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**Working Paper**

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Discussion Paper, No. 284

**Provided in Cooperation with:**

European University Viadrina Frankfurt (Oder), Department of Business Administration and Economics

*Suggested Citation:* Okhrin, Irena; Richter, Knut (2010) : An  $O(T^3)$  algorithm for the capacitated lot sizing problem with minimum order quantities, Discussion Paper, No. 284, European University Viadrina, Department of Business Administration and Economics, Frankfurt (Oder)

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# An $O(T^3)$ Algorithm for the Capacitated Lot Sizing Problem with Minimum Order Quantities

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Discussion Paper No. 284  
April 2010  
ISSN 1860 0921

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# An $O(T^3)$ algorithm for the capacitated lot sizing problem with minimum order quantities

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**Abstract.** This paper explores a single-item capacitated lot sizing problem with minimum order quantity, which plays the role of minor set-up cost. We work out the necessary and sufficient solvability conditions and apply the general dynamic programming technique to develop an  $O(T^3)$  exact algorithm that is based on the concept of minimal sub-problems. An investigation of the properties of the optimal solution structure allows us to construct explicit solutions to the obtained sub-problems and prove their optimality. In this way, we reduce the complexity of the algorithm considerably and confirm its efficiency in an extensive computational study.

*Keywords:* production planning; capacitated lot sizing problem; single item; minimum order quantities; capacity constraints; dynamic programming

## 1 Introduction

The paper explores a single-item capacitated lot sizing problem (CLSP) with minimum order quantity (MOQ). The MOQ constraint plays the role of minor set-up cost (Anderson and Cheah, 1993) and specifies that in every period one can produce either nothing or at least as much as MOQ. Such problems arise in an industrial context where, in order

to derive an optimal production plan, production managers prefer to use a minimum lot size restriction instead of specifying a fixed set-up costs, which are difficult to determine for individual product lines (Porras and Dekker, 2006). In such a way, the managers strive to achieve economies of scale (Zhou et al., 2007) and guarantee full utilization of resources (Constantino, 1998). Recent applications of lot sizing models with minimum order quantity to real-life cases are described in Lee (2004), Porras Musalem and Dekker (2005), Porras and Dekker (2006), Kamath and Bhattacharya (2007) and Zhou et al. (2007), among others.

The deterministic lot sizing problem with capacity constraint considered here was introduced by Manne (1958). Since that time, the capacitated lot sizing problem has been a challenge for many researchers as it is  $\mathcal{NP}$ -hard even in the single-item case (Bitran and Yanasse, 1982). The CLSP with constant capacities, however, can be solved in polynomial time. This was shown by Florian and Klein (1971) who developed an  $O(T^4)$  dynamic programming algorithm and later by van Hoesel and Wagelmans (1996) who improved the complexity of the algorithm to  $O(T^3)$ . Furthermore, an  $O(T^2)$  algorithm was proposed by Chung and Lin (1988) for the case with non-increasing set-up and production costs and non-decreasing capacities. A recent review of the single-item lot sizing problem is presented in Brahimi et al. (2006).

In contrast, the literature on the capacitated lot-sizing problem with minimum order restriction is scarce. The paper by Anderson and Cheah (1993) proposes a heuristics based on a Lagrangean relaxation of the capacity constraints and develops an exact algorithm to solve uncapacitated single-item sub-problems. Next, articles by Mercé and Fontan (2003) as well as by Absi and Kedad-Sidhoum (2007) use an MIP-based heuristics with rolling horizon procedure and solve the sub-problems obtained with the help of an optimization software package or external library. All three papers address a multi-item single-level capacitated lot sizing problem with minimum lot size and none of them have developed an exact algorithm to solve the problem at hand.

The contributions of this paper are threefold. First, we investigate the nature and fundamental properties of the single-item capacitated lot sizing problem with minimum order quantity in great detail. Secondly, we construct explicit solutions to obtain sub-problems and prove that they are optimal. Finally, for the first time, we develop an  $O(T^3)$  exact

algorithm to optimally solve the problem under consideration and prove its efficiency in an extensive computational study. An outline of the remainder of this paper is as follows. Section 2 gives a formulation of the problem and investigates its solvability. In the following section the problem is split into sub-problems and the concept of switching period is introduced. Section 4 elaborates on the basis for construction of an solution to a sub-problem and proves its optimality. The next section develops a dynamic programming algorithm for splitting the problem into sub-problems and for solving them. Section 6 contains an extensive computational study and proves the efficiency of the developed algorithm. Finally, section 7 concludes the paper and the Appendix provides proofs to lemmas and the theorem.

## 2 Problem Formulation and Solvability

Let us consider a dynamic lot sizing problem that aims to work out a production plan by minimizing the sum of set-up cost ( $s_j$ ), production cost ( $p_j$ ) and inventory holding cost ( $h_j$ ) over the planning horizon of  $T$  periods. The typical objective function will look as follows (Brahimi et al., 2006):

$$\min \sum_{j=1}^T (s_j Y_j + p_j X_j + h_j I_j), \quad (1)$$

where  $X_j$  denotes the production quantity in period  $j$ ,  $I_j$  denotes the inventory level at the end of period  $j$  and  $Y_j$  is a binary variable that equals one if production occurs in period  $j$  and zero otherwise.

In this paper we assume that the whole demand over  $T$  periods must be satisfied and the unit production costs are constant. Therefore, the term  $\sum p_j X_j$  can be omitted from the objective function. Furthermore, instead of the set-up cost, we specify the minimum order quantity, thus prohibiting production quantities below some level. If we additionally assume that the holding costs are also constant, then the objective function (1) is reduced to a single term that minimizes the cumulative inventories. In this case, the capacitated

dynamic lot sizing problem with minimum order quantity looks as follows:

$$\min \sum_{j=1}^T I_j \quad (2a)$$

$$I_j = I_{j-1} + X_j - d_j, \quad (2b)$$

$$Y_j L \leq X_j \leq Y_j U, \quad (2c)$$

$$Y_j \in \{0, 1\}, \quad (2d)$$

$$I_j \geq 0, \quad I_0 = I_T = 0, \quad j = 1, \dots, T, \quad (2e)$$

where  $d_j$  denotes the known deterministic demand in period  $j$ . Restriction (2a) is the inventory balance equation and constraint (2b) restricts the production quantities  $X_j$  to the range between the minimum order quantity  $L$  and the capacity level  $U$  for the case when  $Y_j = 1$ . The production quantity  $X_j = L$  will be called the *minimum lot* while production with  $X_j = U$  will be called the *maximum lot*. Alternatively,  $X_j$  and  $Y_j$  can both equal zero. The last restrictions state that no shortages are allowed and there must be no initial and final inventories.

Next, we consider the solvability of the investigated problem. Problem (2) will be called non-trivial if  $L \leq d_{1T} \leq TU$  is fulfilled and trivial otherwise (let  $d_{ij} = d_i + d_{i+1} + \dots + d_j$ ). Henceforward we consider only non-trivial problems. For a non-trivial problem, the greatest number of maximum production lots is bounded by the number  $K = \lfloor d_{1T}/U \rfloor$ . General conditions for estimating the solvability of problem (2) are given in Lemma 1.

**Lemma 1.** The non-trivial problem (2) with  $L < U$  is solvable if and only if

$$d_{1j} \leq jU, \quad j = 1, 2, \dots, K \quad (3)$$

and

$$\begin{aligned} d_{1T} &\geq (K+1)L, \\ d_{j+1,T} &\geq (K+1-j)L \end{aligned} \quad (4)$$

for  $d_{1T} > KU$  and  $\lfloor (d_{1T} - (K+1)L)/(U-L) \rfloor \leq j \leq K$ .

**Note.** A non-trivial problem (2) with  $L = U$  is solvable if and only if  $d_{1T} = KU$  and

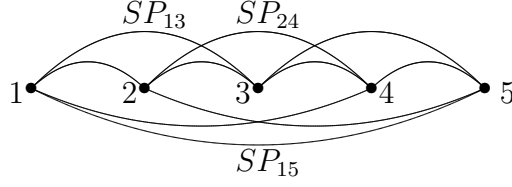


Figure 1: Network representation

conditions (3) are fulfilled.

### 3 Triple Solution

In order to understand the nature of problem (2) and reduce its complexity, we split it into smaller parts and explore them separately. As a *sub-problem*  $SP_{it}$  we define a part of problem (2) on periods  $1 \leq i, i+1, \dots, t \leq T$  with  $I_{i-1} = I_t = 0$ . If (2) is presented as a shortest path problem on the graph with  $T$  vertices, then the objective function value of  $SP_{it}$  represents the weight of the edge  $(i, t)$ , as demonstrated in Figure 1. The solvability of a sub-problem can be examined by the application of Lemma 1 with modified initial and final periods. A sub-problem  $SP_{it}$  is called *minimal* if inventory levels of its optimal solution are positive for all intermediate periods. As shown later, minimal sub-problems are very useful because they represent stable units and do not fall apart when constructing an optimal solution to the initial problem.

Let us now investigate the structure of an optimal solution to a minimal sub-problem. We will prove that there exists such a solution for which (a) at most one production value is strongly greater than the minimum order quantity and, at the same time, strongly smaller than the capacity level, i.e.  $L < X_s < U$ , and (b) all maximum lots follow all minimum lots. These properties are very helpful, but the so-called *switching period*  $s$  where the production value switches from minimum lots to maximum lots is not known in advance. To find it, we have to analyse solutions that satisfy the described properties and correspond to various switching periods and select the best one.

To bound the search space of feasible switching periods, we introduce the *critical period*  $c$  – the first period for which  $d_{i,c-1} \leq (c-1)L$  and  $d_{ic} > cL$  are fulfilled. This means that in the critical period there is a sudden jump in demand value. In order to be able



to satisfy this demand, the production value must be greater than the MOQ at the latest in  $c$ . However, if the production lot in some period is greater than  $L$ , then this period is located after  $s$ . Therefore, a feasible switching period must not be greater than the critical period of the sub-problem. The necessary and sufficient condition for a period to be the feasible switching period is provided below in Theorem 1. Lemma 2 formalizes the described properties of the switching period.

**Lemma 2 (switching period).** Let an optimal solution  $(\hat{I}_j, \hat{X}_j)$  for a minimal sub-problem  $SP_{it}$  with  $L < U$  be given.

- a) Then there is at most one period  $s$ ,  $i \leq s \leq t$  such that  $L < \hat{X}_s < U$ , where  $\hat{X}_j \in \{0, L\}$  for  $j < s$  and  $\hat{X}_j \in \{0, U\}$  for  $j > s$ .
- b) If  $d_{1T} > KU$ , then there is one and only one period  $s \leq c$  such that  $L \leq \hat{X}_s < U$ , with  $\hat{X}_j \in \{0, L\}$  for  $j < s$  and  $\hat{X}_j \in \{0, U\}$  for  $j > s$ .

In the case when  $L = U$ , the switching period is set to  $i - 1$ . Furthermore, if for some  $SP_{it}$  the demand values are smooth and no critical period can be found, we set  $c := t + 1$ . Properties of Lemma 2 generalize the optimality conditions for the unrestricted case when  $U = +\infty$  (Richter and Okhrin, 2007).

**Corollary of Lemma 2 (triple solution).** An optimal solution to a minimal sub-problem  $SP_{it}$  is composed of at most three segments:

- 1)  $\hat{X}_j \in \{0, L\}$ , for  $i \leq j < s$ ;
- 2)  $L \leq \hat{X}_s < U$ ;
- 3)  $\hat{X}_j \in \{0, U\}$ , for  $s < j \leq t$ .

A solution that has the structure as described in the Corollary to Lemma 2 is called the *triple solution* and illustrated in Figure 2. Here the production values for the periods before  $s$  equal either  $L$  or zero while the production values after the switching period are either  $U$  or zero. Besides this, there is at most one period when the production value is strictly between  $L$  and  $U$ . Depending on the demand values, an optimal solution to a sub-problem can consist of any two or even just one segment of the triple solution. It follows from Lemma 2, that for dynamic programming we should consider only triple

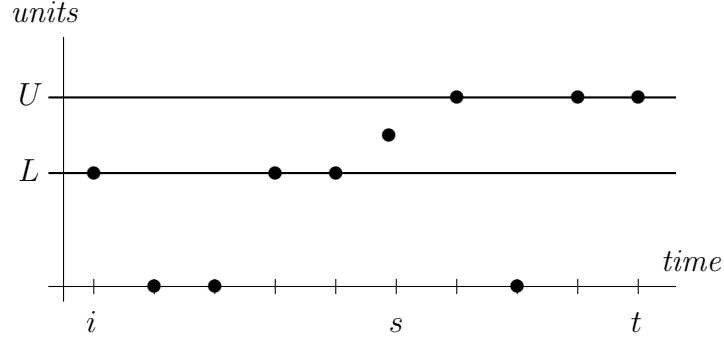


Figure 2: Triple solution

solutions.

## 4 Properties of Optimal Solutions

In the previous section we identified the triple structure of an optimal solution to a minimal sub-problem. To be able to use it, however, we need to know the best switching period  $s$ . Unfortunately, it is not known in advance and must be searched for. In order to find the optimal switching period to a sub-problem, we consider one after another every feasible switching period and calculate the corresponding minimum total inventory  $\hat{I}_{i,s,t}$ . After this, we compute the minimum total inventory  $\hat{I}_{it}$  for the sub-problem  $SP_{it}$  as follows:

$$\hat{I}_{it} = \min\{\hat{I}_{i,s,t} \mid i \leq s \leq t, s - \text{feasible}\}. \quad (5)$$

If no feasible switching periods can be found, the sub-problem is not minimal. In this case, another sub-problem should be selected and solved. If feasible switching periods can be found, then in order to compute  $\hat{I}_{i,s,t}$  for a given and fixed  $s$ , we solve the following *auxiliary problem*, where the first three restrictions resemble the structure of the triple

solution and replace restrictions (2c) and (2d)

$$\begin{aligned}
\hat{I}_{i,s,t} &= \min \sum_{j=i}^{t-1} I_j \\
X_j &\in \{0, L\}, \quad i \leq j < s \\
L &\leq X_s < U, \\
X_j &\in \{0, U\}, \quad s < j \leq t, \\
I_j &= I_{j-1} + X_j - d_j, \quad I_j \geq 0, \quad j = i, \dots, t, \\
I_{i-1} &= I_t = 0.
\end{aligned} \tag{6}$$

What is more, for an optimal solution of the triple structure we can calculate exactly how many times during the span of the sub-problem we produce  $L$  units and how many times we produce  $U$  units. To find this, we first introduce parameters  $K_j$  that denote the greatest feasible number of maximum lots (GrFNMaxL) not satisfying the demand for the last  $t - j + 1$  periods. In other words, for periods  $j, j + 1, \dots, t$  we have to produce  $U$  units at most  $K_j$  times so that we still have the smallest shortage and cannot satisfy the remaining demand fully. If during this time span we produce  $(K_j + 1) \cdot U$  units, we would exceed the demand and have to keep extra units in inventories. However, we are not allowed to produce more than demanded towards the end of the sub-problem, because the final inventory  $I_t$  of the sub-problem must be zero. Therefore, to determine  $K_j$ , we start from the last period and concentrate on “not satisfying” the remaining demand. To close the gap which exists between production values and demand during later periods of the sub-problem, we have to build up some inventories in earlier periods. Parameters  $K_j$  are calculated using the following formulas:

$$\begin{aligned}
K_{t+1} &= 0, \\
K_t &= \min \left\{ 1, \lfloor d_t/U \rfloor \right\}, \\
K_j &= \min \left\{ K_{j+1} + 1, \lfloor d_{jt}/U \rfloor \right\}, \quad s < j < t.
\end{aligned} \tag{7}$$

Next, let  $k_j$  denote the smallest feasible number of minimum lots (SmFNMinL) just sufficient to satisfy the demand for the first  $j < s$  periods. That is,  $k_j$  represents how many times we produce exactly  $L$  units in the time interval  $i, i + 1, \dots, j$ . Contrary to the previous case, at the beginning of the sub-problem we have to produce enough units

| $j$   | 1 | 2 | 3 | 4 | 5  | 6 | 7  | 8  |
|-------|---|---|---|---|----|---|----|----|
| $d_j$ | 2 | 2 | 1 | 0 | 13 | 1 | 10 | 23 |
| $K_j$ |   |   |   |   |    | 3 | 2  | 1  |
| $k_j$ | 1 | 1 | 1 | 2 |    |   |    |    |

Table 1: Computation of parameters  $K_j$  and  $k_j$

to be able to cover the arising demand and additionally to build up inventories that are consumed in later periods. To compute  $k_j$  we use the following formulae:

$$\begin{aligned}
k_{s-1} &= 1 + \max \left\{ \lfloor d_{i,s-1}/L \rfloor, \lfloor \max \{0, d_{it} - (K_{s+1} + 1)U\}/L \rfloor \right\}, \\
k_j &= \max \left\{ k_{s+1} - 1, 1 + \lfloor d_{ij}/L \rfloor \right\}, \quad i \leq j < s-1, \quad s > i, \\
k_{i-1} &= 0, \quad s < i.
\end{aligned} \tag{8}$$

The determination of the parameters  $K_j$  and  $k_j$  is illustrated in Table 1 by example with  $t = 8$  periods. We set  $L = 7$ ,  $U = 10$ ,  $i = 1$  and  $s = 5$ . To determine  $K_j$  we make use of formulae (7) and start from the last period. In the 8<sup>th</sup> period, it would be necessary to produce  $2 \times U = 20$  units to cover the largest possible fraction of demand without building excess inventories. However, in the last period, we do not have enough time to produce twice the number  $U$  of units because only one production lot is allowed during one period. Therefore  $K_8 = 2$  is infeasible; instead  $K_8 = 1$  is true. Considering backwards one after another all other periods  $j > s$ , we compute all values  $K_j$ . By analogy, with the help of recursive formulae (8), we calculate all  $k_j$  starting from  $k_{s-1} = k_4$ . For instance,  $k_2 = 1$  means that till the end of the second period we have to produce once  $L = 7$  units.

**Lemma 3 (properties of  $K_j$  and  $k_j$ ).** For parameters  $K_j$  and  $k_j$ , the following inequalities hold:

$$\begin{aligned}
K_j U &\leq d_{jt}, & s < j \leq t-1, \\
0 \leq K_j - K_{j+1} &\leq 1, & s < j \leq t-1, \\
k_j L &> d_{ij}, & i \leq j < s-1, \\
0 \leq k_{j+1} - k_j &\leq 1, & i \leq j < s-1.
\end{aligned} \tag{9}$$

The above-described properties of parameters  $K_j$  and  $k_j$  are summarized in Lemma 3. The first relation in (9) states that, after the switching period, the production quantities may not exceed demand to ensure zero final inventory. The second inequality shows that parameters  $K_j$  are monotonically non-increasing with a step of one. Analogously, the third relations of Lemma 3 guarantee that in periods before the switching period, we produce enough to cover demand. Finally, according to the last inequalities, parameters  $k_j$  are monotonically non-decreasing also with a step of one.

**Lemma 4 (lot numbers).** Let  $s$  be a feasible switching period for  $SP_{it}$ . Then the auxiliary problem (6) has a unique optimal solution with  $\text{GrFNMaxL} = K_j$  for  $j > s$  and  $\text{SmFNMinL} = k_j$  for  $j < s$ , respectively.

This optimal solution for a given  $s$  can be calculated explicitly using the following formulae:

$$\begin{aligned} \hat{I}_j &= k_j L - d_{ij}, & i \leq j < s, \\ \hat{I}_{j-1} &= d_{jt} - K_j U, & s < j \leq t, \\ \hat{I}_t &= 0, \\ \hat{X}_j &= \hat{I}_j + d_j - \hat{I}_{j-1}, & i \leq j \leq t. \end{aligned} \tag{10}$$

**Theorem 1 (optimal solution to (6)).** Solution (10) to the sub-problem  $SP_{it}$  for a given feasible switching period  $s$  is optimal if and only if  $s \leq c$  and

$$L \leq d_{it} - k_{s-1}L - K_{s+1}U < U. \tag{11}$$

Theorem 1 provides the necessary and sufficient condition for the feasibility of a switching period. Thus, to find all feasible switching periods for the auxiliary problem (6), it is enough to systematically check condition (11) for all possible  $s : i \leq s \leq c$ . Only when (11) holds, does it make sense to solve (6) and compute its minimum total inventory  $\hat{I}_{i,s,t}$  using the relation from Lemma 5 below. Having calculated the minimum total inventory for all feasible switching periods, it is easy to determine the optimal cumulative inventory  $\hat{I}_{it}$  for the sub-problem  $SP_{it}$  by means of formula (5).

Theorem 1 is illustrated in Table 2 by the example which was introduced in Table 1. The body of the table contains values  $k_j$  for  $j < s$  and  $K_j$  for  $j > s$ . The rightmost column

| $j$     | 1 | 2 | 3 | 4 | 5  | 6 | 7  | 8  | $\delta$ |
|---------|---|---|---|---|----|---|----|----|----------|
| $d_j$   | 2 | 2 | 1 | 0 | 13 | 1 | 10 | 23 |          |
| $s = 1$ |   | 5 | 4 | 4 | 4  | 3 | 2  | 1  | 2        |
| $s = 2$ | 1 |   | 4 | 4 | 4  | 3 | 2  | 1  | 5        |
| $s = 3$ | 1 | 1 |   | 4 | 4  | 3 | 2  | 1  | 5        |
| $s = 4$ | 1 | 1 | 1 |   | 4  | 3 | 2  | 1  | 5        |
| $s = 5$ | 1 | 1 | 1 | 2 |    | 3 | 2  | 1  | 8        |
| $s = 6$ | 1 | 1 | 2 | 3 | 4  |   | 2  | 1  | 4        |
| $s = 7$ | 1 | 1 | 2 | 3 | 4  | 5 |    | 1  | 7        |
| $s = 8$ | 1 | 1 | 2 | 3 | 4  | 5 | 6  |    | 10       |

Table 2: Examining the feasibility of switching periods

contains parameter  $\delta := d_{it} - k_{s-1}L - K_{s+1}U$  from (11) which for the given example equals  $\delta = 52 - 7k_{s-1} - 10K_{s+1}$ . The critical period  $c$  equals nine and does not restrict the search space of feasible switching periods. To guarantee the feasibility of a potential switching period, condition  $7 \leq \delta < 10$  must be satisfied. This is true only for two values, namely  $s = 5$  and  $s = 7$ . Indeed, switching period  $s = 1$  cannot be feasible, as starting from the second period we should produce  $K_{s+1}U = 5 \times 10 = 50$  units. As the final inventory  $I_8$  must be zero, we need to produce only two additional units in the first period. However,  $X_1 = 2$  is illegal as it is smaller than the minimum order quantity  $L = 7$ . Therefore,  $s = 1$  is unfeasible and should be discarded.

For switching periods that satisfy the conditions of Theorem 1, we can calculate the total inventory directly, without determining the optimal solution (10). This can be done by means of the formula provided in the following lemma.

**Lemma 5 (minimum total inventory).** Let  $s$  be a feasible switching period for  $SP_{it}$ . Then the minimum total inventory for  $SP_{it}$  is calculated by the formula

$$\hat{I}_{i,s,t} = \sum_{j=i}^t d_j(j-s) + L \sum_{j=i}^{s-1} k_j - U \sum_{j=s+1}^t K_j. \quad (12)$$

## 5 The Algorithm

Summarizing the previous sections, we can now outline the solution procedure for solving a sub-problem  $SP_{it}$ :

- Step 1:* Compute the critical period  $c$ .
- Step 2:* Select a feasible switching period  $s \leq c$  and fix it.
- Step 3:* Calculate  $K_j$  and  $k_j$  using (7) and (8) respectively.
- Step 4:* Check condition (11) from Theorem 1. (13)
- Step 5:* If it holds, compute  $\hat{I}_{i,s,t}$  using (12).
- Step 6:* Select next feasible  $s$  and *return to Step 3*.
- Step 7:* Calculate  $\hat{I}_{it}$  using (5) and determine the optimal  $s$ .
- Step 8:* Compute the optimal solution  $(\hat{X}, \hat{I})$  to  $SP_{it}$  using (10).

In fact, numbers  $K_j$  do not depend on  $s$  and thus can be computed only once for the given  $SP_{it}$ . In contrast, parameters  $k_j$  must be recalculated for every feasible switching period.

Next, we need an algorithm for splitting the initial problem (2) into a series of sub-problems. For this, we make use of the method developed by Florian and Klein (1971) to solve the capacitated single item lot sizing problem. Their algorithm rests upon the fact that the optimal solution between two nearest regeneration periods, i.e. between two periods with zero inventories, has special properties, which help to develop a solution procedure. The algorithm for splitting the initial problem (2) into a series of sub-problems

with smaller horizons looks like the following:

- Step 1:*  $t := h_1$ ;  $F_0 := 0$ ,  $F_t := +\infty$ ,  $t = 1, \dots, T$ .
- Step 2:* If  $(t \leq T)$  then  $i := 1$  else *Stop*.
- Step 3:* If  $(SP_{t+1,T}$  unsolvable) then  $t := t + 1$  and *return to Step 2*.
- Step 4:* If  $(t < h_i)$  then  $t := t + 1$  and *return to Step 2*.
- Step 5:* If  $(F_{i-1} < +\infty)$  and  $(SP_{it}$  solvable) then solve  $SP_{it}$ ; (14)  
     if  $F_{i-1} + \hat{I}_{it} \leq F_t$  then  $F_t := F_{i-1} + \hat{I}_{it}$ , save  $i$  and  $s$ .
- Step 6:* If  $(i < t)$   
     then  $i := 1 + \max\{i, h_1\}$  and *return to Step 4*;  
     else  $t := t + 1$  and *return to Step 2*.

The first step of (14) represents the initialisation of parameters. Value  $h_i = \min\{r \mid d_{ir} \geq L\}$  denotes the lower bound for the horizon of the sub-problem starting with  $i$ . If  $t < h_i$ , then the cumulative demand  $d_{it}$  is smaller than the minimum lot size  $L$  and the sub-problem  $SP_{it}$  is unsolvable. Therefore, it is excluded from consideration. Furthermore, values  $F_t$  denote the minimum cumulative inventories for the first  $t$  periods. Step 2 initialises  $i$  and provides the termination criterion for the algorithm. In the next step, we check if the remaining part of the problem, i.e. the sub-problem  $SP_{t+1,T}$ , is solvable. If it is not, then we would not be able to find its solution in the later iterations and thus considering  $SP_{it}$  makes no sense. Step 3 assures that  $t$  is greater than its lower bound, eliminating thereby unsolvable problems with very small horizons.

Finally, step 5 proves whether the sub-problem  $SP_{it}$  is solvable too and whether we are able to find any solution to the previous part of the problem, which results in a finite value of  $F_{i-1}$ . Only if these two conditions are fulfilled, do we solve  $SP_{it}$  by means of algorithm (13) and remember the beginning of the sub-problem and its optimal switching period. In fact, we do not need to perform step 8 of the algorithm (13) in every iteration of the algorithm (14). It may happen that the solution found until now is sub-optimal and would be outperformed in some further iterations. Therefore, the last step of (13) is superfluous here and should be performed only once at the end of the algorithm (14) when the problem (2) is ultimately and unambiguously split into sub-problems. To accomplish this, however, we must save the regeneration period  $i$  and the optimal switching period



| Mean demand | Demand variance |        |       | Capacity       |      |      |
|-------------|-----------------|--------|-------|----------------|------|------|
|             | small           | medium | large | $\mu/U = 0.75$ | 0.85 | 0.95 |
| 40          | 4               | 8      | 12    | 53             | 47   | 42   |
| 200         | 20              | 40     | 60    | 267            | 235  | 211  |
| 600         | 50              | 100    | 150   | 800            | 706  | 632  |

Table 3: Parameter values

$s$  every time we update  $F_t$ . Step 5 of the algorithm (14) can be applied to all sub-problems, not only minimal ones. If the sub-problem is actually not minimal, then it will be outperformed by others in later iterations of the algorithm. At the end, step 6 updates either  $i$  or  $t$  and initiates a new loop. After completion of algorithm (14), the optimum total inventory for (2) is stored in parameter  $F_T$ . After this, the optimal solution can be found through a backward procedure by means of formulae (10) together with the saved regeneration and switching periods.

The complexity of algorithm (14) is  $O(T^2)$  disregarding the complexity of algorithm (13). Since (13) needs at most  $O(T)$  operations to solve a sub-problem, the complexity of the combined procedure (13)–(14) for solving the initial problem (2) is  $O(T^3)$ .

## 6 Computational Study

To assess the efficiency of the developed procedure (13)–(14), we carried out extensive experiments. Demand values were randomly generated from the normal distribution with three different values of mean  $\mu$  — low, medium and large. For each value of mean, we selected three different levels of variance  $\sigma$  which correspond to small, medium and large fluctuations of the demand values. Furthermore, to determine capacity levels, we set the average target capacity utilization  $\mu/U$  equal to 0.75, 0.85 and 0.95 for every  $\mu$  (Trigeiro et al., 1989). As shortages are not allowed, we considered only cases when there is enough capacity to cover the demand. An overview of parameter values is provided in Table 3.

In addition, we chose the minimum order quantity to be equal to the lower quartile ( $L_1$ ), median ( $L_2$ ) and upper quartile ( $L_3$ ) of the normal distribution with the corresponding

| $\mu, \sigma$ | $L_1$ | $L_3$ |
|---------------|-------|-------|
| 40, 4         | 37    | 43    |
| 40, 8         | 35    | 45    |
| 40, 12        | 32    | 48    |
| 200, 20       | 187   | 213   |
| 200, 40       | 173   | 227   |
| 200, 60       | 160   | 240   |
| 600, 50       | 566   | 634   |
| 600, 100      | 533   | 667   |
| 600, 150      | 499   | 701   |

Table 4: Parameter values for the minimum order quantity

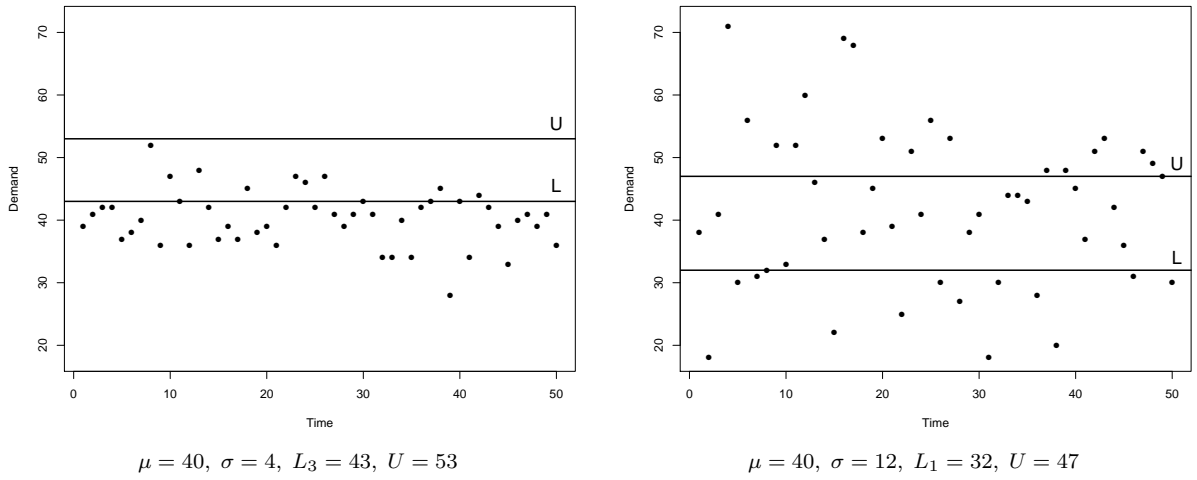


Figure 3: Generated instances with different sets of parameters

$\mu$  and  $\sigma$ . This means that value  $L_1 < \mu$  was selected so that 25 per cent of demand values in the sample were less than  $L_1$ , value  $L_2$  was equal to  $\mu$  and  $L_3 > \mu$  ensured that approximately 25 per cent of demand values were greater than the minimum order quantity (Anderson and Cheah, 1993). The values of  $L_1$  and  $L_3$  are presented in Table 4. Finally, the planning horizon was set to  $T = 50$  for all cases. In total, we received 81 group of problems and for every group we randomly generated ten test instances. Figure 3 illustrates two problems generated for different values of the minimum order quantity, variance and capacity utilization.

The results of the computational study are presented in Table 5 and 6 for  $\mu = 40$  and  $\mu = 600$  respectively. Results for  $\mu = 200$  exhibit similar pattern and thus are omitted here. The body of each table contains the number of iterations necessary to solve the problem (aggregated over ten randomly generated instances). The *number of iterations*

| Capacity utilization | Min order quantity | Number of iterations |                   |                  |
|----------------------|--------------------|----------------------|-------------------|------------------|
|                      |                    | $\sigma$ – small     | $\sigma$ – medium | $\sigma$ – large |
| 0.75                 | $L_1$              | 3036                 | 3044              | 2145             |
|                      | $L_2$              | 7262                 | 6393              | 3449             |
|                      | $L_3$              | 12 877               | 10569             | 6428             |
| 0.85                 | $L_1$              | 2708                 | 2071              | 1009             |
|                      | $L_2$              | 5818                 | 3787              | 1504             |
|                      | $L_3$              | 8078                 | 2078              |                  |
| 0.95                 | $L_1$              | 876                  | 728               | 499              |
|                      | $L_2$              | 762                  | 703               | 447              |

Table 5: Computational results for  $\mu = 40$

| Capacity utilization | Min order quantity | Number of iterations |                   |                  |
|----------------------|--------------------|----------------------|-------------------|------------------|
|                      |                    | $\sigma$ – small     | $\sigma$ – medium | $\sigma$ – large |
| 0.75                 | $L_1$              | 2917                 | 2596              | 2596             |
|                      | $L_2$              | 5711                 | 4905              | 5277             |
|                      | $L_3$              | 10 637               | 11 072            | 8019             |
| 0.85                 | $L_1$              | 2575                 | 1807              | 1705             |
|                      | $L_2$              | 4323                 | 3005              | 3436             |
|                      | $L_3$              | 7335                 | 3091              | 211              |
| 0.95                 | $L_1$              | 839                  | 337               | 285              |
|                      | $L_2$              | 915                  | 350               | 284              |

Table 6: Computational results for  $\mu = 600$

means how many times we perform step 3 in (13) and gives us the empirical complexity of the developed algorithm. As can be seen from the tables, the greatest number of performed iteration for the problems at hand was 12 877. This number is considerably lower than the theoretical complexity of  $O(T^3)$ , which equals 125 000 for  $T = 50$  periods, meaning we have succeed in eliminating a substantial number of hopeless sub-problems and switching periods thereby reducing the size of the problem.

In the tables there are no results provided for  $\mu/U = 0.95$  and  $L_3$ . For this group, the maximum capacity  $U$  was below the minimum order quantity  $L_3$  for all  $\mu$  due to the construction of the test instances. Therefore, all problems from this group were unsolvable. Similarly, the problems from the test group with parameters  $\mu = 40$ ,  $\mu/U = 0.85$ ,  $L_3$ ,  $\sigma$ -large were also unsolvable while the capacity  $U = 47$  was smaller than  $L_3 = 48$ .

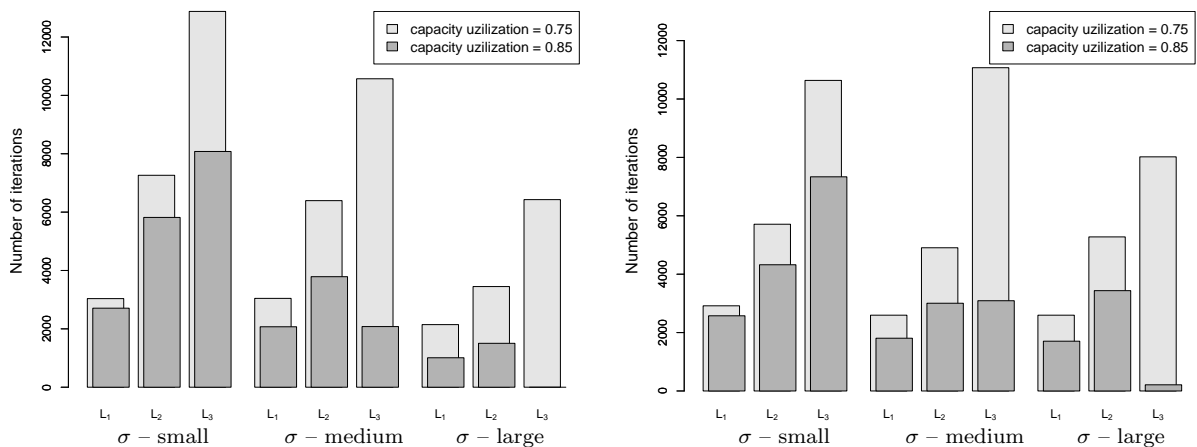


Figure 4: Number of iterations for  $\mu = 40$  and  $\mu = 600$

The relationship between the number of iterations and other parameters for  $\mu = 40$  and  $\mu = 600$  is depicted in Figure 4. It is clear from the histograms that it takes the longest for the algorithm to find the optimal solution in the case with small capacity utilization and high minimum order quantity. This is due to the fact that there are many possibilities for constructing a production plan and the algorithm must analyze many sub-problems to find the minimal one. On the other hand, the solution is found most rapidly for the case with high capacity utilization and low MOQ. This is because we should relatively often produce at capacity with subsequent zero inventory. As a result, there are many short minimum sub-problems which are quickly found by the algorithm. The algorithm was coded in Java 1.6 and run on a 2 GHz Intel Core2 Duo machine with 2 GB memory running Windows Vista. The computational time for solving one instance was below 10 millisecond in all cases.

## 7 Conclusions

The paper investigates a special capacitated single item lot sizing problem, where a minimum order quantity restriction, instead of set-up cost, guarantees a certain level of production lots. We work out the necessary and sufficient solvability conditions and apply the general dynamic programming technique to develop an  $O(T^3)$  exact algorithm that is based on the concept of minimal sub-problems. We investigated the properties of the feasible solution in great detail and were able to elaborate the triple structure of the optimal solution. This allowed us to construct explicit solutions to the obtained sub-problems and

prove their optimality. Thereby we considerably reduced the complexity of the algorithm and confirmed its efficiency in an extensive computational study.

## Appendix

**Proof of Lemma 1.** Let conditions (3) and (4) hold. Since  $L < U$ , then  $KU < d_{1T} < (K + 1)U$  holds. Now we will prove that the following solution is feasible

$$X_j = \begin{cases} U, & j \leq k_U, \\ d_{1T} - k_U U - k_L L, & j = k_U + 1, \\ L, & k_U + 1 < j \leq K + 1, \\ 0, & K + 2 \leq j \leq T, \end{cases} \quad (15)$$

where  $k_u = \lfloor (d_{1T} - (K + 1)L)/(U - L) \rfloor$  and  $k_L = K - k_U$ . It follows from (4) and the definitions of the parameters  $K$  and  $k_U$  that  $k_U \geq 0$  and therefore  $k_U(U - L) + L \leq d_{1T} - KL$ . This means that  $L \leq d_{1T} - (k_U + k_L)L - k_U(U - L) = d_{1T} - k_U U - k_L L$ . Furthermore,  $k_U > (d_{1T} - (K + 1)L - (U - L))/(U - L) = (d_{1T} - KL - U)/(U - L)$ ,  $k_U(U - L) + U > d_{1T} - KL$  and  $U > d_{1T} - k_U U - k_L L$ . Therefore  $L \leq X_{k_U+1} < U$ . The feasibility with respect to the restrictions (2e) can be proved as follows: Due to (3), the inventories  $I_j = jU - d_{1j}$  are nonnegative for the first  $k_U$  periods. Let us consider any period  $j$  such that  $k_U < j \leq K + 1$ . For  $j$  the following relations hold  $I_j = X_{1j} - d_{1j} = X_{1k_U} + X_{k_U+1} + X_{k_U+2,j} - d_{1j} = k_U U + (d_{1T} - k_U U - (K - k_U)L) + (j - k_U - 1)L - d_{1j} = d_{1T} - (K - k_U)L + (j - k_U - 1)L - d_{1j} = d_{j+1,T} - (K - j + 1)L \geq 0$ . Due to (4) the inventories are nonnegative, too. The inventories for the remaining periods are obviously nonnegative. The zero inventory property for the last period is fulfilled due to (15).

If (3) are not fulfilled then the problem (2) is unsolvable. If (3) is fulfilled but (4) does not hold, then either  $KU < d_{1T} < (K + 1)L$  is fulfilled or the relations  $d_{1T} \geq (K + 1)L$  and  $d_{\tau+1,T} < (K + 1 - \tau)L$  are true for some first period  $k_U \leq \tau \leq K + 1$ . In the first case, even though  $K$  maximum lots do not satisfy the total demand,  $K + 1$  production periods will always exceed the demand. Hence, no feasible solution can be found. In the second case, a feasible solution has at least  $K + 1$  setups. The cumulative production rate for the first  $\tau$  periods fulfils  $d_{1\tau} \leq X_{1\tau} \leq d_{1T} - L(K + 1 - \tau)$ , i.e. the demand is

satisfied and the production quantity for the periods  $\tau + 1, \dots, K + 1$  is greater than or equals  $K + 1 - \tau$  times the minimum lot size. This last relation, however, contradicts to the assumption of this case.  $\square$

**Proof of Lemma 2.** a) If there is more than one period of this type, i.e. there exists an optimal solution with  $j_1 < j_2$  and  $L < \hat{X}_{j_1} < U$ ,  $L < \hat{X}_{j_2} < U$ ,  $\hat{I}_j > 0$ ,  $j = j_1, \dots, j_2 - 1$  then at least one unit from  $\hat{X}_{j_1}$  can be moved to  $\hat{X}_{j_2}$ . As a result, the total inventory decreases and the initial solution is not optimal. The same arguments can be applied if minimum lots are produced after maximum lots or after period  $s$ .

b) Let there be two such periods. At least one of them fulfils  $L \leq \hat{X}_j < U$ . Let  $\hat{X}_s$  be the latter of these variables. Due to part a), relation  $\hat{X}_j \leq L$  holds for  $j \leq s - 1$  and therefore  $s \leq c$ .  $\square$

**Proof of Lemma 4.** First we will show that the GrFNMaxL for periods  $j : s < j < t$  equals  $K_j$ . By definition (7),  $K_t = 0$  for an optimal solution to (6) if  $d_t < U$ . In the opposite case,  $K_t = 1$  since otherwise  $\hat{I}_{t-1} \geq U$  and the last setup before  $t$  can be shifted to  $t$ , what contradicts the optimality of the given solution. Let now  $K_{j+1}$  be the GrFNMaxL for time interval  $j + 1, \dots, t$ . Evidently, the GrFNMaxL for  $j : s < j < t$  cannot be greater than  $K_j$  and cannot be smaller than  $K_{j+1}$ . If we assume that  $K_j = K_{j+1}$ , then  $\hat{X}_j = 0$  and  $\hat{I}_{j-1} = \hat{I}_j - K_{j+1}U + d_{jt} \geq U$  and the solution is not optimal. Therefore,  $K_j = K_{j+1} + 1 \leq \lfloor d_{jt}/U \rfloor$ .

Due to the first part of the proof, an optimal solution to (6) fits the property that the total production for the first  $s > 1$  periods has to cover exactly  $d_{it} - K_{s+1}U$  units. Let us denote  $\Delta = d_{it} - (K_{s+1} + 1)U$ . Then  $\hat{X}_{i,s-1} = \Delta + U - \hat{X}_s > \Delta$ . Furthermore, let  $k$  denote the SmFNMinL for period  $s - 1$ . Then  $\hat{X}_{i,s-1} = kL > \Delta$ ,  $k > \Delta/L$  and  $k \geq \lfloor \Delta/L \rfloor + 1$  hold. Since the inventories are positive, more than  $d_{i,s-1}$  units have to be produced in the first  $s - 1$  periods. Then  $kL > d_{i,s-1}$  and therefore the inequality  $k \geq \lfloor d_{i,s-1}/L \rfloor + 1$  is fulfilled. Thus, we have proven that  $k \geq k_{s-1}$ . Now let's assume that  $k > k_{s-1}$ . Then  $(k - 1)L > \Delta$  and thus  $U - L > \hat{X}_s$ . Furthermore, because  $k > d_{i,s-1}/L + 1$ , then  $\hat{I}_{s-1} > L$  and the last production lot before  $s$  can be added to the production value in period  $s$ , what contradicts the optimality of the solution. So, we have shown that the SmFNMinL for  $s - 1$  equals  $k_{s-1}$ .

Now it will be proven that also the SmFNMinL for  $j : i \leq j < s - 1$  are given by  $k_j$ . Let this be true for some period  $j < r < s - 1$ . The SmFNMinL for  $j$  we denoted

again by  $k$ , where  $k > \lfloor d_{ij}/L \rfloor$  since otherwise the inventory  $\hat{I}_j = kL - d_{ij}$  is not positive. Then the relation  $k \geq k_{j+1} - 1$  is fulfilled, since in the opposite case the relation  $\hat{X}_{j+1} = \hat{X}_{i,j+1} - \hat{X}_{ij} = (k_{j+1} - k)L \geq 2L$  holds. Furthermore, the inequality  $k \leq k_{j+1}$  holds. Hence the relation  $k_{j+1} \geq k \geq k_j$  is true. If we assume that  $k = k_{j+1} > \lfloor d_{ij}/L \rfloor + 1$ , then  $\hat{I}_j = kL - d_{ij} \geq (\lfloor d_{ij}/L \rfloor + 2)L - d_{ij} > L$  and the production quantity  $L$  can be shifted forward to period  $j + 1$  what contradicts the optimality of the solution.  $\square$

**Proof of Theorem 1.** If solution (10) is optimal then  $d_{ij} < \hat{X}_{ij} \leq jL$  holds for all periods  $j < s$ . Hence  $s \leq c$  is fulfilled. Due to Lemma 4, numbers  $k_{s-1}$  and  $K_{s+1}$  are valid SmFN-MinL and GrFNMaxL for the feasible switching period. The production quantity for this period fulfils  $L \leq \hat{X}_s < U$ . This means, however, that  $L \leq d_{it} - k_{s-1}L - K_{s+1}U < U$  and that (11) holds.

Let the relations from the statement of the theorem hold. Now we prove that (10) constitute an optimal solution for (6). The following considerations are true: it follows (a) from Lemma 3 that  $\hat{I}_j = k_jL - d_{ij} > 0$ ,  $j < s$  and from  $s \leq c$  that  $\hat{I}_j > 0$ ,  $s \leq j < t$ ; (b) from (10) that  $\hat{X}_j = (k_j - k_{j-1})L$  for  $j < s$  and  $\hat{X}_j = (K_j - K_{j+1})U$  for  $j > s$ . Then due to Lemma 3 the production values equal either zero or  $L$  in case (a) and either zero or  $U$  in case (b). Moreover, from the last equation in (10) it follows also that  $\hat{X}_s = \hat{I}_s + d_s - \hat{I}_{s-1} = d_{s+1,t} - K_{s+1}U + d_s - (k_{s-1}L - d_{i,s-1}) = d_{it} - k_{s-1}L - K_{s+1}U$ . Then due to (11), relations  $L \leq \hat{X}_s < U$  and  $\hat{I}_t = 0$  are fulfilled. Therefore, (10) constitutes a feasible solution to (6) with a given switching period  $s$ . Now it remains to show that it is also optimal. If there is any other better solution  $(\bar{I}_j, \bar{X}_j)$  then  $\bar{I}_j$  will coincide with  $\hat{I}_j$  from (10) for some periods  $j : \tau < j \leq t$  and differ for period  $\tau$ . Firstly, let  $s \leq \tau$ . Then the relation  $\hat{I}_{\tau+1} = \bar{I}_\tau + \bar{X}_{\tau+1} - d_{\tau+1}$  holds. The better solution can only differ if  $\hat{X}_{\tau+1} = U$ ,  $\bar{X}_{\tau+1} = 0$  and  $\bar{I}_\tau > U$ . Then the last setup can be shifted to the period  $\tau + 1$  and the total inventory will decrease. Secondly, let  $\tau < s$ . Then the same arguments with respect to the minimum lots can be applied. Hence solution (10) is optimal.  $\square$

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