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Default Probabilities and Default Correlations

Starting from the Merton framework for firm defaults, we provide the analytics and robustness of the relationship between default probabilities and default correlations. We show that loans with higher default probabilities will not only have higher variances but also higher correlations with other loans. As a consequence, portfolio standard deviation can increase substantially when loan default probabilities rise. This result has two important implications. First, relative prices of loans with different default probabilities should reflect the differential impact on portfolio standard deviation. Second, the standard deviation of loan portfolios and of default rates, as well as the required economic capital will vary significantly over the business cycle.

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Default Probabilities and Default Correlations

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Abstract

Starting from the Merton framework for firm defaults, we provide the analytics and robustness of the relationship between default probabilities and default correlations. We show that loans with higher default probabilities will not only have higher variances but also higher correlations with other loans. As a consequence, portfolio standard deviation can increase substantially when loan default probabilities rise. This result has two important implications. First, relative prices of loans with different default probabilities should reflect the differential impact on portfolio standard deviation. Second, the standard deviation of loan portfolios and of default rates, as well as the required economic capital will vary significantly over the business cycle.

Keywords: Credit portfolio management, Default correlations, Pricing of loans, Macroeconomic risk, Credit risk models.

JEL: G11, G12, G21, G31

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1 Introduction

During the last two decades portfolio considerations have become a central issue in credit risk management. A crucial ingredient for any portfolio consideration in the credit risk context are the correlations of loan returns. They are termed default correlations in the literature since the probabilities of joint defaults are the major building blocks of loan correlations. In this paper we examine the relationship between (univariate) default probabilities and default correlations. The analysis is motivated by two questions.

The first question concerns the pricing of loans with different default probabilities. In current practice, loan prices usually merely reflect the impact of higher default probabilities on expected returns. If, however, loans with a higher default probability also contribute more to the portfolio standard deviation (as we will show to be the case), then the marginal increase of economic capital when adding such a loan to the portfolio will be higher than for loans with lower default probabilities. Moreover, if these differences are substantial (as we will also show), then it is important for loan prices to reflect these differences. By the variance-covariance formula, the contribution of a loan to a portfolio's standard deviation consists of its own standard deviation and of its correlations with other loan returns, i.e. its default correlations. In the simplest setting, the standard deviation of a loan is a multiple of $\sqrt{p(1-p)}$ where p is the firm's default probability.¹ Therefore it is easy to see that the standard deviation will increase in p . This is called the *variance effect* of an increase in default probabilities. However, it is not clear how default correlations will react to changes in p . This effect is called the *correlation effect*.

Second, it has long been recognized that default probabilities change with the state of the economy and that credit risk models should take this into account, since higher default probabilities imply higher expected losses.² However, the way in which these changes in default probabilities affect the second important building block of the loan portfolio distribution - the portfolio standard deviation - has only recently been addressed. Using a simulation approach, GERSBACH AND LIPPONER (2000) have demonstrated that adverse macroeconomic shocks - by increasing default probabilities - can raise default correlations. They show that this effect may account for more than 50% of the increase in the credit risk caused by the shock. In this paper we will provide an analytic foundation for the connection between negative macroeconomic shocks and loan default correlations.

Following the structural approach to credit risk, we construct our model along the same

¹In such a setting, the return on the loan can be described as multiple of a Bernoulli variable that is either equal to 1 (default) with probability p or equal to 0 (no default) with probability $(1-p)$.

²See e.g. WILSON (1998) or, more recently CROUHY, GALAI, AND MARK (2000).

lines as MERTON (1974). The returns on a firm's assets are assumed to be normally distributed and loans are modeled as a claim on the value of the firm. This value is measured by the price at which the firm's total liabilities can be purchased; it is thus equal to the value of the stock and the value of the debt. Default on loans occurs if the market value of the firm falls below a certain threshold which depends on the firm's liability structure. In a sufficiently simple framework, the joint default behavior of two firms can therefore be described by two indicator variables $1\{Z_1 \leq z_1\}$ and $1\{Z_2 \leq z_2\}$.³ Z_1 and Z_2 are two normalized, correlated, jointly normally distributed random variables that describe the firm's standardized returns. If, for example, the standardized returns of the first firm fall below the threshold z_1 , the firm will default. We call z_1 and z_2 the respective default points of the firms. An increase in default probabilities will shift default points to the right. We examine how such a shift changes the default correlation of the two firms.

Our findings are as follows. Default correlations increase under a homogeneous shift to the right (i.e. both default points increase by the same amount). The same is true if the shift is more pronounced for the firm with the lower likelihood of default. Default correlations may only decline if the downward shift for the lower rated firm is significantly higher than that for the higher rated firm.

Furthermore, at a structural level the correlation effect is made up of two intertwined effects. First, when default points move to the right, the skewness of each of the indicator variables will be reduced.⁴ As a consequence, they will reveal more information about the correlated underlying firm returns. Default correlations rise and move closer towards return correlations. We call this phenomenon the *skewness effect* (SE). Second, the distance $d := z_2 - z_1$ between default points may change. Default correlations should decrease in d since it will become harder to infer the state (default/no default) of one firm when observing the state of the other firm. We call this effect the *distance-of-default-points effect* (DDE). However, changing the distance between default points necessarily changes the distance from zero for at least one default point. Hence DDE cannot be completely separated from SE. If the default point of the firm with the lower probability of default increases more than that of the other firm, skewness increases and d decreases, which implies that both effects work in the same direction and default correlations increase. In the

³Note how the indicator function $1\{\cdot\}$ is defined. $1\{A\}$ is equal to 1 if statement A holds and equal to 0 if statement A does not hold.

⁴The term "skewness" in the context of Bernoulli variables refers to the fact that the probability of one outcome is higher than the probability of the other one. In our case both indicator variables would be unskewed for $z_1 = z_2 = 0$. Moreover, note that default probabilities of loans are usually lower than 50%, implying that z_1 and z_2 are smaller than zero and that the binary default variables are skewed towards zero. A shift to the right will therefore reduce skewness.

opposite case, both effects work in different directions and it will depend on the parameters (the location of z_1 and z_2 and the asset correlation) whether the default correlations increase or decrease.

With respect to the two questions posed at the beginning of this introduction, our results suggest that loan prices should reflect the higher contributions to economic capital of loans with a higher default probability. Moreover, we indicate that portfolio standard deviation and hence economic capital can increase significantly under negative macroeconomic shocks. While for the pricing of loans both effects (SE and DDE) are relevant, we argue that, for the impact of macroeconomic shocks on credit portfolios, the distance-of-default-points effect will tend to cancel out while the skewness effect remains. We will also discuss the consequences of these results for the adaptation of credit risk models to the business cycle.

Finally, our results remain robust under various generalizations of our original model. First, we allow for endogenous recovery rates where the severity of the default determines the value of firm assets that can be recovered. Second, by considering loan maturities that are longer than the risk management horizon, we address scenarios where changes in the portfolio value stem from rating migrations rather than from firm defaults. We show that the qualitative nature of our results is robust with respect to such scenarios. Finally, we demonstrate that any alternative distribution for asset returns yields the same results as long as a monotonic transformation into a bivariate normal distribution exists.

The paper is organized as follows. In the next section we introduce the model and present our analytic results. Moreover, we discuss and illustrate the consequences of these results for credit risk management. In section 3 we investigate the robustness of our results with respect to crucial assumptions. Section 4 presents our conclusions.

2 Analytic Results and Applications

In this section we analyze the relationship between default probabilities and default correlations, and discuss the consequences of this analysis for the impact of macroeconomic shocks on portfolio standard deviation and for the pricing of loans. In section 2.1 we present the model, and in section 2.2 we analyze the relationship between default probabilities and default correlations qualitatively, deriving our main analytic results. The applications of these results are discussed in section 2.3.

2.1 The Model

As a starting point, we employ the risk-of-ruin or option-pricing model developed in WILCOX (1973), MERTON (1974) and SCOTT (1981). The probability of a firm going bankrupt depends on both the market value of the firm's assets relative to its outside debt and on the volatility of the market value of the assets.

Using t as time index, we consider a bank holding a credit portfolio consisting of loans to two firms (1 and 2) and undertaking risk management in $t = t_1$. The loans are due in $t = t_2$, and we assume that the bank's risk management horizon is identical with the date at which the loans mature, i.e. the bank is interested in the distribution of the $t = t_2$ value of its portfolio.⁵ We denote the two firms' asset values in t by $V_{1,t}$ and $V_{2,t}$ respectively, and assume that the debt obligations of both firms are due in $t = t_2$ (we denote the sum of these obligations by v_1 and v_2 respectively). According to the option-pricing model, default of firm i in $t = t_2$ occurs if $V_{i,t_2} \leq v_i$. We assume that in this case the firm will repay an exogenously determined fraction of the loan's principal (recovery rate),⁶ while in the other case the complete amount is repaid. Therefore, the stochastics of the portfolio payoff in $t = t_2$ can be characterized by the joint distribution of the binomial random variables $1\{V_{1,t_2} \leq v_1\}$ and $1\{V_{2,t_2} \leq v_2\}$. In the standard framework of the option pricing approach, this distribution is characterized via the distribution of the continuously compounded rates of asset returns $Z_{i,t_2} := \log(V_{i,t_2}/V_{i,t_1})$. The vector (Z_{1,t_2}, Z_{2,t_2}) is assumed to be independent of (V_{1,t_1}, V_{2,t_1}) and bivariate normally distributed with correlation coefficient $\rho > 0$.⁷ Note that the event $V_{i,t_2} \leq v_i$ can be equally well described as

$$Z_{i,t_2} \leq \log(v_i) - \log(V_{i,t_1}). \quad (1)$$

Moreover, from a $t = t_1$ perspective the vector (V_{1,t_1}, V_{2,t_1}) is fixed and the joint distribution of (Z_{1,t_2}, Z_{2,t_2}) does not depend on the realization of this vector. Hence we can normalize equation (1) with respect to mean and variance of Z_{i,t_2} . We conclude that it is sufficient to analyze the joint distribution of the Bernoulli variables $1\{Z_1 \leq z_1\}$ and $1\{Z_2 \leq z_2\}$ where (Z_1, Z_2) are standardized, bivariate normally distributed random

⁵In section 3.2 we consider the case where the loans mature after the risk management horizon.

⁶We consider the case of endogenously determined recovery rates in section 3.1.

⁷Note that this scenario is usually derived from an extension of the MERTON (1974) framework. Asset values in time are described by a two dimensional geometric Wiener process. This model is described in more detail in section 3.2.

variables with correlation ρ and

$$z_i := \frac{\log(v_i) - \log(V_{i,t_1}) - \mathbb{E} Z_{i,t_2}}{\sqrt{\text{Var}(Z_{i,t_2})}} \quad (i = 1, 2). \quad (2)$$

Throughout the paper we will assume that $z_1 \leq z_2$. The correlation between the two Bernoulli variables is termed default correlation and is denoted by $\rho^{\text{def}} = \rho^{\text{def}}(z_1, z_2, \rho)$. Note that since default probabilities are given by $p_i = \Phi(z_i)$,⁸ the relationship between default points (z_1, z_2) and ρ^{def} monotonically translates into a respective relationship between default probabilities (p_1, p_2) and ρ^{def} . We will use either of these representations as convenient.

2.2 Analytic Results

In this section we describe the relationship between default probabilities and default correlations in qualitative terms. If default probabilities change, default points will also change accordingly. Such a shift in default points has two consequences that prove to be important in understanding the relationship between default probabilities and default correlations. First, the distance of the default points from zero will change. Second, the distance $d = z_2 - z_1$ between default points may change. In order to isolate these two effects, we first consider an increase of z_1 and z_2 with the distance between the default points remaining constant. We denote the partial derivatives of ρ^{def} with respect to z_1 and z_2 by ρ_1^{def} and ρ_2^{def} respectively and obtain the following result.

Proposition 1

Consider a homogeneous move of both default points to the right (i.e. a move where the distance between default points remains constant).

- (i) *If $p_1, p_2 < 50\%$, then default correlations **increase** ($\rho_1^{\text{def}} + \rho_2^{\text{def}} > 0$);*
- (ii) *If $p_1, p_2 > 50\%$, then default correlations **decrease** ($\rho_1^{\text{def}} + \rho_2^{\text{def}} < 0$).*

As for all other propositions, the proof of proposition 1 is given in appendix A. The reasoning behind proposition 1 runs as follows: If $z_i < 0$ ($i = 1, 2$), then a homogeneous shift of z_1 and z_2 to the right will reduce the skewness of the binary variables $1\{Z_1 \leq z_1\}$ and $1\{Z_2 \leq z_2\}$. The less skewed these binary variables are, the more information they reveal about the correlated underlying variables Z_1 and Z_2 . Accordingly, default correlations increase towards the higher correlations of returns. If $z_1, z_2 > 0$, skewness increases

⁸ $\Phi(\cdot)$ denotes the cumulative standard normal density function.

when default points shift to the right and hence default correlations decrease. We call this phenomenon the skewness effect (SE). In practical applications we are mainly interested in the case $z_1, z_2 < 0$ since default probabilities higher than 50% are not relevant. Nevertheless, the result for $z_1, z_2 > 0$ confirms the reasoning we propose. For the rest of the analysis we will focus on the case $z_1, z_2 < 0$.⁹

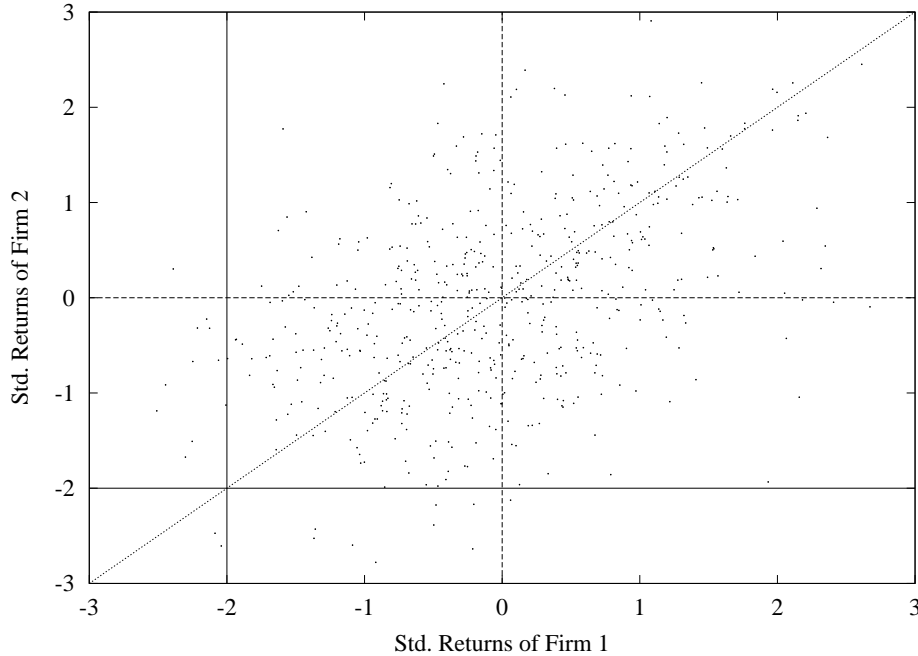


Figure 1: Scatter plot of two standardized, bivariate normally distributed random variables with correlation $\rho = 50\%$.

Why does a reduction of skewness increase default correlations? Figure 1 shows the scatter plot of two normally distributed correlated random variables describing realizations of the pair (Z_1, Z_2) . Note that the corresponding scatter plot for the derived binary variables $1\{Z_i \leq z_i\}$ ($i = 1, 2$) would depict only four points $((0, 0), (0, 1), (1, 0)$ and $(1, 1))$. The frequency with which each of these four possible realizations occurs can be inferred from figure 1 by counting the number of points in the respective quadrants of the two “coordinate systems” inserted in the figure. The system depicted with solid lines illustrates the case $z_1 = z_2 = -2$. In this case the distribution of the indicator variables is strongly asymmetric: nearly all data points lie in $(0, 0)$ (both firms survive) while there are only very few points in $(1, 1)$ (both firms default).

⁹Note that results for $z_1, z_2 > 0$ and for negative correlations are also available in most cases. But since these parameter constellations are not relevant in practice we will not discuss them.

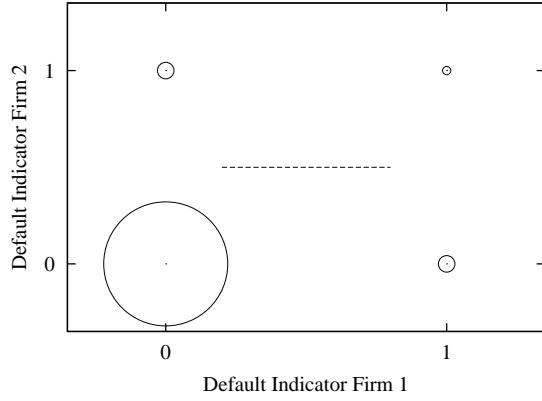


Figure 2: Stylized “Scatter Plot” of the indicator variables: asymmetric (skewed) case. Almost no upward sloping tendency in the data.

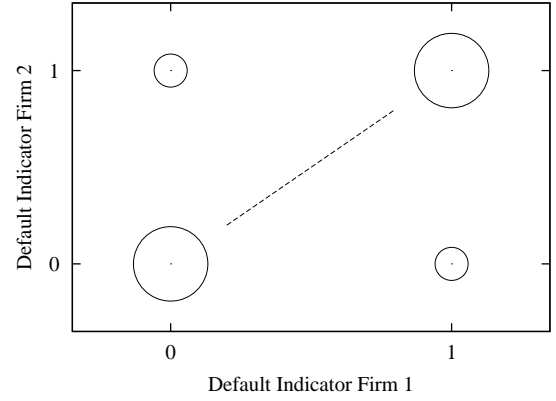


Figure 3: Stylized “Scatter Plot” of the indicator variables: symmetric (unskewed) case. Strong upward sloping tendency in the data.

What happens to the frequencies when we move the origin of the coordinate system along the (broken) 45° line from $(-2, -2)$ to $(0, 0)$ (the origin of the system with the broken lines)? The major effect is that point mass is shifted from $(0, 0)$ to $(1, 1)$ which reduces the asymmetry of the distribution. As a consequence, the upward sloping tendency in the data increases and this is a manifestation of a higher correlation coefficient ρ^{def} . This is illustrated in figures 2 and 3. The size of the circles around the four possible realizations is used to illustrate the number of observations (big circle - many observation, small circle - few observation).

Finally, note that the skewness effect consists of two counteracting effects concerning the information revealed about $1\{Z_2 \leq z_2\}$ when observing $1\{Z_1 \leq z_1\}$ (or vice versa). The information content of the event $1\{Z_1 \leq z_1\} = 1$ decreases when default points increase jointly, while the information content of the event $1\{Z_1 \leq z_1\} = 0$ increases. This is because the information about the underlying return realizations decreases (increases), which in turn impacts on the information revealed about the other binary variable. If firm 1 defaults, then one can infer that $Z_1 \in (-\infty, z_1)$. This interval increases with z_1 decreasing the information available about Z_1 from the default event. This in turn implies that less information about Z_2 and, hence, about $1\{Z_2 \leq z_2\}$ is obtained. The opposite is true for the non-default event. If firm 1 does not default, then $Z_1 \in (z_1, \infty)$, an interval decreasing in size if z_1 increases.

We illustrate this point for fully correlated firm returns ($\rho = 100\%$). Conditional on the default of firm 2, the probability that firm 1 will also default is $\Phi(z_1)/\Phi(z_2)$, and the

ratio of conditional and unconditional default probability for firm 1 is therefore given by $1/\Phi(z_2)$. Hence, this ratio (and therefore the information content of firm 2's default) decreases if z_1 and z_2 increase. The contrary is true for the information content of the event where firm 1 has not defaulted. Conditional on the information that firm 1 has survived, the probability that firm 2 has also survived is $[1 - \Phi(z_2)]/[1 - \Phi(z_1)]$. Hence the ratio of the conditional and unconditional probability that firm 2 will survive is given by $1/[1 - \Phi(z_1)]$ and increases if skewness is reduced. As can be seen from the arguments in the previous paragraphs, the effect of non-default event information increasing dominates the effect of default event information decreasing.

We now turn to the second consequence of a change in default probabilities, namely that the distance d between default points can change. The reasoning based on information revelation about return realizations implies that default correlations should decrease in d . We call this effect the distance-of-default-points effect (DDE). Unfortunately, DDE cannot be completely separated from SE since changing the distance between default points necessarily changes the distance from zero for at least one default point. Suppose, for example, that one default point is fixed while the other one moves to the right. If the smaller of the two points, z_1 , moves, then both effects should work in the same direction. Skewness is reduced and the distance between default points decreases, which should increase default correlations. If, however, z_2 moves, then the two effects work in opposite directions and it is no longer clear which one dominates the other. Proposition 2 shows that whether ρ^{def} decreases or increases depends on the default point ratio $\lambda := z_1/z_2$ and on the asset-return correlation ρ . To prepare for the formulation of proposition 2, we define

$$\rho_+(\lambda) := (25/32) \left\{ \lambda - \sqrt{\lambda^2 + 24/25} \right\}.$$

Proposition 2

Suppose that $p_1, p_2 < 50\%$ and consider a move of only one default point to the right.

- (i) If z_1 moves, then default correlations **increase** ($\rho_1^{\text{def}} > 0$) if $\rho < \rho_+(\lambda)$. Note that this inequality is fulfilled if $\lambda \leq 96\%$ or if $\rho \leq 56\%$.
- (ii) If z_2 moves, then default correlations
 - **increase** ($\rho_2^{\text{def}} > 0$) if $\rho < \rho_+(1/\lambda)$
 - **decrease** ($\rho_2^{\text{def}} < 0$) if $\rho > 2\lambda/(1 + \lambda^2)$.

Figures 4 and 5 visualize the parameter sets for which, according to our theoretical results, default correlations will increase (DC+) or decrease (DC-). Figure 4 depicts the case where

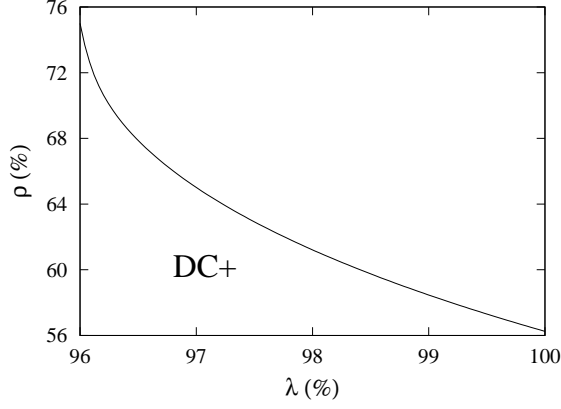


Figure 4: The set DC+ contains all points below the solid line and visualizes all combinations (λ, ρ) that fulfill the inequality $\rho < \rho_+(\lambda)$.

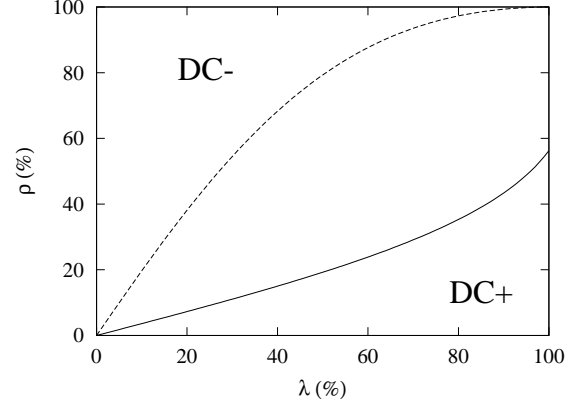


Figure 5: The set DC+ contains all points below the solid line and visualizes all combinations (λ, ρ) that fulfill the inequality $\rho < \rho_+(1/\lambda)$. The set DC- contains all points above the broken line and visualizes all combinations (λ, ρ) that fulfill the inequality $\rho > 2\lambda/(1 + \lambda^2)$.

z_1 moves. In this case, default correlations increase if (λ, ρ) lies in DC+. The intuition developed for SE and DDE would imply the more general statement that $\rho_1^{\text{def}} > 0$ for all $0 < \lambda < 1$ and $0 < \rho \leq 1$. While a formal proof is not in reach yet, our simulation exercises have confirmed this conjecture. Figure 5, on the other hand, depicts the case where z_2 moves. In this case, default correlations increase if (λ, ρ) lies in DC+ and decrease if (λ, ρ) lies in DC-.

Figures 6 and 7 illustrate the two effects identified in propositions 1 and 2. Figure 6 demonstrates the skewness effect for the symmetric case where both firms have the same default probability. Default correlations are maximum when the binary variables are unskewed, i.e. when default probabilities are equal to 50%. Figure 7 visualizes the distance-of-default-points effect for the case where firm 1 has a default probability of 0.05% and the default probability of firm 2 varies from 0.05% to 50%. In this case default correlations first increase (high λ) and then decrease (low λ).

Finally, figure 8 summarizes the major insights from propositions 1 and 2 and our conjecture. Default correlations increase if default points move to the right and if the move of the default point associated with the lower default probability is more pronounced. An analogous result holds if default points move to the left. In the question-mark ranges both default points move to the left or to the right but it depends on the parameter vector (λ, ρ) whether default correlations will increase or decrease.

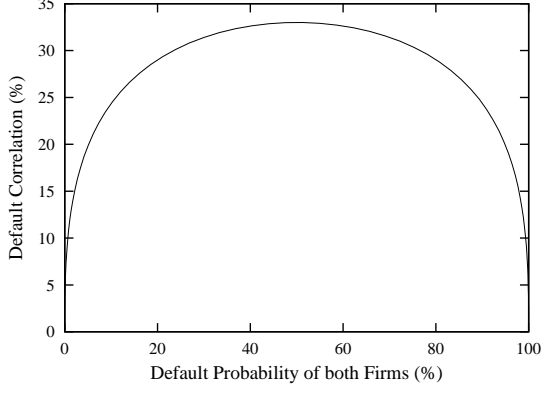


Figure 6: The skewness effect. The figure depicts the default correlation of two firms with the same default probability p when p ranges from 0 to 100% ($\rho = 50\%$).

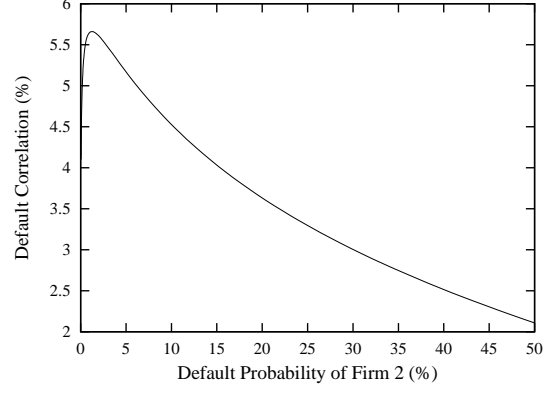


Figure 7: The distance-of-default-points effect. The figure depicts the default correlation between firm 1 ($p_1 = 0.05\%$) and firm 2 (p_2 ranging from 0 to 50%) ($\rho = 50\%$).

The next proposition completes our picture of ρ^{def} by exploring the boundary cases $z_1 = z_2 = 0$ and $z_1, z_2 \rightarrow -\infty$.

Proposition 3

- (i) ρ^{def} has a local maximum in $z_1 = z_2 = 0$ and

$$\rho^{\text{def}}(0, 0, \rho) = \frac{2}{\pi} \arcsin(\rho).$$

- (ii) If $\rho < 1$, then $\lim_{z \rightarrow -\infty} \rho^{\text{def}}(z, z, \rho) = 0$. Moreover, $\rho^{\text{def}}(z, z, 1) = 1$ for all z .

The intuition of the SE and DDE effects developed above suggests that $(0, 0)$ is also a global maximum, which is confirmed by our simulation results. Hence, $(2/\pi) \arcsin(\rho)$ can be used as an upper boundary for default correlations. We conclude this section by stating a result expressing how default correlation changes due to shifts in default points depend on the return correlation ρ .

Proposition 4

Suppose that $z_1, z_2 < 0$. Then there are real numbers $\bar{\rho}_L = \bar{\rho}_L(z_1, z_2)$ and $\bar{\rho}_H = \bar{\rho}_H(z_1, z_2)$ with $\bar{\rho}_L < \bar{\rho}_H$ such that the following statements hold:

- (i) $(\rho_1^{\text{def}} + \rho_2^{\text{def}})$ is increasing in ρ for $\rho < \bar{\rho}_H$ and decreasing in ρ for $\rho > \bar{\rho}_H$.
- (ii) ρ_2^{def} is increasing in ρ for $\rho < \bar{\rho}_L$ and decreasing in ρ for $\rho > \bar{\rho}_L$.

Moreover, if $10^{-6} \leq p_1, p_2 \leq 0.46$, then $\bar{\rho}_H \in [0.53, 0.89]$.

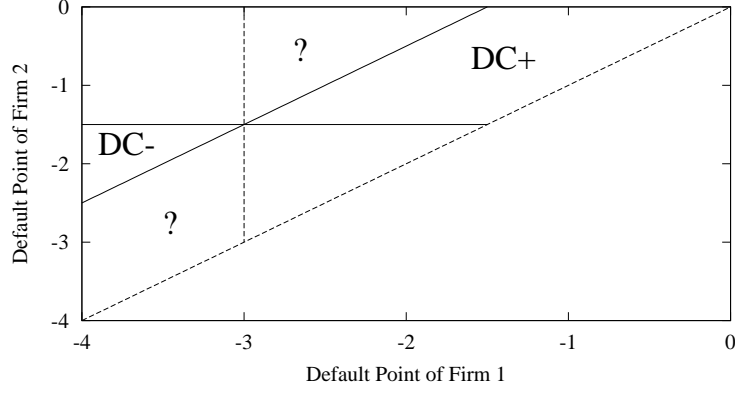


Figure 8: For a given combination of default points $(-3, -1.5)$, the figure depicts the default-point range in which default correlations are higher than $\rho^{\text{def}}(-3, -1.5)$ (i.e. DC+) and the range in which they are lower (DC-).

2.3 Applications

After stating our analytic results, we will now discuss their potential applications.

2.3.1 Default Correlations and Macroeconomic Shocks

In this section we explore how macroeconomic shocks impact on credit portfolios. In particular, we address the following questions:

1. Can default correlations of two firms decrease after a negative macroeconomic shock?
2. Which correlation effect (SE or DD) is most relevant at the portfolio level?
3. What can be said about the size of the effects?

We start with the first question. Suppose that a macroeconomic shock scales down the $t = t_1$ asset value of firm i by the factor Δ_i ($V_{i,t_1} \rightarrow \Delta_i V_{i,t_1}$). Then, according to equation (2), z_i increases by

$$\delta_i := \frac{-\log(\Delta_i)}{\sqrt{\text{Var}(Z_{i,t_2})}}, \quad (3)$$

i.e. $z_i \rightarrow z_i + \delta_i$.¹⁰ Hence, such a macroeconomic shock will move both default points to the right, but generally by different magnitudes. We can describe this shift by a move

¹⁰Note that $\delta_i > 0$ since $\Delta_i < 1$.

of δ_1 units in direction $(1, \delta)$ where $\delta := \delta_2/\delta_1$. Note that according to propositions 1 and 2 and our conjecture, the default correlation between firm 1 and 2 can only decrease if $\delta > 1$. Moreover, we can describe the marginal change of default correlations by the derivative in the direction of $(1, \delta)$, i.e. by

$$\rho_1^{\text{def}} + \delta \rho_2^{\text{def}} = (\rho_1^{\text{def}} + \rho_2^{\text{def}}) + (\delta - 1)\rho_2^{\text{def}}. \quad (4)$$

Since $\rho_1^{\text{def}} + \rho_2^{\text{def}} > 0$, we obtain that $\rho_1^{\text{def}} + \delta \rho_2^{\text{def}}$ can only be negative if $\rho_2^{\text{def}} < 0$. Note that for arbitrary asset correlations ρ we can find a pair (z_1, z_2) for which $\rho_2^{\text{def}} < 0$ by simply choosing z_1 close enough to zero (see figure 5). However, if we restrict our analysis to default points associated with default probabilities lower than, say, 10 percent and higher than 10^{-4} percent we obtain that $\lambda \geq 27\%$ and can infer from figure 5 that ρ_2^{def} can only be negative if $\rho \geq 9.9\%$. Moreover, our simulation results suggest that for fixed δ and fixed default points z_1 and z_2 , there is a critical value $\bar{\rho} = \bar{\rho}(\delta, z_1, z_2)$ so that $\rho_1^{\text{def}} + \delta \rho_2^{\text{def}}$ is increasing in ρ for $\rho < \bar{\rho}$ and decreasing in ρ for $\rho > \bar{\rho}$.¹¹ Hence, default correlations will only decrease if ρ exceeds a certain threshold $\tilde{\rho}(\delta, z_1, z_2)$.

Summing up, we have found that default correlations will only increase if both variables δ and ρ are high. In order to assess which combinations of δ and ρ can occur, it is important to keep in mind that the firms' exposure to the macroeconomic shock in $t = t_1$ might explain quite a large part of the correlation between returns in $t = t_2$. For greater concreteness, we add a third point in time, t_0 , define the asset returns from t_0 to t_1 by $Z_{i,t_1} := \log(V_{i,t_1}/V_{i,t_0})$, and assume that the firms' normalized returns depend linearly on a macroeconomic factor Z^S and on idiosyncratic components ϵ_i :

$$\frac{Z_{i,t} - \mathbb{E} Z_{i,t}}{\sqrt{\text{Var}(Z_{i,t})}} = \theta_i Z_t^S + \epsilon_{i,t} \quad (i = 1, 2; t = t_1, t_2). \quad (5)$$

The random variables $(\epsilon_{i,t}, Z_i^S)_{i=1,2; t=t_1, t_2}$ are assumed to be mutually stochastically independent, and θ_1 and θ_2 are positive real numbers. Inserting equation (5) in equation (2) and using the fact that $\log(V_{i,t_1}) = \log(V_{i,t_0}) + Z_{i,t_1}$, we finally obtain that the default points z_1 and z_2 can be written as

$$z_i = \bar{z}_i - \theta_i Z_{t_1}^S \quad (i = 1, 2).$$

¹¹Note that for $\rho < \bar{\rho}_L$ and $\rho > \bar{\rho}_H$ this observation is backed by proposition 4. For $\bar{\rho}_L < \rho < \bar{\rho}_H$, we only know that $(\rho_1^{\text{def}} + \rho_2^{\text{def}})$ increases in ρ while $(\delta - 1)\rho_2^{\text{def}}$ decreases.

\bar{z}_1 and \bar{z}_2 are constants that depend on (V_{1,t_0}, V_{2,t_0}) , (v_1, v_2) , $(\epsilon_{1,t_1}, \epsilon_{2,t_1})$, and on the means and variances of Z_{1,t_2} and Z_{2,t_2} respectively.¹² By normalizing the variance of $Z_{t_2}^S$ to 1 we obtain that $\rho = \theta_1\theta_2$, $\delta = \theta_2/\theta_1$ and that $0 \leq \theta_i \leq 1$ ($i = 1, 2$). This in turn implies that $\delta \leq 1/\rho$.

What can we learn from this exercise? First, since the macroeconomic factor responsible for the scale-down in asset values will also explain a certain fraction of the asset-return correlation ρ ,¹³ there is a structural relationship between ρ and δ . Second, for fixed ρ this relationship limits the possible values for δ from above, since ρ determines a minimal *joint* exposure to the macroeconomic factor. Hence, an increase in ρ will be associated with a decrease of the upper boundary for δ . Whether a macroeconomic shock can decrease default correlations at all will therefore depend on how strongly the upper bound for δ is depressed when ρ rises, i.e. on the extent to which the asset correlations are explained by the macroeconomic factor that has triggered the shock. In our example, where the macroeconomic factor fully explains the correlation, simulation results suggest that default correlations always increase if the default probabilities p_1 and p_2 lie between 10^{-4} and 10 percent.

We now turn to the portfolio level. In an average portfolio, the DD effect will tend to cancel out, since there should be as many pairs of loans where d decreases as pairs where d increases. As a benchmark, consider a portfolio where for each loan in the portfolio with default points (z_1, z_2) and exposure direction $(1, \delta)$ there is another loan with (z_1, z_2) and the opposite exposure direction $(\delta, 1)$. If we add correlation effects pairwise across the portfolio, the marginal effect of each pair is given by

$$(\rho_1^{\text{def}} + \delta\rho_2^{\text{def}}) + \delta\rho_1^{\text{def}} + \rho_2^{\text{def}} = (1 + \delta)(\rho_1^{\text{def}} + \rho_2^{\text{def}}) > 0.$$

On a more practical level, suppose that loans are subdivided into classes according to their default probability (for example by rating classes). Now calculate the δ of each class by summing all single exposures of loans in that class. If the values for δ in all classes are about the same, then DDE should approximately cancel out and the whole effect of the shock on portfolio correlations can be described by SE. If, however, class exposures differ significantly, then DDE might modify or intensify the rise of default correlations caused by SE. However, even if DDE is significant at portfolio level and thus modifies SE,

¹²Here we have assumed that $\text{Var}(Z_{i,t_1}) = \text{Var}(Z_{i,t_2})$ ($i = 1, 2$). In the Merton framework, which is more precisely described in section 3.2, this is equivalent to the assumption that $(t_2 - t_1) = (t_1 - t_0)$. Hence we assume that the risk management horizon and the period during which the macroeconomic shock is analyzed have the same length.

¹³In our example the macroeconomic factor fully explains ρ .

then the reasoning above suggests that the overall effect should still be a rise in default correlations. Moreover, in terms of the whole economy, the argument that DDE cancels out could be made even stronger, since if credit portfolios of all banks are considered, the δ for each loan class is averaged over a much higher number of firms. This in turn should imply that the values of δ are much more similar among classes. Therefore, as we turn to the third question posed at the beginning of this section, we will concentrate on SE.

In order to give an impression of the size of the effects at work, we have calculated default correlations for different loan types and different macroeconomic scenarios. The default probabilities associated with these scenarios are shown in table 1.

	AV	A	B	C
Initial		.05	.20	7.12
Shock 1	-15%	.12	.43	11.20
Shock 2	-30%	.31	1.01	17.96
Shock 3	-50%	1.35	3.56	34.52

Table 1: Default probabilities in percent for different loan types and different shock scenarios.

The first scenario (initial scenario) serves as a benchmark and the three others describe deviations from the initial scenario induced by macroeconomic shocks of different intensities. We have chosen three different loan types (labeled A, B and C) which have different initial default probabilities. The initial default probabilities for each loan type have been chosen as average one-year default rates of the high, medium and low rating segments of Moody's.¹⁴ In the three shock scenarios, asset values will be reduced by 15, 30 and 50 percent (i.e. $\Delta = 0.85, 0.7$ and 0.5 respectively). In order to calculate the resulting changes in default points we finally need to fix the standard deviation of returns (see equation (3)). We have chosen an average value 0.64 for yearly returns from BERNDT (1991). We have then calculated default correlations in the initial scenario and derived the factors by which these correlations increase under the three shock scenarios. All calculations have been made for all possible combinations of the three loan types and for different asset correlations ($\rho = 10\%$, $\rho = 30\%$ and $\rho = 50\%$). Table 2 shows the results of these calculations.

¹⁴More precisely, we have divided all firms that were rated by Moody's in 2000 into three equal-sized groups. The highly rated segment (labeled A) includes firms with ratings from Aaa to A3, the medium-rated segment (labeled B) includes firms with ratings from Baa1 to Ba2 and the low-rated segment (labeled C) includes firms with ratings from Baa3 to C. See MOODY'S (2000).

	A			B			C		
A	.11	.98	4.21	.18	1.38	5.18	.53	2.35	5.02
	1.89	1.58	1.36	1.80	1.53	1.33	1.54	1.39	1.28
	3.75	2.62	1.91	3.38	2.44	1.84	2.43	1.98	1.67
	10.11	5.46	3.17	8.28	4.76	2.93	4.48	3.18	2.43
B				.30	2.03	6.89	.92	3.92	8.45
				1.72	1.49	1.31	1.47	1.34	1.24
				3.04	2.27	1.75	2.18	1.81	1.56
				6.77	4.13	2.66	3.65	2.70	2.10
C							3.10	11.30	22.65
							1.26	1.19	1.13
							1.56	1.39	1.27
							1.95	1.65	1.43

Table 2: Default correlations (bold) in percent and scaling factors by which default correlations increase under the shock scenarios. The scaling factor describes the multiplier by which default correlations change compared to the initial scenario. In each cell of the matrix, the columns correspond to the different asset correlations chosen. Asset correlations ρ increase from left to right ($\rho = 10, 30$, and 50 percent). The rows correspond to the different shock scenarios. The intensity of the shock increases from the top to the bottom row.

In each cell of the matrix, initial default correlations are bold and are followed by the scaling factors by which default correlations increase under the different shock scenarios. The different columns correspond to the different asset correlations chosen. For example, the third line in each cell shows the factors by which default correlations increase under the shock scenario 2 for the different values of ρ . The second entry in this line presents the factor for $\rho = 50\%$. The reported results suggest that the changes in default correlations may be substantial. Moreover, default correlations rise most strongly relative to their pre-shock value if return correlations and initial default probabilities are low. The effect becomes less pronounced if those parameters increase.

At this point we have concluded that at portfolio level, the increase in variances caused by higher default probabilities after a macroeconomic shock will be reinforced by an increase in default correlations. We now give an impression of the size of these effects for an average loan portfolio. Table 3 shows the characteristics of the portfolios used in order to illustrate our theoretical results. Note that the principal is chosen such that all loans have an expected repayment of 1.

Principal	Recovery rate	ρ	Number of firms
$1/(1 - p_i)$	0	50%	300

Table 3: Characteristica of the portfolios used for illustration.

We have considered three different types of such portfolios: portfolios only consisting of type A loans (labeled AAA), portfolios only consisting of type C loans (labeled CCC), and portfolios with 100 loans of each type (labeled ABC). Table 4 illustrates how the standard deviations (Std) and the economic capital (EC) of an ABC portfolio increase under the different shock scenarios we have considered. For example, if asset values decrease by 15%, then portfolio standard deviation will increase by 33%. In order to identify the size of the correlation effect, we have calculated the increase in portfolio standard deviation that is achieved when only the increase in loan variances is considered and default correlations are fixed to their pre-shock levels. These results are reported in column (Std-Corr).

AV	Std-Corr	Std	EC
-15%	24	33	24
-30%	55	79	51
-50%	105	167	100

Table 4: Percentage increase in standard deviation (Std) and economic capital (EC) after macroeconomic shocks for an ABC portfolio. The column (Std-Corr) gives the percentage increase in portfolio standard deviation when the increase in default correlations is not taken into account.

A more exhaustive study of the relevance of the correlation effect for standard deviation and economic capital of credit portfolios under macroeconomic shocks has been provided by GERSBACH AND LIPPONER (2000) and LIPPONER (2000). Using a simulation approach, they isolate the correlation effect of a macroeconomic shock. After the shock has occurred, they scale down return correlations until default correlations are at the same level as before the shock. Comparing standard deviation and economic capital obtained with the correct and the scaled-down return correlations enables them to measure the size of the correlation effect relative to the other effects. They show that the correlation effect may account for more than 50% of the increase in credit risk induced by the macroeconomic shock.

An important further implication of our results is that the standard deviation of default rates will vary throughout the business cycle, as is the case with the standard deviation of credit portfolios. This observation is important since the currently proposed reduced-form credit-risk models use default rate distributions as input for the value-at-risk analysis of credit portfolios.¹⁵ In table 5 we illustrate how default-rate standard deviations vary throughout the business cycle. Considering an industry with an ABC firm portfolio, we have calculated the percentage increase of the standard deviation when the economy moves from the expansion to the recession state of the business cycle.¹⁶ Three different expansion/recession scenarios - where the asset values increase (decrease) by 10, 20 and 30 percent compared to the average case - have been evaluated. Our results suggest that - when using default rate distributions as an input for credit risk models - the standard deviations of default rates should be adapted to the business cycle. For example the simulation results in GORDY (2000) show that the percentile values calculated by reduced-form models are very sensitive to changes in default-rate standard deviation. For the portfolios considered by Gordy, an increase of variances by 100% increases percentiles by two to three times.

AV	Std-Corr	Std
$\pm 10\%$	34	47
$\pm 20\%$	78	114
$\pm 30\%$	135	214

Table 5: Percentage increase of default-rate standard deviation when the economy moves from the expansion to the recession state of the business cycle. We have assumed that the firm portfolio of the sector or industry under consideration is of type ABC.

2.3.2 Pricing of Loans

The price of a loan should reflect the costs of the additional amount of capital that has to be held against the credit portfolio when the specific loan is added to the portfolio. In current practice, interest rates on loans in most cases merely reflect the impact of higher default probabilities on *expected* returns. Our results suggest that the impact on portfolio standard deviation should also be taken into account. We believe that the following two

¹⁵Among them are Credit Risk⁺ from Credit Suisse Group and Credit Portfolio View from McKinsey. See CREDIT SUISSE (1997) and MCKINSEY (1998)

¹⁶Employing an ABC portfolio, we have in particular assumed that the number of firms in the industry is 300. However, the figures in table 5 do not change substantially for a larger number of firms.

observations are especially important in this respect. First, as can be seen from table 2, the contribution of a loan to the standard deviation of the credit portfolio varies with the composition of the portfolio. For example, the default correlation of a C loan with loans in an AAA portfolio is 5.02%, while default correlations will rise to 22.65% if the considered portfolio is of type CCC (distance-of-default-points effect). Second, the variance effect (loans with higher default probability have a higher standard deviation than loans with lower default probability) is reinforced by the correlation effect (they also have a higher correlation with other loans).

Table 6 illustrates that it is important to recognize that a loan’s contribution to the portfolio standard deviation varies strongly with its default probability. We compare standard deviation and economic capital of an AAA and a CCC portfolio by calculating the ratio by which both of these measures increase when moving from the analysis of a CCC portfolio to the respective values for an AAA portfolio. For example, the CCC standard deviation is 25.28 times higher than the AAA standard deviation. As before, we have calculated the increase in portfolio standard deviation when the increase of default correlations is not taken into account in column (Std-Corr). Note that all of these ratios also reflect differences in the principals of the loans in both portfolios (see table 3). Since we are mainly interested in the variance and correlation effect, the second row of the table calculates the respective ratios when the principals of the loans in both portfolios are normalized to 1.

Principals	Std-Corr	Std	EC
Heterogeneous	12.10	27.20	57.13
Homogeneous	11.24	25.28	53.1

Table 6: Ratios by which standard deviation (Std) and economic capital (EC) increase from AAA to CCC portfolios. The first column (Std-Corr) displays the ratio for portfolio standard deviation when the increase of default correlations is not taken into account. The second row shows the ratios in the case where the loans’ principals are normalized to 1.

3 Robustness

In the preceding section we established the theoretical foundation of the correlation effect and illustrated its importance for credit portfolio management. In this section we explore the robustness of our results with respect to the assumptions made during the specification

of the model. In section 3.1 we consider endogenous recovery rates, in section 3.2 we examine the case where the risk management horizon is not identical with the maturity of the loans, and in section 3.3 we discuss how alternative distributional assumptions would affect our results.

3.1 Endogenous Recovery Rates

In this section we uphold the assumption of bivariate normally distributed returns but consider endogenous recovery rates. To account for endogenous recovery rates, we assume that a certain fraction β of the asset value V_2 can be recovered in the event of default. Recall that $Z_{i,t_2} = \log(V_{i,t_2}/V_{i,t_1})$ and hence $V_{i,t_2} = V_{i,t_1} \exp(Z_{i,t_2})$. Normalizing the loan repayment in the case of non-default to 1, the payoff of a portfolio of two loans is given by $2 - I_1 - I_2$ where

$$I_i := 1\{Z_{i,t_2} \leq z_i\} \left(1 - \beta V_{i,t_1} \exp(Z_{i,t_2})\right) \quad (i = 1, 2).$$

We standardize V_{1,t_1} to 1 and adjust β and V_{2,t_1} such that a certain fixed fraction $\tilde{\beta}$ of the loan is recovered for $Z_{1,t_2} = z_1$ and $Z_{2,t_2} = z_2$ respectively. Hence,

$$\beta = \tilde{\beta} \exp(-z_1) \quad \text{and} \quad V_{2,t_1} = \tilde{\beta} \exp(-z_2)/\beta.$$

Finally, by normalizing the mean of (Z_{1,t_2}, Z_{2,t_2}) to zero and by assuming, for simplicity, that Z_{1,t_2} and Z_{2,t_2} have the same variance σ^2 , we can calculate the building blocks of the correlation between loan repayments. Variances can be derived from¹⁷

$$\begin{aligned} \mathbb{E} I_i^2 &= \int_{-\infty}^{z_i/\sigma} \left(1 - \tilde{\beta} \exp(\sigma\zeta - z_i)\right)^2 \varphi(\zeta) d\zeta \\ \mathbb{E} I_i &= \int_{-\infty}^{z_i/\sigma} \left(1 - \tilde{\beta} \exp(\sigma\zeta - z_i)\right) \varphi(\zeta) d\zeta, \end{aligned}$$

and the covariance from the formulas for $\mathbb{E} I_i$ ($i = 1, 2$) and from

$$\begin{aligned} \mathbb{E} I_1 I_2 &= \int_{-\infty}^{z_1/\sigma} \int_{-\infty}^{z_2/\sigma} \left(1 - \tilde{\beta} \exp(\sigma\zeta_1 - z_1)\right) \left(1 - \tilde{\beta} \exp(\sigma\zeta_2 - z_2)\right) \varphi_\rho(\zeta_1, \zeta_2, \rho) d\zeta_1 d\zeta_2 \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{z_1/\sigma} \left(1 - \tilde{\beta} \exp(\sigma\zeta - z_1)\right) \exp\left\{-\frac{\zeta^2}{2(1-\rho^2)}\right\} g(\zeta) d\zeta \end{aligned}$$

¹⁷ $\varphi(\cdot)$ denotes the density function of the standard normal distribution.

where

$$\varphi_\rho(\zeta_1, \zeta_2, \rho) := \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{\zeta_1^2 - 2\rho\zeta_1\zeta_2 + \zeta_2^2}{2(1-\rho^2)}\right\}$$

and

$$g(z) := \int_{-\infty}^{z_2/\sigma} \left(1 - \tilde{\beta} \exp(\sigma\zeta - z_2)\right) \exp\left\{-\frac{\zeta^2 - 2\rho z\zeta}{2(1-\rho^2)}\right\} d\zeta.$$

For the robustness analysis we have relied on numerical results for three reasons. First, the analytic tractability of the problem seems questionable. Second, there is a quite clear-cut intuition of how default-correlation changes for the endogenous case relate to such changes in the exogenous case. If recovery rates are endogenous, loan repayments in the event of default provide full information about realized returns. This implies that default correlations in general should be higher than in the exogenous case, where the default event only reveals that the firm's standardized returns are below the default point. Moreover, with higher default probabilities, the event of full information revelation is more likely which increases the information about joint returns available from loan repayments. This adds to the skewness effect and default correlations should therefore increase even more if recovery rates are endogenous. Third, the log-normal specification of asset values implies that the amount that can be recovered in the event of default decreases exponentially when returns decrease (see e.g. the equations for $\mathbb{E} I_i^2$ or $\mathbb{E} I_1 I_2$). This suggests that the difference between the endogenous and the exogenous case should decrease rapidly when returns decrease from default points, which in turn implies that the correlation measures should not differ very strongly.

We have calculated default correlations for a wide variety of parameter constellations. The results obtained confirm the intuition outlined above. Table 7 documents the case $\sigma = 1$ and $\tilde{\beta} = 0.8$.

3.2 Rating Migration

In the last section we suggested that endogenous recovery rates do not affect our qualitative results. We now return to the assumption of exogenous recovery rates in examining the robustness of our results with respect to the relationship between risk management horizon and loan maturity. In this section we investigate how our results are affected when loans do not mature at the end of the risk management horizon. In order to conduct this analysis we add an additional point in time, t_3 . We assume that the bank does risk management in $t = t_1$, uses the risk management horizon $t = t_2$ and holds zero-coupon loans maturing in $t = t_3$. As in the previous section, the loans default if the firms' asset values

	A			B			C		
A	.10	1.03	4.80	.18	1.46	5.91	.54	2.61	6.20
	1.89	1.58	1.35	1.81	1.54	1.33	1.56	1.41	1.29
	3.78	2.62	1.90	3.42	2.45	1.83	2.51	2.03	1.70
	10.36	5.50	3.14	8.54	4.83	2.91	4.84	3.40	2.56
B				.30	2.14	7.81	.93	4.29	10.11
				1.73	1.49	1.30	1.49	1.35	1.25
				3.09	2.28	1.75	2.26	1.86	1.58
				7.03	4.21	2.66	3.97	2.88	2.21
C							3.20	12.32	25.83
							1.28	1.20	1.14
							1.64	1.45	1.29
							2.20	1.81	1.52

Table 7: Default correlations for endogenous recovery rates ($\sigma = 1$ and $\tilde{\beta} = 0.8$). The table can be read in the same way as table 2.

fall below the respective default points, i.e. if $V_{t_3,i} \leq v_i$ ($i = 1, 2$). Note that in such a setting the loans have to be reevaluated in $t = t_2$. In order to derive a complete valuation framework, we make the following assumptions:

1. The asset values of both firms follow a two-dimensional geometric Brownian motion, i.e.:

$$\begin{pmatrix} dV_{1,t}/V_{1,t} \\ dV_{2,t}/V_{2,t} \end{pmatrix} = dW_t.$$

$W := (W_t)_{t \in [0, \infty)}$ is a two-dimensional Brownian motion with mean vector $\mu = (\mu_1, \mu_2)^T$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho \\ \sigma_1 \sigma_2 \rho & \sigma_2^2 \end{pmatrix}.$$

μ_i , σ_i and ρ describe the mean, variance and correlation of the two firms' instantaneous assets returns respectively ($i = 1, 2$).

2. The instantaneous risk-free interest rate is r , i.e. a dollar invested from time t to time s in risk-free securities yields $e^{r(s-t)}$ dollars.

When pricing both loans in $t = t_2$, we work with the standard risk-neutral or martingale probability measure.¹⁸ We use p_{it}^* to denote the risk-neutral conditional probability that firm i defaults in $t = t_3$, given all the information available in t . Since under the risk-neutral probability measure, the expected return on all securities is the risk-free rate r , the risk-neutral default probabilities are given by the probabilities of the corresponding risk-neutral asset value processes $V_i^* := (V_{i,t}^*)_{t \in [0, \infty)}$ falling below the firms' respective default points v_1 and v_2 in $t = t_3$.¹⁹

Note that V_i^* is defined in the same way as the asset value process V_i except that the return rates μ_i are replaced by the riskless return r . Hence, the log increments

$$\left(\log(V_{i,s}^*) - \log(V_{i,t}^*) \right)_{i=1,2} \quad (s > t)$$

of the risk neutral processes are distributed jointly normally with means $(s-t)(r - \sigma_i^2/2)$, variances $(s-t)\sigma_i^2$ and correlation ρ , and are independent of $V_{i,t}^*$.²⁰ For $t < s = t_3$ we therefore obtain that

$$\begin{aligned} p_{i,t}^* &= \mathbb{P} \left\{ \frac{\log(V_{i,t_3}^*) - \log(V_{i,t}^*) - (r - \sigma_i/2)(t_3 - t)}{\sigma_i \sqrt{t_3 - t}} \leq \frac{\log(v_i) - \log(V_{i,t}^*) - (r - \sigma_i/2)(t_3 - t)}{\sigma_i \sqrt{t_3 - t}} \right\} \\ &= \Phi \left\{ \frac{\log(v_i) - \log(V_{i,t_1}^*) - (r - \sigma_i/2)(t_3 - t_1)}{\sigma_i \sqrt{t_3 - t_1}} - \frac{\log(V_{i,t}^*) - \log(V_{i,t_1}^*) - (r - \sigma_i/2)(t - t_1)}{\sigma_i \sqrt{t_3 - t_1}} \right\} \\ &= \Phi \left\{ \sqrt{\frac{t_3 - t_1}{t_3 - t}} z_i^* - \sqrt{\frac{t - t_1}{t_3 - t_1}} \frac{\log(V_{i,t}^*) - \log(V_{i,t_1}^*) - (r - \sigma_i/2)(t - t_1)}{\sigma_i \sqrt{t - t_1}} \right\} \end{aligned}$$

where

$$z_i^* := \frac{\log(v_i) - \log(V_{i,t_1}^*) - (r - \sigma_i/2)(t_3 - t_1)}{\sigma_i \sqrt{t_3 - t_1}} \quad (i = 1, 2).$$

For $t = t_1$ we obtain $p_{i,t_1}^* = \Phi(z_i^*)$, which allows us to fix the $t = t_1$ risk-neutral default probabilities by choosing appropriate values for z_i^* . Once these values are fixed, the $t = t_2$

¹⁸For a description of the martingale-measure approach to the pricing of securities see e.g. JARROW AND TURNBULL (1996), ch. 5 and 6. Note that this approach is the state of the art for repricing loans before maturity. It does, however, not take into account problems arising from the fact that - due to asymmetric information - the bank might not be able to sell its loans to the market in $t = t_2$ at the martingale measure price. To our knowledge there is currently no risk management approach that combines risk neutral pricing and asymmetric information among market participants to reprice loans.

¹⁹See CROUHY, GALAI, AND MARK (2000), p.100.

²⁰See e.g. JARROW AND TURNBULL (1996). Of course the same is true for the asset value processes themselves. The only difference is that in this case the means for the log increments are given by $(s-t)(\mu_i - \sigma_i^2/2)$.

risk neutral default probabilities are given by

$$p_{i,t_2}^* = \Phi\left(\sqrt{\tau+1} z_i^* - \sqrt{\tau} Z_i\right)$$

where

$$Z_i := \frac{\log(V_{i,t_2}^*) - \log(V_{i,t_1}^*) - (r - \sigma_i/2)(t_2 - t_1)}{\sigma_i \sqrt{t_2 - t_1}} \quad (i = 1, 2)$$

and

$$\tau := \frac{t_2 - t_1}{t_3 - t_2}.$$

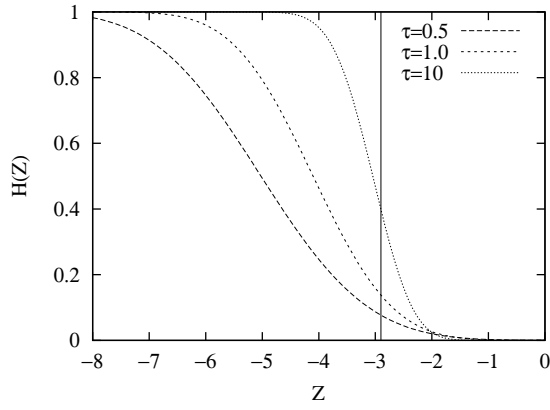


Figure 9: The function $H_{-2.9,\tau}(\cdot)$ for different values of τ .

ing that

$$\begin{aligned} \rho^{\text{mig}} &:= \text{Corr}(V_{1,t_2}^L, V_{2,t_2}^L) \\ &= \text{Corr}(p_{1,t_2}^*, p_{2,t_2}^*) \\ &= \text{Corr}\left(H_{z_1,\tau}(Z_1), H_{z_2,\tau}(Z_2)\right). \end{aligned}$$

$H_{z,\tau}$ is defined by

$$H_{z,\tau}(Z) := \Phi(\sqrt{\tau+1} z - \sqrt{\tau} Z)$$

and (Z_1, Z_2) is a standard bivariate normally distributed random vector with correlation ρ . Comparing ρ^{mig} with ρ^{def} , we observe that the functions $1\{\cdot \leq z_i\}$ are replaced by $H_{z_i,\tau}(\cdot)$ ($i = 1, 2$). Figure 9 depicts the indicator function and the function H for $z_1 = z_2 = -2.9$ (which corresponds to a risk-neutral default probability of 0.2%, i.e. a B loan) and for

Note that τ describes the relative sizes of the risk management period and the loan duration after the risk management horizon. Using the $t = t_2$ risk-neutral default probabilities, we can describe the t_2 value V_{i,t_2}^L of loan i in the following way:

$$V_{i,t_2}^L = L_i e^{-r(t_3-t_2)} \left[1 - p_{i,t_2}^* (1 - \beta)\right].$$

L_i is the principal that has to be paid back in t_3 and β is the (exogenous) recovery rate. Note that from a $t = t_1$ perspective, only the probabilities p_{1,t_2}^* and p_{2,t_2}^* are random, implying

different values of τ . It can be seen that H is a “continuous version” of the indicator function and that H converges towards the indicator form for $\tau \rightarrow \infty$. The intuition about the relationship between default probabilities and default correlations that led to the analytic results in section 2.2 should therefore still apply qualitatively. In order to confirm this reasoning we have calculated ρ^{mig} for different values of τ . The results for $\tau = 1$ are displayed in table 8.²¹

	A			B			C		
A	.11	1.05	4.81	.18	1.48	5.91	.55	2.59	6.00
	1.90	1.58	1.35	1.81	1.54	1.33	1.56	1.40	1.28
	3.79	2.62	1.89	3.42	2.45	1.82	2.48	1.99	1.67
	10.37	5.48	3.11	8.51	4.80	2.88	4.68	3.26	2.42
B				.31	2.18	7.80	.96	4.29	9.83
				1.73	1.49	1.30	1.48	1.34	1.24
				3.09	2.28	1.74	2.23	1.83	1.55
				6.99	4.17	2.63	3.82	2.77	2.10
C							3.28	12.32	25.29
							1.27	1.19	1.13
							1.60	1.42	1.27
							2.06	1.71	1.45

Table 8: Loan correlations ρ^{mig} for $\tau = 1$. The table can be read in the same way as table 2.

Note that the results do not differ substantially from the ones obtained in section 2.3. Moreover, the differences should be most pronounced for low τ , since $H_{z_i, \tau}(\cdot)$ will differ the stronger from the indicator functions the lower the values for τ are. But even for $\tau = 0.05$ (which implies that loan duration is 20 times longer than the length of the risk management period)²² no significant deviations from the values in table 8 result.

3.3 Distributional Assumptions

In the last two sections we derived the stability of our results with respect to recovery rates and the relationship between risk management horizon and loan duration. We now

²¹Note that the labeling of the different loan types in this section refers to the risk neutral default probabilities (e.g. an A loan has an initial risk neutral default probability of 0.05%).

²²The typical length of a risk management period in the banking industry is one year (see e.g. JP MORGAN (1997)).

return to the assumptions that loans mature at the end of the risk management horizon and that recovery rates are exogenous, and examine the robustness of our results with respect to the assumptions about the distribution of asset returns.

We will show that the crucial point of our assumptions is in fact not that returns are bivariate normally distributed but that an arbitrary monotonic transformation of the asset value process has this property. Let us describe the general framework we have in mind in more detail. Suppose that there is a monotonic transformation \mathcal{T} of the asset values V_{1,t_2} and V_{2,t_2} which may depend on some aspects \mathcal{H}_{i,t_1} of the asset value history up to time t_1 ,²³ so that the transformed variables $Z_{i,t_2} := \mathcal{T}(V_{i,t_2}, \mathcal{H}_{i,t_1})$ are bivariate normally distributed. The transformation has to fulfill the following two conditions:

1. Z_{1,t_2} and Z_{2,t_2} are independent of \mathcal{H}_{1,t_1} and \mathcal{H}_{2,t_1} ;
2. $\mathcal{T}(\cdot, \mathcal{H}_{i,t_1})$ is strictly increasing for all realizations of \mathcal{H}_{i,t_1} .

In this case the random variable $1\{V_{i,t_2} \leq v_i\}$ that describes the default behavior of firm i can be equivalently written as $1\{Z_{i,t_2} \leq z_i\}$ where $z_i = z_i(v_i, \mathcal{H}_{i,t_1})$ is increasing in v_i . We call the transformation \mathcal{T} the “correlation model”. Note that the model we have formulated in section 2.1 is a special case of this general framework. The transformation used in section 2.1 was

$$\mathcal{T}(V_{i,t_2}, V_{i,t_1}) := \log(V_{t_2}/V_{t_1}),$$

and hence \mathcal{H}_{i,t_1} is the σ - algebra generated by V_{i,t_1} ($i = 1, 2$).

As can be seen from the preceding arguments, the robustness of our analysis with respect to the underlying distributional assumptions boils down to the robustness of the correlation model chosen and has nothing to do with assumptions about the univariate distribution of returns. If the correlation model is a good approximation for the correlation structure, then our whole analysis applies, since arbitrary univariate distributions are supported by our model.²⁴

Moreover, we conjecture that the intuition developed in section 2.2 about the relationship between default points and default correlations should apply more generally for linear correlation models, i.e. for models where asset-return correlations are derived from a common

²³More technically, \mathcal{H}_{i,t_1} is assumed to be an arbitrary sub σ - algebra of $\sigma\left((V_{i,s})_{s \leq t_1}\right)$.

²⁴Note that, given a continuously distributed random variable R with distribution function F and given any other distribution function G , we can find a monotonic transformation of R which is distributed according to G . This transformation \mathcal{T} is given by $\mathcal{T} = G^{-1} \circ F$.

linear dependency on some independent factors.²⁵ For example, it has long been argued that stable-law distributions might provide a better description of actual returns than the normal distribution.²⁶ Moreover, since the sum of independent stable-law-distributed random variables also follows a stable law law, a straightforward specification of a linear correlation model is possible. It might therefore be insightful to test our conjecture for such random variables.

The most important empirical objection against the linear correlation model for asset returns is that correlations of large negative returns seem to be much greater than expected under bivariate normality.²⁷ Moreover, there are quite a few theoretical arguments of why this may be the case, ranging from contagion from some markets to others, joint credit constraints, to changes in market structures and practices.²⁸ In an extended version of this paper (ERLENMAIER AND GERSBACH 2001) we explore the consequences of this (potential) deviation from a linear correlations model for our results. We find that higher return correlations for low returns will tend to moderate the skewness effect or even reverse it. Whether this mechanism has a significant impact on default correlations will have to be assessed by empirical research.

4 Conclusions

In this paper we have established the structural relationship between default probabilities and default correlations. Loans with higher default probabilities will not only have higher variances (variance effect) but also higher correlations with other loans (correlation effect). Hence, the variance effect (which is an obvious consequence of higher default probabilities)

²⁵Note that multivariate normally distributed random variable are a special case of linear correlation models.

²⁶See FAMA (1970) for a discussion of the literature or RACHEV, SCHWARTZ, AND KHINDANOVA (2000) for a more recent contribution. Stable-law distributions can capture two important empirical deviation from normality (thick tails and excess kurtosis).

²⁷See LONGIN AND SOLNIK (1999). Note that LONGIN AND SOLNIK (1999) investigate *equity* returns. Of course it is not clear whether these results can be extended to asset returns. However, since asset returns are not directly observable, the results for equity returns can be seen as a first indication that similar results may be obtained for asset returns, especially since for highly rated firms, asset and equity returns should exhibit similar patterns if the value of debt does not vary too much.

²⁸Recent discussions of possible routes of contagion include DRAZEN (1998), EICHENGREEN, ROSE, AND WYPLOSZ (1996), and GERLACH AND SMETS (1995). The CGFS report (COMMITTEE ON THE GLOBAL FINANCIAL SYSTEM 1999) on the events following the Russian default in August 1998 presents a narrative account of how the effects of shocks were reinforced and spread to other markets by market practices. Concerning the literature on credit constraints see HOLMSTRÖM AND TIROLE (1997) and references therein.

is reinforced by the correlation effect. We have seen that due to these effects, portfolio standard deviation can increase substantially with higher loan default probabilities. These results have important implications for banks and regulators.

First, when determining relative prices of loans with high and loans with low default probability, banks should take into account the differences in the contribution to the overall standard deviation of their portfolio and hence to the economic capital needed to be held against the credit portfolio.

Second, during economic downturns default probabilities will increase. This will not only increase expected losses but also the standard deviation of loan portfolios. Hence the increase of required economic capital during downturns will stem from at least these two sources. We feel that this observation is important for regulators in seeking to gain a more complete picture of the consequences of macroeconomic shocks for the banking system. It will also be important for banks which attempt to hedge against fluctuations in required economic capital caused by macroeconomic risk.

Third, consider the consequences for current credit risk models.²⁹ Academics and regulators have pointed out that credit risk models should take into account the fact that default probabilities increase during economic downturns, since the banks' expected losses will increase. Our results emphasize this point, showing that not only expected losses but also default correlations and, accordingly, portfolio standard deviation will increase. Moreover, the correlation effect has different consequences for structural models on the one hand and reduced-form models on the other. The structural approach to credit events is used by Credit Portfolio Manager from KMV and (indirectly) by Credit Metrics (CM) from JP Morgan. This approach essentially employs the framework outlined in section 2.1 to derive joint default probabilities. While KMV uses a firm's stock market value and its debt structure to derive default points (a method sensitive to changes in asset values), CM uses historical rating class default frequencies to calibrate default points. Our results emphasize that adjusting the CM default points to the business cycle is important. Once these default points have been appropriately adjusted to changed default probabilities, CM will also take the variance and correlation effect into account.

Reduced-form models, on the other hand, are used by Credit Risk⁺ (CR) from the Credit Suisse Group and Credit Portfolio View from McKinsey (McK). A major building block of these models is the distribution of default rates. While BÄR (2000) has demonstrated that default rates can be predicted quite well using macroeconomic variables, thereby presenting a method of how to adjust default rate means to the current economic envi-

²⁹A review of these models is given in CROUHY, GALAI, AND MARK (2000).

ronment, it is much less clear how to adjust the standard deviation of default rates. Our results suggest that default-rate standard deviation will vary significantly throughout the business cycle and that reduced-form models should reflect such changes. How to estimate default rate variances conditional on the business cycle therefore emerges as an important empirical issue that still awaits a satisfactory answer.³⁰

We have derived our results for a fairly simple joint default model. However, we have shown that they remain robust under endogenous recovery rates, loan reevaluation at the end of the credit risk horizon and under alternative distributional assumptions for asset returns (as long as a monotonic transformation into a bivariate normal distribution exists). To gain a more complete picture, two issues have to be taken up in future research. First, rather than assuming that default can only occur at the date of the loan’s maturity, it should be more realistic to model default as an absorbing barrier to a firm’s asset value process, which can be reached at *any* point before the loan’s maturity. The robustness of our results in such a modeling framework should be investigated.

An equally important issue is whether the linear correlation model used by KMV and CM is appropriate. We have indicated that for correlation models where the return correlation is higher for low returns than suggested by the linear correlation model, the relationship between default probabilities and default correlations might even be reversed. These results suggest that a detailed empirical assessment may become necessary of whether other correlation models should enter credit risk frameworks of the structural type. In the interim, simple regime-switching correlation models could be used for stress testing.³¹

Finally, while the impact of the macroeconomic environment on expected default rates (and hence on average default probabilities) has been studied quite carefully, changes in default correlations are much more difficult to handle empirically. We have highlighted the difficulties of both modeling approaches in assessing the changes of default correlations due to macroeconomic shocks. When determining the bank capital needed to hold against a credit portfolio, it might therefore be useful to increase the default correlations calculated by the models by a “security factor”. The expected losses produced should be more stable across time.

³⁰Note that formally, McK does not need default rate distributions as input. These distributions are derived from the distributions of macroeconomic variables that are supposed to explain default rate changes. However, in order to justify that the implied default rate variances not only reflect the variances of the underlying macroeconomic variables, conditional default rate variances need to be predicted accurately. But this again rises the question of how to estimate these conditional variances.

³¹See e.g. ANG AND BEKAERT (1999).

A Proofs

In this part of the appendix we prove our main results concerning the relationship between default probabilities and default correlations. First of all note that

$$\rho^{\text{def}} = \rho^{\text{def}}(z_1, z_2, \rho) = \frac{\text{Cov}\left(1\{Z_1 \leq z_1\}, 1\{Z_2 \leq z_2\}\right)}{\sqrt{\text{Var}(1\{Z_1 \leq z_1\}) \text{Var}(1\{Z_2 \leq z_2\})}} =: \frac{D}{N}.$$

In section A.1 we derive some useful expressions for D and N and for their first and second derivatives. This will be helpful in proving our main results in section A.2. The following notation will be used throughout the appendix: Φ^ρ for the distribution function of (Z_1, Z_2) , $\varphi(\cdot)$ and $\Phi(\cdot)$ for the one-dimensional standard normal density and distribution function respectively. Moreover, throughout the appendix we will use subscripts to indicate partial derivatives. D_1 for example will denote the partial derivative of D with respect to z_1 and D_{12} is a short form for $\partial^2 D / \partial z_1 \partial z_2$. Finally note that z_1 always indicates the default point associated with the lower default probability ($z_1 \leq z_2$).

A.1 The Building Blocks D and N

Since $D = \Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2)$, we will first derive a formula for $\Phi^\rho(z_1, z_2)$. Note that (Z_1, Z_2) has the same distribution as (Z_1, \tilde{Z}_2) where $\tilde{Z}_2 = \rho Z_1 + \epsilon$, and ϵ is distributed according to $\mathcal{N}(0, 1 - \rho^2)$ and independent of Z_1 .³² Hence,

$$\Phi^\rho(z_1, z_2) = \int_{-\infty}^{z_1} \Phi\left(\frac{z_2 - \rho\zeta}{\sqrt{1 - \rho^2}}\right) \varphi(\zeta) d\zeta. \quad (6)$$

Note that by the theorem of Lebesgue, formula (6) also applies for the case $\rho = \pm 1$. In this case the limit of the right-hand side of the equation is taken. In the next sections we will take first and second derivatives of the right-hand side of equation (6). Again by the theorem of Lebesgue, the respective derivatives for $\rho = \pm 1$ can be obtained by taking the limit of the formulas derived for $|\rho| < 1$.

³² $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2 .

A.1.1 The Functions D and N

From equation (6) we obtain

$$D = \left[\int_{-\infty}^{z_1} \Phi\left(\frac{z_2 - \rho\zeta}{\sqrt{1-\rho^2}}\right) \varphi(\zeta) d\zeta \right] - \Phi(z_1)\Phi(z_2). \quad (7)$$

The function N can be represented as

$$N = \sqrt{\Phi(z_1)(1-\Phi(z_1))} \sqrt{\Phi(z_2)(1-\Phi(z_2))} \quad (8)$$

and³³

$$D|_{z_1=z_2=0} = \frac{1}{2\pi} \arcsin(\rho) \text{ and } N|_{z_1=z_2=0} = \frac{1}{4}.$$

A.1.2 The First Derivatives of D and N

$$D_1 = \varphi(z_1) \left[\Phi\left(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\right) - \Phi(z_2) \right] \quad (9)$$

$$N_1 = \frac{1}{2} \left[\Phi(z_2)(1-\Phi(z_2)) \right]^{1/2} \left[\Phi(z_1)(1-\Phi(z_1)) \right]^{-1/2} \varphi(z_1)(1-2\Phi(z_1)). \quad (10)$$

Moreover,

$$D_1|_{z_1=z_2=0} = 0 \text{ and } N_1|_{z_1=z_2=0} = 0.$$

A.1.3 The Second Derivatives of D and N

$$D_{11} = -z_1 \varphi(z_1) \left[\Phi\left(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\right) - \Phi(z_2) \right] - \varphi(z_1) \frac{\rho}{\sqrt{1-\rho^2}} \varphi\left(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\right)$$

$$D_{12} = \varphi(z_1) \left[\frac{1}{\sqrt{1-\rho^2}} \varphi\left(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\right) - \varphi(z_2) \right]$$

$$\begin{aligned} N_{11} = & \frac{1}{2} \left[\Phi(z_2)(1-\Phi(z_2)) \right]^{1/2} \left\{ -\frac{1}{2} \left[\Phi(z_1)(1-\Phi(z_1)) \right]^{-3/2} \left[\varphi(z_1)(1-2\Phi(z_1)) \right]^2 \right. \\ & \left. + \left[\Phi(z_1)(1-\Phi(z_1)) \right]^{-1/2} \left[-z_1 \varphi(z_1)(1-2\Phi(z_1)) - 2\varphi(z_1)^2 \right] \right\} \end{aligned}$$

³³The proof for the arcsin representation of D is given in GERSBACH AND LIPPONER (2000).

$$N_{12} = \frac{1}{4} \left[\Phi(z_2) (1 - \Phi(z_2)) \right]^{-1/2} \varphi(z_2) (1 - 2\Phi(z_2)) \left[\Phi(z_1) (1 - \Phi(z_1)) \right]^{-1/2} \varphi(z_1) (1 - 2\Phi(z_1)).$$

Moreover,

$$D_{11} \big|_{z_1=z_2=0} = -\frac{\rho}{2\pi\sqrt{1-\rho^2}} \text{ and } D_{12} \big|_{z_1=z_2=0} = \frac{1}{2\pi} \left(\frac{1}{\sqrt{1-\rho^2}} - 1 \right),$$

$$N_{11} \big|_{z_1=z_2=0} = -\varphi(0)^2 = -\frac{1}{2\pi} \text{ and } N_{12} \big|_{z_1=z_2=0} = 0.$$

A.1.4 Derivatives of Fractions

The following general formulas provide the link between the derivatives of D and N and those of ρ^{def} . We define

$$F := F(z_1, z_2) = \frac{D(z_1, z_2)}{N(z_1, z_2)}.$$

and observe that

$$F_1 = \frac{D_1 N - D N_1}{N^2},$$

$$F_{11} = \frac{D_{11} N - D_1 N_1}{N^2} - \frac{(D_1 N_1 + D N_{11}) N^2 - D N_1 2 N N_1}{N^4} \quad (11)$$

$$= \frac{D_{11} N - D N_{11} - 2 D_1 N_1}{N^2} + \frac{2 D N_1^2}{N^3} \quad (12)$$

and

$$F_{12} = \frac{D_{12} N - D_1 N_2}{N^2} - \frac{(D_2 N_1 + D N_{12}) N^2 - D N_1 2 N N_2}{N^4} \quad (13)$$

$$= \frac{D_{12} N - D N_{12} - D_1 N_2 - D_2 N_1}{N^2} + \frac{2 D N_1 N_2}{N^3}. \quad (14)$$

A.2 Proofs

In this section we present the proofs of propositions 1, 2 and 4, all of which cover the case $z_1, z_2 < 0$. The proof of proposition 3, which is concerned with the boundary cases $z_1 = z_2 = 0$ and $z_1 = z_2 \rightarrow \infty$ is given in an extended version of this paper (ERLENMAIER AND GERSBACH 2001).

To derive our results we first of all calculate the derivative of ρ^{def} with respect to z_1 and z_2 .

Lemma 1

If $|\rho| < 1$, then

$$\rho_1^{\text{def}} = \frac{\varphi(z_1)}{N} \left\{ \left(\Phi\left(\frac{z_2 - \rho z_1}{\sqrt{1 - \rho^2}}\right) - \Phi(z_2) \right) - \psi(z_1) \left(\Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2) \right) \right\} \quad (15)$$

$$\rho_2^{\text{def}} = \frac{\varphi(z_2)}{N} \left\{ \left(\Phi\left(\frac{z_1 - \rho z_2}{\sqrt{1 - \rho^2}}\right) - \Phi(z_1) \right) - \psi(z_2) \left(\Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2) \right) \right\} \quad (16)$$

where

$$\psi(z) := \frac{1 - 2\Phi(z)}{2\Phi(z)(1 - \Phi(z))}.$$

For $\rho = 1$ we obtain

$$\begin{aligned} \rho_1^{\text{def}} &= \frac{\varphi(z_1)(1 - \Phi(z_2))}{2N(1 - \Phi(z_1))} \\ \rho_2^{\text{def}} &= -\frac{\varphi(z_2)\Phi(z_1)}{2N\Phi(z_2)}. \end{aligned}$$

Remark 1

The formulas for $\rho = 1$ can be derived as limits of the formulas (15) and (16) for $\rho \rightarrow 1$.

Corollary 1

If

$$\rho > \frac{2\lambda}{1 + \lambda^2},$$

then $\rho_2^{\text{def}} < 0$.³⁴

Proof of corollary 1.

Use equation (16), and note that $\Phi^\rho(z_1, z_2) > \Phi(z_1)\Phi(z_2)$ ³⁵ and that $\psi(z) > 0$ if $z < 0$.

³⁴Remember that $\lambda = z_2/z_1$.

³⁵This inequality follows from the more general inequality, $\mathbb{E}(f(Z)g(Z)) \geq \mathbb{E}(f(X))\mathbb{E}(g(Z))$, which applies for arbitrary real-valued random variables Z and monotonically increasing functions f and g (see e.g. HARDY, LITTLEWOOD, AND PÓLYA (1991)). In our case we can choose $f(z) := 1\{z \leq z_1\}$ and $g(z) := \Phi((z_2 - \rho z)/(\sqrt{1 - \rho^2}))$. Note also that in this case the strict inequality applies, which can be seen by examining the proof in HARDY, LITTLEWOOD, AND PÓLYA (1991). We thank Lutz Duembgen for this suggestion.

Hence, $\rho_2^{\text{def}} < 0$ if

$$\frac{z_1 - \rho z_2}{\sqrt{1 - \rho^2}} < z_1,$$

which is equivalent to $(1 - \rho\lambda)^2 > (1 - \rho^2)$.³⁶

□

Proof of lemma 1.

First of all note that by the theorem of Lebesgue the formulas for $\rho = 1$ can be derived as limits of the respective formulas for $|\rho| < 1$ and that by symmetry $\rho_2^{\text{def}}(z_1, z_2, \rho) = \rho_1^{\text{def}}(z_2, z_1, \rho)$. It is therefore sufficient to derive the formula for ρ_1^{def} if $|\rho| < 1$. But this formula follows directly from

$$\rho_1^{\text{def}} = \frac{D_1}{N} - \frac{DN_1}{N^2},$$

equations (8) - (10) and the fact that $D = \Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2)$.

□

To derive our main results, we draw on the fact that ρ_1^{def} and ρ_2^{def} are increasing (decreasing) functions of ρ in certain areas. Therefore, we now examine how ρ_1^{def} and ρ_2^{def} depend on ρ . The derivatives of ρ_1^{def} and ρ_2^{def} with respect to ρ will be denoted by $\rho_{1,\rho}^{\text{def}}$ and $\rho_{2,\rho}^{\text{def}}$ respectively.

Lemma 2

$$\begin{aligned}\rho_{1,\rho}^{\text{def}} &= \xi \left\{ \frac{\rho z_2 - z_1}{1 - \rho^2} - \psi(z_1)\varphi(z_1) \right\} \\ \rho_{2,\rho}^{\text{def}} &= \xi \left\{ \frac{\rho z_1 - z_2}{1 - \rho^2} - \psi(z_2)\varphi(z_2) \right\}\end{aligned}$$

where

$$\xi := \frac{\varphi(z_1)\varphi\left((z_2 - \rho z_1)/\sqrt{1 - \rho^2}\right)}{N\sqrt{1 - \rho^2}}.$$

Hence

$$\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}} = \xi \left\{ \frac{-(z_1 + z_2)}{1 + \rho} - \psi(z_1)\varphi(z_1) - \psi(z_2)\varphi(z_2) \right\}.$$

³⁶Note that $z_1 < 0$.

Proof.

First we calculate the derivative of Φ^ρ with respect to ρ . In order to do so we use that

$$\frac{\partial}{\partial \rho} \left(\frac{z_2 - \rho \zeta}{\sqrt{1 - \rho^2}} \right) = \frac{\rho z_2 - \zeta}{(1 - \rho^2)^{3/2}}. \quad (17)$$

Hence

$$\begin{aligned} \frac{\partial}{\partial \rho} \Phi^\rho(z_1, z_2) &= \int_{-\infty}^{z_1} \varphi(\zeta_1) \frac{\partial}{\partial \rho} \int_{-\infty}^{\frac{z_2 - \rho \zeta_1}{\sqrt{1 - \rho^2}}} \varphi(\zeta_2) d\zeta_1 d\zeta_2 \\ &= \frac{1}{(1 - \rho^2)^{3/2}} \int_{-\infty}^{z_1} (\rho z_2 - \zeta_1) \varphi(\zeta_1) \varphi\left(\frac{z_2 - \rho \zeta_1}{\sqrt{1 - \rho^2}}\right) d\zeta_1. \end{aligned}$$

But since $\sqrt{2\pi}\varphi(\zeta) = e^{-\zeta^2/2}$, we obtain

$$2\pi\varphi(\zeta)\varphi\left(\frac{z_2 - \rho \zeta}{\sqrt{1 - \rho^2}}\right) = \exp\left\{-\frac{\zeta^2(1 - \rho^2) + z_2^2 - 2z_2\rho\zeta + \rho^2\zeta^2}{2(1 - \rho^2)}\right\} =: \exp(T).$$

Moreover, T can be written as

$$\begin{aligned} T &= -\frac{\zeta^2 + z_2^2 - 2z_2\rho\zeta}{2(1 - \rho^2)} \\ &= -\frac{(\zeta - z_2\rho)^2 + z_2^2(1 - \rho^2)}{2(1 - \rho^2)} \\ &= -\frac{(\zeta - z_2\rho)^2}{2(1 - \rho^2)} - \frac{z_2^2}{2}. \end{aligned}$$

Hence

$$\varphi(\zeta)\varphi\left(\frac{z_2 - \rho \zeta}{\sqrt{1 - \rho^2}}\right) = \varphi\left(\frac{\zeta - z_2\rho}{\sqrt{1 - \rho^2}}\right)\varphi(z_2) \quad (18)$$

and therefore

$$\frac{\partial}{\partial \rho} \Phi^\rho = \frac{\varphi(z_2)}{(1 - \rho^2)^{3/2}} \int_{-\infty}^{z_1} (\rho z_2 - \zeta) \varphi\left(\frac{\zeta - z_2\rho}{\sqrt{1 - \rho^2}}\right) d\zeta.$$

But since the integrand can be integrated analytically to

$$(1 - \rho^2)\varphi\left(\frac{\zeta - z_2\rho}{\sqrt{1 - \rho^2}}\right),$$

we obtain

$$\frac{\partial}{\partial \rho} \Phi^\rho = \frac{\varphi(z_2)}{\sqrt{1 - \rho^2}} \varphi\left(\frac{z_1 - \rho z_2}{\sqrt{1 - \rho^2}}\right).$$

We are now able to calculate the derivative of ρ_1^{def} with respect to ρ . Recall that

$$\rho_1^{\text{def}} = \frac{\varphi(z_1)}{N} \left\{ \left(\Phi\left(\frac{z_2 - \rho z_1}{\sqrt{1 - \rho^2}}\right) - \Phi(z_2) \right) - \psi(z_1) \left(\Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2) \right) \right\}.$$

Therefore, using equations (17) and (18), we find:

$$\begin{aligned} \rho_{1,\rho}^{\text{def}} &= \frac{\varphi(z_1)}{N\sqrt{1 - \rho^2}} \left\{ \frac{\rho z_2 - z_1}{1 - \rho^2} \varphi\left(\frac{z_2 - \rho z_1}{\sqrt{1 - \rho^2}}\right) - \psi(z_1)\varphi(z_2)\varphi\left(\frac{z_1 - \rho z_2}{\sqrt{1 - \rho^2}}\right) \right\} \\ &= \frac{\varphi(z_1)\varphi\left((z_2 - \rho z_1)/\sqrt{1 - \rho^2}\right)}{N\sqrt{1 - \rho^2}} \left\{ \frac{\rho z_2 - z_1}{1 - \rho^2} - \psi(z_1)\varphi(z_1) \right\}. \end{aligned}$$

Moreover, since $\rho_2^{\text{def}}(z_1, z_2, \rho) = \rho_2^{\text{def}}(z_2, z_1, \rho)$, a formula for $\rho_{2,\rho}^{\text{def}}$ is obtained by interchanging the roles of z_1 and z_2 in the formula for $\rho_{1,\rho}^{\text{def}}$. The expression for $\rho_{2,\rho}^{\text{def}}$ given in the lemma can then be derived by using that - according to equation (18) - the following statement holds:

$$\varphi(z_2)\varphi\left(\frac{z_1 - \rho z_2}{\sqrt{1 - \rho^2}}\right) = \varphi(z_1)\varphi\left(\frac{z_2 - \rho z_1}{\sqrt{1 - \rho^2}}\right).$$

□

Lemma 2 provides the key to our main results. Note that according to lemma 2 the following holds:

$$\rho_{1,\rho}^{\text{def}} = \xi H\left(g_\lambda(\rho), z_1\right) \tag{19}$$

$$\rho_{2,\rho}^{\text{def}} = \xi H\left(g_{1/\lambda}(\rho), z_2\right) \tag{20}$$

$$\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}} = \xi \left[H\left(\frac{1}{1 + \rho}, z_1\right) + H\left(\frac{1}{1 + \rho}, z_2\right) \right] \tag{21}$$

where

$$\begin{aligned} H(\alpha, z) &:= -\alpha z - \psi(z)\varphi(z) = -\alpha z - \frac{(1 - 2\Phi(z))\varphi(z)}{2\Phi(z)(1 - \Phi(z))}, \\ g_\lambda(\rho) &:= \frac{1 - \rho\lambda}{1 - \rho^2}. \end{aligned}$$

Since $\xi > 0$, the signs of $\rho_{1,\rho}^{\text{def}}$, $\rho_{2,\rho}^{\text{def}}$ and $\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}}$ are equal to the signs of the right-hand sides of equations (19), (20) and (21) respectively. To prepare for the proofs, we will therefore - in the next two lemmata - derive some properties of the functions H and g .

Lemma 3

Suppose that $z < 0$. Then the following statements hold.

(i) If $\alpha > 0.5$, then $H(\alpha, z) > 0$ for all

$$z < -\sqrt{\frac{4\alpha - 1}{2\alpha - 1}} =: \bar{z} = \bar{z}(\alpha).$$

(ii) $H(\alpha, z) > 0$ for all $\alpha \geq 0.65$.

(iii) If $10^{-6} \leq \Phi(z) \leq 0.46$, then $H(\alpha, z) < 0$ for all $\alpha \leq 0.52$.

Proof.

(i) First we observe that

$$1 - \frac{\Phi(z)}{1 - \Phi(z)}$$

is strictly decreasing in z and converges to 1 for $z \rightarrow -\infty$. Hence

$$H(\alpha, z) \geq -\alpha z - \frac{\varphi(z)}{2\Phi(z)} =: h_1(z).$$

Note that the anti-derivative of h_1 is given by

$$\begin{aligned} \int h_1 &= -\frac{1}{2} \left[\alpha z^2 + \log(\Phi(z)) \right] \\ &= \frac{1}{2} \log\left(\frac{e^{-\alpha z^2}}{\Phi(z)}\right). \end{aligned}$$

We will show that $\tilde{h}_1(z) := e^{-\alpha z^2}/\Phi(z)$ is strictly increasing in z for $z < \bar{z}$, which implies that $\int h_1$ is strictly increasing in z ; this in turn leads to $h_1(z) > 0$ for $z < \bar{z}$. To show that $\tilde{h}_1(\cdot)$ is strictly increasing in z , note that

$$\begin{aligned} \tilde{h}_1'(z) &= \frac{-e^{-\alpha z^2} [2\alpha z \Phi(z) + \varphi(z)]}{\Phi(z)^2} \\ &=: -\frac{e^{-\alpha z^2}}{\Phi(z)^2} \tilde{h}_2(z). \end{aligned}$$

Moreover,

$$\tilde{h}_2'(z) = (2\alpha - 1)z\varphi(z) + 2\alpha\Phi(z)$$

and $\tilde{h}_2''(z) = \varphi(z)[(4\alpha - 1) - (2\alpha - 1)z^2]$. Hence, if $\alpha > 0.5$, then $\tilde{h}_2''(z) < 0$ if $z < \bar{z}$ and therefore $\tilde{h}_2'(z) < \lim_{z \rightarrow -\infty} \tilde{h}_2'(z) = 0$. Therefore, $\tilde{h}_2(z) < \lim_{z \rightarrow -\infty} \tilde{h}_2(z) = 0$, implying that $\tilde{h}_1'(z) > 0$ for $z < \bar{z}$.

(ii) Note that H is increasing in α and that it is therefore sufficient to show that $H(0.65, z) > 0$ for all $z < 0$. Moreover, according to (i), $H(0.65, z) > 0$ for $z < \bar{z}(0.65)$. But since $\bar{z}(0.65) \geq -2.31$, we only need to consider the interval $[-2.31, 0)$. The proof will proceed in two steps. For $z \in [-2.31, 0.1]$ we have relied on numerical methods. These methods use an approximation of the second term of H , which is no longer feasible when z approaches zero. The area $[0.1, 0)$ is therefore treated in the second step.

For the numerical analysis we have chosen a grid of 40,000 equidistant points and have evaluated the function

$$\psi(z)\varphi(z) = \frac{(1 - 2\Phi(z))\varphi(z)}{2\Phi(z)(1 - \Phi(z))}.$$

The values on the grid have been approximated by standard numerical integration (for $\Phi(z)$) and Taylor series expansion (for $\varphi(z)$). The values between grid points have been approximated from above by the mean value theorem of differential calculus, using that

$$\left| \frac{d}{dz}(\psi(z)\varphi(z)) \right| \leq \frac{\varphi(z_2)}{2\Phi(z_1)} \left\{ \frac{|z_1| + 2\varphi(z_2)}{1 - \Phi(z_2)} + \frac{\varphi(z_2)}{\Phi(z_1)} \right\}$$

for $z \in [z_1, z_2]$.³⁷

Now we turn to the second step of the proof, where we show that $H(0.65, z) > 0$ for $z \in [-0.1, 0)$. We use the fact that $\tilde{h}_3(z) := \varphi(z)/\Phi(z)$ is strictly decreasing in z for $z < 0$.³⁸ This implies that if $z \geq z_1$ then

$$H(z) \geq -\alpha z - \frac{c}{2} \left[1 - \frac{\Phi(z)}{1 - \Phi(z)} \right] =: h_2(z)$$

³⁷We denote the nominator of $2\psi(z)\varphi(z)$ by f_1 and the denominator by f_2 . Note that $f_2' = f_1$ and that we can therefore write the derivative of $2\psi(z)\varphi(z)$ as $(f_1'/f_2) - (f_1/f_2)^2$. But $f_2(z) \geq \Phi(z_1)(1 - \Phi(z_2))$ and

$$f_1'(z) = -z(1 - 2\Phi(z))\varphi(z) - 2\varphi(z)^2.$$

Finally, $f_1/f_2 \leq \varphi(z_2)/\Phi(z_1)$.

³⁸Note that

$$\tilde{h}_3'(z) = -\frac{\varphi(z)}{\Phi(z)^2} [z\Phi(z) + \varphi(z)] =: -\frac{\varphi(z)}{\Phi(z)^2} \tilde{h}_4(z).$$

$\tilde{h}_4(z)$ is strictly increasing in z since $\tilde{h}_4'(z) = z\varphi(z) + \Phi(z) - z\varphi(z) = \Phi(z) > 0$. Therefore $\tilde{h}_4(z) > \lim_{z \rightarrow -\infty} \tilde{h}_4(z) = 0$ and hence $\tilde{h}_3'(z) < 0$.

where $c := \varphi(z_1)/\Phi(z_1)$. On the other hand,

$$\begin{aligned} h'_2(z) &= -\alpha + \frac{c}{2(1 - \Phi(z))^2} \left[\varphi(z)(1 - \Phi(z)) + \varphi(z)\Phi(z) \right] \\ &= -\alpha + \frac{c\varphi(z)}{2(1 - \Phi(z))^2} \end{aligned}$$

and hence we obtain that $h'_2(z) \leq -\alpha + cd$ for $z \leq z_2$ where

$$d := \frac{\varphi(z_2)}{2(1 - \Phi(z_2))^2}.$$

For $z_1 = -0.1$ and $z_2 = 0$ we have $c \leq 0.74$ and $d \leq 0.8$. Hence $h'_2(z) < 0$ if $\alpha \geq 0.65$ and therefore $h_2(z) > h_2(0) = 0$.

(iii) Since H is increasing in α , it is sufficient to show that $H(0.52, z) < 0$ for all $\Phi^{-1}(10^{-6}) \leq z \leq \Phi^{-1}(0.46)$. This result has been derived by the same numerical method as used in the proof of (ii).

□

Lemma 4

(i) If $\lambda > 1$, then $g_\lambda(\cdot)$ is strictly decreasing in ρ .

(ii) If $0 < \lambda \leq 1$, then $g_\lambda(\cdot)$ is strictly decreasing for $0 < \rho < \rho_{\min}$ and strictly increasing for $\rho_{\min} < \rho < 1$ where

$$\rho_{\min} := \frac{1 - \sqrt{1 - \lambda^2}}{\lambda}.$$

(iii) The equation $g_\lambda(\rho) = c$ has the following solutions:

$$\rho_{1/2}(c) = \frac{1}{2c} \left\{ \lambda \mp \sqrt{\lambda^2 - 4c(1 - c)} \right\}.$$

(iv) For all $\lambda > 0$ we have $g_\lambda(\rho) \geq .64$ for $0 < \rho < \rho_1(.64)$. Moreover,

$$\rho_1(.64) = \frac{25}{32} \left\{ \lambda - \sqrt{\lambda^2 + 24/25} \right\} =: \rho_+(\lambda).$$

Proof.

Note that

$$g'_\lambda(\rho) = -\frac{\lambda\rho^2 - 2\rho + \lambda}{(1 - \rho^2)^2}.$$

All results can be obtained by straightforward calculations.

□

Proof of proposition 1.

First of all note that by symmetry the statement for $z_1, z_2 > 0$ can be deduced from the statement for $z_1, z_2 < 0$. If $z_1, z_2 > 0$, we can use that

$$1\{Z_i \leq z_i\} = 1 - 1\{Z_i > z_i\} \quad (i = 1, 2)$$

and that

$$\text{Corr}\left([1 - 1\{Z_1 > z_1\}], [1 - 1\{Z_2 > z_2\}]\right) = \text{Corr}\left(1\{Z_1 > z_1\}, 1\{Z_2 > z_2\}\right).$$

But by the symmetry of the bivariate normal distribution we have³⁹

$$\left(1\{Z_1 > z_1\}, 1\{Z_2 > z_2\}\right) \stackrel{\mathcal{D}}{=} \left(1\{Z_1 \leq -z_1\}, 1\{Z_2 \leq -z_2\}\right)$$

and thus it is sufficient to consider the case $z_1, z_2 < 0$ for the rest of the proof.

Recall from lemma 2 that

$$\frac{\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}}}{\xi} = -\frac{z_1 + z_2}{1 + \rho} - \psi(z_1)\varphi(z_1) - \psi(z_2)\varphi(z_2) =: h_1(\rho),$$

which implies that h_1 determines the sign of $\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}}$. First of all note that h_1 is strictly decreasing in ρ . Hence, h_1 is either always higher or always lower than zero or there is a $\rho_0 = \rho_0(z_1, z_2)$ such that h_1 is positive for $\rho < \rho_0$ and negative for $\rho > \rho_0$. Concluding that the same statement applies for $\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}}$, we obtain that

$$\rho_1^{\text{def}}(\rho) + \rho_2^{\text{def}}(\rho) > \min\left\{[\rho_1^{\text{def}}(0) + \rho_2^{\text{def}}(0)], [\rho_1^{\text{def}}(1) + \rho_2^{\text{def}}(1)]\right\}$$

for $0 < \rho < 1$. But $\rho_1^{\text{def}}(0) = \rho_2^{\text{def}}(0) = 0$ and it remains to show that $\rho_1^{\text{def}}(1) + \rho_2^{\text{def}}(1) > 0$.

³⁹By $X \stackrel{\mathcal{D}}{=} Y$ we mean that X and Y have the same distribution.

Recall from lemma 1 that for $\rho = 1$

$$\begin{aligned}\rho_1^{\text{def}} &= \frac{\varphi(z_1)(1 - \Phi(z_2))}{2N(1 - \Phi(z_1))} \\ \rho_2^{\text{def}} &= -\frac{\varphi(z_2)\Phi(z_1)}{2N\Phi(z_2)}.\end{aligned}$$

Hence:

$$\begin{aligned}[\rho_1^{\text{def}} + \rho_2^{\text{def}}]\{2N(1 - \Phi(z_1))\Phi(z_2)\} &= \varphi(z_1)\Phi(z_2)(1 - \Phi(z_2)) - \varphi(z_2)\Phi(z_1)(1 - \Phi(z_1)) \\ &=: h_2(z_1)\end{aligned}$$

and

$$h'_2(z_1) = \varphi(z_1)\left\{-z_1\Phi(z_2)(1 - \Phi(z_2)) - \varphi(z_2)(1 - 2\Phi(z_1))\right\}.$$

The following three statements imply that $h_2(z_1) > 0$ for all $-\infty < z_1 < z_2$, which in turn proves that $\rho_1^{\text{def}} + \rho_2^{\text{def}} > 0$.

1. There is a \bar{z} ($-\infty \leq \bar{z} \leq z_2$) such that $h'_2(z_1) > 0$ for $z_1 < \bar{z}$ and $h'_2(z_1) < 0$ for $z_1 > \bar{z}$.
2. $\lim_{z_1 \rightarrow -\infty} h_2(z_1) = 0$.
3. $h_2(z_2) = 0$.

Statements 2 and 3 are obvious. To prove statement 1 it is sufficient to show that

$$h_3(z_1) := -z_1\Phi(z_2)(1 - \Phi(z_2)) - \varphi(z_2)(1 - 2\Phi(z_1))$$

has the proposed property. But

$$h'_3(z_1) := -\Phi(z_2)(1 - \Phi(z_2)) + 2\varphi(z_2)\varphi(z_1)$$

and statement 1 follows from

1. $h_3(z_2) < 0$ and
2. there is a \tilde{z} ($-\infty < \tilde{z} \leq z_2$) so that $h'_3(z_1) < 0$ for all $z_1 < \tilde{z}$ and $h'_3(z_1) > 0$ for all $z_1 > \tilde{z}$.

Point 2 is obvious and 1 is equivalent to

$$-\frac{z_2}{2} - \frac{\varphi(z_2)(1 - 2\Phi(z_2))}{2\Phi(z_2)(1 - \Phi(z_2))} = H\left(\frac{1}{2}, z_2\right) \stackrel{!}{<} 0,$$

which is true according to lemma 3.

□

Proof of proposition 2.

The second part of statement (ii) has already been derived as corollary to lemma 1. Concerning the other statements, recall that, according to lemma 2, $\rho_{1,\rho}^{\text{def}}(\rho) > 0$ as long as $H(g_\lambda(\rho), z_1) > 0$, and that $\rho_{2,\rho}^{\text{def}}(\rho) > 0$ as long as $H(g_{1/\lambda}(\rho), z_2) > 0$. By lemma 3 this is the case if $g_\lambda(\rho) \geq .64$ and $g_{1/\lambda}(\rho) \geq .64$ respectively. But according to lemma 4, the latter conditions are fulfilled for all $\rho < \rho_+(\lambda)$ and $\rho < \rho_+(1/\lambda)$ respectively. For ρ in these areas we therefore obtain that $\rho_1^{\text{def}}(\rho) > \rho_1^{\text{def}}(0) = 0$.

□

Proof of proposition 4.

(i) Recall that the sign of $\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}}$ is the same as the sign of

$$H\left(\frac{1}{1+\rho}, z_1\right) + H\left(\frac{1}{1+\rho}, z_2\right).$$

Obviously, $1/(1+\rho)$ is decreasing in ρ which implies the existence of $\bar{\rho}_H$. Moreover, if both summands are positive, which is the case for $0.64 \cdot (1+\rho) < 1$, $\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}}$ will also be positive. The opposite is true if they are negative, which is the case for $0.52 \cdot (1+\rho) > 1$. These conditions translate into $\rho < 0.54$ and $\rho > 0.89$ respectively.

(ii) By lemma 2, the sign of $\rho_{2,\rho}^{\text{def}}$ is the same as that of $H(g_{1/\lambda}(\rho), z_2)$. But from lemma 4 we know that $g_{1/\lambda}(\cdot)$ is strictly decreasing for $\lambda \leq 1$, which implies the existence of $\bar{\rho}_L$. Moreover, since $g_1(\rho) = 1/(1+\rho)$ and $g_{1/\lambda}(\rho) < g_1(\rho)$ for $\lambda < 1$, we obtain that $\bar{\rho}_L < \bar{\rho}_H$.

□

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