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Working Paper

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Wirtschaftstheoretische Diskussionsbeiträge, No. 05-07

Provided in Cooperation with:
Lehrstuhl für Volkswirtschaftslehre, Technische Universität Dortmund

Suggested Citation: Konrad, Kai A.; Leininger, Wolfgang (2005) : The generalized Stackelberg equilibrium of the all-pay auction with complete information, Wirtschaftstheoretische Diskussionsbeiträge, No. 05-07, Universität Dortmund, Wirtschafts- und Sozialwissenschaftliche Fakultät, Dortmund

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https://hdl.handle.net/10419/39437

WIRTSCHAFTSTHEORETISCHE
DISKUSSIONSBEITRÄGE

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DISCUSSION PAPERS
IN ECONOMICS

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The generalized Stackelberg equilibrium of the all-pay auction with complete information

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September 1, 2005

Abstract

We characterize the equilibrium of the all-pay auction with general convex cost of effort and sequential effort choices. We consider a set of n players who are arbitrarily partitioned into a group of players who choose their efforts 'early' and a group of players who choose 'late'. Only the player with the lowest cost of effort has a positive payoff in any equilibrium. This payoff depends on his own timing vis-a-vis the timing of others. We also show that the choice of timing can be endogenized, in which case the strongest player typically chooses 'late', whereas all other players are indifferent with respect to their choice of timing. In the most prominent equilibrium the player with the lowest cost of effort wins the auction at zero aggregate cost.

Keywords: Sequential all-pay auction, complete information, generalized cost, generalized Stackelberg game.

JEL classification code: D72, D74

1 Introduction

The analytics of the all-pay auction with complete information with simultaneous choices of efforts by several contestants is well understood. Hillman and Riley (1989) were first to characterize an equilibrium, and Baye, Kovenock and de Vries (1996) provide a complete characterization of the set of equilibria for the case in which contestants have linear cost of expending effort. It is clear from the analysis in Kaplan, Luski and Wettstein (2003) that the solution concept of the case with linear cost carries over to cases in

which contestants have non-linear cost of expending effort. Other analyses contributing to the solution of the all-pay auction with complete information are Hillman and Samet (1987) who, among other aspects, consider the problem of entry fees or minimum contest efforts, and Baye, Kovenock and de Vries (2005) who work out the complete analytic solution for all-pay auctions with two players and various cost of effort functions, in which a player's total cost of effort may generally depend on the own effort expended, on the effort that is expended by the rival player, and on the outcome of the contest. All these analyses consider simultaneous effort choices.

Simultaneity of effort choices cannot be taken for granted for situations in which all-pay auctions take place. Particularly when the conflict has some asymmetry properties, sequential effort choices may emerge naturally from the institutional environment, or even endogenously. Leininger (1991) models a patent race between incumbent and entrant, who value the patent differently, as a sequential all-pay auction with possibly repeated bidding under complete information. Konrad (2002) considers an all-pay auction with two asymmetric players. He mostly concentrates on symmetric moves, but also discusses the implications of sequential effort choices for the equilibrium outcome in a specific analytic context, allowing for two types of incumbency advantages. Jost and Kräkel (2005) also look at this specific problem and compare the outcomes for sequential and simultaneous choices for the case with two players. Moreover, they consider a contest success function with noise. The more general case in which there is a set with more than two players who are partitioned in two disjoint subsets of which the members of one subset choose effort prior to the members of the other subset has not been considered. This paper solves for this generalized Stackelberg equilibrium, and with a generalized cost function.

We also discuss the issue of endogenous sequentiality that emerges in a contest environment. In the context of the Tullock (1980) contest, Baik and Shogren (1992), Leininger (1993), Baik (1994, 2005) and Wärneryd (2000) provide analyses of this problem. Suppose there are two points in time at which contest effort can be expended. One point is called e(arly), the other is called l(ate). In a stage prior to e, contestants can choose whether to make their contest effort choices at e or at l. It turns out that, if the effort choices at e are irreversibly made, the "weaker" player has an incentive to choose e and the "stronger" player has an incentive to choose l. In the end, both contestants are better off. We analyse in this paper how the result on endogenous timing generalizes to a larger set of asymmetric players for the

all-pay auction with complete information.

2 Equilibrium

A prize of given size $V = 1$ is allocated among a set N of players $i = 1, ...n$ in an all-pay auction with complete information. Let $\mathbf{x} = (x_1, x_2, ...x_n)$ denote the vector of players' efforts. Each player chooses his effort irreversibly at a particular point of time. The players in a subset $E \subset N$ choose their efforts simultaneously at a point e(arly) and players from set $L = N - E$ choose their efforts at point l(ate), where l occurs after e, such that players who are in set L can observe the effort choices made by the players who are in set E.

Players are risk neutral. A player's payoff is

$$\pi_i(\mathbf{x}) = p_i(\mathbf{x}) \cdot 1 - C_i(x_i). \tag{1}$$

Here the cost $C_i(x_i)$ of expending a given effort x_i is a function of this effort, and we will assume throughout that $C_i(0) = 0$, $C_i'(x_i) > 0$, and $C_i''(x_i) \geq 0$. The standard cases with linear cost, $C_i(x_i) = c_i x_i$, and with quadratic cost, $C_i(x_i) = c_i(x_i)^2$ are special cases. We assume that players are asymmetric and can be sorted according to their effort cost. Without loss of generality we consider them numbered such that, for two players i and j with $i < j$, it holds that $C_i'(\ x) < C_j'(x)$ for all $x \in (0, K]$. This global asymmetry rules out a number of equilibria that may occur if some players are identical. Note that, according to this sorting assumption, player 1 has the lowest cost for expending a given effort, player 2 has the second lowest cost, etc. The sorting of players in N according to their cost of effort also induces a similar sorting of players in E and in L, and we re-number the players in E and L according to their cost functions as $e_{(1)}, e_{(2)}, ..., e_{(\#E)}$ with $C_{e_{(i)}}(x) < C_{e_{(j)}}(x)$ and $l_{(1)}, l_{(2)}, ..., l_{(\#L)}$ with $C_{l_{(i)}}(x) < C_{l_{(j)}}(x)$ for all $i < j$ for all $x > 0$.

The probability that player i wins the prize is denoted as p_i, and is a function of $\mathbf{x}$, as follows. Denote $\bar{x} \equiv \max_{k \in N}\{x_k\}$. If $x_i = \bar{x} > x_j$ for all $j \in N - \{i\}$, then $p_i = 1$ and $p_j = 0$ for all $j \neq i$. If there are several players who have chosen the same, highest effort, $\bar{x}$, we assume the following tie-breaking rule. Let M be the set of players who choose $\bar{x}$. If $M \subset E$, or $M \subset L$, then each $i \in M$ wins the prize with the same probability equal to $1/\#M$, where $\#M$ is the cardinality of M. If $M \cap E \neq \varnothing$ and $M \cap L \neq \varnothing$ then the allocation of the prize among the players in M depends on their cost of effort. If $C_i(\bar{x}) \geq 1$ for all $i \in M \cap L$, then $p_i = 1/\#(M \cap E)$

for $i \in M \cap E$ and zero for all other players. Denote $(M \cap L)^+(\bar{x})$ the subset of $(M \cap L)$ with players for which $C_i(\bar{x}) < 1$ holds. If $(M \cap L)^+(\bar{x}) \neq \varnothing$, then $p_i = 1/\#(M \cap L)^+(\bar{x})$ for $i \in (M \cap L)^+(\bar{x})$ and $p_i = 0$ for all other players.

Strategies are defined as follows. For a player $i \in E$, a pure strategy is an effort choice x_i and the strategy set of player i is the set of feasible efforts $[0, K]$, with K being a finite number that is sufficiently large to be never binding.[1] We also define a mixed strategy F_i for player $i \in E$ as an element of the set of probability distributions over the effort levels from the set $[0, K]$. A player $i \in L$ observes the effort choices of all players $j \in E$. We denote the vector of these effort choices by all players from E as $\mathbf{x}_E$. As it turns out only $\bar{x}_E$, the maximal component of $\mathbf{x}_E$, is payoff relevant for the game in period l. Hence we can denote a pure strategy of player $i \in L$ as a function $x_i(\bar{x}_E) : [0, K] \to [0, K]$, and a mixed strategy of this player as a function $F_i(\bar{x}_E) : [0, K] \to \Sigma_{[0,K]}$, where $\Sigma_{[0,K]}$ denotes the set of probability distributions over the effort levels from the set $[0, K]$.

Proposition 1 *For any given sets L and E there exists a unique subgame perfect equilibrium of the (sequential) all-pay auction. Equilibrium pay-offs are given by:*

$$\pi_j = 0 \text{ for all } j = 2, ...n, \text{ and} \tag{2}$$

$$\pi_1 = \begin{cases} 1 & \text{if } L = \{1\} \\ 1 - C_1(\bar{x}_{l_{(2)}}) & \text{if } 1 \in L \text{ and } \#L > 1 \quad \begin{cases} \text{with } \bar{x}_{l_{(2)}} \text{ the solution} \\ \text{of } C_{l_{(2)}}(\bar{x}_{l_{(2)}}) = 1 \end{cases} \\ 1 - C_1(\bar{x}_2) & \text{if } 1 \in E \quad \text{with } \bar{x}_2 \text{ the solution of } C_2(\bar{x}_2) = 1 \end{cases} \tag{3}$$

Proof. Consider first stage l. Denote $\bar{x}_E = \max_{e_{(i)} \in E}\{x_{e_{(i)}}\}$. Recall that $e_{(i)}$ are the players of set E ordered along ascending cost functions. Let $L^+(\bar{x}_E)$ be the set of players $k \in L$ for which $C_k(\bar{x}_E) < 1$, and $L^-(\bar{x}_E)$ the set of the players $k \in L$ for which $C_k(\bar{x}_E) \geq 1$.

Consider the following candidate equilibrium of the subgame at stage l. (i) All players $k \in L^-(\bar{x}_E)$ choose $x_k(\mathbf{x}_E) = 0$. (ii) For players $k \in L^+(\bar{x}_E)$, if $\#L^+(x_E) = 1$ then $x_k(\mathbf{x}_E) = \bar{x}_E$. If $\#L^+(x_E) > 1$, then $x_k(\mathbf{x}_E) = 0$ for all $k \in L$ with $k \notin \{l_{(1)}, l_{(2)}\}$,

$$F_{l_{(1)}}(\mathbf{x}_E) = \begin{cases} 0 & \text{for} \quad x \in [0, \bar{x}_E) \\ C_{l_{(2)}}(x) & \text{for} \quad x \in [\bar{x}_E,\ \bar{x}_{l_{(2)}}] \\ 1 & \text{for} \quad x > \bar{x}_{l_{(2)}} \end{cases} \tag{4}$$

[1] For instance, it will be sufficient to define K as the solution of $C_1(K) = 1$.

and

$$F_{l_{(2)}}(\mathbf{x}_E) = \begin{cases} 1 - C_{l_{(1)}}(\bar{x}_{l_{(2)}}) + C_{l_{(1)}}(\bar{x}_E) & \text{for} \quad x \in [0, \bar{x}_E] \\ 1 - C_{l_{(1)}}(\bar{x}_{l_{(2)}}) + C_{l_{(1)}}(x) & \text{for} \quad x \in (\bar{x}_E,\ \bar{x}_{l_{(2)}}] \\ 1 & \text{for} \quad x > \bar{x}_{l_{(2)}}. \end{cases} \tag{5}$$

To confirm that this constitutes an equilibrium we show that these strategies are mutually optimal replies. (i) For given $\bar{x}_E$, $x_k = 0$ uniquely maximizes the payoff of players $k \in L^-(\bar{x}_E)$, independent of the choices of other players in L. (ii) Consider $k \in L^+(x_E)$ with $k \notin \{l_{(1)}, l_{(2)}\}$. The payoff in the candidate equilibrium is $\pi_k(0) = 0$ and, given (4), $\pi_k(x_k) < 0$ for any $x_k > 0$. Thus $x_k = 0$ is the unique best reply. Consider next $l_{(1)}$. The payoff of this player given the candidate equilibrium strategies of other players is $\pi_{l_{(1)}}(0) = 0$, $\pi_{l_{(1)}}(x) = -C_{l_{(1)}}(x)$ for $x \in (0, \bar{x}_E)$ and

$$\pi_{l_{(1)}} = F_{l_{(2)}}(x) \cdot 1 - C_{l_{(1)}}(x) \tag{6}$$

for $x_{l_{(1)}} \geq \bar{x}_E$. This payoff is equal to $1 - C_{l_{(1)}}(\bar{x}_{l_{(2)}}) > 0$ for all $x_{l_{(1)}} \in [\bar{x}_E, \bar{x}_{l_{(2)}}]$, because of (5), and smaller than this for all x outside this interval. Consider finally the payoff of $l_{(2)}$. This payoff is $\pi_{l_{(2)}} = 0$ for $x_{l_{(2)}} = 0$, $\pi_{l_{(2)}}(x) = -C_{l_{(2)}}(x)$ for $x \in (0, \bar{x}_E)$, and

$$\pi_{l_{(2)}} = F_{l_{(1)}}(x) \cdot 1 - C_{l_{(2)}}(x) \tag{7}$$

for $x_{l_{(2)}} > \bar{x}_E$. This payoff (7) is equal to zero for all $x_{l_{(2)}} \in (\bar{x}_E,\ \bar{x}_{l_{(2)}}]$, because of (4), and negative for all x outside this interval. Accordingly, any $x \in \{0\} \cup (\bar{x}_E, \bar{x}_{l_{(2)}}]$ is an optimal reply for $l_{(2)}$.

Note that the equilibrium in the subgame among players from set L is the equilibrium of a simultaneous all-pay auction with a minimum bid of $\bar{x}_E$. The uniqueness of this equilibrium can be shown following the line of arguments in Baye, Kovenock and de Vries (1996).

Consider now stage e. Define $\max_{l_{(i)} \in L}\{\bar{x}_{l_{(i)}}\} \equiv \bar{x}_L$. An implication of the subgame equilibrium is that a player from the set E who made the highest bid $\bar{x}_E$ among all players from this group wins the prize *if and only if* $\bar{x}_E > \bar{x}_L$. Intuitively, in the characterization of the equilibrium strategies of players from the set L, this $\bar{x}_L$ plays a similar role for players from the set E when they choose their efforts, as $\bar{x}_E$ does in the subgame when players from the set L make their choices. Note, that $\bar{x}_L$, unlike $\bar{x}_E$, is not determined by the actual effort choices of players, but is uniquely determined by the cost

structure of the players in L. Hence, somewhat surprisingly, the game among players at stage e, too, has the structure of a simultaneous all-pay auction with a minimum bid, which now is given by $\bar{x}_L$. $\bar{x}_L$ results from *foreseeing* the optimal behaviour of players moving at l. Uniqueness of this equilibrium then again follows from Baye, Kovenock and de Vries (1996) for any set E and any $\bar{x}_L$.

Let $E^+(\bar{x}_L)$ be the set of players k in E for which $C_k(\bar{x}_L) < 1$, and $E^-(\bar{x}_L)$ the set of players k in E for which $C_k(\bar{x}_L) > 1$. Note that $C_k(\bar{x}_L) = 1$ for $k \in E$ is ruled out by $C_k(\bar{x}_L) = 1$ for some $k \in L$ and the fact that all players in N differ in their cost functions. We consider the following candidate equilibrium for effort choices among players from the set E. (i) All players $k \in E^-(\bar{x}_L)$ choose $x_k = 0$. (ii) If $\#E^+(\bar{x}_L) = 1$, then this player $k \in E^+(\bar{x}_L)$ chooses $x_k = \bar{x}_L$. (iii) If $\#E^+(\bar{x}_L) > 1$, then all players $e_{(k)} \in E^+(\bar{x}_L)$ with $k > 2$ choose $x_{e_{(k)}} = 0$, player $k = e_{(1)}$ chooses a mixed strategy that is described by the cumulative distribution function

$$F_{e_{(1)}} = \begin{cases} 0 & \text{for} \quad x \in [0, \bar{x}_L) \\ C_{e_{(2)}}(x) & \text{for} \quad x \in [\bar{x}_L,\ \bar{x}_{e_{(2)}}] \\ 1 & \text{for} \quad x > \bar{x}_{e_{(2)}} \end{cases} \tag{8}$$

and player $k = e_{(2)}$ chooses a mixed strategy that is described by the cumulative distribution function

$$F_{e_{(2)}} = \begin{cases} 1 - C_{e_{(1)}}(\bar{x}_{e_{(2)}}) + C_{e_{(1)}}(\bar{x}_L) & \text{for} \quad x \in [0, \bar{x}_L] \\ 1 - C_{e_{(1)}}(\bar{x}_{e_{(2)}}) + C_{e_{(1)}}(x) & \text{for} \quad x \in (\bar{x}_L,\ \bar{x}_{e_{(2)}}] \\ 1 & \text{for} \quad x > \bar{x}_{e_{(2)}} \end{cases} \tag{9}$$

We show that this constitutes equilibrium strategies by showing that the candidate equilibrium strategies are mutually optimal replies given the equilibrium of the subgame among players in L. A key aspect of the subgame equilibrium is that players from L will overbid any effort x made in stage e that is smaller than $\bar{x}_L$. (i) Consider $k \in E^-(\bar{x}_L)$. A choice of $x_k = 0$ yields $\pi_k(0) = 0$. Effort levels $0 < x_{e_{(i)}} < \bar{x}_L$ yield a payoff $\pi_k = -C_k(x_k) < 0$ and effort levels $x_k \geq \bar{x}_L$ have cost $C_k(x_k)$ that exceed the value of the prize for $k \in E^-(\bar{x}_L)$, leading to a negative payoff, even if k wins the prize with this effort choice. Note that this also holds for the possible effort choice case $x_k = \bar{x}_L$: by the definition of $\bar{x}_L$ as the solution to $C_{l_{(1)}}(\bar{x}_L) = 1$, the definition of $E^-(\bar{x}_L)$ and the strict monotonicity of cost functions of all players in N, strict inequality $C_{e_{(i)}}(\bar{x}_L) > 1$ must hold for all $e_{(i)} \in E^-(\bar{x}_L)$.

Accordingly, $x_k = 0$ is the unique optimal reply for $k \in E^-(\bar{x}_L)$. (ii) Consider $k \in E^+(\bar{x}_L)$. If $\#E^+(\bar{x}_L) = 1$, the only player in this set has to be player 1, i.e., the player with the lowest cost among all players in N. Moreover, $\#E^+(\bar{x}_L) = 1$ implies that the player with the second lowest cost is in the set L, i.e., $2 = l_{(1)} \in L$. Therefore, $\bar{x}_L = \bar{x}_2$. Player 1's payoff is $\pi_1(\bar{x}_2) = 1 - C_1(\bar{x}_2) > 0$ if $x_1 = \bar{x}_2$. Player 1's payoff from $x_1 < \bar{x}_2$ is equal to $-C_1(x_1) \leq 0$, and $\pi_1(x_1) = 1 - C_1(x_1) < 1 - C_1(\bar{x}_2)$ for $x_1 > \bar{x}_2$. Accordingly, $x_1 = \bar{x}_2$ is the optimal reply. (iii) Consider $\#E^+(\bar{x}_L) > 1$. The payoff of player $k \in E^+(\bar{x}_L)$ with $k \notin \{e_{(1)}, e_{(2)}\}$ is $\pi_k(0) = 0$ for $x_k = 0$, $\pi_k(x_k) = -C_k(x_k) < 0$ for all $0 < x_k < \bar{x}_L$, and, given (8), $\pi_k(x_k) < 0$ also for all $x_k \geq \bar{x}_L$. Hence $x_{l(k)} = 0$ is the optimal play. Note that $\#E^+(\bar{x}_L) > 1$ implies that $e_{(1)} = 1$ and $e_{(2)} = 2$. Player 2's payoff is equal to $p_2(x) \cdot 1 - C_2(x_2)$. Using the candidate equilibrium strategies of all other players in E and the properties of the unique equilibrium of the subgame, $\pi_2(0) = 0$ for $x_2 = 0$, $\pi_2(x_2) = -C_2(x_2) < 0$ for $0 < x_2 < \bar{x}_L$, $\pi_2(x_2) = 0$ for $x_2 \in [\bar{x}_L, \bar{x}_2]$, and $\pi_2(x_2) = 1 - C_2(x_2) < 0$ for $x_2 > \bar{x}_2$. Accordingly, any bid from the support $\{0\} \cup (\bar{x}_L, \bar{x}_2]$ is an optimal reply for player 2. Finally, player 1's payoff is $\pi_1(x) = p_1(x) \cdot 1 - C_1(x_1)$. Given the candidate equilibrium strategies of other players $k \in E$ and the effort choices in the subgame perfect equilibrium this payoff is equal to $1 - C_1(\bar{x}_2) > 0$ for all $x_1 \in [\bar{x}_L, \bar{x}_2]$ and smaller for all other x_1. Accordingly, any bid from the interval $[\bar{x}_L, \bar{x}_2]$ is an optimal reply for player 1. Hence, the mixed strategies $F_{l(1)}$ and $F_{l(2)}$ are best replies to each other.

In the candidate equilibrium, $\pi_k = 0$ for all $k \neq 1$. Consider π_1. If $1 \in L$, then $x_k = 0$ for all $k \in E$. Hence, $\bar{x}_E = 0$. If $L = \{1\}$, then 1 wins with no effort and has payoff $\pi_1 = 1$. If $\#L > 1$, then $\pi_1 = 1 - C_1(\bar{x}_{l_{(2)}})$. If $1 \in E$, it has been shown that $x_1 = \bar{x}_2$ is within player 1's equilibrium support and yields $\pi_1 = 1 - C_1(\bar{x}_2)$. This concludes the proof. ■

Proposition 1 shows, that - given our tie-breaking rule - for any partition of players into the sets E and L, there is a *unique* subgame perfect equilibrium. Since the number of different partitions of n players into two sets is 2^n, we have characterized 2^n possible equilibrium configurations. Note however that these result in only n different equilibrium payoff vectors $(\pi_1, 0, .., 0)$, where π_1 can assume the values 1 and $1 - C_1(\bar{x}_i), i = 2, .., n$. Adding another player $(n+1)$ would double the number of feasible partitions (as this player could "join" any previous partition in either E or L), but add only one further equilibrium payoff vector $\boldsymbol{\pi} = (1 - C_1(\bar{x}_{n+1}), 0, ..., 0)$ with a new positive payoff for player 1. If the new player would not be the one with the highest

cost of effort (and hence change our labelling of players), it would still be true that the *number* of different equilibrium payoffs for player 1 only increases by one to $(n+1)$. Moreover, n of those are identical to the previous ones unless the new player happens to be the -new- strongest one with the lowest cost of effort. In this latter case all n feasible equilibrium payoffs for player 1, which are smaller than 1, can be different from the previous feasible equilibrium values with only n players. With just 2 players the equilibrium payoffs for three player partitions $(E_1, L_1) = (\{1\}, \{2\}), (E_2, L_2) = (\{1,2\}, \{\emptyset\})$ and $(E_3, L_3) = (\{\emptyset\}, \{1,2\})$ are identically equal to $(1 - C_1(\bar{x}_2), 0)$, whereas $(E_4, L_4) = (\{2\}, \{1\})$ yields the payoffs (1,0).

Proposition 1 also shows that the equilibrium of the generalized Stackelberg equilibrium has similar properties as the solution with two players. Only the player with the strongest cost advantage has a positive payoff, and this payoff is larger if this player chooses his effort last. An interesting aspect is that the all-pay auction that takes place at stage l and the competition it involves for those players, does not benefit the players who choose at stage e compared to the situation in which there is only one player who chooses late. The reason is that $\bar{x}_L$, the minimal bid for a player in E in order to win, does not depend on actual behavior of players in the competition at L, but their *potential* limit behavior in rational play. This largest possible rational counter bid, however, is determined by exogenous data, namely the cost parameters of the players moving at L. Also similar to the two-player case, the player with the lowest cost is best off if he moves later than all other players. For all other players, timing does not affect their payoff. They all cannot expect to gain something from participating in the contest and from making positive bids. These results do not depend in an essential way on our choice of tie-breaking rule. This rule was chosen in order to arrive at *strict* subgame perfect equilibria. Choice of other tie-breaking rules; i.e. the probably more "'obvious"' one, which stipulates the winning probability of a player from M as always equal to $\frac{1}{\#M}$ *regardless* of his membership in E or L, would only produce $\varepsilon-$equilibria. Moreover, these $\varepsilon-$equilibria would all lie in $\varepsilon-$neighborhoods of our strict equilibria.

3 Endogenous timing

Let N continue to be the set of players, but consider a game that is "enlarged" by a decision stage 1 that occurs prior to stage e. In stage 1 each player $k \in N$

chooses whether to choose effort at stage e or at stage l, and may randomize, in which case $q_k \in [0,1]$ denotes the probability by which k chooses e and $(1-q_k)$ the probability by which k chooses l. Once all players have chosen their q_k, the actual timing $t_k \in \{e,l\}$ of each player k's effort choice is determined and observed by all players. Let $\mathbf{t} = (t_1, ...t_n)$ denote the vector describing this set of actual timing, which emerges as a draw from $\mathbf{Q}$ that denotes the joint probability distribution over (independent) timing choices.

Note that a given $\mathbf{t}$ sorts the players into sets $E(\mathbf{t})$ and $L(\mathbf{t})$ which constitute a partition of N, and that there is a one-to-one mapping between feasible $\mathbf{t}$ and feasible partitions of N into two sets E and L.

At stage e, the history of the game is described by ($\mathbf{Q}$ and) $\mathbf{t}$. If $t_k = e$, then player k must choose his effort $x_k \in [0,K]$, and this choice can be a function of this history. He may choose to randomize and choose effort as a draw from a cumulative distribution function F_k as a function which generally can be a function of the history at stage e. Players with $t_k \neq e$ cannot take any action at this point. At the end of stage e, all players k with $t_k = e$ have chosen their actions, and their actual efforts x_k are observed by all players. The vector $\mathbf{x}_E$ describes these actual efforts.

The history of the game at stage l is characterized by ($\mathbf{Q}$,$\mathbf{F}_E$,) $\mathbf{t}$ and $\mathbf{x}_E$, (where $\mathbf{F}_E$ denotes the joint distribution function from which actual effort choices $\mathbf{x}_E$ have been drawn independently.) Players k with $t_k = l$ must choose an effort level $x_k \in [0,K]$ and may choose effort as a function of the history. They may also randomize and choose according to a cumulative distribution function F_k. Players k with $t_k \neq l$ cannot assume any action at this point.

At the end of stage l the prize is awarded to the player who expended the highest effort, and the same tie-breaking rule as in section 2 is used if several players expended the same highest amount of effort.

Note that because of uniqueness of equilibrium at stage l *given* x_E (and t), it is irrelevant, which distributions $\mathbf{Q}$ resp. $\mathbf{F}_E$ have generated these realizations. I.e. a player cannot condition his behavior at l on $\mathbf{Q}$ resp. $\mathbf{F}_E$ *given* the realizations x_E and t, which are observed by all players. The same argument then applies for players at stage e: *given* t *and* given the knowledge, that players at l cannot condition behavior on $\mathbf{Q}$ and their choices of $\mathbf{F}_E$, a player at e cannot condition his strategy choice on $\mathbf{Q}$ as t already determines the unique subgame perfect equilibrium in the continuation game (see the proof of Proposition 1).

The following Proposition characterizes subgame-perfect equilibrium in

the game with endogenous timing of effort decisions.

Proposition 2

i) $q_1 = 0$ *is an almost strictly dominant strategy for player 1; i.e. it is the unique best reply to any choice of* $(q_2, ..., q_n)$ *by players 2 to n, for which* $q_2 > 0$. *For any* $(q_2, ..., q_n)$ *with* $q_2 = 0$ *any choice of* $q_1 \in [0, 1]$ *is a best reply for player 1.*

ii) Let $q_1 = 0$*; then any choice of timing strategies from* $[0, 1]$ *by players 2,...,n is (part of) a subgame perfect equilibrium in the game with endogenous timing of effort decisions.*

Proof. Note first that, by the one-to-one mapping between $\mathbf{t}$ and $E(\mathbf{t})$ and $L(\mathbf{t})$, $\mathbf{Q}$ induces a probability distribution over partitions (E, L) of N. Note further that the equilibrium that is characterized in the proof of Proposition 1 for any given partition and with payoffs as in Proposition 1 for this partition, is the subgame perfect equilibrium of the continuation game of the "enlarged" game for which this partition emerges at the end of stage 1. And only $\mathbf{t}$ (or, equivalently, the partition that is described by $\mathbf{t}$) constitutes the payoff-relevant history.

Hence players anticipate that the equilibrium that is characterized in Proposition 1 is played in the continuation games for any partition of N into $E(\mathbf{t})$ and $L(\mathbf{t})$. We now show that for a player k, $k > 1$, any $q_k \in [0, 1]$ is an optimal choice *regardless* of the chosen q_l by players $l \in \{1, ..., k-1, k+1, ..., n\}$. We then show, that $q_k = 0$ in turn is an optimal reply for player 1 to any of these choices by players 2 to n, and *uniquely* so, if $q_2 > 0$.

Consider first $k > 1$. Given any joint distribution over $(t_1, ..., t_{k-1}, t_{k+1}, ..., t_n)$, any q_k leads to a probability distribution over partitions E and L from which one actual partition results. The payoff of player k in the subgame perfect equilibrium for each of these partitions is $\pi_k = 0$ by Proposition 1. Accordingly, any q_k is an optimal reply to any joint distribution over $(t_1, ..., t_{k-1}, t_{k+1}, ..., t_n)$.

Turn now to $k = 1$. Define $\mathbf{t}_{-1} \equiv (t_2, ..., t_n)$. For a given $\mathbf{t}_{-1}$, let $j_{\min}(\mathbf{t}_{-1}) = \min[j \in \{2, ..., n\} \,|\, t_j = l\ ]$. Then the payoff of player 1 from choosing q_1 given $\mathbf{t}_{-1}$ is $(1 - q_1)(1 - C_1(\bar{x}_{j_{\min}(\mathbf{t}_{-1})}) + q_1(1 - C_1(\bar{x}_2))$ and this payoff is non-increasing in q_1 and strictly decreasing if $j_{\min}(\mathbf{t}_{-1}) \neq 2$. Player 1 is always strictly better off by choosing l instead of e, if player 2's choice q_2 results in the realization of $t_2 = e$. Player 1 is indifferent between e and

l if and only if $t_2 = l$. This, however, is *ex ante*; i.e. before the q's are (simultaneously) chosen, only assured, if $q_2 = 0$. As soon as q_2 leads with positive probability to the realization $t_2 = l$, player 1 is ex ante better off by choosing l with certainty; i.e. $q_1 = 0$. ■

The intuition for why other players are essentially indifferent about when to choose their effort is straightforward. They have a payoff of zero in the continuation game that starts with stage e, regardless of the partition of players between E and L. This makes them indifferent regarding their own choice of timing. This argument does not apply in general for player 1. If this player chooses his effort late, he essentially competes only with the players who also choose their effort late, as all players in E who choose their effort early are induced to choose zero effort in the equilibrium that is characterized in Proposition 1. However, if player 1 chooses his effort early, he essentially competes with all other players, either simultaneously, or via the fact that they can react to his bid and consider whether to make a higher bid. The advantage of choosing late therefore is that all competitors who choose early drop out of the competition, and this advantage is bigger if the group of players who choose early includes the particularly strong players. In the extreme case in which all players $k \neq 1$ belong to E and choose early, if player 1 chooses late, there is no competitor left and player 1 receives the prize without any effort, whereas, if he chooses early, he competes with the whole group of other players. This may explain why player 1 has a preference for moving late. The only case in which this preference is not strict is when player 2 also chooses late. In this case the show-down between these two strongest players is inevitable, and player 1 is indifferent whether to choose early and preemt player 2, or whether he enters into a simultaneous all-pay auction with this second-strongest player. But note that any offer of a small "premium" payment $\epsilon > 0$ for moving early -either by the rules of the game or by player 1- would select the efficient equilibrium as the unique equilibrium (for all tie-breaking rules).

4 Conclusions

The equilibrium in an all-pay auction with complete information has some interesting properties if one allows for some generalizations that naturally extend the results on the two-player all-pay auction with sequential bids. We consider partitions of a larger group of players into two subsets. Play-

ers in one subset choose their effort individually and simultaneously, prior to the individual and simultaneous choices of members of the other group, and we consider the endogenous formation of such groups and their timing of bids. Main results are: the sequential structure of bids favors the strongest player, particularly if some of his strongest competitors have to make their bids prior to him. For an appropriate partition of players the prize is efficiently allocated to the strongest bidder, with aggregate bid cost in this equilibrium of zero. Moreover, this partition of players into groups can emerge as an equilibrium outcome if all players choose their timing of bids simultaneously in a stage prior to the actual bidding stages. Hence our simple redesign of the simultaneous all-pay auction as a Stackelberg game yields an extremely efficient allocation mechanism for competitions, which involve (partially) wasteful bids or effort choices. In the prominent equilibrium, in which player 1 is the only one to move late, the prize is allocated efficiently with no "rent dissipation" at all!

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