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**Working Paper**

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IWQW Discussion Papers, No. 08/2010

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*Suggested Citation:* Tinkl, Fabian (2010) : A note on Hadamard differentiability and differentiability in quadratic mean, IWQW Discussion Papers, No. 08/2010, Friedrich-Alexander-Universität Erlangen-Nürnberg, Institut für Wirtschaftspolitik und Quantitative Wirtschaftsforschung (IWQW), Nürnberg

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# IWQW

Institut für Wirtschaftspolitik und Quantitative  
Wirtschaftsforschung

Diskussionspapier  
Discussion Papers

No. 8/2010

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ISSN 1867-6707

# A note on Hadamard differentiability and differentiability in quadratic mean

Working Paper, 21. Oktober 2010

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*Keywords and phrases:* Hadamard differential, Differentiability in quadratic mean

## Abstract

We prove that Hadamard differentiability in addition with usual assumptions on the loss function for  $M$  estimates implies differentiability in quadratic mean. Thus both concepts are exchangeable.

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# 1 Introduction

As it was pointed out by [Pollard, 1997] there is a connection between differentiability in an  $L^2$  sense, differentiability in quadratic mean and local asymptotic normality. The idea of local asymptotic normality (LAN) goes back to [LeCam, 1970], who showed that differentiability in quadratic mean implies a quadratic approximation for the log-likelihood function, without using the 2nd derivative. We will show, how a different concept of differentiability, namely the Hadamard differentials, are under mild regularity conditions equivalent to the definition of differentials in quadratic mean, and therefore the results of [Pollard, 1997], [LeCam, 1970] and [Fernholz, 1983] will coincide for  $M$  estimates. Thus it can be seen, that it is also sufficient for  $M$  estimates to be Hadamard differentiable to enjoy the LAN property.

## 2 The equivalence Theorem

Throughout this article we regard an  $\sigma$ -finite probability space and a sampling  $X_i$  (not necessarily iid.) from a measurable random variable  $X : \Omega \rightarrow \mathbb{R}$ , where  $X$  has a distribution function, which is absolutely continuous w.r.t. some  $\sigma$ -finite measure. Define a  $M$  estimate as the solution of the following minimization problem

$$\hat{\theta}_n = \arg \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \rho(X_i, \theta), \quad (1)$$

where  $\theta$  is some unknown parameter, where  $\theta \in \Theta$ , and  $\Theta$  is some metric space with the uniform metric. Under standard conditions, that  $\theta \rightarrow \rho(\theta)$  is continuous and  $\Theta$  is compact as well as  $\rho$  is uniformly integrable w.r.t.  $\theta$ , we get that  $\hat{\theta}_n \xrightarrow{P} \theta_0$ , where

$$\theta_0 = \arg \min_{\theta \in \Theta} E[\rho(X, \theta)]. \quad (2)$$

We suppress the dependence of  $\rho$  on  $\omega \in \Omega$  from now on, and treat  $\rho$  as a function of the parameter(-vector)  $\theta$ . If  $\rho$  is differentiable w.r.t. to  $\theta$ , one may take a first-order Taylor series expansion around  $\hat{\theta}_n$  in neighborhood of  $\theta_0$ , where (2) attains its minimum:

$$\rho(\hat{\theta}_n) = \rho(\theta_0) + (\hat{\theta}_n - \theta_0)^T \rho'_\theta(\theta_0) + Rem(\theta)(\hat{\theta}_n - \theta_0). \quad (3)$$

Suppose that  $Rem \rightarrow 0$  in some proper sense. If  $Rem \xrightarrow{L^2} 0$ , i.e.  $E[|Rem(\hat{\theta}_n - \theta_0)|^2] = 0$ , we call  $\rho$  differentiable in quadratic mean (DQM). If  $\rho$  is Hadamard differentiable,

then  $Rem \xrightarrow{P} 0$ , see [Fernholz, 1983]. Under usual regularity conditions on  $\rho$  and its derivative we can conclude, that  $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow N(0, \sigma^2)$ , with  $\sigma^2 = \text{var}(\phi(X))$ , and  $\phi$  is the influence function. Define Hadamard differentials as follows (see [van der Vaart, 1998]):

**Definition 1** A function  $\rho : D_\theta \rightarrow W$  is Hadamard differentiable (or compact differentiable) at  $\theta \in D_\theta$  if there exists  $\rho'_\theta \in L_1(D_\theta, W)$  such that for any compact set  $\Gamma$  of  $D_\theta$ ,

$$\lim_{t \rightarrow 0} \left\| \frac{\rho(\theta + th_t) - \rho(\theta)}{t} - \rho'_\theta(h) \right\|_W \rightarrow 0, \quad (4)$$

for every  $h_t \rightarrow h$ , s.t.  $\theta + th_t \in D_\theta$  for  $t > 0$  and  $t$  small enough.

One may write instead of (4):

$$Rem_1(\hat{\theta}_n - \theta_0) = \rho(\hat{\theta}_n) - \rho(\theta_0) - (\hat{\theta}_n - \theta_0)^T \rho'_\theta(\theta_0) \quad (5)$$

as the remainder term of (3). Note that  $\rho'_\theta$  indicates the Hadamard derivative. It is shown by [Fernholz, 1983], that if

$$\lim_{\theta \rightarrow \theta_0} \frac{Rem_1(\theta - \theta_0)}{|\theta - \theta_0|} = 0,$$

then:  $\sqrt{n}Rem_1(\theta - \theta_0) = o_P(1)$ . DQM can be defined using (3):

**Definition 2**  $\rho(\theta)$  is said to be DQM if there exists a function  $D\rho_\theta \in L^2(P)$ , s.t.

$$\lim_{\theta \rightarrow \theta_0} \frac{\|\rho(\theta) - \rho(\theta_0) - (\theta - \theta_0)^T D\rho_\theta(\theta_0)\|_2}{\|\theta - \theta_0\|} = 0 \quad (6)$$

Once again we write for (3):

$$Rem_2(\hat{\theta}_n - \theta_0) = \rho(\hat{\theta}_n) - \rho(\theta_0) - (\hat{\theta}_n - \theta_0)^T D\rho_\theta(\theta_0) \quad (7)$$

We now state the result:

**Theorem 1** Suppose  $\rho(\theta)$  is Hadamard differentiable w.r.t.  $\theta$  in a neighborhood  $N_\theta$  of  $\theta_0$ ,  $\sup E[|\rho(\theta)|] < \infty$  and  $\sup E[\rho^2(\theta)] < \infty$  for  $\theta \in N_\theta$  and  $\hat{\theta}_n$  is a consistent estimate for  $\theta_0$ , then  $\rho(\theta)$  is also differentiable in quadratic mean, and both derivative coincide on  $N_\theta$ .

**Proof** Write  $t = \hat{\theta}_n - \theta_0$ ,  $\theta = \theta_0$  and  $h_t \rightarrow 0$  in (4), so we can rewrite (4):

$$\lim_{\hat{\theta}_n \rightarrow \theta_0} \left\| \frac{\rho(\hat{\theta}_n) - \rho(\theta_0) - (\hat{\theta}_n - \theta_0)^T \rho'_\theta(0)}{\hat{\theta} - \theta_0} \right\|_W = 0 \quad (8)$$

Because  $\hat{\theta}_n \xrightarrow{P} \theta_0$  is a Cauchy sequence we have for  $\hat{\theta}_n \in N_\theta$ , that  $|\hat{\theta}_n - \theta_0| \rightarrow 0$  and we get expression (6). We now have to show, that both remainder terms converge in a  $L^2$  sense, as well as in probability to zero. Then from (3) we can deduce that  $Rem_1 = Rem_2$  a.s. and therefore both derivatives must coincide a.s.. From the assumptions imposed on  $\rho$  we can conclude that  $\rho$  is uniformly integrable (see corollary 6.22 in [Klenke, 2006]). If  $\hat{\theta}_n \xrightarrow{P} \theta_0$  we have that  $\sqrt{n}Rem_1(\hat{\theta}_n - \theta_0) = o_P(1)$  and  $\sqrt{n}Rem_2(\hat{\theta}_n - \theta_0) \xrightarrow{L^2} 0$ . As  $\rho(\theta) \in L^2(P)$  it follows with continuity of  $\rho(\theta)$  w.r.t.  $\theta$  that  $h = |\rho'_\theta| \in L^2$ , where  $\rho'_\theta$  is the Hadamard derivative. Therefore we have  $Rem_1(\hat{\theta}_n - \theta_0) \in L^2$  for  $n$  large enough. We now have:  $Rem_1$  is uniformly integrable, and  $Rem_1 \xrightarrow{P} 0$  and thus

$$|\sqrt{n}Rem_1(\hat{\theta}_n - \theta_0)|^2 \xrightarrow{n \rightarrow \infty} |0|^2$$

Write  $g = |\sqrt{n}Rem_1(\hat{\theta}_n - \theta_0) - 0|^2$ , then  $g_n \xrightarrow{P} 0$  and  $g_n$  is uniformly integrable, as

$$g_n = |\sqrt{n}Rem_1(\hat{\theta}_n - \theta_0)|^2 \leq 4 \cdot |o_P(1)|^2.$$

Therefore  $g_n \in L^2(P)$  and thus  $D\rho_\theta = \rho'_\theta$  a.s. in  $N_\theta$ .

□

### 3 Conclusion

We proved the equivalence of Hadamard differentiability and differentiability in quadratic mean for  $M$  estimates usually considered in robust statistics. This result can be used to deduce alternative assumptions imposed on  $M$  estimators to be consistent and asymptotically normally distributed.

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