guarantee that (1) is well defined. Define the *excess payoff* σ as follows

$$\sigma(x,P) = \Pi(\delta_x, P) - \Pi(P,P) \tag{2}$$

where δ_x is the Dirac measure that puts probability of 1 at x.

Consider a given initial population be P_0. Let P_t be the population at time $t > 0$. Denote the Borel σ-algebra on S by $\mathscr{S}$. We will say that P_t *evolves according to the replicator dynamic* if for all sets $A \in \mathscr{S}$, we have

$$\frac{dP}{dt}(A) = \int_A \sigma(x,P_t)\, P_t(dx) \tag{3}$$

Under Assumption 1, π is also bounded on S. Therefore, (3) is well defined and has a unique solution.[4]

An important property of the replicator dynamic is that the support of the population is time invariant. Formally, let $C(P)$ denote the support of a population P. If $\{P_t\}$ evolves according to (3), then $C(P_t) = C(P_0)$ for all $t > 0$.

3 Iteratively Dominated Strategies

It is well-known that iteratively (strictly) dominated strategies become extinct under the replicator dynamic in finite games (see Samuelson and Zhang (1992)). Unfortunately, there is currently no equivalent result for continuous strategy games. Here we will present a particular notion of iterated dominance that will allow us extend the result of Samuelson and Zhang and to prove our main result.

Definition 1. A set of strategies D is *iteratively ε-dominated* if there is a sequence of sets $\{A_n, B_n\}_{n=0}^N \subseteq S^{2(N+1)}$ with $A_n, B_n \in \mathscr{S}$ and $P_0(A_n), P_0(B_n) > 0$ such that $D \subseteq \bigcup B_n$ and for all n, all $x \in A_n$, $y \in B_n$ and $z \in R \setminus \bigcup_{m<n} B_m$

$$\pi(x,z) - \pi(y,z) \geq \varepsilon \tag{4}$$

[4]See Oechssler and Riedel (2001).

Our definition of $\varepsilon-$dominance and the traditional definition of iterated dominance differ in two aspects. First, we will typically need the payoff difference to be bounded away from zero. Second, we need that the dominance occurs not between individual strategies but by sets of strategies of positive measure. Finally, notice that this definition considers only a finite number of iterations. While a more general definition could be considered, this one suffices to prove our results.

Our first result uses a similar argument than the one in Cressman and Hofbauer (2005) for the extinction of (iteratively) dominated strategies under the replicator dynamic for some quadratic payoff functions. Lemma 1 shows that if D is iteratively ε-dominated, then $P_t(D) \to 0$ as $t \to \infty$ at an exponential rate. Notice that for this result we don't need Assumptions 1 and 2, since it is assumed that the sequence $\{A_n, B_n\}_{n=0}^N$ exists. However, π is required to be bounded in order to have the replicator equation well defined. Throughout the paper the replicator dynamic is convenient since the support of the population is time invariant.

Lemma 1. *Suppose that π is bounded and that the population evolves according to the replicator dynamic. If D is iteratively ε-dominated for some $\varepsilon > 0$ then, for all $a \in (0,1)$, there exist $k, T < \infty$ such that for all $t > T$:*

$$P_t(D) \leq ke^{-a\varepsilon t} \tag{5}$$

In fact, the proof of Lemma 1 shows that the weight of $\bigcup_{n=0}^m B_n$ goes to zero at an exponential rate for all $m \in \{0, \ldots, N\}$.

4 Static solution concepts

The concept of a CSS was introduced by Eshel and Motro (1981). A strategy x^* is a CSS if a monomorphic population using a strategy x in a neighborhood of x^* can be outperformed by some other (nearby) strategy x' if and only if the distance from x' to x^* is smaller than the distance from x to x^*. It tries to capture the idea that if a population uses strategies only in the neighborhood of a CSS, mutations could only drive it closer to the CSS. Formally, $x^* \in S$ is a *Continuously Stable Strategy* if the following two conditions are verified:

1. δ_{x^*} is an ESS.

2. There exists an $\varepsilon > 0$ such that for all x with $|x - x^*| < \varepsilon$ there exists $\eta > 0$ so that for all x' with $|x - x'| < \eta$

$$\pi(x', x) > \pi(x, x) \text{ if and only if } |x - x^*| > |x' - x^*| \tag{6}$$

See Section 7 for the definition of an ESS.

Unfortunately, it is known that this concept does not guarantee convergence. This is because CSS determines that there is a locally superior strategy to an invader in the direction of the equilibrium. However, the convergence fails since this strategy might not be in the support of the population. The related concept of NIS (see Section 7) requires that the equilibrium itself is locally superior to potential invaders. As we will show in Section 7, the NIS condition is not sufficient for convergence either.

It turns out that there is a related concept, which we will refer to as a Dominant Solvable Strategy (DSS), that gives the desired convergence result. This concept captures the conditions that guarantee dominance solvability in the traditional sense (see Moulin, 1984).

Definition 2. Strategy $x^* \in S$ is a *Dominance Solvable Strategy* (DSS) if the following two conditions are verified:

1. δ_{x^*} is an ESS.
2. There exists an $\eta > 0$ such that if $|x - x^*| < \eta$, then

$$|x - x^*| > |BR(x) - x^*| \tag{7}$$

There is a close relation between these two concepts. In general, DSS imposes stronger conditions. On the other hand, if the game exhibits strategic complementarities, then both concepts are equivalent.

Lemma 2. *Let x^* be a DSS. Under Assumptions 1 and 2, then x^* is a CSS. Moreover, if G exhibits strategic complementarities, then the converse is also true.*

5 Convergence

Before we can prove our main result, we have to specify a notion of convergence. As Oechssler and Riedel (2002) advocate, the appropriate notion of distance between distributions for our model is the Prohorov metric. With this metric, distribution can be close both if (i) most probability is assigned to strategies that are arbitrarily close (although not necessarily exactly the same ones) and (ii) small enough probability is assigned to other strategies.

Formally, denote $D(\varepsilon, A)$ an ε-neighborhood of set $A \in \mathscr{S}$. Namely,

$$D(\varepsilon, A) = \{x : \exists y \in A, |y - x| < \varepsilon\} \tag{8}$$

The Prohorov distance ρ is given by

$$\rho(P,Q) = \inf\{\varepsilon > 0, Q(A) \leq P(D(\varepsilon, A)) + \varepsilon \text{ and } P(A) \leq Q(D(\varepsilon, A)) + \varepsilon \text{ for all } A \in \mathscr{S}\} \tag{9}$$

For the purposes of this paper, it is of special interest the Prohorov distance to a Dirac measure δ_x. In this case, $\rho(P, \delta_x) < \varepsilon$ if and only if $P[S \setminus (x - \varepsilon, x + \varepsilon)] < \varepsilon$.

In other words, two measures are close if for every set the weight on that set under one measure is close to the weight of a neighborhood of that set under the other measure. In the case of convergence to a Dirac measure on a strategy x, this implies that the measure puts most weight on a neighborhood of x. Convergence in the Prohorov metric is equivalent to weak convergence.

We are now ready to state our main result:

Theorem 3. *Let G be a game that satisfies Assumptions 1 and 2. Consider a population P_t that evolves according to the replicator dynamic. If x^* is a DSS and $C(P_0)$ is an interval, then there exists $\eta > 0$ such that for every initial population with $\{x^*\} \subset C(P_0) \subseteq [x^* - \eta, x^* + \eta]$, $P_t \to \delta_{x^*}$ as $t \to \infty$ in the Prohorov metric.*

Figure 1 illustrates the proof of Theorem 3, which relies on Lemma 1. Let R be the strategies that have not been eliminated after some interations. The sequence of elimination we construct is such that R is an interval. Let $\overline{R} > x^*$ be the largest strategy in R. Define $z \equiv \text{argmax}_{y \in R} BR(y)$. The DSS property implies that the best response to a strategy is closer to the equilibrium than the strategy itself. Without loss of generality, assume that $z < \overline{R}$.[5] The quasiconcavity assumption gives that, for any y, $\pi(x, y)$ is strictly decreasing in x for all $x > BR(y)$. Therefore, $\pi(x, y)$ is strictly decreasing in the first argument for all $x > BR(z)$ and all $y \in R$. This implies that $BR(z)$ strictly dominates $\overline{R}$. So far, this dominance is pointwise only. Lemma 1, however, requires that a set of positive measure is dominated by another set of positive measure. The rest of the proof consist of constructing sets A and B such that for some $\varepsilon > 0$, $\pi(x, y) - \pi(x', y) \geq \varepsilon$ for all $x \in A$, $x' \in B$ against all y. This can be done as shown in Figure 1.

The converse to Theorem 3 is not true. For example, Cressman and Hofbauer (2005) show that if the payoff function is quadratic, an equilibrium that satisfies the NIS condition (which is stronger than CSS) is sufficient for convergence even when there are no dominated strategies at all.

On the other hand, Section 7 presents an example in which an equilibrium that is NIS (and therefore CSS) that is unstable under the replicator dynamic. This equilibrium is not a DSS, so Theorem 3 does not apply.

Theorem 3 shows that the DSS condition is sufficient for dominance solvability. Moulin (1984) shows also that it is almost necessary for (pointwise) dominance, and therefore

[5] If this is not true, then $\text{argmin}_{y \in R} BR(y) > \underline{R}$ and the symmetric argument is valid.

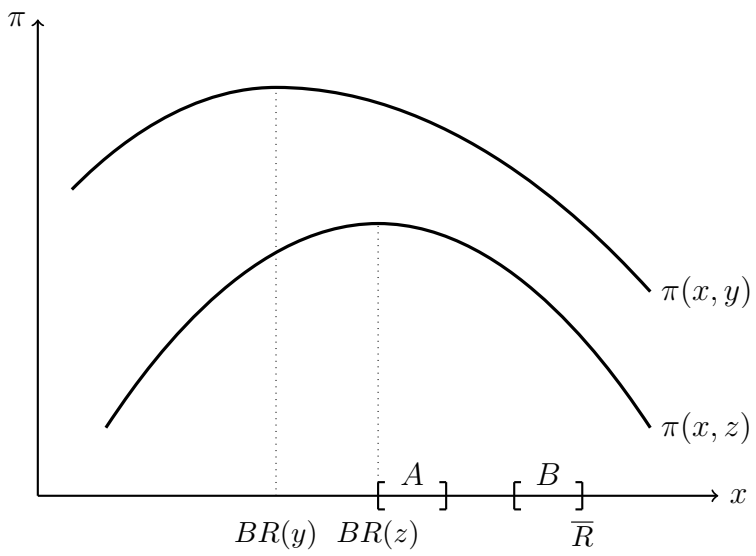

Figure 1: Graphical interpretation of Theorem 3. Payoff function for strategy x against $z \equiv \text{argmax}_{y' \in R} BR(y')$ and $y \in R$. Since $\pi(x, y)$ is strictly decreasing on the interval $[BR(z), \overline{R}]$ for all y, set A dominates B. By Lemma 1, $P_t(B) \to 0$ as $t \to \infty$.

for our setwise definition. In particular, if the payoff function is differentiable, then it is necessary that condition (7) is satisfied with equality or strict inequality.

The result from Theorem 3 can be extended to initial populations with large supports under strategic complementarities. Remember that in this case, by Lemma 2, CSS and DSS are equivalent. Despite these conditions being local, these properties are extend beyond a neighborhood of the equilibrium. In the extreme case where the equilibrium is unique, the 'DSS property' applies to the entire strategy space: the best response strategy is always closer to the equilibrium than the original strategy.

Proposition 4. *Suppose that G satisfies Assumptions 1 and 2 and that the population evolves according to the replicator dynamic. Assume additionally that G exhibits strategic complementarities and that there is a unique Nash equilibrium x^* that is also a CSS. If $C(P_0)$ is an interval and $\{x^*\} \subseteq C(P_0)$ then $P_t \to \delta^*$ as $t \to \infty$ in the Prohorov metric.*

A final corollary of our result is the relationship between convergence in the replicator dynamic and the Cournot dynamic. The (discrete time) Cournot dynamic consists of players choosing the best response to the opponent's strategy in the previous period. Moulin (1984) showed that, for the class of games we consider, (pointwise) dominance solvability is equivalent to convergence in the Cournot dynamic. Our results show that convergence in the Cournot dynamic implies convergence in the replicator dynamic. The converse is not true, however, since our results give sufficient but not not necessary conditions.

6 Applications

In this section we will discuss some applications of Theorem 3 and Proposition 4 to known economic models. Since our assumptions are fairly general, most models with continous payoffs fall in the class of games we study.

6.1 First price auction

Consider an evolutionary model of a first price sealed bid auction with independent values, as in Louge and Riedel (2010). There are 2 bidders and valuations, v, are distributed uniformly on $[0, 1]$. The class of bidding functions allowed are $b(v) = \frac{v^x}{2}$, where $x \in S$ is the bidder's strategy. We assume $S = [\alpha, \beta]$ with $0 < \alpha < 1$ and $\beta > 1$. The unique Nash equilibrium of the auction is $b^*(v) = E[v' \mid v > v'] = \frac{v}{2}$ (i.e. $x^* = 1$). The expected payoff of strategy x against strategy y can be expressed as:

$$\pi(x, y) = \frac{xy(1 + 2y)}{2x^2 + 2x^2y + 6xy + 4xy^2 + 4y^2} \tag{10}$$

which is continuous and concave in x. Additionally, x^* is a CSS (but not a NIS) with respect to the class of functions allowed. This game exhibits strategic complementarities.[6] All the above facts are obtained from Louge and Riedel (2010).

All the conditions for Proposition 4 are verified and we can obtain the following result:

Corollary 5. *Let G describe a first price sealed bid auction, with bidding strategies $b(v) = \frac{v^x}{2}$. Let $x^* = 1$ be the unique Nash equilibrium of this game. Then, for any initial population such that $C(P_0) \supseteq \{x^*\}$ is an interval, the replicator dynamic converges to δ_1.*

6.2 Contests

We introduce here a contest game à la Tullock (1980). Let x be the effort chosen by a player. The payoff function is $\pi(x, y) = p(x, y) - c(x)$, where p is the probability of winning the contest when the opponent chooses effort y and c the cost. The value of winning is normalized to 1. Assume $c(x) = x$ and $p(x, y) = \frac{x}{x+ky}$ with $k > 0$. The strategy set is $S = [\alpha, \beta]$, with $\alpha > 0$ (although small). The best response is $BR(y) = (ky)^{\frac{1}{2}} - ky$, which is strictly increasing if $y < \frac{1}{4k}$ and strictly decreasing if $y > \frac{1}{4k}$. The unique Nash equilibrium of this game is $x^* = \frac{k}{(1+k)^2}$ is also a CSS and a NIS.

Consider first the case where $k < 1$. Then $x^* < \frac{1}{4k}$. If $\beta \in \left(x^*, \frac{1}{4k}\right]$, then this game is of strategic complementarities. On the other hand, if $k > 1$ we have that $x^* > \frac{1}{4k}$ and $\frac{d}{dy}BR(x^*) < 0$. In this case, we will assume $\beta \geq \frac{k}{(1+k)^2}$ in order to have $x^* \in S$. The equilibrium is a DSS if $k < 3$.

[6] This property is straightforward, since the best response function $BR(y) = \left[\frac{2y^2}{1+y}\right]^{\frac{1}{2}}$ is strictly increasing.

We can now apply both Theorem 3 and Proposition 4 to this game and conclude:

Corollary 6. *Let G be a contest and $x^* = \frac{k}{(1+k)^2}$ the unique Nash equilibrium of this game. Let $C(P_0)$ be an interval. If $k < 1$ then, for any initial population such that $C(P_0) \supseteq \{x^*\}$, the replicator dynamic converges to δ_{x^*}. If $1 < k < 3$, then there exists $\mu > 0$ such that if $\{x^*\} \subset C(P_0) \subseteq [x^* - \eta, x^* + \eta]$, the replicator dynamic converges to δ_{x^*}.*

6.3 Cournot duopoly

Two firms compete in a market setting quantities. The profit function is $\pi(x, y) = xd(x + y) - c(x)$ for a firm that produces quantity x against a firm that produces y. It is assumed that c and d are differentiable, with $d' < 0$, $d'' \leq 0$, $c' > 0$ and $c'' \geq 0$. It is well known that. under these assumptions, quantities are strategic substitutes and that there is a unique equilibrium x^*. Because of the strategic substitutes property, the equilibrium is automatically a CSS and a NIS. Also, under our assumptions, $\frac{dBR}{dy}(x^*) > -1$. This inequality implies also that x^* is a DSS. Theorem 3 implies the following result.

Corollary 7. *Let G be a Cournot duopoly. There exists $\mu > 0$ such that if $C(P_0) \subseteq [x^* - \mu, x^* + \mu]$ is an interval, the replicator dynamic converges to δ_{x^*}.*

It is well known that the Cournot dynamic converges in this game as well. As noted above, convergence in the replicator dynamic implies convergence in the Cournot dynamic.

It can also be remarked that in this model $\frac{dBR}{dy}(y) > -1$ for all y. This suggests that the 'DSS property' can be extended to the full strategy space and give a global convergence result in the spirit of Proposition 4. Unfortunately, the DSS solution concept in its current form is only local and does not capture this feature of the model.

6.4 Bertrand duopoly with differentiated products

Consider a duopoly with differentiated products. There is a linear demand $D(x, y) = a - bx + dy$ for a firm that sets price x and a competitor that sets price y, where $a, b, d > 0$ and constant marginal cost $c > 0$. The strategy space is $S = [0, \beta]$, for large enough β. The profit function is then $\pi(x, y) = (a - bx + dy)(x - c)$, which is continuous and concave. The best response function is $BR(y) = (2b)^{-1}(a + bc + dy)$. Since all parameters are positive, this is a game of strategic complementarities. An equilibrium x^* exists if and only if $d < 2b$. When it exists, this equilibrium is unique and also a DSS (and consequently a CSS). It is not a NIS unless $d < b$. We can summarize our conclusions:

Corollary 8. *Let G describe a Bertrand duopoly with differentiated products with linear demand and cost functions. Let x^* be the unique Nash equilibrium of this game. Then, for any initial population such that $C(P_0) \supseteq \{x^*\}$ is an interval, the replicator dynamic converges to δ_{x^*}.*

This result can be trivially extended to general payoff functions. However, when there are multiple equilibria, not all of them are CSS. In particular, only the ones such that the best response crosses the 45 degree line from above are CSS. Proposition 4 implies that the replicator dynamic converges to the equilibria that satisfy the CSS requirement, provided that initial distribution has support in an interval where no other equilibrium exists.

7 Related literature

In this section we will compare our results above to others in the literature. First, we introduce the solution concepts that were mentioned. Second, we present several examples to clarify to which classes of games our results and the literature's apply.

7.1 Other solution concepts

In addition to the concept of a CSS introduced in Section 4, there are two other relevant static solution concepts for evolution in games with continuous strategies. First, the classic concept of ESS of Maynard Smith and Price (1973) is usually formulated in the following way. A population $P \in \Delta(S)$ is an *Evolutionarily Stable Strategy* (ESS) if for every population $Q \neq P$ there exists $\varepsilon > 0$ such that

$$\Pi(P, (1-\varepsilon)P + \varepsilon Q) > \Pi(Q, (1-\varepsilon)P + \varepsilon Q) \tag{11}$$

This concept tries to capture the idea of an incumbent population P invaded by a small fraction ε of mutants. The incumbents form an ESS if they outperform the mutants in the new (mixed) population. Notice that every strict Nash Equilibrium is also a ESS. The main problem in this definition is that there are some types of mutations that are not taken into account. This concept is reasonable in finite games, since the only way to be "close" to a population is to put arbitrarily high probability on it. In games of continuous strategies this is no longer the case. In particular, an invading population can be close to the incumbent because the supports are close, even though the distributions might be dramatically different. Due to the linearity of Π, all strict equilibria are ESS. If not strict, an ESS needs to be superior to all potential invaders.

Second, there is the concept of Neighborhood Invader Strategy (NIS). It was introduced by Apaloo (1997) and it implies CSS. Formally, a strategy x^* is a *NIS* if there

exists an $\varepsilon > 0$ such that for all x with $|x - x^*| < \varepsilon$

$$\pi(x^*, x) > \pi(x, x) \tag{12}$$

In words, NIS requires all populations using strategies in the neighborhood of x^* to be outperformed by x^* itself, and not simply by a strategy *closer* to x^*. It has been shown to be a necessary condition for general convergence under the replicator dynamic.[7]

In the case of strategic complementarities, a NIS is also a DSS. In general games, no relation can be establish. The example below has an equilibrium that is a NIS but not a DSS. Our application to Bertrand competition with differentiated products above have equilibria that are DSS but not NIS.

7.2 An Example With Nondominance and Nonconvergence

This game was introduced in Eshel and Sansone (2003). Consider $\pi(x, y) = (x - y)^4 - 2x^4 - 2y^4$. We have that $x^* = 0$ is strict Nash equilibrium (and therefore an ESS). It is also a CSS and NIS, since for all y

$$\pi(0, y) = -2y^4 > -4y^4 = \pi(y, y) \tag{13}$$

However, as we will see below, this equilibrium not stable under the replicator dynamic. This function is strictly quasiconcave in x, although it is not concave (see Figure 2 below). The best response function is $BR(y) = -\left[\sqrt[3]{2} - 1\right]^{-1} y$. Therefore, strategies are substitutes. Suppose that the initial population is a uniform distribution Q_ε on $[-\varepsilon, \varepsilon]$. It can be seen that

$$\Pi(\delta_0, Q_\varepsilon) = -\frac{1}{5}\varepsilon^4 < \frac{4}{15}\varepsilon^4 = \Pi(Q_\varepsilon, Q_\varepsilon) \tag{14}$$

In other words, a neighborhood around the equilibrium obtains an expected payoff below the population average when the population is uniform. Moreover, using the replicator dynamic and the initial population Q_ε, consider set $A = \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$. We obtain

$$\left.\frac{dP}{dt}(A)\right|_{t=0} = -\frac{5}{32}\varepsilon^4 < 0 \tag{15}$$

Therefore, δ_{x^*} cannot be stable under the replicator dynamic.

Figure 2 illustrates why the argument of Theorem 3 is not valid in this example. The strategy set is $[-0.05, 0.05]$. The functions depicted are the payoff of strategy x against -0.05, 0 and 0.05. We have that $BR(0.05) \approx -0.1925$ and $BR(-0.05) \approx 0.1925$, violating the DSS requirement. Since BR is continuous, $[-0.05, 0, 0.05] \subset BR([-0.05, 0, 0.05])$. Therefore, no strategy in $[-0.05, 0.05]$ is dominated since all of them are best response

[7] See Eshel and Sansone (2003).

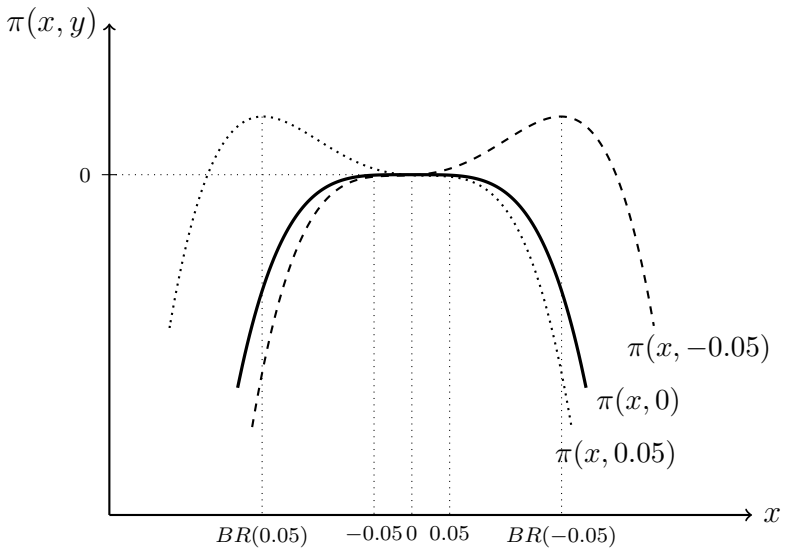

Figure 2: Example with payoff function $\pi(x,y) = (x-y)^4 - 2x^4 - 2y^4$ for $y \in [-0.05, 0, 0.05]$. The unique equilibrium $x^* = 0$ is a CSS and a NIS, but unstable under the replicator dynamic. However, $[-0.05, 0, 0.05] \subset BR([-0.05, 0, 0.05])$. Therefore, Theorem 3 does not apply since the equilibrium does not verify the DSS requirement.

to some strategy.

As inequality (15) shows, the set $[-0.025, 0.025]$ obtains a lower expected payoff than $[-0.05, -0.025] \cup [0.025, 0.05]$ when the population is uniform on $[-0.05, 0.05]$. This is because, while $\pi(-0.025, -0.025) < \pi(0, -0.025) < \pi(0.025, -0.025)$, the difference in the terms in the first inequality is much smaller than the difference for the second inequality. On average, a neighborhood around $x^* = 0$ does worse than the strategies on the extremes of $[-0.05, 0.05]$.

8 Summary

We have shown that when the initial population has full support around an equilibrium, positive results regarding convergence of the replicator dynamic can be obtained. We have introduced the concept of a Dominant Solvable Strategy (DSS) and showed that under reasonable conditions on the payoff function (i.e. continuity and quasiconcavity), a DSS is asymptotically stable. The result is even stronger in games with strategic complementarities with a unique equilibrium. In this case, the replicator dynamic converges to a DSS from any initial distribution with full support in an interval containing the equilibrium.

A DSS is a more restrictive concept than a Continuously Stable Strategy (CSS). In the case of strategic complementarities, however, they coincide. As shown above, there are relevant economic applications with strategic complementarities where Nash equilibria are

CSS but not NIS. Previous work had obtained a negative result in that NIS is a necessary condition for convergence of the replicator dynamic from arbitrary populations. Here we show that if the strategies used in the population are disperse enough (i.e. they are a continuum), then the milder CSS (or DSS) condition can be sufficient for stability.

On the other hand, CSS and NIS where also not sufficient conditions for convergence, as illustrated in the example of Section 7. For the examples known to us where CSS and NIS do not converge, these equilibria are not DSS and Theorem 3 does not apply.

Our convergence result relies on the elimination of iteratively dominated strategies. Despite this being a recurrent result in the literature, we had to modify it slightly in order to extend it to continuous strategy games. Our iteratively ε-dominated concept needs to include sets of strategies of positive measure and a difference in payoffs bounded away from zero.

There is still an open question regarding the necessary conditions for convergence of populations with continuous support. For example, positive results have been obtained for NIS in quadratic payoff functions, even when these are not dominance solvable (see Cressman and Hofbauer, 2005).

There is an interesting connection between the DSS concept and convergence of the Cournot dynamic. Moulin (1984) showed that, for the class of games we consider, the Cournot dynamic converges if and only if the game is dominance solvable. Together with the results in this paper, we can conclude that convergence in the replicator dynamic implies convergence in the Cournot dynamic.

Our DSS concept, as well as CSS and NIS are local conditions. Bomze (1990, 1991) propose solution concepts that capture small invasions of mutants using strategies far from the equilibrium. There is still progress to be made in combining the features of both the local concepts with his *uninvadability* requirement.

There are some economic applications that do not fit in our class. In particular, some continuous strategy games have discontinuous payoff functions such as the Nash demand and ultimatum games. There are still few results for this important group of models.

Appendix

Proof of Lemma 1. Suppose that D is dominated via some sequence $\{A_n, B_n\}_{n=0}^N$ that verifies (4). Fix some $a \in (0,1)$. We will show that for all $n \in \{0, \ldots, N\}$ there exist k_n, T_n such that for all $t > T_n$

$$\sum_{m=0}^{n} P_t(B_m) \leq k_n e^{-a\varepsilon(t-T_n)} \tag{16}$$

Consequently,

$$P_t(D) \leq P_t(\bigcup_{n=0}^{N} B_n) \leq \sum_{n=0}^{N} P_t(B_n) \leq k_N e^{-a\varepsilon(t-T_N)} = \left[k_N e^{a\varepsilon T_N}\right] e^{-a\varepsilon t} \tag{17}$$

for all $t > T_N$.

The proof is by induction. Consider first A_0 and B_0. Since π is bounded, (3) is well defined and it gives:

$$\frac{d}{dt}\left(\frac{P_t(A_0)}{P_t(B_0)}\right) = \frac{1}{(P_t(B_0))^2}\left[P_t(B_0)\int_{A_0}\sigma\left(x,P_t\right)P_t(dx) + P_t(A_0)\int_{B_0}\sigma\left(x,P_t\right)P_t(dx)\right] \tag{18}$$

$$= \frac{1}{(P_t(B_0))^2}\left[P_t(B_0)\int_S\int_{A_0}\pi\left(x,z\right)P_t(dx)P_t(dz)+\right. \tag{19}$$

$$\left.+\ P_t(A_0)\int_R\int_{B_0}\pi\left(x,z\right)P_t(dx)P(_tdz)\right] \tag{20}$$

$$= \frac{1}{(P_t(B_0))^2}\int_R\int_{A_0}\int_{B_0}\left[\pi(x,z)-\pi(y,z)\right]P_t(dx)P_t(dy)P_t(dz) \tag{21}$$

$$\geq \frac{\varepsilon}{(P_t(B_0))^2}\int_{A_0}\int_{B_0}P_t(dx)P_t(dy) \tag{22}$$

$$= \frac{P_t(A_0)}{P_t(B_0)}\varepsilon > 0 \tag{23}$$

Integrating we obtain

$$P_t(B_0) \leq P_t(A_0)\frac{P_0(B_0)}{P_0(A_0)}e^{-\varepsilon t} \leq \frac{1}{P_0(A_0)}e^{-a\varepsilon t} \tag{24}$$

Therefore, (16) is verified for $k_0 = \frac{1}{P_0(A_0)}$ and $T_0 = 0$.

Now assume that there exist k_{n-1}, T_{n-1} such that $\sum_{m=0}^{n-1}P(B_m) \leq k_{n-1}e^{-a\varepsilon(t-T_{n-1})}$ for $t > T_{n-1}$. We will show that there exists k_n, T_n such that $\sum_{m=0}^{n}P(B_m) \leq k_{n+1}e^{-a\varepsilon(t-T_n)}$ $t > T_n$. Define $\bar{\pi} = \max_{x,y\in S}|\pi(x,y)| < \infty$. The replicator equation gives:

$$\frac{d}{dt}\left(\frac{P_t(A_n)}{P_t(B_n)}\right) = \frac{1}{(P_t(B_n))^2}\int_R\int_{A_n}\int_{B_n}\left[\pi(x,z)-\pi(y,z)\right]P_t(dx)P_t(dy)P_t(dz) \tag{25}$$

$$= \frac{1}{(P_t(B_n))^2}\int_{R\setminus\cup_{m=0}^{n-1}B_m}\int_{A_n}\int_{B_n}\left[\pi(x,z)-\pi(y,z)\right]P_t(dx)P_t(dy)P_t(dz)+ \tag{26}$$

$$+\frac{1}{(P_t(B_n))^2}\int_{\cup_{m=0}^{n-1}B_m}\int_{A_i}\int_{B_n}\left[\pi(x,z)-\pi(y,z)\right]P_t(dx)P_t(dy)P_t(dz) \tag{27}$$

$$\geq \frac{P_t(A_n)}{P_t(B_n)}\left[\varepsilon\left(1-P\left(\bigcup_{m=0}^{n-1}B_m\right)\right)-2\bar{\pi}P_t\left(\bigcup_{m=0}^{n-1}B_m\right)\right] \tag{28}$$

$$\geq \frac{P_t(A_n)}{P_t(B_n)}\left[\varepsilon-(2\bar{\pi}+\varepsilon)\sum_{m=0}^{n-1}P_t(B_m)\right] \tag{29}$$

Define T_n such that

$$\frac{(1-a)\varepsilon}{2\bar{\pi}+\varepsilon} = k_{n-1}e^{-a\varepsilon(T_n-T_{n-1})} \geq \sum_{m=0}^{n-1} P_{T_n}(B_m) \tag{30}$$

Then for all $t > T_n$

$$\frac{d}{dt}\left(\frac{P_t(A_n)}{P_t(B_n)}\right) \geq \frac{P_t(A_n)}{P_t(B_n)}\left[\varepsilon - (2\bar{\pi}+\varepsilon)\frac{(1-a)\varepsilon}{2\bar{\pi}+\varepsilon}\right] = \frac{P_t(A_n)}{P_t(B_n)}a\varepsilon \tag{31}$$

Integration gives for all $t > T_n$

$$P_t(B_n) \leq P_t(A_n)\frac{P_{T_n}(B_n)}{P_{T_n}(A_n)}e^{-a\varepsilon(t-T_n)} \leq \frac{1}{P_{T_n}(A_n)}e^{-a\varepsilon t-T_{n+1}} \tag{32}$$

Finally,

$$\sum_{m=0}^{n} P(B_m) \leq k_{n-1}e^{-a\varepsilon(t-T_{n-1})} + \frac{1}{P_{T_n}(A_n)}e^{-a\varepsilon(t-T_n)} \tag{33}$$

$$= \left[\frac{(1-a)\varepsilon}{2\bar{\pi}+\varepsilon} + \frac{1}{P_{T_n}(A_n)}\right]e^{-a\varepsilon(t-T_n)} \tag{34}$$

$$\equiv k_n e^{-a\varepsilon(t-T_n)} \tag{35}$$

□

Proof of Lemma 2. Suppose that (7) is verified for $\varepsilon > 0$. Without loss of generality, assume $y \in (x^*, x^*+\varepsilon]$. Then $BR(y) < y$. By strict quasiconcavity, $\pi(x,y)$ is strictly decreasing for all $x > BR(y)$. Therefore, in a neighborhood around y, $\pi(x,y) > \pi(y,y)$ if and only if $x < y$. Since $y > x^*$, then this is equivalent to $|y-x^*| > |x-x^*|$.

Suppose now that BR is strictly increasing. Take $y > x^*$. Then, $BR(y) > BR(x^*) = x^*$. Since x^* is a CSS, then $\pi(x,y)$ is strictly decreasing in the first argument at y. Strict quasiconcavity implies then that $BR(y) < y$. Putting both inequalities together, we get $|y-x^*| > |BR(y)-x^*|$. The same argument applies to $y < x^*$. □

Proof of Theorem 3. Let $\eta > 0$ be such that (7) is satisfied. We will show that for every $\eta' \in (0,\eta)$ there exists $\varepsilon > 0$ such that $C(P_0) \setminus [x^*-\eta', x^*+\eta']$ is iteratively ε-dominated via a sequence $\{A_n, B_n\}_{n=0}^N$.

Define $R_0 = C(P_0)$ and $R_n = R_0 \setminus \cup_{m=0}^{n-1} B_m$. Since R is an interval, we have $R_0 = [\underline{R}_0, \overline{R}_0]$. We will construct our sequence so that R_n is an interval for all n.

Consider first the case where

$$0 < x^* - \underline{R}_n < \eta' < \overline{R}_n - x^* \tag{36}$$

for some n. Since x^* is a DSS, it follows that $\overline{r}_n \equiv \text{argmax}_{y\in R_n} BR(y) \in [x^*, \overline{R}_n)$.

Under Assumptions 1 and 2, $\pi(x,y)$ is strictly decreasing in the first argument for all $x \geq BR(y)$. This argument implies that

$$\pi(\overline{r}_n, y) > \pi(\overline{R}_n, y) \tag{37}$$

for all $y \in R_n$.

Since $R_0 \setminus [x^*-\eta', x^*+\eta']$ is compact and $|y-BR(y)|$ is continuous and strictly positive on this set, there exists $\gamma > 0$ such that $|y-BR(y)| \geq \gamma$ for all $y \in R_0 \setminus [x^*-\eta', x^*+\eta']$.

Define the sets

$$A_n = [\overline{r}_n, \overline{r}_n + \frac{\gamma}{3}] \tag{38}$$

$$B_n = [\underline{B}_n, \overline{R}_n] \tag{39}$$

where $\underline{B}_n = \max\{\overline{R}_n - \frac{\gamma}{3}, x^* + \frac{\eta'}{2}\}$. Set

$$\varepsilon_n = \min_{y \in R_n} \left[\pi(\overline{r}_n + \frac{\gamma}{3}, y) - \pi(\overline{R}_n - \frac{\gamma}{3}, y)\right] > 0 \tag{40}$$

Therefore, every strategy in A_n obtains a payoff at least ε_n higher that any strategy in B_n. Notice that A_n and B_n have positive measure. Additionally, if (36) if verified for some R_n, then it is verified for all R_m with $m > n$.

Fix N so that $\overline{R}_0 - N\frac{\gamma}{3} \leq x^* + \eta' < \overline{R}_0 - (N-1)\frac{\gamma}{3}$. Since γ is bounded away from zero, N is finite.

Define $\varepsilon = \min_{n \leq N} \varepsilon_n > 0$. Therefore, we have shown that $[x^* + \eta', \overline{R}_0] \subset \cup_{n=0}^{N} B_n$ is ε-dominated. By Lemma 1, then there exists a finite t' such that $P_t\left([x^* + \eta', \overline{R}_0]\right) < 1 - \eta'$ for all $t' > t$.

The second case, where

$$0 < \overline{R}_n - x^* < \eta' < x^* - \underline{R}_n \tag{41}$$

for some n is exactly symmetric to the previous case. The same proof can be contructed for $\underline{r}_n \equiv \text{argmin}_{y \in R_n} BR(y) \in (\underline{R}_n, x^*]$ and the sets

$$A_n = [\underline{r}_n - \frac{\gamma}{3}, \underline{r}_n] \tag{42}$$

$$B_n = [\underline{R}_n, \overline{B}_n] \tag{43}$$

where $\overline{B}_n = \min\{\underline{R}_n + \frac{\gamma}{3}, x^* - \frac{\eta'}{2}\}$.

Finally, suppose

$$\eta' < \min\{x^* - \underline{R}_n *, \overline{R}_n - x^*\} \tag{44}$$

In this case, the previous argument can be modified in the following way. If $|x^* - \underline{r}_n| \leq |x^* - \overline{r}_n|$, then A_n, B_n are given by (38)-(39). Otherwise, if $|x^* - \underline{r}_n| > |x^* - \overline{r}_n|$, then A_n, B_n are given by (42)-(43).

This construction continues as long as n satisfies (44). After a finite number of periods, either (36) or (41) will hold. Then, sets A_n, B_n are constructed according to (38)-(39) or (42)-(43), respectively, for all subsequent iterations.

□

Proof of Proposition 4. Let $C(P_0) = \left[\underline{R}_0, \overline{R}_0\right]$. In the case of strategic complementarities, we can consider the intervals $[\underline{R}_0, x^*]$ and $\left[x^*, \overline{R}_0\right]$ independently. Without loss of generality, we will prove the result for $C(P_0) = R_0 = \left[x^*, \overline{R}_0\right]$.

Since x^* is a Nash equilibrium, $BR(x^*) = x^*$. Under strategic complementarities, $BR(x) > x^*$ for all $x > x^*$. By Lemma 2, x^* is a DSS. That is, for some $\eta > 0$, $x \in (x^*, x^* + \eta)$ implies $BR(x) < x$. Since the equilibrium is unique and BR is continuous, then $x^* < BR(x) < x$ for all $x > x^*$.

For a fixed $\eta' > 0$, there exists $\gamma > 0$ such that $|y - BR(y)| \geq \gamma$ for all $y \in [x^* + \eta', \overline{R}_0]$.

Similarly to the proof of Theorem 3, define $R_n = \left[x^*, \overline{R}_n\right] = R_0 \setminus \cup_{m=0}^{n-1} B_m$ and $\overline{r}_n = BR(\overline{R}_n) < \overline{R}_n$ to construct the sequence of sets $\{A_n, B_n\}$ as in (38)-(39). Define $\varepsilon = \min_{n \leq N} \varepsilon_n > 0$, where ε_n is given by (40). Therefore, we have shown that $[x^* + \eta', \overline{R}_0] \subset \cup_{n=0}^{N} B_n$ is ε-dominated. By Lemma 1, then there exists a finite t' such that $P_t\left([x^* + \eta', \overline{R}_0]\right) < 1 - \eta'$ for all $t' > t$. □

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