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**Thünen-Series of Applied Economic Theory**  
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Working Paper No. 113

**Safety-First Portfolio Optimization:**  
**Fixed versus Random Target**

by

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# Safety-First Portfolio Optimization: Fixed versus Random Target<sup>\*</sup>

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## Abstract

Safety-first portfolio optimization is concerned with maximizing the expected portfolio return subject to a safety-first constraint, which is defined as the probability of failing to achieve a specified target. Commonly the target is assumed to be fixed, which, however, leads to significant conceptual disadvantages, e.g. when an actively managed portfolio seeks to track or outperform a random benchmark, such as a stock index. In this paper we consider both a fixed and a random target strategy and give an answer to the question which strategy is better off. In detail, we compare optimal expected returns of both strategies under the normal assumption and obtain following results: The random target strategy outperforms the fixed target strategy if the portfolio return and the random target are positively correlated and riskless investing is prohibited, whereas the fixed target strategy outperforms the random target strategy if the portfolio return and the random target are non-positively correlated. Furthermore, we show that these results also hold when relaxing the normal assumption.

*Keywords:* safety-first, portfolio optimization, random target, benchmarking

*JEL Classification:* G11

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# 1 Introduction

Portfolio optimization under the safety-first criterion is concerned with maximizing the expected portfolio return, while, simultaneously, the probability of failing to achieve a specified (fixed) target must fall below a critical level and has its origins in the early papers of Roy (1952), Telser (1955) and Kataoka (1963). The safety-first risk measure is commonly expressed by a probability statement as  $P(Z < T) \leq \alpha$  where  $Z$  is a random variable, e.g. portfolio return,  $T$  is a fixed target, e.g. a minimum desired portfolio return and  $\alpha$  is a critical level on the probability of failing to achieve the target.

A comparative advantage of the safety-first criterion over deviation risk measures, such as the variance, is its consistency with the way investors actually perceive risk (see for example Atwood et al., 1988; Harlow, 1991; Brogan and Stidham, 2005). Empirically, this was shown for example by Lopes (1987), Kroll et al. (1988), DeBondt (1998) and Neugebauer (2008). Thus, since its first formalization by Shefrin and Statman (2000) the safety-first criterion has played a key role in behavioral portfolio theory, which studies the asset allocation with respect to an appropriate risk perception and other behavioral considerations. In a recent paper, Das et al. transformed the utility based interpretation of mean-variance portfolio theory to the more appealing target based interpretation of safety-first portfolio theory under the assumption of normal distributed asset returns. More generally, Kalin and Zagst (1999) showed the equivalence of mean-variance and safety-first portfolio theory for a wide class of probability distributions with shift and scale parameter.

What all the above cited papers have in common is the assumption of a fixed target  $T$ , which, however, leads to significant conceptual disadvantages: Suppose an investment funds which seeks to achieve a fixed return  $T$  for the next period. According to this target the funds manager purchases and sells assets. What happen when the market return within the next period is greater than  $T$  though? The funds performed rather poorly. This situation could have been avoided if the manager had reallocated the assets according to the expected performance of the market, which is common practice in passive portfolio management. Or, suppose a funds which seeks to outperform the market (active portfolio management), i.e. the target for the next investment period is the sum of the expected market performance and some extra return.<sup>1</sup> Or, more generally, from an individual perspective the target may not even be known. Many individuals have the target of “being successful”, but only a very few know precisely which selection of money, leisure time, culture etc. must be attained to achieve this target (Bordley and LiCalzi, 2000). Hence, in many applications of the safety-first framework an unknown or random target,  $T$ , seems quite reasonable. But what is *ceteris paribus* better, a fixed or a random target? As the safety-first model calculates an optimal return, the question is, which target choice generates greater returns? In this paper we suggest a first answer to that question.

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<sup>1</sup>For financial risk management with benchmarking see for example Basak et al. (2006), Browne (2000) and Gaivoronski et al. (2005).

In detail, we assume normal distributed asset returns. Then, we know for example from Kalin and Zagst (1999) that the (probabilistic) safety-first risk measure can easily be transformed in a deterministic risk measure in terms of standard deviation. In section 2, we use this result to transform the safety-first portfolio model in an equivalent deterministic version, which is general enough to consider both, fixed and random targets. In section 3, we compare optimal expected portfolio returns of the fixed and random target strategy. We obtain following results: The random target strategy outperforms the fixed target strategy if the portfolio return and the random target are positively correlated and riskless investing is prohibited, whereas the fixed target strategy outperforms the random target strategy if the portfolio return and the random target are not positively correlated. The normal assumption is, indeed, common practice in the financial literature, but it is questionable whether return distributions are normal in general. Hence, if the return distribution is unknown, but good estimates of the mean and the variance can be calculated, then Chebyshev's inequality provides an upper bound for the safety-first criterion. In section 4, we show that all results from section 3 hold when employing Chebyshev's inequality. Section 5 offers a discussion and concludes the paper.

## 2 Safety-First Portfolio Optimization with Normal Distributed Asset Returns

Consider an investment universe of  $n$  different financial assets with  $\mathbf{R} := (R_1, \dots, R_n)^T$  presenting the vector of random asset returns. A portfolio where short sales are prohibited is defined as a vector  $\mathbf{x} \in [0, 1]^n$  with  $x_i$  being the proportion invested in asset  $i$  and the proportions sum to one, which is also known under the “fully invested constraint”. Let the product  $\mathbf{x}^T \mathbf{R}$  be the random portfolio return,  $T$  be a fixed or random target and  $\alpha$  be a critical probability, then the safety-first portfolio model which maximizes the expected portfolio return subject to a safety-first constraint can be expressed as

**Problem 1.**

$$\begin{aligned} \max_{\mathbf{x} \in [0, 1]^n} \mathbb{E}(\mathbf{x}^T \mathbf{R}) \quad & s.t. \\ P(\mathbf{x}^T \mathbf{R} < T) & \leq \alpha, \\ \mathbf{1}^T \mathbf{x} & = 1. \end{aligned} \tag{1}$$

For a numerical treatment of problem 1 it is useful to provide a deterministic rather than a probabilistic expression of the safety-first constraint (1), which can be easily achieved under the normal assumption (see Kalin and Zagst, 1999). Because of its universality and simplicity the normal distribution is commonly assumed to describe asset returns. Leibowitz and Henriksson (1989), Leibowitz and Kogelman (1991), Albrecht (1993), among others, studied the safety-first framework with fixed target under the normal assumption. Define therefore  $Z := \mathbf{x}^T \mathbf{R} - T$  with  $Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$  as a normal distributed random variable with expected value  $\mu_Z$  and variance  $\sigma_Z^2$ . Then, employing

the usual textbook transformation for the normal distribution, safety-first constraint (1) can be equivalently expressed as

$$\begin{aligned}
P(Z < 0) &\leq \alpha \Leftrightarrow \\
P\left(\frac{Z - \mu_Z}{\sigma_Z} < -\frac{\mu_Z}{\sigma_Z}\right) &\leq \alpha \Leftrightarrow \\
\Phi\left(-\frac{\mu_Z}{\sigma_Z}\right) &\leq \alpha \Leftrightarrow \\
\Phi^{-1}(\alpha)\sigma_Z + \mu_Z &\geq 0,
\end{aligned} \tag{3}$$

where  $\Phi^{-1}(\alpha)$  is the  $\alpha$ -quantile of the standard normal distribution and  $\sigma_Z$  is the standard deviation obtained from drawing the positive square root of the variance,  $\sigma_Z^2$ . The following theorem, found in a slightly modified version in Kall and Mayer (2005, p. 103f) and Ruszczyński and Shapiro (2003, p. 10), provides precise expressions for  $\mu_Z$  and  $\sigma_Z^2$  (and  $\sigma_Z$ ).

**Theorem 1.** *If the random return vector  $\mathbf{R} = (R_1, \dots, R_n)^T$  is multivariate normal distributed with expected value vector  $\boldsymbol{\mu} \in \mathbb{R}^n$  and covariance matrix  $\Sigma \in \mathbb{R}^{n \times n}$ ,  $\mathbf{R} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ , and the random target  $T$  is normal distributed with expected value  $\mu_T \in \mathbb{R}$  and variance  $\sigma_T^2 \in \mathbb{R}_+$ ,  $T \sim \mathcal{N}(\mu_T, \sigma_T^2)$ , then the random variable*

$$Z = \mathbf{x}^T \mathbf{R} - T \sim \mathcal{N}(\boldsymbol{\mu}^T \mathbf{x} - \mu_T, \|L^T \mathbf{x} - \mathbf{b}\|^2)$$

*is normal distributed with expected value  $\boldsymbol{\mu}^T \mathbf{x} - \mu_T \in \mathbb{R}$  and variance  $\|L^T \mathbf{x} - \mathbf{b}\|^2 \in \mathbb{R}_+$  where  $L \in \mathbb{R}^{n \times n}$  is a lower triangular matrix and  $\mathbf{b} \in \mathbb{R}^n$ , both obtained from the Cholesky-factorization of the covariance matrix*

$$\text{COV}(\mathbf{R}^T, T) = \text{COV}(R_1, \dots, R_n, T) = \begin{pmatrix} L \\ \mathbf{b}^T \end{pmatrix} \begin{pmatrix} L \\ \mathbf{b}^T \end{pmatrix}^T,$$

and  $\|\cdot\|$  denotes the Euclidean norm.

□<sup>2</sup>

Applying then the deterministic safety-first constraint (3) with  $\mu_Z = \boldsymbol{\mu}^T \mathbf{x} - \mu_T$  and  $\sigma_Z = \|L^T \mathbf{x} - \mathbf{b}\|$ , problem 1 can be reformulated as

**Problem 2.**

$$\begin{aligned}
\max_{\mathbf{x} \in [0,1]^n} \mathbb{E}(\mathbf{x}^T \mathbf{R}) &= \max_{\mathbf{x} \in [0,1]^n} \mathbf{x}^T \boldsymbol{\mu} \quad s.t. \\
\Phi^{-1}(\alpha) \|L^T \mathbf{x} - \mathbf{b}\| + \boldsymbol{\mu}^T \mathbf{x} &\geq \mu_T, \\
\mathbf{1}^T \mathbf{x} &= 1.
\end{aligned} \tag{4}$$

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<sup>2</sup>Those readers who are interested in the proof, please consult Kall and Mayer (2005, p. 103f).

Problem 2 is the deterministic equivalent of problem 1 and can now be solved numerically. It is linear in its objective but quadratic in its constraints.<sup>3</sup> It therefore relates to the class of quadratic optimization problems. If  $\alpha \in (0, \frac{1}{2})$  the deterministic safety-first constraint (4) is concave in  $\mathbf{x}$  as was first shown by Kataoka (1963). Under this assumption problem 2 can easily be solved by concave optimization methods.<sup>4</sup> As  $\alpha$  represents the maximum probability of failing to achieve the target and is moreover specified by the investor herself, it is usually chosen to be low, e.g. 5% or 10%. The restriction of  $\alpha$  not exceeding 50% does not limit the practical value of our results. We therefore stick throughout our analysis to this assumption.

Note, modeling a riskfree asset  $j$  can be easily achieved by setting the  $j$ -th row of  $L$  to the zero-vector. Then, we have  $R_j = \mu_j$  with  $\mu_j$  being the riskfree rate. Analogously, a fixed target instead of a normal distributed target can be achieved by setting  $\mathbf{b}$  to the zero-vector. Then, the correlation between the target and the asset returns is zero and  $T$  collapses to a fixed target measured as  $\mu_T$ .

### 3 Fixed versus Benchmark Target

This section concerns the comparison of the fixed target strategy ( $S_1$ ) with the random target strategy ( $S_2$ ). The fixed and the random target is denoted as  $T_1$  and  $T_2$ , respectively. Both strategies face the same investment universe and the same estimates for the expected returns,  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$ , and covariances,  $\Sigma$ . Additionally, the covariations between the random target and the asset returns are denoted by the vector  $(L\mathbf{b}) = \mathbb{COV}(\mathbf{R}, T_2)$ . Note that  $L$  is a lower triangular matrix and  $\mathbf{b}$  is a vector, both obtained from the Cholesky-factorization of the overall covariance matrix  $\mathbb{COV}(R_1, \dots, R_n, T_2)$  (see Theorem 1). The vector  $\mathbb{COV}(\mathbf{R}, T_2)$  denotes therefore the cross-covariances between  $T_2$  and  $R_i$ ,  $i = 1, \dots, n$ .

As assumed in Theorem 1,  $T_2$  is normal distributed with expected value  $\mu_{T_2}$  and variance  $\sigma_{T_2}^2$ . It appears reasonable to assume,  $\mu_{T_2} > T_1$ , so that the expected target return of  $S_2$  is greater than  $T_1$ . This is due to the higher risk of strategy  $S_2$ , which must be compensated by a greater expected target return. However, to keep our results as simple as possible we assume throughout this section that  $\mu_{T_2} = T_1$ . But, all results presented here can be straightforwardly modified such that  $\mu_{T_2} > T_1$ , for example by defining  $\mu_{T_2} := T_1 + \epsilon$ ,  $\epsilon > 0$  and adapting the calculations.

In the following, we compare the performance of  $S_1$  and  $S_2$  while comparing optimal expected portfolio returns. In subsection 3.1 the practical more interesting case where the random target and the asset returns are crosswise positively correlated is discussed, whereas, in subsection 3.2 the opposite case is discussed.

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<sup>3</sup>More precisely, (4) is a second-order cone constraint, also called ice-cream cone or Lorentz cone. For a detailed discussion on that, please consult Kall and Mayer (2005, p. 273f) and the references therein.

<sup>4</sup>If the feasible domain is concave and not empty, there exists a unique maximum. For an overview of convex optimization see Boyd and Vandenberghe (2007)

### 3.1 The Positive Correlated Case

Consider strategy  $S_2$  where a portfolio is managed subject to the performance of a benchmark, such as a stock index like the S&P 500, without directly investing into the benchmark.<sup>5</sup> There are at least three situations where this appears reasonable: First, the portfolio seeks to outperform the benchmark, which is typically for actively traded funds. Second, from an individual perspective, an investor might be attracted by the performance of a foreign market, but she does not want to invest directly into the market. She rather seeks to track the performance of the foreign market with only investing in domestic stocks. This situation is derived from a behavioral phenomenon called *home bias*, which was first documented by French and Poterba (1991). Third, the situation where a foreign market should be tracked by only investing in the domestic market could also be interesting for institutional investors, for example to avoid exchange risks on foreign currencies. Reducing all three examples to the stock market, they have in common to track or outperform one stock market by investing in similar but different assets from another stock market. We therefore assume that asset returns are positively correlated with the benchmark return, i.e. the cross-covariance vector between  $\mathbf{R} = (R_1, \dots, R_n)^T$  and  $T_2$  is

$$(L\mathbf{b}) = \mathbb{COV}(\mathbf{R}, T_2) > \mathbf{0}. \quad (5)$$

This assumption is generally justified for most of the risky financial assets, in particular for stock markets. Later in this subsection we show three stock market examples, which provide support that even a stronger version of (5) is justified. Remark that assumption (5) excludes riskless investing because the covariance between the return of the riskless asset and the benchmark target must be zero. We have  $S_1$  with fixed target,  $T_1$ , and  $S_2$  with normal distributed target,  $T_2 \sim \mathcal{N}(T_1, \sigma_{T_2}^2)$ . Applying theorem 1 yields

$$\begin{aligned} Z_1 &= \mathbf{x}_1^T \mathbf{R} - T_1 \quad \text{with} \quad Z_1 \sim \mathcal{N}(\boldsymbol{\mu}^T \mathbf{x}_1 - T_1, \|L^T \mathbf{x}_1\|^2), \\ Z_2 &= \mathbf{x}_2^T \mathbf{R} - T_2 \quad \text{with} \quad Z_2 \sim \mathcal{N}(\boldsymbol{\mu}^T \mathbf{x}_2 - T_1, \|L^T \mathbf{x}_2 - \mathbf{b}\|^2), \end{aligned}$$

where  $\mathbf{x}_1$  and  $\mathbf{x}_2$  represents the portfolio vector of  $S_1$  and  $S_2$ , respectively. Note, the expected values of  $Z_1$  and  $Z_2$  coincide, but the variances differ. Thus, in the normal distributed case, the question, whether  $S_1$  outperforms  $S_2$  or vice versa is simply the question of comparing variances. The following theorem shows that, under a weak additional assumption, the variance of  $Z_2$  is smaller than the variance of  $Z_1$  and therefore,  $S_2$  outperforms  $S_1$ . To prove this theorem we make use of

**Lemma 1.** *Provided assumption (5) holds. If*

$$\mathbb{COV}(R_i, T_2) > \frac{1}{2} \sigma_{T_2}^2, \quad i = 1, \dots, n, \quad (6)$$

---

<sup>5</sup>A direct investment in an index can be obtained by purchasing an exchange traded funds (ETF) on the index, which explicitly tracks the index and is, moreover, attractive because of low transactions costs and tax efficiency.



holds, then for any critical probability  $\alpha \in (0, \frac{1}{2})$  the following inequality is true:

$$\Phi^{-1}(\alpha)\sigma_{Z_2} + \mu_{Z_2} > \Phi^{-1}(\alpha)\sigma_{Z_1} + \mu_{Z_1}$$

**Proof:** From the fully invested constraint (2) together with (6) follows

$$\mathbb{COV}(\mathbf{R}, T_2)^T \mathbf{x} > \frac{1}{2}\sigma_{T_2}^2,$$

which is equivalent to

$$\begin{aligned} 0 &> -2\mathbb{COV}(\mathbf{R}, T_2)^T \mathbf{x} + \sigma_{T_2}^2 && \Leftrightarrow \\ \|L^T \mathbf{x}\|^2 &> \|L^T \mathbf{x}\|^2 - 2(L\mathbf{b})^T \mathbf{x} + \mathbf{b}^T \mathbf{b} && \Leftrightarrow \\ \|L^T \mathbf{x}\| &> \|L^T \mathbf{x} - \mathbf{b}\| && \Leftrightarrow \\ \Phi^{-1}(\alpha)\|L^T \mathbf{x}\| + \boldsymbol{\mu}^T \mathbf{x} - T_1 &< \Phi^{-1}(\alpha)\|L^T \mathbf{x} - \mathbf{b}\| + \boldsymbol{\mu}^T \mathbf{x} - T_1 && \Leftrightarrow \\ \Phi^{-1}(\alpha)\sigma_{Z_1} + \mu_{Z_1} &< \Phi^{-1}(\alpha)\sigma_{Z_2} + \mu_{Z_2}. \end{aligned}$$

Notice that the second last inequality reverses because  $\Phi^{-1}(\alpha) < 0 \ \forall \alpha \in (0, \frac{1}{2})$ . □

**Theorem 2.** *Provided (5) and (6) hold. For any critical probability  $\alpha \in (0, \frac{1}{2})$  the optimal expected portfolio return of  $S_2$  is larger or equal than the optimal expected portfolio return of  $S_1$ , i.e.  $\boldsymbol{\mu}^T \mathbf{x}_2^* \geq \boldsymbol{\mu}^T \mathbf{x}_1^*$ .*

**Proof:** It is sufficient to show that the set of feasible portfolios of  $S_1$  is a subset of feasible portfolios of  $S_2$ . In other words, it is to show that any feasible portfolio for  $S_1$  is also feasible for  $S_2$ , but at least one portfolio exists which is feasible for  $S_2$ , but not for  $S_1$ . We first show, if the feasible domain for  $S_2$  is not empty, there exists a portfolio  $\mathbf{x}$  which is feasible for  $S_2$ , but not for  $S_1$ . For any feasible  $\alpha \in (0, \frac{1}{2})$  choose  $\mathbf{x}$  so that

$$\alpha = P(\mathbf{x}^T \mathbf{R} < T_2) = P(Z_2 < 0) \quad \Leftrightarrow \quad \Phi^{-1}(\alpha)\sigma_{Z_2} + \mu_{Z_2} = 0.$$

Applying lemma 1 yields  $\Phi^{-1}(\alpha)\sigma_{Z_1} + \mu_{Z_1} < 0$ , which is equivalent to

$$P(Z_1 < 0) = P(\mathbf{x}^T \mathbf{R} < T_1) > \alpha.$$

Thus, the safety-first constraint for  $S_1$  is not met. With the same arguments it follows that any portfolio  $\mathbf{x}$ , which is feasible for  $S_1$  is also feasible for  $S_2$ . If there exists no portfolio  $\mathbf{x}$  such that the safety-first constraint for  $S_2$  is satisfied, i.e.  $P(\mathbf{x}^T \mathbf{R} < T_2) > \alpha \ \forall \mathbf{x} \in [0, 1]^n$ , then the portfolio problem of  $S_2$  is infeasible and thus also the portfolio problem for  $S_1$ . □

Remark, the special case where the safety-first constraint for  $S_1$  is satisfied for all portfolios  $\mathbf{x}$ , i.e.  $P(\mathbf{x}^T \mathbf{R} < T_1) < \alpha \ \forall \mathbf{x} \in [0, 1]^n$ , then the entire wealth is invested in the the asset with the highest expected return and thus the same is true for  $S_2$ . In

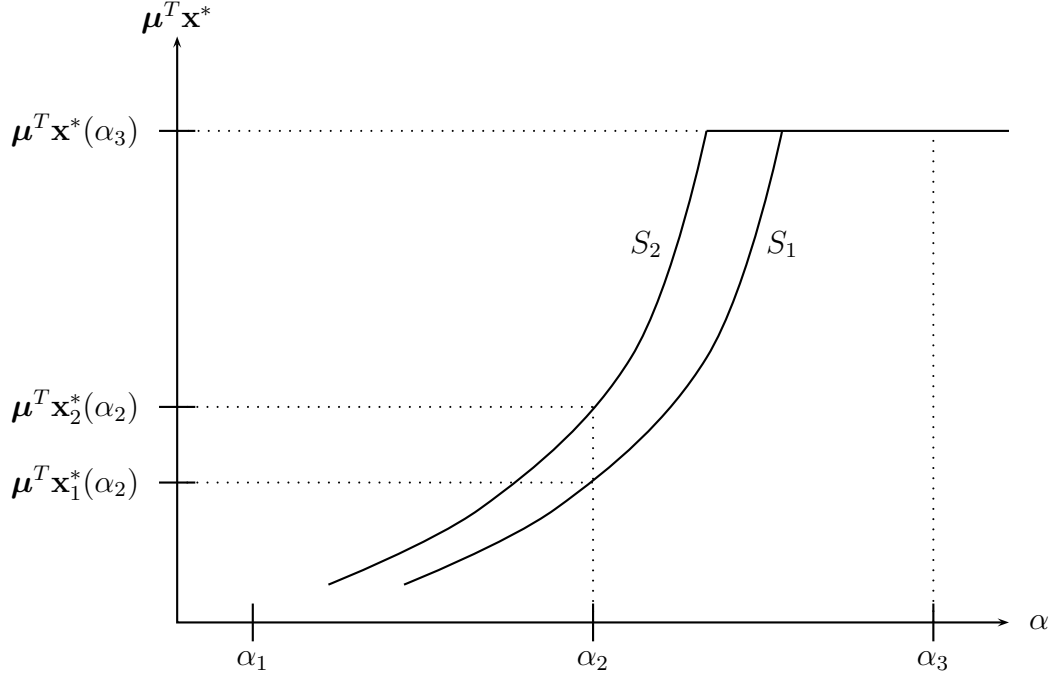


Figure 1: Safety-first efficient frontiers for  $S_1$  and  $S_2$ .

this case both investors obtain the same optimal expected portfolio return and only one asset is hold. This special but trivial case is considered in theorem 1, but it is not explicitly mentioned in the proof thereafter.

In figure 1, where the safety-first efficient frontiers<sup>6</sup> for  $S_1$  and  $S_2$  are sketched, we illustrate the previous result: Suppose, both investors choose a critical probability of  $\alpha_1$ , then the portfolio problem is neither feasible for  $S_1$  nor feasible for  $S_2$ . Suppose, both choose  $\alpha_2$ , then  $S_2$  outperforms  $S_1$  as  $\mu^T \mathbf{x}_2^* > \mu^T \mathbf{x}_1^*$ . Finally, suppose that they choose  $\alpha_3$ , then their optimal expected portfolio returns coincide and their entire wealth is invested in the asset with the highest expected return.

Remark, if inequality (6) reverses, the result clearly reverses, i.e.  $S_1$  outperforms  $S_2$ . But this is practically not the case as supported by table 1, which summarizes cross-covariances of four domestic (German) stocks from different industries and three important stock indices. Table 1 shows that condition (6) is satisfied in all cases, which indicates a strong relevance of this result for stock market applications, e.g. for actively managed stock portfolios. Suppose exemplary two investment funds, both  $S_1$  and  $S_2$  facing the same investment universe, but  $S_1$  guarantees a certain minimum return,  $T_1$ ,  $S_2$  guarantees nothing. Now, the question which funds is better off in terms of optimal expected portfolio returns could be answered with the help of theorem 2.

<sup>6</sup>Shefrin (2005) uses the term *SP/A efficient frontier*, which is the same as the safety-first efficient frontier, plotted in  $(\mu^T \mathbf{x}^*, \alpha)$ -space. The safety-first efficient frontier is monotone non-decreasing as investors prefer higher portfolio returns ( $\mu^T \mathbf{x}^*$ ) but lower risk ( $\alpha$ ).

Table 1: Cross-covariances between the returns of (*B*) Bayer (pharmaceuticals and chemicals), (*C*) Commerzbank (financial services), (*D*) Dt. Telekom (telecommunications), (*E*) E.ON (energy) and three important stock indices were estimated on the basis of monthly returns for the period January 2003 to December 2008.

$T_2$	$\frac{1}{2}\text{VAR}(T_2)$	$\text{COV}(B, T_2)$	$\text{COV}(C, T_2)$	$\text{COV}(D, T_2)$	$\text{COV}(E, T_2)$
DAX	16.2	34.2	62.0	19.8	30.7
S&P 500	7.8	18.8	40.4	9.9	16.5
FTSE 100	6.9	14.8	40.3	9.5	11.3

### 3.2 The Non-Positive Correlated Case

This case which assumes a non-positive correlation between the random target and the asset returns, i.e. the cross-covariance vector between  $T_2$  and  $\mathbf{R}$  is non-positive,

$$(L\mathbf{b}) = \text{COV}(\mathbf{R}, T_2) \leq \mathbf{0}, \quad (7)$$

comprises two situations or a mix of both: First, from an individual perspective the target may not even be known. Many individuals have the target of “being successful”, but only a very few know precisely which selection of money, leisure time, culture etc. must be attained to achieve this target (Bordley and LiCalzi, 2000). Or, individuals may follow a group target, which can be interpreted as *herd behavior* (see for example Shiller, 2005, p. 157-172). In these situations the correlation between  $T_2$  and  $\mathbf{R}$  is zero. Second, there exists a negative correlation between  $T_2$  and  $\mathbf{R}$ , which for example occurs when  $T_2$  expresses the return of a bond market whereas  $\mathbf{R}$  are stock returns.<sup>7</sup> We argue that a benchmark (random target) is chosen inappropriately if there exists a negative correlation between the benchmark and the portfolio return.

As in the previous subsection, for the strategies  $S_1$  and  $S_2$  we have the corresponding random variables

$$Z_1 = \mathbf{x}_1^T \mathbf{R} - T_1 \quad \text{with} \quad Z_1 \sim \mathcal{N}(\boldsymbol{\mu}^T \mathbf{x}_1 - T_1, \|L^T \mathbf{x}_1\|^2) \quad \text{and}$$

$$Z_2 = \mathbf{x}_2^T \mathbf{R} - T_2 \quad \text{with} \quad Z_2 \sim \mathcal{N}(\boldsymbol{\mu}^T \mathbf{x}_2 - T_2, \|L^T \mathbf{x}_2 - \mathbf{b}\|^2),$$

respectively, obtained from theorem 1. Notice, in the case where  $T_2$  and  $\mathbf{R}$  are uncorrelated, the variance of the sum,  $\sigma^2(Z_2) = \sigma^2(\mathbf{x}_2^T \mathbf{R} - T_2)$ , is the sum of variances,  $\sigma^2(Z_2) = \sigma^2(\mathbf{x}_2^T \mathbf{R}) + \sigma^2(T_2) = \|L^T \mathbf{x}_2\|^2 + \sigma_{T_2}^2$ . In this case it can be shown graphically that  $S_1$  outperforms  $S_2$ : From figure 2, which shows the probability density functions of  $Z_1$  ( $f_1(z)$ ) and  $Z_2$  ( $f_2(z)$ ), it is easy to verify that  $S_2$  is riskier than  $S_1$ . In this context risk is the probability of not achieving the target, i.e. the probability of shortfall,  $P(Z < 0) = P(\mathbf{x}^T \mathbf{R} < T)$ , which is for  $S_2$  the shaded area under  $f_2(z)$  and for  $S_1$  the

<sup>7</sup>We tested this case while comparing the covariations between the German REXP, which measures the performance of 30 representative German sovereign bonds, and the same four stocks from table 1 for the same time period. In all cases we calculated a negative correlation.

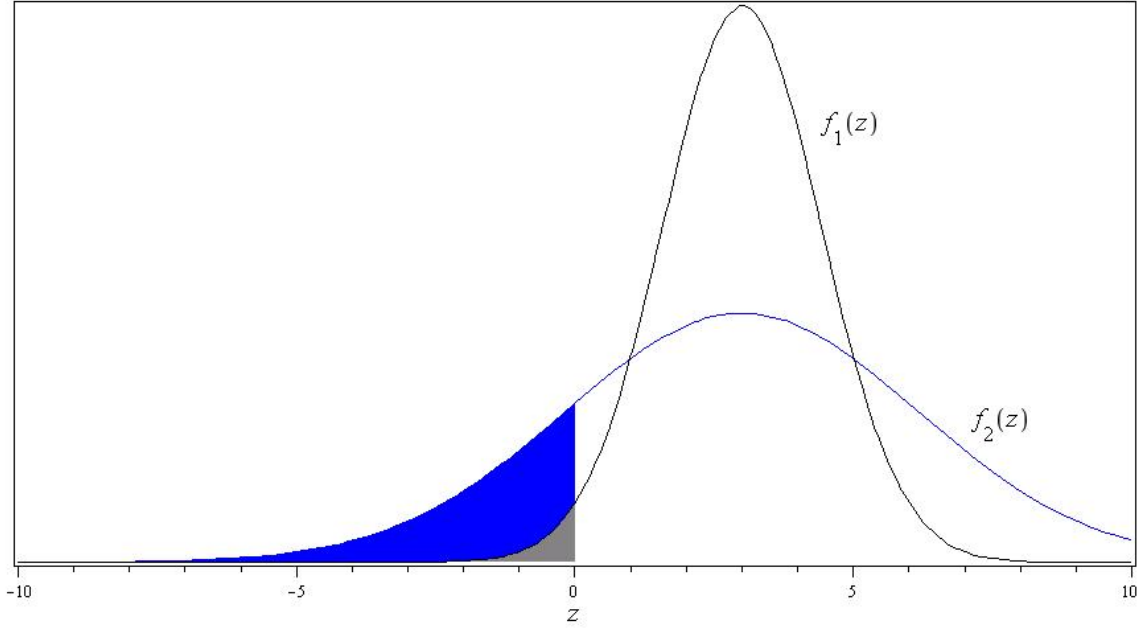


Figure 2: Probability density function of  $S_1$  ( $f_1(z)$ ) and  $S_2$  ( $f_2(z)$ )

shaded area under  $f_1(z)$ . As the shaded area under  $f_2(z)$  is larger than under  $f_1(z)$ ,  $S_2$  is riskier than  $S_1$ . We now show formally that under the more general assumption (7),  $S_1$  outperforms  $S_2$ . To prove the following theorem we use

**Lemma 2.** *Provided (7) holds. For any critical probability  $\alpha \in (0, \frac{1}{2})$  the following inequality holds:*

$$\Phi^{-1}(\alpha)\sigma_{Z_2} + \mu_{Z_2} < \Phi^{-1}(\alpha)\sigma_{Z_1} + \mu_{Z_1}$$

**Proof:**

$$\begin{aligned} 0 &< \underbrace{-2\text{COV}(\mathbf{R}, T_2)^T \mathbf{x}}_{\leq 0} + \underbrace{\sigma_{T_2}^2}_{> 0} \Leftrightarrow \\ \|L^T \mathbf{x}\|^2 &< \|L^T \mathbf{x}\|^2 - 2(L\mathbf{b})^T \mathbf{x} + \mathbf{b}^T \mathbf{b} \Leftrightarrow \\ \|L^T \mathbf{x}\| &< \|L^T \mathbf{x} - \mathbf{b}\| \Leftrightarrow \\ \Phi^{-1}(\alpha)\|L^T \mathbf{x}\| + \boldsymbol{\mu}^T \mathbf{x} - T_1 &> \Phi^{-1}(\alpha)\|L^T \mathbf{x} - \mathbf{b}\| + \boldsymbol{\mu}^T \mathbf{x} - T_1 \Leftrightarrow \\ \Phi^{-1}(\alpha)\sigma_{Z_1} + \mu_{Z_1} &> \Phi^{-1}(\alpha)\sigma_{Z_2} + \mu_{Z_2} \end{aligned}$$

Notice that the second last inequality reverses because  $\Phi^{-1}(\alpha) < 0 \ \forall \alpha \in (0, \frac{1}{2})$ . □

**Theorem 3.** *Provided (7) holds. For any critical probability  $\alpha \in (0, \frac{1}{2})$  the optimal expected portfolio return of  $S_1$  is larger or equal than the optimal expected portfolio return of  $S_2$ , i.e.  $\boldsymbol{\mu}^T \mathbf{x}_1^* \geq \boldsymbol{\mu}^T \mathbf{x}_2^*$ .*

**Proof:** Employing lemma 2 instead of 1 and redoing the proof of theorem 2 yields the proposition.  $\square$

This result requires, compared to the positive correlated case discussed in subsection 3.1, no additional assumption. Furthermore, it allows riskless investing, which is prohibited in the previous case. As  $S_1$  outperforms  $S_2$  it reveals, however, that general uncertainty about the target and the choice of an inappropriately benchmark should be avoided in this context.

## 4 Distribution is unknown

Empirically, it was shown that return distributions are fat-tailed (see for example Adler, 1998, and the references therein) and skewed to the left, i.e. losses weigh heavier than gains, discussed for example in Harlow (1991). These findings indicate that the normal assumption does not necessarily hold in general. We therefore relax this assumption and only assume that good estimates for the first two moments of the return distributions exist. Then we can use a textbook probability inequality, the Chebyshev inequality.<sup>8</sup> In the following we apply Chebyshev's inequality to the safety-first constraint (1) and obtain a stronger but deterministic version of (1).<sup>9</sup> Providing this, it is easy to verify that under unknown return distributions the results from section 3 hold analogously.

Again, let  $Z = \mathbf{x}^T \mathbf{R} - T$  be a random variable with  $\mathbf{x}^T \mathbf{R}$  being the random portfolio return and  $T$  being a target, either fixed or random. For the expected value and the variance of  $Z$  we get

$$\begin{aligned}\mu_Z &= \boldsymbol{\mu}^T \mathbf{x} - \mu_T, \\ \sigma_Z^2 &= \|L^T \mathbf{x} - \mathbf{b}\|^2.\end{aligned}$$

The following inequalities provide an upper bound for the safety-first constraint (1):

$$\begin{aligned}P(\mathbf{x}^T \mathbf{R} < T) &= P(Z < 0) \\ &\leq P(Z \leq 0) = P(Z - \mu_Z \leq -\mu_Z) = P(\mu_Z - Z \geq \mu_Z) \\ &\leq P(|\mu_Z - Z| \geq \mu_Z) \\ &\leq \frac{\sigma_Z^2}{\mu_Z^2} = \frac{\|L^T \mathbf{x} - \mathbf{b}\|^2}{(\boldsymbol{\mu}^T \mathbf{x} - \mu_T)^2},\end{aligned}$$

where the last inequality is obtained from Chebyshev's rule.<sup>10</sup> Instead of (1) the stronger inequality

$$\frac{\|L^T \mathbf{x} - \mathbf{b}\|^2}{(\boldsymbol{\mu}^T \mathbf{x} - \mu_T)^2} \leq \alpha$$

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<sup>8</sup>This idea has been first suggested by Roy (1952).

<sup>9</sup>For a detailed discussion on the application of Chebyshev's inequality to the safety-first criterion Kall and Mayer (2005, see) or Birge and Louveau (1997).

<sup>10</sup>A detailed illustration for the univariate case is provided in Breuer et al. (2006, p. 119f). For the Chebyshev inequality in general consult a textbook on probability theory, such as Behnen and Neuhaus (1995).

can be applied. Drawing the square root and rearranging yields

$$-\alpha^{-\frac{1}{2}}\|L^T\mathbf{x} - \mathbf{b}\| + \boldsymbol{\mu}^T\mathbf{x} \geq \mu_T. \quad (8)$$

Comparing (8) and the deterministic safety-first constraint (4) obtained for the multivariate normal case (see section 2), the sole difference is the multiplier for the term  $\|L^T\mathbf{x} - \mathbf{b}\|$ . In (4) the multiplier is  $\Phi^{-1}(\alpha)$  whereas it is  $-\alpha^{-\frac{1}{2}}$  for (8), both are negative for all  $\alpha \in (0, \frac{1}{2})$ . Thus, redoing the proofs from section 3 with (8) instead of (4) yields the same results. Moreover, as (8) is concave for all  $\alpha$  the same efficient solving methods as for problem 2 can be used.

Note, applying the stronger inequality (8) instead of (1) reduces the number of feasible portfolios, i.e. the feasible domain of problem 1 with (8) instead of (1) is a subset of the feasible domain of the original problem 1. Thus, choosing  $\alpha$  very small may lead to infeasibility of the safety-first problem under (8), but not necessarily under the true safety-first constraint (1). Nevertheless, this approach provides a useful and tractable alternative to the multivariate normal case.

## 5 Conclusion

This paper attempted to analyze safety-first portfolio optimization under two different target assumptions, the fixed target, which is widely assumed in the literature, and the random target, which describes the uncertain return of a benchmark and is commonly applied for managing institutional portfolios. The main focus of this paper was on the question which strategy is better off. Assuming multivariate normality the answer is: (1) The random target strategy outperforms the fixed target strategy if the portfolio return and the random target are positively correlated and riskless investing is prohibited, (2) the fixed target strategy outperforms the random target strategy if the portfolio return and the benchmark target are not positively correlated. The first result is useful for institutional funds management, which seeks to track or outperform a benchmark, but also for the skilled private investor. It suggests how to find a portfolio which may outperform a benchmark. The second result suggests that general uncertainty about the target and an inappropriate target choice should be avoided. As the normal assumption is hotly debated in the financial literature we relaxed this assumption and showed that both results hold when approximating the safety-first statement by the well known Chebyshev inequality.

The normal distribution and Chebyshev's inequality are on the one hand very tractable and easy to implement, but on the other hand not very accurate. To overcome this limitation several extensions are possible: First, the normal assumption can be relaxed to the general distribution family depending on a shift and a scale parameter (see Kalin and Zagst, 1999). Second, a *copula* function, which provides a general technique for formulating a multivariate distribution, can be used. Third, considering higher order moments a more accurate probability inequality can be used. We recommend this issues for further research.

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