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## The over-concentrating nature of simultaneous ascending auctions

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THE CENTER FOR THE STUDY  
OF INDUSTRIAL ORGANIZATION  
AT NORTHWESTERN UNIVERSITY

Working Paper #0064

The Over-Concentrating Nature of  
Simultaneous Ascending Auctions<sup>\*</sup>

By

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## Abstract

This paper analyzes simultaneous ascending auctions of two different items, viewed as complements by multi-item bidders. The finding is that such auctions overly concentrate the goods to a multi-item bidder and never overly diffuse them to single-item bidders. The main reason is that some bidders strictly want to jump-bid and jump-bidding allows the game to mimic a package auction, where single-item bidders cannot fully cooperate among themselves to bid against multi-item bidders. The second reason is that over-concentration causes resale and there is an equilibrium where a multi-item bidder becomes the reseller and chooses to under-sell the goods.

# 1 Introduction

Simultaneous ascending auctions of heterogeneous items have caught much attention from researchers ever since the United States government, influenced by economists, started using these auctions to allocate radio frequencies in early 1990s. Even before that major application, economists had taken simultaneous ascending auctions as natural conceptual constructs to understand decentralized markets. Indeed, when there is no central coordination on the sales of multiple goods separately owned by different entities, the efficient Vickrey-Clarke-Groves mechanisms are unlikely to be used, and it is natural to assume that separate initial owners sell their goods separately. To capture the interactions among different sectors of an economy without artificially ranking one sector over another, it is natural to assume that these separate auctions start simultaneously. The open-outcry ascending-bid feature of these auctions provides a transparent setup to understand the process of price formation.

Researchers have found that simultaneous ascending auctions can achieve efficient outcomes if the items for sale are substitutes (Gul and Stacchetti [9] and Milgrom [15]). However, when the items may be complements, these auctions are not found to achieve efficiency (Gul and Stacchetti [10] and Milgrom [15]), although efficiency can be achieved by a centralized bidding process (Ausubel [2] and Bikhchandani, de Vries, Schummer and Vohra [5]). To capture the decentralized nature of markets, we need a theory of simultaneous ascending auctions of possibly complementary goods without central coordination. Although these auctions are already known to be probably inefficient, researchers have not found a pattern of the inefficiency. The hurdle is that inefficiency may take various forms, all parameter-dependent, so it is difficult to make predictions. These auctions are known to suffer an *exposure problem*: a bidder may have bought an item at a price above its standalone value and fail to acquire its complements (e.g., Bykowsky, Cull, and Ledyard [7], and Milgrom [15]). Worried by this problem, a bidder who considers multiple items as complements may underbid before he acquires any item and overbid for the rest after he has acquired some. Then the goods may be *over-concentrated* to a single bidder while efficiency requires that they go to different bidders, or the goods may be *over-diffused* to separate owners while efficiency requires that a single bidder should own them. Both kinds of inefficiency seem to be probable and we may not know which one is dominant without knowing specific parameters.

This paper, in contrast, gives an unambiguous prediction that the inefficiency of simultaneous ascending auctions takes only the form of probable over-concentration and never over-diffusion. This prediction is obtained through modifying the traditional model to capture the dynamic transparent interactions among auctions: When a multi-item bidder  $i$  is about to buy an item A without knowing the future price of its complement B, his rivals for B wish to finish the auction of B before bidder  $i$  buys A, because  $i$  will become more aggressive for B once he has bought A. To expedite the ascending auction, the rivals jump-bid for B. From these jump bids, player  $i$  infers his rivals' values and hence the future price for B. Then, before committing to buying any item, the multi-item bidder knows immediately whether he can profitably acquire the entire package. If the answer is No, he immediately withdraws his bids from all items to avoid the loss (which will be unavoidable if he buys A now). Provided that players react quickly enough, these can be done in a split second. Hence the exposure problem vanishes, and the only remaining source of inefficiency is that bidders who value only single items cannot fully cooperate with each other to compete against multi-item bidders. This kind of inefficiency is the well-known *threshold problem* for package auctions, where bids are contingent on packages of items (e.g., [7] and [15]). That leads to probable over-concentration and never over-diffusion.

Over-concentration creates a strict incentive for resale. Hence the model is further modified to allow a winner of all items to resell them. This paper finds that the same kind of inefficiency persists in spite of resale: Being a monopolist, a middleman chooses to under-sell the goods and hence over-concentrates them in his own hands.

The primitives are listed in §2. There are two items for sale, A and B, and three bidders, a *local* bidder who values only A, another who values only B, and a *global* bidder who values both as complements. Bidders commonly know who is global and who is local but do not know others' valuations. Results in this paper can be extended to the more general case where there are multiple i.i.d. clones of each bidder (§6.4). Extension to the case with more than two items is left outside this paper: The intuition appears to be extendable, but formalizing the extended intuition requires a fixed point argument (§7.1).

The traditional model, which implicitly assumes away jump-bidding, is analyzed in §3. It illustrates the aforementioned ambiguity in the traditional model. This section is related

to the asymmetric-information analysis of simultaneous auctions in the literature such as Krishna and Rosenthal [13] (sealed-bid second-price), Rosenthal and Wang [17] (sealed-bid first-price), Albano, Germano and Lovo [1] (ascending-bid, two items, and uniformly distributed values), and Brusco and Lopomo [6] (tacit collusion between bidders who can reenter an auction to retaliate a non-colluder; not about exposure or threshold problem). None of them consider jump-bidding.

In §4, the traditional model is modified to allow jump bidding and bid withdrawal (with penalty) during the course of an auction. The amendments can be enforced individually within each auction without any centralized intervention. The crucial step in this section is to construct a continuation equilibrium in the event that a local bidder is the first to drop out. One can view this continuation game as an English auction that proceeds arbitrarily fast due to jump bidding. The construction (Proposition 1) is nontrivial in spite of the seeming simplicity of English auctions, because the bids in this continuation game ascend through open outcries instead of the traditional modeling devices of price clocks and fixed increments. The upshot is that, once a local bidder becomes the first dropout, the two items are sold almost immediately, within two consecutive jump bids. Then Proposition 2 says that the simultaneous ascending auctions can replicate any equilibrium allocation of an ascending package auction (defined in §4.4), which is over-concentrating. This result is relevant to the ongoing policy debate on the necessity of package auctions in allocating radio frequencies. This section is slightly related to the jump-bidding literature such as Avery [4] and Gunderson and Wang [11], which have shown that jump-bidding may reduce the demand from one's rival. None of them consider multiple heterogeneous items. The timing of jump-bidding is exogenous in those papers and is endogenous in this paper.

In §5, the model is further modified to allow *cross-bidding*, bidding for an unvalued item. We need to consider cross-bidding because, conditional on the equilibrium in the traditional model (Lemma 1), a local bidder wishes to bid for his unvalued item in order to prevent the global bidder from becoming more aggressive after winning it. Cross-bidding opens the possibility of reviving the exposure problem as a self-fulfilling prophecy: The global bidder may fear that a cross-bidder will quit the two auctions one by one at separate times and hence may fear the possible event that he will have to buy one item without knowing the price of its complement; then a cross-bidder may find it unnecessary to jump-bid given certain prior

distributions. However, such exposure problem vanishes, for any prior distribution, if the global bidder engages the local player into a jump-bidding continuation equilibrium similar to the one without cross-bidding. Consequently, the over-concentration prediction in §4 is extended (Proposition 3).

In §6, the model is further modified to allow resale. Then Proposition 4 extends the over-concentration prediction. The necessity and intuition for the extension have been mentioned previously. The paper ends with a speculative remark (§7.2) on the self-emergence of mechanisms. The index at the end of the paper lists all the special symbols and terms.

## 2 The primitives

There are two items, A and B. There are three bidders: a local bidder  $\alpha$  who values only item A, a local bidder  $\beta$  who values only item B, and a global bidder  $\gamma$  who views both items as complements. The following table lists their valuations:

	$\emptyset$	A	B	A & B
local $\alpha$	0	$t_\alpha$	0	$t_\alpha$
local $\beta$	0	0	$t_\beta$	$t_\beta$
global $\gamma$	0	0	0	$t_\gamma$

For each  $i \in \{\alpha, \beta, \gamma\}$ ,  $t_i$  is a random variable whose realized value is bidder  $i$ 's the private information and is independently drawn from a distribution  $F_i$ , with continuous positive density  $f_i$  and support  $[0, \bar{t}_i]$ . A bidder's payoff is equal to his valuation of the package he acquires minus his total payment.

The solution concept is perfect Bayesian equilibrium, briefly called *equilibrium*.

A random variable and its realized value are denoted by the same letter, with the former boldfaced. For example, if  $g(x, y)$  and  $\psi(z)$  are real functions of variables  $x$ ,  $y$ , and  $z$ ,  $E[g(\mathbf{x}, y) \mid \psi(\mathbf{z}) \geq 0]$  denotes the expected value of  $g(x, y)$ , with  $(\mathbf{x}, \mathbf{z})$  random, conditional on  $\psi(\mathbf{z}) \geq 0$ . Let  $1_S(\cdot)$  denote the indicator function for event  $S$ . Let  $z^+ := \max\{z, 0\}$ .

## 3 A basic analysis of the exposure problem

### 3.1 The basic mechanism

The two items are auctioned off via separate clock auctions that start simultaneously. Prices start at zero. For each item  $k$ , the price  $p_k$  for item  $k$  rises continuously at an exogenous positive speed  $\dot{p}_k$  until  $k$  is sold. Bidder  $\alpha$  can bid only for item A, bidder  $\beta$  only for B, and  $\gamma$  can bid for both items. Ties are broken by coin toss.

To be eligible for an item, a bidder needs to participate in its auction from the start. Once he *quits* (drops out) from an item, a bidder cannot raise his bid for that item any more. If a bidder does not quit from an item, we say he *continues* or *stays* or *remains* for it. The auction of an item ends when all but one bidder has quit the item; immediately the remaining bidder buys the item at its current price.<sup>1</sup> The good cannot be returned for refund. Bidders' actions are commonly observed.

### 3.2 The equilibrium

Restricted to bidding only for his valued item, the only undominated strategy for a local bidder is to bid for his desired item up to its true value. The undominated strategy for global bidder  $\gamma$ , in contrast, is to quit before the total price reaches the value of having both items. That is because he takes into account the exposure problem that he may buy an item at a price above its standalone value and fail to acquire its complement at a low enough price.

**Lemma 1** *For any  $(p_A, p_B) \in [0, \bar{t}_\alpha] \times [0, \bar{t}_\beta]$  and type  $t_\gamma \in [0, \bar{t}_\gamma]$ , define*

$$v_A(t_\gamma, p_B) := \mathbb{E} [(t_\gamma - \mathbf{t}_\beta)^+ | \mathbf{t}_\beta \geq p_B]; \quad (1)$$

$$v_B(t_\gamma, p_A) := \mathbb{E} [(t_\gamma - \mathbf{t}_\alpha)^+ | \mathbf{t}_\alpha \geq p_A]. \quad (2)$$

*If cross-bidding and jump-bidding are banned, straightforward bidding is weakly dominant for each local bidder. Given any current  $(p_A, p_B)$ , the best reply from the global bidder  $\gamma$  is:*

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<sup>1</sup>This decentralized closing rule is aligned with this paper's focus on the decentralized nature of markets. The simultaneous auctions used by FCC have a centrally coordinated closing rule (Milgrom [15]).



1. If neither A nor B has had a winner, continue bidding for both items if  $v_A(t_\gamma, p_B) > p_A$  and  $v_B(t_\gamma, p_A) > p_B$ , and quit from both auctions if one of the inequalities fails.
2. If item A or B has been won by someone else, quit from both auctions immediately.
3. If item A (or B) has been won by bidder  $\gamma$ , continue bidding for item B (or A) until its current price  $p_B$  (or  $p_A$ ) reaches  $t_\gamma$ .

**Proof** Strategy 2 is obvious: the price for an item say A is for sure higher than its standalone value 0, since local bidder  $\alpha$ 's value is for sure positive. Strategy 3 simply follows from the fact that the payment for the already acquired item is sunk.

Consider the event for strategy 1, with current prices  $(p_A, p_B)$  and both local bidders remaining active. Suppose bidder  $\gamma$  has bought A. Then we are in the event for strategy 3; thus, if  $t_\gamma > t_\beta$  he wins item B and his profit is equal to  $t_\gamma - t_\beta - p_A$  (since bidder  $\beta$  is straightforward); else  $\gamma$  loses B and his profit is  $-p_A$ . Thus, when both local bidders are still active, bidder  $\gamma$ 's expected profit from buying item A at the current instant is equal to

$$v_A(t_\gamma, p_B) - p_A, \tag{3}$$

and analogously his expected profit from buying item B at the current instant is equal to

$$v_B(t_\gamma, p_A) - p_B. \tag{4}$$

Note: as type distributions have no atom or gap, (3) and (4) are continuous and strictly decreasing functions of  $(p_A, p_B)$  and hence shrink continuously with time.

Let us prove the optimality of strategy 1. At any instant in the event for strategy 1, either (a) both (3) and (4) are positive, or (b) one of them is nonpositive. In case (a), by continuity of (3) and (4) with respect to time, these inequalities continue to hold for a sufficiently short interval. Recall that (3) stands for bidder  $\gamma$ 's expected profit from buying A conditional on not yet quitting B, and recall the analogous interpretation for (4). Thus, at the current instant it is dominated to quit from one item and continue with the other. It is also dominated to quit both items, because doing so gives zero payoff while not doing so ensures a positive expected payoff. Hence bidder  $\gamma$  continues on both items in this case.

In case (b), (3) or (4) is nonpositive. Without loss, say it is (3). As (3) is strictly decreasing in time, bidder  $\gamma$ 's expected profit from buying item A is negative from now on if he does not quit B. If he quits B, then he also quits A by strategy 2. Thus, he quits at least A; then by strategy 2 he quits B at the same time. Hence strategy 1 is optimal. ■

### 3.3 Various kinds of inefficiency

Let us examine the allocation induced by the above equilibrium. By its definition (1) and the atomless and gapless type distributions, the function  $v_A(t_\gamma, \cdot)$  is continuous and strictly decreasing; when  $p_B$  decreases from  $\min\{t_\gamma, \bar{t}_\beta\}$  to zero,  $v_A(t_\gamma, p_B)$  rises from  $[t_\gamma - \bar{t}_\beta]^+$  to  $E[t_\gamma - \mathbf{t}_\beta]^+$  (Figure 1). Thus, in  $R^2$ , given any  $t_\gamma \in [0, \bar{t}_\gamma]$ , the ray

$$\{(p_A, p_B) \in [0, \infty)^2 : p_B = (\dot{p}_B/\dot{p}_A)p_A\} \quad (5)$$

and the continuous path

$$\{(v_A(t_\gamma, p_B), p_B) : p_B \in [0, \min\{t_\gamma, \bar{t}_\beta\}]\} \cup \{(p_A, \min\{t_\gamma, \bar{t}_\beta\}) : p_A \in [0, (t_\gamma - \bar{t}_\beta)^+]\} \quad (6)$$

have exactly one common point, denoted by  $(p'_A(t_\gamma), p'_B(t_\gamma))$  (Figure 1). Analogously, (5) and the path

$$\{(p_A, v_B(t_\gamma, p_A)) : p_A \in [0, \min\{t_\gamma, \bar{t}_\alpha\}]\} \cup \{(\min\{t_\gamma, \bar{t}_\alpha\}, p_B) : p_B \in [0, (t_\gamma - \bar{t}_\alpha)^+]\} \quad (7)$$

have exactly one common point, denoted by  $(p''_A(t_\gamma), p''_B(t_\gamma))$  (Figure 1). Note that (5) represents the ray along which  $(p_A, p_B)$  rises when both auctions are still going on. Hence at the point  $(p'_A(t_\gamma), p'_B(t_\gamma))$ , either bidder  $\gamma$  becomes indifferent between winning and losing A conditional on staying for B, or the price of B for sure stops rising ( $p'_B(t_\gamma) = \bar{t}_\beta$ ). Likewise, at  $(p''_A(t_\gamma), p''_B(t_\gamma))$ , either bidder  $\gamma$  becomes indifferent about winning B conditional on staying for A, or  $p_A$  for sure stops rising ( $p''_A(t_\gamma) = \bar{t}_\alpha$ ). Let

$$p_A^*(t_\gamma) := \min\{p'_A(t_\gamma), p''_A(t_\gamma)\} \quad \& \quad p_B^*(t_\gamma) := \min\{p'_B(t_\gamma), p''_B(t_\gamma)\}.$$

Since the slope of the price ray  $p_B = (\dot{p}_B/\dot{p}_A)p_A$  is positive,

$$(p_A^*(t_\gamma), p_B^*(t_\gamma)) = (p'_A(t_\gamma), p'_B(t_\gamma)) \quad \text{or} \quad (p_A^*(t_\gamma), p_B^*(t_\gamma)) = (p''_A(t_\gamma), p''_B(t_\gamma)). \quad (8)$$

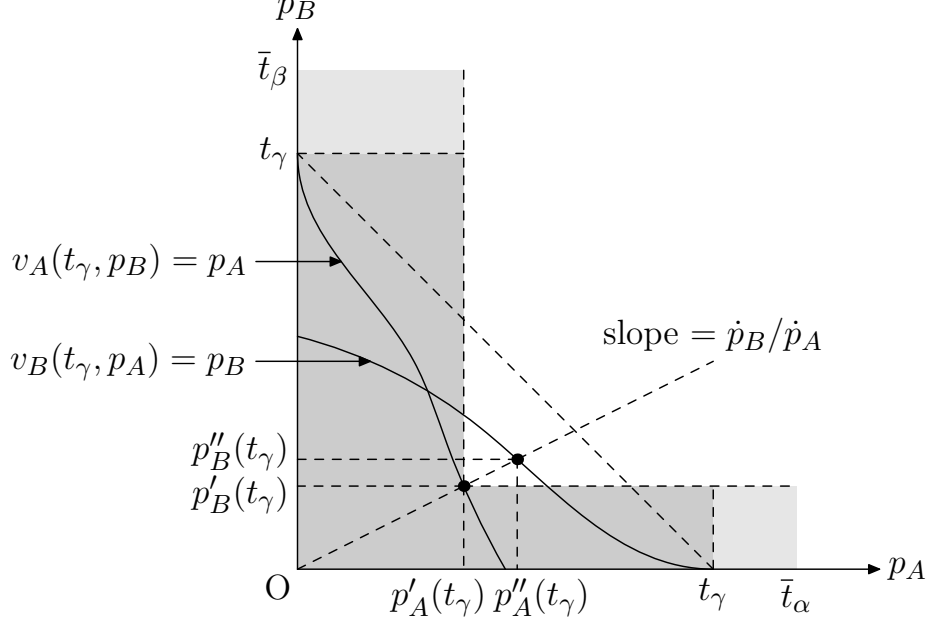


Figure 1: Dark:  $\{A, B\} \rightarrow \gamma$ ; grey:  $\{A, B\} \rightarrow \alpha$  or  $\beta$ ; white:  $A \rightarrow \alpha$  &  $B \rightarrow \beta$ .

Note that  $(p_A^*(t_\gamma), p_B^*(t_\gamma))$  is the instant at which global bidder  $\gamma$  quits both items, unless he has already won an item.

The equilibrium allocation is: If  $t_\alpha > p_A^*(t_\gamma)$  and  $t_\beta > p_B^*(t_\gamma)$ , item A goes to local bidder  $\alpha$  and item B goes to local  $\beta$ . If  $t_\alpha < p_A^*(t_\gamma)$  and  $t_\beta < t_\gamma$ , both items go to global bidder  $\gamma$  (plans 1 and 3 of Lemma 1). If  $t_\alpha < p_A^*(t_\gamma)$  and  $t_\beta > t_\gamma$ , item A goes to  $\gamma$  and B goes to  $\beta$ . If  $t_\beta < p_B^*(t_\gamma)$ , then  $\gamma$  wins both items if  $t_\alpha < t_\gamma$  and wins only B and loses A to  $\alpha$  if  $t_\alpha > t_\gamma$ . Ties occur with zero probability, as type distributions are atomless and functions  $v_A(\cdot, p_B)$  and  $v_B(\cdot, p_A)$  are continuous.

**Lemma 2** *If  $t_\gamma > 0$ , then  $p_A^*(t_\gamma) > 0$  and  $p_B^*(t_\gamma) > 0$ ; if also  $t_\gamma \neq \bar{t}_\alpha + \bar{t}_\beta$ , then  $t_\gamma > p_A^*(t_\gamma) + p_B^*(t_\gamma)$ .*

**Proof** Since  $0 < \dot{p}_B / \dot{p}_A < \infty$ , it is obvious that  $p_A^*(t_\gamma) > 0$  and  $p_B^*(t_\gamma) > 0$  for all  $t_\gamma > 0$ . To prove the rest of the lemma, recall definition (1) and the assumption that the distribution of  $t_\gamma$  has no gap. Then  $v_A(t_\gamma, p_B) < t_\gamma - p_B$  unless  $p_B = \bar{t}_\beta$ , and  $v_B(t_\gamma, p_A) < t_\gamma - p_A$  unless  $p_A = \bar{t}_\alpha$ . Thus, by (8), the desired inequality  $t_\gamma > p_A^*(t_\gamma) + p_B^*(t_\gamma)$  follows unless

$$(p'_A(t_\gamma), p'_B(t_\gamma)) = (t_\gamma - \bar{t}_\beta, \bar{t}_\beta) = (\bar{t}_\alpha, t_\gamma - \bar{t}_\alpha) = (p''_A(t_\gamma), p''_B(t_\gamma)),$$

which implies  $t_\gamma = \bar{t}_\alpha + \bar{t}_\beta$ . ■

Inefficiency of the equilibrium takes three different forms, each probable. One is *over-diffusion*: item A goes to local bidder  $\alpha$  and B goes to local  $\beta$ , while efficiency requires that both items go to the global bidder. This is the event

$$t_\alpha > p_A^*(t_\gamma) \ \& \ t_\beta > p_B^*(t_\gamma) \ \& \ t_\alpha + t_\beta < t_\gamma,$$

which occurs with a positive probability because  $t_\gamma > p_A^*(t_\gamma) + p_B^*(t_\gamma)$  (Lemma 2) and type distributions have no gap. The second kind of inefficiency is *over-concentration*: one bidder wins both items while efficiency requires that they go to different bidders, i.e.,

$$[t_\alpha < p_A^*(t_\gamma) \ \& \ t_\beta < t_\gamma \ \& \ t_\alpha + t_\beta > t_\gamma] \quad \text{or} \quad [t_\beta < p_B^*(t_\gamma) \ \& \ t_\alpha < t_\gamma \ \& \ t_\alpha + t_\beta > t_\gamma].$$

This event occurs with a positive probability because  $p_A^*(t_\gamma) > 0$  and  $p_B^*(t_\gamma) > 0$  (Lemma 2). The third kind of inefficiency is incomplete diffusion: the global bidder wins exactly one item while efficiency requires both items go to local bidders. This is the event

$$[t_\alpha < p_A^*(t_\gamma) \ \& \ t_\beta > t_\gamma] \quad \text{or} \quad [t_\beta < p_B^*(t_\gamma) \ \& \ t_\alpha > t_\gamma],$$

which occurs with a positive probability, again because  $p_A^*(t_\gamma) > 0$  and  $p_B^*(t_\gamma) > 0$ .

Thus, the exposure problem leads to various kinds of inefficient outcomes. Such ambiguity, however, is only because our analysis so far has not fully exploited the transparent nature of simultaneous ascending auctions. With actions commonly observed, bidders might be able to avoid the exposure problem via signaling such as jump-bidding.

## 4 Jump bidding eliminates the exposure problem

### 4.1 The self-emergence of jump-bidding

Conditional on the equilibrium of the basic mechanism and the event that a local bidder is the first to quit, it is Pareto improving to allow jump-bidding and bid withdrawal. To see that, consider the moment when local bidder  $\alpha$  is quitting at  $p_A$ . Now global bidder  $\gamma$  is on

the verge of buying A without knowing how much he will have to pay for its complement B. Suppose the other local bidder  $\beta$  could credibly reveal his value  $t_\beta$  to bidder  $\gamma$  at this moment. Then bidder  $\gamma$  would know that the price for item B will be  $t_\beta$ . If his value is less than  $p_A + t_\beta$ ,  $\gamma$ 's profit will be negative if he is to buy both items, and he would not be able to avoid such loss if he buys A now, because he will bid for B up to  $t_\gamma$  once he has bought A. Thus, if  $t_\gamma < p_A + t_\beta$ , bidder  $\gamma$  wishes to withdraw from both items immediately. If allowed, such arrangement not only helps the global bidder to avoid the exposure problem, it also expands local bidder  $\beta$ 's winning event from  $\{t_\gamma : t_\beta > t_\gamma\}$  to  $\{t_\gamma : t_\beta > t_\gamma - p_A\}$ , because  $\gamma$ 's maximum willingness-to-pay for item B jumps from  $t_\gamma - p_A$  to  $t_\gamma$  once  $\gamma$  has bought A. Now that bidder  $\gamma$  may probably withdraw from A, the first dropout local bidder  $\alpha$  may probably get to buy A at his highest bid price  $p_A$ . Hence this arrangement is Pareto improving.

Furthermore, each remaining bidder has a unilateral interest in this arrangement. More precisely, conditional on the equilibrium of the basic mechanism and the event that a local bidder say  $\alpha$  is the first to quit, it is dominant for local bidder  $\beta$  to submit a jump bid for B, and it is dominant for global bidder  $\gamma$  to expedite the auction of B before buying or withdrawing from A. Say  $\alpha$  is the first to quit when the current prices are  $(p_A, p_B)$ . With global bidder  $\gamma$  sticking to the basic mechanism equilibrium so far, the fact that he has not quit implies that  $v_A(t_\gamma, p_B) > p_A$  and  $v_B(t_\gamma, p_A) > p_B$  (Lemma 1); thus, unless interrupted,  $\gamma$  would immediately buy item A and then bid for B until its price reaches  $t_\gamma$ . Therefore, no matter how  $\gamma$  will infer from others' actions, local bidder  $\beta$  gains from bumping up the price for B before  $\gamma$  buys A so that the higher price for B might exceed  $\gamma$ 's currently low maximum willingness-to-pay for B. Specifically, one can show that bidder  $\beta$  gains at least

$$p_A \mathbb{E} \left[ \mathbf{1}_{p_A + p'_B > \mathbf{t}_\gamma}(\mathbf{t}_\gamma) \mid v_A(\mathbf{t}_\gamma, p_B) > p_A; v_B(\mathbf{t}_\gamma, p_A) > p_B \right]$$

by submitting a jump bid

$$p'_B := \mathbb{E} \left[ \mathbf{t}_\gamma \mid t_\beta > \mathbf{t}_\gamma; v_A(\mathbf{t}_\gamma, p_B) > p_A; v_B(\mathbf{t}_\gamma, p_A) > p_B \right] - p_A.$$

For bidder  $\gamma$ , if he sticks to the basic mechanism equilibrium, his payoff is  $(t_\gamma - t_\beta)^+ - p_A$  (as explained previously, he will buy A and then bid for B up to  $t_\gamma$ ). If he can somehow expedite the price ascension of B for a while before deciding on A, then his payoff will be the same if the faster rising price does not reach  $t_\beta$ , and his payoff will be  $(t_\gamma - t_\beta - p_A)^+$  if the price reaches  $t_\beta$ . Note that  $(t_\gamma - t_\beta - p_A)^+ \geq (t_\gamma - t_\beta)^+ - p_A$  for all possible  $t_\beta$  and

strictly so for some  $t_\beta$ . Hence each bidder gains from unilaterally triggering a jump-bidding and withdrawal arrangement.

This arrangement, however, is not admissible in the basic mechanism, which restricts bidders' choice to deciding when to release the buttons of the price clocks. If all but one bidder has quit an item, the remaining bidder automatically buys it without any chance to react. It is not natural to insist that such restrictive basic mechanism be the model of simultaneous ascending auctions, for at least two reasons. First, bids in many actual ascending auctions are submitted through open outcries, so it is unnatural to assume that bidders have no pausing time to react to each other during the intermission between outcries. Second and more importantly, even if the auctioneers wish to implement the basic mechanism, bidders strictly prefer, both collectively and unilaterally, breaking its rules, as demonstrated above.

Thus, we amend the mechanism: First, if someone drops out from an auction, the remaining bidders in that auction are given a brief moment to adjust their actions including withdrawing their bids (subject to possible penalty). Second, bidders are allowed to submit jump-bid. These new rules are formalized in §4.2.

## 4.2 A model that allows jump-bidding

During the auction of an item, an active bidder in the auction has four admissible actions:

- Non-dropout actions:
  - *continue*: keep pressing his button for the item;
  - *jump-bid*: cry out a bid higher than the item's current price;
- Dropout actions (both are called *drop out* or briefly *quit*):
  - *stop*: release the button and forever forfeit the right to raise his bid for the item;
  - *withdraw*: release the button and forever forfeit the eligibility to buy the item.

A bidder is called *the first dropout* if he quits (stops or withdraws) all the item(s) for which he has been bidding while none other bidders have quit.

The events in one auction has no effect on the admissibility of a bidder's actions in the other auction. Here are the rules within each auction (for a particular item):

1. Once an active bidder has quit, he is no longer active and has no admissible action.
2. If all active bidders continue, the price rises continuously in the exogenous speed.
3. If a bidder  $i$  quits when the item's current price is  $p$ , the price clock for this item pauses for at most  $\delta$  seconds to give each active bidder a short *decisive moment* to decide whether to withdraw from or continue with the item. Once they have all decided, the decisive moment ends; else if  $\delta$  seconds has passed, the decisive moment ends and all active bidders who have not withdrawn are treated as having chosen "continue".
  - a. If all active bidders withdraw and—
    - i. if the dropout bidder  $i$  did not withdraw, the item is sold to him at the price  $p$ ;
    - ii. if  $i$  did withdraw, then the good is not sold and each withdrawing bidder ( $i$  and those who withdraw in the decisive moment) pays an equal share of the *withdrawal penalty* equal to  $p$ .
  - b. If exactly one active bidder  $j$  continues, the good is sold to this  $j$  at price  $p$ . If at least two active bidders continue, the price clock resumes from the level  $p$ .
4. A jump-bidder picks two positive numbers  $x, y \in (0, \eta]$ , with  $\eta > 0$  being a parameter.
  - a. For  $x$  seconds, the price clock for the item pauses for his jump bid, say  $b$ , to be submitted; at the end of this interval, the item's price is bumped up to  $b$ .
  - b. Then the price pauses for at most  $y$  seconds for the other active bidders to respond; if a bidder does not respond by the deadline, his action is treated as "stop". Once they have all responded, the pause ends.
    - i. If all but the jump-bidder quit, the jump-bidder buys the item at the price  $b$ .
    - ii. If some other bidder continues and no one submits a higher jump bid, the price clock resumes from the level  $b$  once the pause ends.
    - iii. If another bidder responds with a higher jump bid  $b'$ , the submission process (step 4.a) for  $b'$  needs to be completed within the  $y$ -second pause. Then the procedure repeats with the new jump bid  $b'$ .

The *submission time*  $x$  and *response time*  $y$  in rule 4 are to formalize the meaning of “immediate” in continuous time. Immediately submitting a jump bid means picking a tiny  $x \in (0, \eta]$  so that the price is bumped up to the jump-bid level in  $x$  seconds. To expedite the auction process, a jump-bidder can set a tiny  $y > 0$  as the response deadline. Even if the deadline is far in the future, the other bidder can respond “immediately” by picking a tiny submission time  $x'$  for his replying jump bid. Why not simply assume that jump-bidding takes zero time? Such an assumption would allow an unreal arrangement where bidders jump-bid and respond arbitrarily often at a single instant.

Note that these amendments can be enforced individually within each auction without any intervention from outside the auction. Hence the model is still aligned with the decentralized nature of markets. Should outside intervention be used, the exposure problem can be eliminated trivially: when local bidder  $\alpha$  drops out, pause the auction of item A until the auction of B ends and then let the global bidder decide whether to buy A or not. A main point of the next subsection is that outside intervention is completely unnecessary.

### 4.3 Jump bidding in the decisive moment

Interestingly, although the above amendments are mainly driven by the fact in §4.1 that each active bidder finds it weakly dominant to jump-bid conditional on the basic mechanism equilibrium, such unilateral incentive may vanish in the amended mechanism. To see that, suppose local bidder say  $\alpha$  becomes the first dropout when the current prices are  $(p_A, p_B)$ . During the decisive moment (rule 3, §4.2) triggered by this dropout, not expecting the basic mechanism equilibrium, bidder  $\beta$  might think it probable that  $v_B(t_\gamma, p_A) < p_B$  and hence that global bidder  $\gamma$  would have quit by now had  $\gamma$  been worried by the exposure problem. Then  $\beta$  might want to continue without jump-bidding, so that  $\gamma$ , receiving no signal from  $\beta$ , would quit B when the decisive moment ends. Differently, global bidder  $\gamma$  still wants to finish up the auction of item B within the decisive moment, since during this moment he still has the option of withdrawing from A. However, the only way to expedite the auction is to jump-bid successively. Since he has to pay his jump bid if the jump bid wins the good for him, bidder  $\gamma$  needs to avoid overshooting the target. Thus, his jump bids depend on his belief about bidder  $\beta$ 's type and hence are not necessarily dominant actions.



Therefore, the actions during a decisive moment necessarily involve signaling and belief updating and hence possibly multiple equilibria. In this subsection, we construct a continuation equilibrium where jump-bidding eliminates the exposure problem: Say local bidder  $\alpha$  becomes the first dropout. If the other local bidder  $\beta$  does not immediately jump-bid, global bidder  $\gamma$  will jump-bid successively so that the price for B will rise up to  $\gamma$ 's maximum willingness-to-pay for B before the moment ends, unless bidder  $\beta$  has quit by then. Expecting this reply, bidder  $\beta$  does not gain from delaying the auction. Hence he immediately jump-bids for B. Having to pay this bid if it wins B for him, bidder  $\beta$  does not overbid. From the jump bid, global bidder  $\gamma$  correctly infers about  $\beta$ 's value. If this inferred value is not less than  $\gamma$ 's maximum willingness-to-pay for B when  $\gamma$  still has the withdrawal option for item A, then global bidder  $\gamma$  immediately withdraws from both items; else bidder  $\gamma$  tops  $\beta$ 's bid with a jump bid equal to the inferred value, which leads to bidder  $\beta$ 's immediately dropout. Thus, the winner of item B is determined with at most two jump bids. Since the time it takes to submit a jump bid can be arbitrarily short (rule 4.a, §4.2) and a jump-bidder can set an arbitrarily close deadline for rivals to respond (rule 4.b, §4.2), this two-bid process is completed within the decisive moment. Thus, the exposure problem vanishes: global bidder  $\gamma$  faces no price uncertainty about item B when he can buy item A at its current price, as he knows the prices of both items by the end of the decisive moment.

Without loss of generality, suppose throughout the rest of this subsection that the first dropout is bidder  $\alpha$  from item A if global bidder  $\gamma$  is not the first dropout. Conditional on  $\alpha$  being the first dropout, a remaining bidder  $i$ 's maximum willingness-to-pay for item B during the decisive moment triggered by the dropout is the bidder's *interim value*, denoted by  $w_i$ . Still able to withdraw from A during the decisive moment, the global bidder's interim value is less than his value. Obviously for a local bidder, value is the same as interim value.

**Lemma 3** *If a local bidder say  $\alpha$  is the first dropout when the current price for item A is  $p_A$ , then the remaining bidders' interim values for B, given realized types  $(t_\beta, t_\gamma)$ , are:*

$$\begin{aligned} w_\beta &= t_\beta; \\ w_\gamma &= \tilde{w}_\gamma(t_\gamma, p_A, \lambda) := t_\gamma - \lambda p_A, \end{aligned} \tag{9}$$

where

$$\lambda := \begin{cases} 1 & \text{if } \alpha \text{'s action is "stop"} \\ 1/2 & \text{if } \alpha \text{'s action is "withdraw"}. \end{cases}$$

**Proof** As bidder  $\alpha$  has dropped out, global bidder  $\gamma$  can buy A at its current price  $p_A$  by taking the action “continue” at the end of the decisive moment. Suppose, during the decisive moment, item B is available for  $\gamma$  to buy at some price  $p_B$ . If he buys it,  $\gamma$  also buys A and gets a total profit  $t_\gamma - p_A - p_B$ . If he does not buy it,  $\gamma$  needs to withdraw from A since his standalone value of A is zero and the penalty for withdrawal is less than  $p_A$  (rule 3.a, §4.2). If  $\alpha$ 's dropout action is not “withdraw”,  $\gamma$ 's withdrawal from A gives him zero payoff, for then item A will be sold to bidder  $\alpha$  at its current price and so  $\gamma$  does not need to pay any withdrawal penalty; thus, bidder  $\gamma$  is willing to buy item B if and only if  $p_B < t_\gamma - p_A = \tilde{w}_\gamma(t_\gamma, p_A, 1)$  as in Eq. (9). If bidder  $\alpha$ 's dropout action is “withdraw”,  $\gamma$ 's payoff from withdrawing from A is  $-p_A/2$ , since he needs to pay half of the bid  $p_A$  that bidder  $\alpha$  and he both withdraw; thus, bidder  $\gamma$  is willing to buy B if and only if  $p_B < t_\gamma - p_A/2 = \tilde{w}_\gamma(t_\gamma, p_A, 1/2)$  as in Eq. (9). ■

Note that the continuation game given  $\alpha$ 's being the first dropout is simply a private-value English auction of item B, with the only twist that bidder  $\gamma$ 's maximum willingness-to-pay jumps from  $w_\gamma$  to  $t_\gamma$  at the end of the decisive moment if he has not quit by then. Then naturally a bidder  $i$  may expect that the best possible outcome for him is the *Vickrey outcome* with interim values  $(\mathbf{w}_\beta, \mathbf{w}_\gamma)$ , i.e., he wins item B if and only if  $w_i \geq w_{-i}$  ( $-i := \{\beta, \gamma\} \setminus \{i\}$ ), and his payment in expectation is equal to  $w_{-i}$  conditional on winning and is equal to zero conditional on losing. With this expectation, bidder  $i$  finds it optimal to submit a jump bid equal to the expected value of  $\mathbf{w}_{-i}$  conditional on the event  $w_{-i} \leq w_i$ : if  $i$ 's interim value is  $w_i$  and the history up to the current instant is  $h$ , this jump bid is equal to

$$\mathcal{P}_{i,h}(w_i) := \mathbb{E}[\mathbf{w}_{-i} \mid \mathbf{w}_{-i} \leq w_i; h]. \quad (10)$$

If  $\beta$  does that, bidder  $\gamma$  can infer about  $\beta$ 's type from  $\beta$ 's jump bid, as long as the posteriors up to current history  $h$  and hence the functional forms of  $\mathcal{P}_{i,h}$  are common knowledge. Furthermore, the next lemma implies that, from  $\beta$ 's jump bid, bidder  $\gamma$  almost surely knows immediately who has the higher interim value.

**Lemma 4** For each  $i \in \{\beta, \gamma\}$  and for any history  $h$  in the continuation game, the function  $\mathcal{P}_{i,h}$  defined in (10) is weakly increasing; furthermore, for any jump bid  $x_i$  whose inverse image  $\mathcal{P}_{i,h}^{-1}(x_i)$  is nonempty and for almost every possible  $w_{-i}$  (relative to the posterior given  $h$ ),

$$\inf \mathcal{P}_{i,h}^{-1}(x_i) \geq w_{-i} \quad \text{or} \quad \sup \mathcal{P}_{i,h}^{-1}(x_i) \leq w_{-i}. \quad (11)$$

**Proof** By definition (10), the function  $\mathcal{P}_{i,h}$  is weakly increasing. If it is not strictly increasing, then the posterior distribution of  $\mathbf{w}_{-i}$  conditional on history  $h$  has a gap: By (10),  $\mathcal{P}_{i,h}(w_i) = \mathcal{P}_{i,h}(w'_i)$  if and only if this distribution has zero weight strictly between  $w_i$  and  $w'_i$ ; i.e., for any  $x_i$  in the range of  $\mathcal{P}_{i,h}$ , the event “ $w_{-i}$  belongs to the interior of  $\mathcal{P}_{i,h}^{-1}(x_i)$ ” has zero probability. Hence (11) is true almost surely conditional on  $h$ . ■

A technical issue is how players coordinate on who should jump-bid first and how long one should wait for the jump-bid from others. Hence I include a protocol of endogenous sequencing as part of the equilibrium. Recall the maximum duration  $\delta$  of the decisive moment. Let  $(i_n)_{n=1}^\infty$  be a sequence such that every  $i_n$  is the name of an active bidder. The bidders follow an *endogenous sequence*  $(i_n)_{n=1}^\infty$  if the following procedure is commonly expected:

- I. One by one in sequence  $(i_n)_{n=1}^\infty$ , bidders jump-bid or drop out. The instant when the decisive moment starts is the starting point when player  $i_1$  is expected to act. For any  $n = 1, 2, \dots$ , the instant when player  $i_n$ 's action is completed is the starting point when player  $i_{n+1}$  is expected to act.
- II. Let  $\Delta$  denote the state variable, representing the remaining time from the current instant to the end of the decisive moment. Initialize its value by  $\Delta := \delta$ .
- III. If  $\Delta \leq 0$ , then the decisive moment has ended and each active player bids straightforwardly (continuing up to one's value) in the clock auction unless the game is over.
- IV. At the starting instant when player  $i_n$  is expected to act, given  $\Delta > 0$ :
  - i. the other players will wait for  $i_n$ 's action for  $\Delta/2$  seconds;
  - ii. if bidder  $i_n$  jump-bids, he picks its submission time (rule 4.a, §4.2) to be  $\Delta/2$  and sets the response time (rule 4.b, §4.2) to be  $\Delta/4$ ;

- iii. if bidder  $i_n$  neither jump-bids nor drops out within this  $\Delta/2$  seconds, the other bidders, surprised, regard bidder  $i_n$ 's action as jump-bidding to the current price of the item; then the procedure continues according to the remainder of the sequence;
- iv. if player  $i_m$  for some  $m > n$  jump-bids or drops out before player  $i_n$  does in  $i_n$ 's  $\Delta/2$ -second interval, then the procedure continues according to the remainder of the sequence starting from  $i_{m+1}$ ;
- v. at the end of this  $\Delta/2$ -second interval, update the state variable: if all but one bidder have dropped out, then  $\Delta := 0$ ; else  $\Delta := \Delta/2$ .

At any instant during the decisive moment, even if bidders have deviated from the protocol, as long as the remaining time  $\Delta$  from now to the end of the moment is positive and bidders follow this protocol from now on, the bidders can proceed arbitrarily far along the endogenous sequence during the remain of the decisive moment, as  $\sum_{k=1}^{\infty} (\Delta/2^k) = \Delta$ . By the same token, if a bidder occurs infinitely often in the sequence, he can jump-bid arbitrarily often in the decisive moment, unless the auction is over. In the equilibrium constructed next, bidders  $\beta$  and  $\gamma$  act alternately, following the endogenous sequence  $(\beta, \gamma, \beta, \gamma, \beta, \dots)$ . There is no incentive to deviate from the sequence: each bidder expects the Vickrey outcome and hence is indifferent about who gets the first move.

**Proposition 1** *If a local bidder say  $\alpha$  is the first dropout (from item A) and if the posterior beliefs up to this point are common knowledge, then there exists a continuation equilibrium on whose path the winner of item B is determined during the decisive moment; in this continuation equilibrium, if the global bidder wins an item, then he wins its complement and, before buying any of them, he knows the total price for both items.*

**Proof** In this proof,  $h$  denotes the history up to the instant when a bidder is about to act, and  $W_i(h)$  denotes the support of  $\mathbf{w}_i$  according to the other bidder's belief at that instant. Initialize  $h$  and  $W_i(h)$  to be respectively the history and support at the start of the decisive moment triggered by  $\alpha$ 's dropout. Starting from this instant, we construct an equilibrium:

- a. Bidders  $\beta$  and  $\gamma$  follow the protocol of endogenous sequence  $(\beta, \gamma, \beta, \gamma, \beta, \dots)$  specified by rules I–IV.

- b. At any instant from which there is a positive duration before the decisive moment ends, if it is bidder  $i$ 's turn to act given current history  $h$  and bidder  $-i$  still active, then:
  - i. if  $w_i \leq \inf W_{-i}(h)$ ,  $i$  immediately withdraws from any item for which he is bidding;
  - ii. else  $i$  immediately makes a jump bid for B equal to  $\mathcal{P}_{i,h}(w_i)$  defined by (10).
- c. If bidder  $i$  has just submitted a jump bid  $x_i$  for B given commonly known current history  $h$  and support  $W_i(h)$ , then update  $p_B := x_i$  and:
  - i. if  $x_i$  belongs to the range  $\mathcal{P}_{i,h}(W_i(h))$  of the jump-bid function  $\mathcal{P}_{i,h}$ , the current history is updated to include the inferred information  $w_i \in \mathcal{P}_{i,h}^{-1}(x_i)$ ;
  - ii. if  $x_i \notin \mathcal{P}_{i,h}(W_i(h))$ , which is off path, then bidder  $-i$  adopts the posterior that  $w_i$  is drawn from the exponential distribution supported by  $[x_i, \infty)$ , and the history  $h$  is updated by including this posterior and removing anything contradicting the posterior; note that the updated  $h$  and  $W_i(h)$  remain commonly known.
- d. If bidder  $-i$  drops out, then bidder  $i$  buys item B at its current price.

*Step 1:* The jump bids prescribed above are admissible because they are always higher than the current price  $p_B$ : For the initial jump bid, note from the current history  $h$  that  $p_B < w_i$  for each  $i \in \{\beta, \gamma\}$  (else bidder  $i$  would have quit), hence  $\mathcal{P}_{\beta,h}(w_\beta) > p_B$  by Eq. (10). For subsequent jump bids, let the current jump bid be  $x_i$ . If  $x_i \in \mathcal{P}_{i,h}(W_i(h))$ ,  $x_i < E[\mathbf{w}_i \mid \mathbf{w}_i \in \mathcal{P}_{i,h}^{-1}(x_i); h]$  by Eq. (10); thus, if bidder  $-i$  jump-bids next, his jump bid is higher than  $x_i$  (strategy b and Eq. (10)). If  $x_i \notin \mathcal{P}_{i,h}(W_i(h))$ , then  $x_i < E[\mathbf{w}_i \mid \mathbf{w}_i \in [x_i, \infty); h]$  by the off-path updating rule c.ii; hence any subsequent jump bid from bidder  $-i$  is above  $x_i$ .

*Step 2:* The on-path actions of this proposed equilibrium: Once the decisive moment starts, bidder  $\beta$  immediately submits a jump bid equal to  $\mathcal{P}_{\beta,h}(w_\beta)$ ; Eq. (11) says that almost surely (relative to  $h$ ) there are only two alternatives and bidder  $\gamma$  can tell which alternative is true: either  $\beta$ 's  $\mathbf{w}_\beta$  revealed through the jump bid is greater than or equal to  $w_\gamma$ , or  $\beta$ 's revealed  $\mathbf{w}_\beta$  is less than or equal to  $w_\gamma$ . In the first case, bidder  $\gamma$  immediately withdraws from both items (strategy b.i) and the game ends. In the second case,  $\gamma$  submits a higher jump bid according to strategy b.ii, with history  $h$  updated. Since  $w_\beta \leq w_\gamma$  in this case,  $\gamma$ 's jump bid is equal to the expected value of  $\mathbf{w}_\beta$  conditional on its being in the inverse image

of  $\beta$ 's jump bid (Eq. (10)); seeing this jump bid, bidder  $\beta$  infers that  $w_\gamma \geq w_\beta$  and hence drops out immediately (strategy b.i) and the game ends. Thus, the game ends within two consecutive jump bids and, by Eq. (10), achieves the Vickrey outcome (relative to  $(\mathbf{w}_\beta, \mathbf{w}_\gamma)$ ).

*Step 3.* Claim: Given any history of this continuation game, if the decisive moment has not ended and if both bidders will follow the proposed equilibrium from now on, the winner of item B is determined before the moment ends, and the Vickrey outcome is achieved: By Step 2, it takes at most two consecutive jump bids to complete the auction if the bidders follow the proposed strategy from now on, no matter what the history is. There is still time for such two jump bids within the decisive moment, due to the choice of submission and response deadlines in the endogenous sequence protocol (rule IV).

*Step 4:* , By Step 3, we are done if the proposed strategies are sequentially rational for each bidder  $i \in \{\beta, \gamma\}$ . Strategy d is obviously so: It is strictly dominated for bidder  $i$  to stay active when  $p_B > w_i$ ; thus, if bidder  $-i$  drops out, the current price  $p_B$  cannot exceed  $w_i$ , which, as a private value, is unaffected by  $-i$ 's dropout. Hence it is optimal for  $i$  to buy B. To prove the sequential rationality of strategies a and b, we prove the following—

**Lemma 5** *Given any history  $h$  before the end of the decisive moment, if bidder  $-i$  will follow the proposed equilibrium from now on, then, from the viewpoint of  $h$ , the best possible outcome for bidder  $i$  is the Vickrey outcome with interim values  $(\mathbf{w}_\beta, \mathbf{w}_\gamma)$ .*

By step 3, given any history  $h$  before the end of the decisive moment, if both bidders follow the proposed strategy from now on, the Vickrey outcome is achieved during the moment. Then this lemma implies the desired sequentially rationality of strategies a and b. We first prove Lemma 5 in three special cases and then combine them into the general case:

*Case 4.1:* Suppose bidder  $-i$ 's  $w_{-i}$  has been fully revealed. It suffices Lemma 5 to show that bidder  $i$ 's best reply is to bid  $w_{-i}$  if  $w_i > w_{-i}$  and to drop out if  $w_i \leq w_{-i}$ . Obviously it is weakly dominated for him to bid above  $w_{-i}$ . We need only to show that he cannot win with a bid less than  $w_{-i}$ . If his bid  $b_i$  is below  $w_{-i}$  (including the case that he fails to act before the deadline in the endogenous sequence protocol IV.iii), then  $b_i$  is outside the range of  $i$ 's bids, which is the singleton  $\{w_{-i}\}$  (Eq. (10)), hence the other bidder  $-i$  will adopt the posterior in c.ii and will submit a bid strictly between  $b_i$  and  $w_{-i}$ , so that bidder  $i$  cannot

win immediately. If  $i$ 's next bid is still less than  $w_{-i}$ , bidder  $-i$ 's response is again higher. During the decisive moment, as long as bidder  $i$  continues bidding below  $w_{-i}$ , this process repeats for indefinitely many times by the endogenous sequence protocol IV; furthermore, since the off-path posterior distribution in plan c.ii is atomless, bidder  $-i$ 's replying bids converge to his interim value  $w_{-i}$  before the decisive moment ends, unless bidder  $i$  has quit. Thus,  $i$  cannot win with a bid less than  $w_{-i}$ , as claimed.

*Case 4.2:* Suppose, given current history  $h$ , bidder  $i$  submits a jump bid  $x_i$  and bidder  $-i$  follows the proposed strategies and does not quit immediately. Then  $-i$ 's action is an immediate jump bid. If  $x_i$  is outside the range  $\mathcal{P}_{i,h}(W_i(h))$ , we are in case 4.1: bidder  $-i$  adopts the posterior in c.ii, based on which he submits  $x_{-i}$ ; then  $x_{-i}$  is strictly increasing in  $w_{-i}$  (Eq. (10)). Thus, suppose  $x_i \in \mathcal{P}_{i,h}(W_i(h))$ . Then, in bidding  $x_i$ , bidder  $i$  knows:

Conditional on the event that bidder  $-i$ 's immediate response is a jump bid  $x_{-i}$ , Eq. (11) says that  $w_{-i} \geq \sup \mathcal{P}_{i,h}^{-1}(x_i)$  is true with probability one, and hence the following event has probability one:

Bidder  $-i$  expects me (bidder  $i$ ) to drop out immediately after his bid  $x_{-i}$  ( $-i$  expects the on-path events in Step 2). Consequently, if I immediately drop out, I will get zero payoff. Otherwise, I submit a bid  $x'_i \in (x_{-i}, w_i]$  and  $-i$  will be surprised and will adopt the posterior in c.ii. Then he will either (i) submit a higher bid  $x'_{-i}$  if  $w_{-i} > x'_i$  or (ii) drop out if  $w_{-i} \leq x'_i$ . In subcase (i),  $w_{-i}$  is fully revealed and I am in case 4.1. In subcase (ii), I buy item B at the price  $x'_i$ , which is no less than  $w_{-i}$ ; and subcase (ii) occurs only if  $w_{-i} \leq x'_i \leq w_i$ . Thus, I (bidder  $i$ ) cannot do better than the Vickrey outcome.

*Case 4.3:* Suppose, given current history  $h$ , bidder  $i$  jump-bids to  $x_i$ . If  $x_i \notin \mathcal{P}_{i,h}(W_i(h))$ , bidder  $-i$  is surprised and we are back to a subcase of the event analyzed by bidder  $i$  in case 4.2. Hence suppose  $x_i \in \mathcal{P}_{i,h}(W_i(h))$ . Partition the current support  $W_i(h)$  of  $\mathbf{w}_i$  into the inverse images of function  $\mathcal{P}_{i,h}$ ; let  $[\hat{w}_i]$  denote a partition cell containing  $\hat{w}_i$ . For bidder  $i$ , submitting a bid  $x_i \in \mathcal{P}_{i,h}(W_i(h))$  is equivalent to the action of picking a  $\hat{w}_i \in W_i(h)$  and announcing: “ $w_i \in [\hat{w}_i]$  and my payment is  $\mathcal{P}_{i,h}(\hat{w}_i)$  if I win immediately.” Let  $u_i(\hat{w}_i, w_i)$

denote bidder  $i$ 's expected payoff from this action, conditional on current history  $h$  and his true  $w_i$ . Note that  $u_i(w_i, w_i)$  is equal to bidder  $i$ 's expected payoff from the Vickrey outcome. Hence it suffices Lemma 5 to prove  $u_i(\hat{w}_i, w_i) \leq u_i(w_i, w_i)$ . Before that, we need a notation: Let  $[\hat{w}_i] \geq w_{-i}$  denote “ $z \geq w_{-i} \forall z \in [\hat{w}_i]$ ” and let  $[\hat{w}_i] < w_{-i}$  denote “not  $[\hat{w}_i] \geq w_{-i}$ ”. Eq. (11) implies: for every  $\hat{w}_i \in W_i(h)$  and for almost every  $w_{-i} \in W_{-i}(h)$  (relative to  $h$ ),

$$1_{[\hat{w}_i] \geq w_{-i}}(w_{-i}) = 1_{\hat{w}_i \geq w_{-i}}(w_{-i}); \quad 1_{[\hat{w}_i] < w_{-i}}(w_{-i}) = 1_{\hat{w}_i < w_{-i}}(w_{-i}). \quad (12)$$

If  $[\hat{w}_i] < w_i$ , then item B goes to  $i$  if and only if (a) either the other bidder  $-i$  quits immediately or (b) bidder  $-i$  does not quit immediately but will quit later. By strategy b.i, event (a) is  $[\hat{w}_i] \geq w_{-i}$ . By case 4.2, event (b) is contained by  $\{w_{-i} : [\hat{w}_i] < w_{-i} \leq w_i\}$  and  $i$ 's payment in event (b) is at least  $\mathbf{w}_{-i}$  in expectation. Thus,

$$\begin{aligned} u_i(\hat{w}_i, w_i) &\leq \mathbb{E} \left[ 1_{[\hat{w}_i] \geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) (w_i - \mathcal{P}_{i,h}(\hat{w}_i)) + 1_{[\hat{w}_i] < \mathbf{w}_{-i} \leq w_i}(\mathbf{w}_{-i}) (w_i - \mathbf{w}_{-i})^+ \mid h \right] \\ &= \mathbb{E} \left[ 1_{\hat{w}_i \geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) (w_i - \mathcal{P}_i(\hat{w}_i)) + 1_{\hat{w}_i < \mathbf{w}_{-i} \leq w_i}(\mathbf{w}_{-i}) (w_i - \mathbf{w}_{-i})^+ \mid h \right] \\ &= w_i \mathbb{E} \left[ 1_{w_i \geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h \right] - \mathbb{E} \left[ \mathbf{w}_{-i} 1_{w_i \geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h \right] \\ &= u_i(w_i, w_i), \end{aligned}$$

where the first equality uses Eq. (12) and the second uses (10).

If  $[\hat{w}_i] \geq w_i$ , then item B goes to  $i$  if and only if the other bidder  $-i$  immediately drops out after  $i$ 's current jump-bid: If  $i$  cannot outbid  $-i$  with this  $\hat{w}_i$ , then the best possible outcome for  $i$  will be the Vickrey outcome (case 4.2), which requires  $w_i \geq w_{-i}$  for  $i$  to win B; hence  $i$  will not win because  $w_{-i} > [\hat{w}_i] \geq w_i$ , as bidder  $i$  does not win with  $\hat{w}_i$  right now. Hence  $i$ 's winning event is  $\{w_{-i} : [\hat{w}_i] \geq w_{-i}\}$ , which is  $\{w_{-i} : \hat{w}_i \geq w_{-i}\}$  by (12). Thus,

$$\begin{aligned} u_i(\hat{w}_i, w_i) &= (w_i - \mathcal{P}_{i,h}(\hat{w}_i)) \mathbb{E} \left[ 1_{\hat{w}_i \geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h \right] \\ &= w_i \mathbb{E} \left[ 1_{\hat{w}_i > \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h \right] - \mathbb{E} \left[ \mathbf{w}_{-i} 1_{\hat{w}_i \geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h \right] \\ &= \mathbb{E} \left[ (w_i - \mathbf{w}_{-i}) 1_{\hat{w}_i > \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h \right] \\ &\leq \mathbb{E} \left[ (w_i - \mathbf{w}_{-i}) 1_{w_i > \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h \right] \\ &= u_i(w_i, w_i), \end{aligned}$$

where the second equality uses (10). This proves Lemma 5 for case 4.3.



*Case 4.4* (the general case): Suppose, given current history  $h$ , bidder  $i$  chooses any admissible action. (a) Suppose first that bidder  $i$  is expected to act at the current instant. If he jump-bids, then we are in case 4.3; if his action is “continue” throughout the minute interval during which bidder  $-i$  is waiting, then bidder  $i$ ’s action is taken as a jump bid (endogenous sequence protocol IV.iii) and again we are in case 4.3; if  $i$ ’s action is dropout then he gets zero. Thus, the best possible for him is the Vickrey outcome. (b) Next suppose that bidder  $i$  is expected to wait for bidder  $-i$  at the current instant. If  $i$  deviates from the sequencing protocol, then he jump-bids now and we are in case 4.3. The only alternative left is for him to wait as expected; then by part (a) of this paragraph, when it is his turn to act, bidder  $i$  will follow the proposed strategy to achieve the Vickrey outcome from the viewpoint in that future; hence this alternative means following the proposed strategy starting from the current instant. That proves Lemma 5 and hence Proposition 1. ■

**Corollary 1** *If a local bidder is to become the first dropout and expects the continuation equilibrium in Proposition 1, then his dropout action is “stop” and not “withdraw”.*

**Proof** Without loss of generality, let  $\alpha$  be the first dropout when the current price for item A is  $p_A$ . By Proposition 1, item B is won by either  $\beta$  or  $\gamma$  during the decisive moment. If global bidder  $\gamma$  wins B, he continues on A, so “stop” and “withdraw” both yield zero payoff for bidder  $\alpha$ . If  $\gamma$  loses B, he withdraws from A, so bidder  $\alpha$  gets a nonnegative payoff from “stop” and gets a negative payoff (penalty  $-p_A/2$ ) from “withdraw”. ■

#### 4.4 An over-concentrating ascending package auction

If the continuation equilibrium in Proposition 1 is expected, the global bidder in the simultaneous auctions wins either all or none of the items, and he has no price uncertainty when buying them. Hence the simultaneous auctions with this continuation equilibrium resembles a package auction where a bidder can submit bids for package  $\{A, B\}$ . Thus, let us consider the following ascending package auction:

1. Local bidder  $\alpha$  bids only for package  $\{A\}$ ,  $\beta$  only for  $\{B\}$ , and global bidder  $\gamma$  bids

only for  $\{A, B\}$ . The price for a package starts at zero and rises continuously, with jump-bidding banned, until all but one bidder for the package have quit.

2. If no one has quit, the price  $p_A$  (or  $p_B$ ) for package  $\{A\}$  (or  $\{B\}$ ) rises at exogenous speed  $\dot{p}_A$  (or exogenous  $\dot{p}_B$ ), and the price  $p_{AB}$  for  $\{A, B\}$  rises at speed  $\dot{p}_A + \dot{p}_B$ .
3. If global bidder  $\gamma$  quits before local bidders  $\alpha$  and  $\beta$ , item A is sold to bidder  $\alpha$  at the current price  $p_A$ , and B is sold to  $\beta$  at the current price  $p_B$ . (Note that  $p_A + p_B$  is equal to bidder  $\gamma$ 's dropout price  $p_{AB}$ .)
4. If bidder  $\alpha$  quits at price  $p_A$  before bidders  $\beta$  and  $\gamma$ , stop raising  $p_A$ , and raise  $p_B$  and  $p_{AB}$  at the same speed.
  - a. If subsequently bidder  $\beta$  quits at price  $p_B$  before the global bidder  $\gamma$ , sell both items to  $\gamma$  at a total price equal to  $p_A + p_B$ .
  - b. If subsequently bidder  $\gamma$  quits at price  $p_{AB}$  before local bidder  $\beta$ , sell item B to bidder  $\beta$  at its current price  $p_B$ , and sell A to bidder  $\alpha$  at his dropout price  $p_A$ . (Note that  $p_{AB} = p_A + p_B$ .)
5. If bidder  $\beta$  quits at price  $p_B$  before bidders  $\alpha$  and  $\gamma$ , do the same thing as in provision 3 by switching between A and B and between  $\alpha$  and  $\beta$ .
6. If bidders  $\alpha$  and  $\beta$  quit simultaneously before global bidder  $\gamma$ ,  $\gamma$  wins.

In this package auction, the global bidder's weakly dominant strategy is to bid for the entire package until  $p_{AB} = t_\gamma$ , since the game to him is an English auction of a single bundle  $\{A, B\}$ . Once a local bidder say  $\alpha$  has quit at  $p_A$ , local bidder  $\beta$  finds it dominant to bid for item B up to its true value; if he wins,  $\beta$  buys B at price  $t_\gamma - p_A$  (as  $\gamma$  bids up to  $t_\gamma$ ) and  $\alpha$  buys A at his dropout price  $p_A$ . The case where local bidder  $\beta$  quits first is symmetric.

A *dropout price* means the price for a local bidder's valued item at which the bidder quits immediately unless someone has already quit. An *undominated strategy equilibrium* in the package auction corresponds to a pair  $(s_\alpha, s_\beta)$  of dropout strategies such that (i)  $s_i$  tells local bidder  $i$  what his dropout price is given his type and the current history of the game and (ii)  $s_\alpha$  best replies  $s_\beta$  and vice versa, with the global bidder being straightforward.

Contrary to the straightforward global player, the local bidders do not always bid up to their true values: A local bidder who drops out may win his desired item because the other local bidder may outbid the global bidder. Hence a local bidder may free ride the other. This *threshold problem* is exactly the same as the classic public goods problem with private information. It cannot be eliminated no matter how bidders signal to each other. Now that local bidders probably underbid while the global bidder bids up to his value, we have—

**Lemma 6** *In any undominated strategy equilibrium of the package auction, over-concentration occurs with a positive probability and over-diffusion never occurs.*

**Proof** Pick any undominated strategy equilibrium. For each local bidder  $i$ , let  $P_i(t_i)$  denote the expected value of  $i$ 's equilibrium payment, viewed at the start of the game, given his type  $t_i$ . Since global bidder  $\gamma$  bids straightforwardly and a local bidder  $i$ 's dropout price is never greater than  $i$ 's value by  $i$ 's rationality, over-diffusion never occurs. For over-concentration, it suffices to show that there is a positive probability with which some local bidder's dropout price is less than his value. Suppose not, then this equilibrium achieves ex post allocation almost surely. By the definition of the package auction, the total payment from both local bidders is equal zero if global bidder  $\gamma$  wins and is equal to  $\gamma$ 's truthfully reported value  $t_\gamma$  if  $\gamma$  loses. Thus, with “ $\gamma$  loses” equivalent to “ $t_\gamma < t_\alpha + t_\beta$ ” in this allocation,

$$\mathbb{E}[P_\alpha(\mathbf{t}_\alpha) + P_\beta(\mathbf{t}_\beta)] \geq \mathbb{E}[\mathbf{t}_\gamma \mathbf{1}_{\mathbf{t}_\gamma < \mathbf{t}_\alpha + \mathbf{t}_\beta}(\mathbf{t}_\gamma, \mathbf{t}_\alpha, \mathbf{t}_\beta)]. \quad (13)$$

But this contradicts the fact that it is impossible to have an equilibrium-feasible allocation that is almost surely ex post efficient and satisfies (13). This fact is simply a reinterpretation of the impossibility of efficient provision of public goods (Krishna and Perry [12, §8.2]), with the cost of public goods being the global bidder's value. ■

## 4.5 Simultaneous auctions mimic the package auction

Concerned about the exposure problem of simultaneous auctions, researchers have considered replacing them by a package auction to allocate possibly complementary goods. The

discussion has been going on for a decade<sup>2</sup> and is still unsettled, because a package auction has the shortcomings of threshold problem and combinatorial complexity. Adding to this discussion, the next proposition says that, at least in our simplistic model, there is no loss of generality to simply use simultaneous ascending auctions.<sup>3</sup>

**Proposition 2** *For any undominated strategy equilibrium of the package auction in §4.4, there is an equilibrium in the simultaneous auctions that generates the same allocation, where over-concentration occurs with a positive probability and over-diffusion never occurs.*

**Proof** Pick any equilibrium of the package auction. As noted in §4.4, the equilibrium corresponds to a pair  $(s_\alpha, s_\beta)$  of local bidders' dropout strategies that best reply each other, with the global bidder straightforward. We need only to construct an equilibrium for the simultaneous auctions that replicates this equilibrium's allocation. Then Lemma 6 implies its property regarding over-concentration and over-diffusion. Here is the construction:

- â. If both local bidders are still active, global bidder  $\gamma$  stays active for both items if their total price is less than  $t_\gamma$  and quits from both if otherwise.
- â. A local bidder  $i$  stays active until someone else quits or his dropout price (defined in §4.4) prescribed by  $s_i$  has been reached.
- â. If a local bidder is the first dropout, his dropout action is “stop” and not “withdraw”.
- â. If a local bidder  $i$  becomes the first dropout with the other local bidder  $j$  still active, then the remaining bidders  $j$  and  $\gamma$  play the continuation equilibrium in Proposition 1.

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<sup>2</sup>The discussion dates back to at least as early as January 1995, when a version of Bykowsky, Cull, and Ledyard [7] was drafted. A recent paper leading this discussion is Ausubel and Milgrom [3].

<sup>3</sup>A caveat of this message is that our model of package auctions does not allow global bidder  $\gamma$  to jump-bid. Hence the model might preclude a possible equilibrium where  $\gamma$  makes a jump bid for bundle  $\{A, B\}$  without specifying the jump on each item; that may impose an additional threshold problem on the local bidders, who have to divide the jump between themselves if they do not want to lose right away. This additional problem might not be replicable in simultaneous auctions, because bidder  $\gamma$  in simultaneous auctions has to submit bids for each item and hence he instead of the local bidders divides the jump between the items. This caveat, however, does not undermine the normative implication of our message, because this possible additional threshold problem would make package auctions even more over-concentrating.

- ê. Given any (commonly known) current history  $h$  (including current prices  $(p_A, p_B)$ ):
  - i. if no one has quit yet, then local bidder  $\alpha$ 's type-distribution is updated conditional on  $s_\alpha(t_\alpha, h) > p_A$  and likewise for local bidder  $\beta$ , and global bidder  $\gamma$ 's distribution is updated conditional on  $t_\gamma > p_A + p_B$ ;
  - ii. if a local bidder is the first to quit, updating follows the continuation equilibrium in Proposition 1; if the global bidder is the first to quit, the game ends.

We first prove a claim: Given any pair of dropout prices, the outcome in the simultaneous auctions according to plans  $\hat{a}$ ,  $\hat{c}$ , and  $\hat{d}$  is identical to the outcome in the package auction defined in §4.4. Given any pair of dropout prices and the exogenous price speeds, the same bidder becomes the first dropout at the same instant in both mechanisms. Let the current prices be  $(p_A, p_B)$  at that instant. If global bidder  $\gamma$  is the first dropout, then in both mechanisms item A goes to bidder  $\alpha$  at price  $p_A$  and B goes to  $\beta$  at price  $p_B$ . Else a local bidder is the first dropout. Without loss of generality, let it be bidder  $\alpha$ :

- i. In the package auction, the remaining local bidder  $\beta$  would bid for item B up to  $t_\beta$ , hence his winning event is  $\{t_\gamma : t_\gamma - p_A \leq t_\beta\}$ . If he wins,  $\beta$  buys B at price  $t_\gamma - p_A$  and  $\alpha$  buys A at his dropout price  $p_A$ . If  $\beta$  loses,  $\gamma$  buys A at price  $p_A$  and B at price  $t_\beta$ .
- ii. In the simultaneous auctions carrying out plans  $\hat{a}$ ,  $\hat{c}$ , and  $\hat{d}$ , the Vickrey outcome with interim values  $(\mathbf{w}_\beta, \mathbf{w}_\gamma)$  is achieved (Proposition 1), where  $w_\beta = t_\beta$  and  $w_\gamma = t_\gamma - p_A$  (plan  $\hat{c}$  and Eq. (9)). Thus, local bidder  $\beta$ 's winning event is “ $w_\gamma \leq w_\beta$ ”, i.e., “ $t_\gamma - p_A \leq t_\beta$ ”. If he wins,  $\beta$  buys B at a price equal to the expected value of  $t_\gamma - p_A$  and  $\alpha$  buys A at his dropout price  $p_A$ . If  $\beta$  loses, item A goes to  $\gamma$  at price  $p_A$  and B goes to  $\gamma$  at a price equal to  $t_\beta$  in expectation (Proposition 1).

Thus, the two mechanisms have the same the winning event payment rule, as claimed.

It follows from the above claim that the simultaneous auctions according to plans  $\hat{a}$ – $\hat{e}$  replicate the package auction according to dropout strategies  $(s_\alpha, s_\beta)$ .

Thus, we complete the proof by showing that plans  $\hat{a}$ – $\hat{e}$  constitute an equilibrium of the simultaneous auctions. By the updating rule  $\hat{e}$ , when a local bidder becomes the first dropout,

the posteriors conditional on this event are common knowledge. Thus, Proposition 1 is applicable and plan  $\hat{d}$  is a continuation equilibrium here. With plan  $\hat{d}$  expected, strategy  $\hat{a}$  is optimal for the global bidder and strategy  $\hat{c}$  is optimal for each local bidder (Corollary 1). The only remaining step is to check the dropout strategies in plan  $\hat{b}$ . By the claim proved above, the simultaneous auctions game, conditional on plans  $\hat{a}$ ,  $\hat{c}$ , and  $\hat{d}$ , is strategically equivalent to the package auction. As the dropout strategies  $(s_\alpha, s_\beta)$  constitute an equilibrium in the package auction, they best reply each other in the simultaneous auctions, given  $\hat{a}$ ,  $\hat{c}$ , and  $\hat{d}$ . Thus, plans  $(\hat{a})$ – $(\hat{e})$  constitute an equilibrium in the simultaneous auctions, as desired. ■

As any undominated strategy equilibrium of the package auction leads to probable over-concentration but never over-diffusion (Lemma 6), Proposition 2 implies that the ascending simultaneous auctions game has an over-concentrating equilibrium if undominated strategy equilibrium exists in the package auction.<sup>4</sup>

## 5 Extension to cross-bidding

*Cross-bidding* means bidding for an item which always has zero value for the bidder, e.g., bidder  $\alpha$  bidding for B or  $\beta$  bidding for A. The previous sections assume that cross-bidding is not allowed. This assumption is not innocuous, because a local bidder has a strict incentive to cross-bid conditional on the equilibrium in the basic mechanism (§3.1): Before winning any item, the global bidder's highest total bid for both items is less than his valuation of the whole package ( $t_\gamma > p_A^*(t_\gamma) + p_B^*(t_\gamma)$  in Lemma 2). But once he has won an item, his highest bid for its complement jumps to his valuation of both items (plan 3 in Lemma 1). Hence a local bidder say  $\alpha$  wishes to bid for the zero-value item B in order to prevent the global bidder from becoming aggressive after winning B when local bidder  $\beta$  quits. Thus, in the

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<sup>4</sup> When bidders' types and bids are continuous variables, existence of equilibrium in the package auction is an open nontrivial question. The complexity comes from the fact that a local bidder's payoff is discontinuous in his dropout price and the discontinuity contains both a drop and a jump: if local bidder  $\alpha$  is quitting at instant  $x$ , local bidder  $\beta$  gets zero payoff if he also quits at  $x$  (both losing to the global bidder); whereas,  $\beta$  gets positive payoffs if he quits slightly before instant  $x$  (then  $\alpha$  will bid up to  $t_\alpha$ ) or quits slightly after  $x$  (then  $\beta$  will bid up to  $t_\beta$ ). We of course can ensure existence of equilibrium by assuming that types and bids are finite discrete, bearing the cost of messy calculations due to positive probabilities of ties.

basic mechanism, local bidders have an incentive to break its rule that bans cross-bidding.

Hence we modify the basic mechanism in §3.1 to allow cross-bidding. By assumption, to be eligible for an item, a bidder needs to participate in its auction from the start. Hence a bidder not cross-bidding currently cannot switch to cross-bidding.

This section extends the analysis in §4 to the modified model that allows cross-bidding. Does jump-bidding self-emerge from an equilibrium of this modified model? If Yes, does the over-concentration prediction remain true in the mechanism that allows both jump- and cross-bidding? We will see that the answers to both questions are affirmative.

## 5.1 Self-emergence of jump-bidding despite cross-bidding

Let us consider the modified mechanism, the basic mechanism in §3.1 plus the option of cross-bidding. Crossing-bidding may undermine the self-emergence of jump-bidding because, when cross-bidding is allowed, a local bidder may find jump-bidding unnecessary: When a local bidder say  $\beta$  is alone competing against global bidder  $\gamma$  for both items,  $\gamma$  may think that  $\beta$  will quit item A and stay for B until  $p_B$  reaches  $t_\beta$ . Then  $\gamma$  would expect himself to be uncertain about  $p_B$  in the future possible event that he wins A, hence he may quit both items before their total price reaches  $t_\gamma$ . If  $\gamma$  is expected to think so when local bidder  $\alpha$  becomes the first dropout, local bidder  $\beta$  may stay cross-bidding without jump-bidding.

However, cross-bidding has a mitigating effect opposite to the above *sequential dropout effect*: When a local bidder say  $\beta$  is cross-bidding A, global bidder  $\gamma$ 's winning A implies that both local bidders quit A, so  $\beta$ 's value is not that high and hence  $\gamma$ 's subsequent competition with  $\beta$  for item B will not be that severe. Thus, the global bidder is less likely to be the first dropout when someone is cross-bidding than when no one is cross-bidding. If this mitigating effect outweighs the sequential dropout effect, local bidders would rather not cross-bid at all and then the equilibrium would be on-path identical to the basic mechanism equilibrium (Lemma 1), conditional on which jump-bidding and bid withdrawal self-emerge.

Furthermore, as proved next in Lemma 7, there is always an equilibrium where local bidders do not cross-bid at all and the on-path actions are identical to those in the basic

mechanism equilibrium (Lemma 1). The possibility of sequential dropout by a cross-bidder is eliminated: the global bidder expects that a cross-bidder always quits both items simultaneously if quitting any at all, and a cross-bidder does so as expected because his payoff is the same whether he stays cross-bidding or not (the first paragraph of the proof of Lemma 7). Thus, when local bidders cross-bid, the simultaneous auctions to the global bidder become an English auction with a single “item”  $\{A, B\}$ . Hence, to suppress the global bidder’s maximum bid by the exposure problem, local bidders would rather not cross-bid.

With the equilibrium on-path identical to the basic mechanism equilibrium in Lemma 1, the reasoning in §4.1 can be reused. Thus, despite cross-bidding, jump-bidding is both Pareto improving and individually weakly dominant conditional on an equilibrium in the mechanism that bans jump-bidding. Hence the tendency for jump-bidding to self-emerge remains.

**Lemma 7** *If the basic mechanism in §3.1 is modified just to allow cross-bidding (but not jump-bidding or bid withdrawal), the following strategies constitute an equilibrium:*

- a. *Each local bidder bids only for his valued item up to its value and does not cross-bid; if no one is cross-bidding, global bidder  $\gamma$  follows the strategy (plans 1–3) in Lemma 1.*
- b. *If local bidder  $\alpha$  is cross-bidding, his strategy is to keep bidding for both items until—*
  - i. *if the global bidder quits before others, then quit B and continue A; or*
  - ii. *if  $p_A \geq t_\alpha$ , then quit both items immediately; or*
  - iii. *if the other local bidder  $\beta$  has quit and  $p_A + p_B \geq t_\alpha$ , quit both items immediately;*
  - iv. *if having quit A somehow, quit B immediately.*

*The case of local bidder  $\beta$  is symmetric with the substitutions  $A \leftrightarrow B$  and  $\alpha \leftrightarrow \beta$ .*

- c. *If cross-bidding is still going on, bidder  $\gamma$  follows the strategy in Lemma 1 with this revision: if  $\alpha$  is cross-bidding, replace  $v_B(t_\gamma, p_A)$  in plan 1 by  $t_\gamma - p_A$ ; if  $\beta$  is cross-bidding, replace  $v_A(t_\gamma, p_B)$  in plan 1 by  $t_\gamma - p_B$ .*

**Proof** First, we claim: if a local bidder say  $\alpha$  has to cross-bid, strategy (b) is optimal for him. Strategy b.iv is obvious since  $\alpha$  values only item A. Strategy b.i is obvious since it



implies that  $\alpha$  wins A and  $\beta$  wins B immediately after  $\gamma$  quits. Strategy b.ii is also obvious because  $\alpha$  can only lose profit when  $p_A \geq t_\alpha$ . For strategy b.iii, suppose bidder  $\alpha$  is cross-bidding and local  $\beta$  has quit. Global bidder  $\gamma$  either bids for both items until  $p_A + p_B = t_\gamma$  (if  $\alpha$  keeps cross-bidding) or bids for A until  $p_A = t_\gamma$  (if  $\alpha$  quits cross-bidding, i.e., quits B and stays for A).<sup>5</sup> Hence bidder  $\alpha$  is indifferent between staying for both items versus quitting B and staying for A. Thus, it is optimal for him to stay cross-bidding until  $p_A + p_B$  reaches his value  $t_\alpha$  and then quit both items immediately.

Next, we show that global bidder  $\gamma$ 's strategy (c) best replies (b). Say  $\alpha$  has been cross-bidding, with current prices  $(p_A, p_B)$ . As  $\alpha$  follows strategy (b), if  $\gamma$  wins item B now,  $\alpha$  must have quit both items, hence  $\gamma$  immediately wins item A and gets a profit  $t_\gamma - p_A - p_B$ . Thus, conditional on staying for A, bidder  $\gamma$  keeps bidding for B if and only if  $t_\gamma - p_A > p_B$ . Hence  $v_B(t_\gamma, p_A)$  in plan 1 is replaced by  $t_\gamma - p_A$ . In contrast, if  $\beta$  is bidding only for B, it is possible that  $\beta$  stays for B after bidder  $\gamma$  has won A. Hence  $\gamma$ 's expected profit from buying A at the current instant is equal to  $v_A(t_\gamma, p_B) - p_A$ , as in Lemma 1. Hence (c) is a best reply. Plan (c) implies  $\gamma$ 's on-path behavior prescribed in (a).

Given (c), let us find  $\gamma$ 's *dropout point*, the first instant where he quits both items when each local bidder is bidding for some item. If both local bidders cross-bid, then by (c-ii)  $\gamma$  continues for both items if  $t_\gamma > p_A + p_B$  and else quits both, so  $\gamma$ 's dropout point is the unique intersection between the line  $p_A + p_B = t_\gamma$  and the price path  $p_B = (\dot{p}_B/\dot{p}_A)p_A$ ; denote this point by  $(p_A^o(t_\gamma), p_B^o(t_\gamma))$ . If  $\alpha$  cross-bids and  $\beta$  does not, then  $\gamma$  continues for both items if  $v_A(t_\gamma, p_B) > p_A$  and  $t_\gamma > p_A + p_B$  (note that the first inequality implies the second) and else quits both, hence his dropout point is  $(p_A'(t_\gamma), p_B'(t_\gamma))$  (intersection of (5) and (6)). If  $\alpha$  does not cross-bid and  $\beta$  does,  $\gamma$  continuous for both items if  $v_B(t_\gamma, p_A) > p_B$  (and hence  $t_\gamma > p_A + p_B$ ) and else quits both, hence his dropout point is  $(p_A''(t_\gamma), p_B''(t_\gamma))$  (intersection of (5) and (7)). If neither local bidders cross-bid, then  $\gamma$ 's dropout point is the same as in Lemma 1,  $(p_A^*(t_\gamma), p_B^*(t_\gamma))$ . Note:  $(p_A''(t_\gamma), p_B''(t_\gamma)) < (p_A^o(t_\gamma), p_B^o(t_\gamma))$  unless the two points

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<sup>5</sup>Here we have used the assumption that bid withdrawal is prohibited: If the cross-bidder quits item B and stays for A, global bidder  $\gamma$ , without an option to withdraw his bid, automatically buys B and hence will bid for A up to  $t_\gamma$ . This assumption, however, is dispensable: Bidder  $\gamma$ , as before, expects the lone cross-bidder  $\alpha$  to quit from both items simultaneously if  $\alpha$  quits at all; if  $\alpha$  surprises  $\gamma$  by deviating to quitting only item B and staying for item A, bidder  $\gamma$  adopts a posterior belief that  $\alpha$  will quit A sufficiently soon so that it is profitable for  $\gamma$  to buy B and continue for A. Hence  $\gamma$  would not withdraw.

coincide, and  $(p_A^*(t_\gamma), p_B^*(t_\gamma)) < (p'_A(t_\gamma), p'_B(t_\gamma))$  unless they coincide.

Finally, we show: given (b) and (c), cross-bidding is worse than not cross-bidding for each local bidder, say  $\alpha$ . First consider the case where bidder  $\beta$  cross-bids. If  $(p''_A(t_\gamma), p''_B(t_\gamma)) = (p^o_A(t_\gamma), p^o_B(t_\gamma))$ ,  $\alpha$  is indifferent about cross-bidding, because his winning event and payment are unaffected. Suppose the two points do not coincide. When  $t_\beta \leq p''_B(t_\gamma)$ , bidder  $\alpha$  is again indifferent: whether  $\alpha$  cross-bids or not,  $\beta$  quits before  $\gamma$  and then  $\gamma$  will bid up to  $t_\gamma$ , so  $\alpha$ 's winning event is  $t_\alpha > t_\gamma$  and he pays  $t_\gamma$  if he wins. When  $p''_B(t_\gamma) < t_\beta \leq p^o_B(t_\gamma)$ , bidder  $\alpha$  is better-off not cross-bid than cross-bid: if  $\alpha$  cross-bids,  $\beta$  quits before  $\gamma$ , and so  $\alpha$ 's winning event is  $t_\alpha > t_\gamma$  and he pays  $t_\gamma$  if he wins; if  $\alpha$  does not cross-bid,  $\gamma$  quits before  $\beta$ , so  $\alpha$ 's winning event is  $t_\alpha > p'_A(t_\gamma)$  and he pays  $p'_A(t_\gamma)$  if he wins. Since  $t_\gamma > p'_A(t_\gamma)$ , cross-bidding is worse. When  $t_\beta > p^o_B(t_\gamma)$ , bidder  $\alpha$  again prefers not cross-bidding:  $\gamma$ 's maximum bid for A is  $p^o_A(t_\gamma)$  if  $\alpha$  cross-bids and is a less amount  $p''_A(t_\gamma)$  if  $\alpha$  does not cross-bid. Thus, bidder  $\alpha$  prefers not cross-bidding to cross-bidding when bidder  $\beta$  cross-bids. The case when bidder  $\beta$  does not cross-bid is similar: by switching from cross-bidding to not cross-bidding,  $\alpha$  moves  $\gamma$ 's dropout point from  $(p'_A(t_\gamma), p'_B(t_\gamma))$  down to  $(p^*_A(t_\gamma), p^*_B(t_\gamma))$ ; as in the previous case, this increases  $\alpha$ 's winning probability and decreases his payment if he wins. Thus, cross-bidding is suboptimal for a local bidder, and his strategy in (a) is a best reply. ■

## 5.2 Over-concentration with jump- and cross-bidding

Now that the self-emerging tendency of jump-bidding remains when cross-bidding is allowed, let us revise the mechanism to allow both jump- and cross-bidding. More precisely, let us take the model in §4.2 and add the assumption that a local bidder can cross-bid at the start of the game and can continue doing so as long as he has not stopped cross-bidding.

Compared to the analysis in §4, the main difference due to cross-bidding is: when a local bidder becomes the first dropout, the other local bidder may still be cross-bidding. In this case, global bidder  $\gamma$  is currently free from the exposure problem, because he does not need to buy an item right now without knowing the price of its complement. He will continue to be free from the exposure problem unless the lone cross-bidder quits only one item and stays active for the other. Even if the cross-bidder does so, bidder  $\gamma$  can still be

free from the exposure problem by playing the jump-bidding continuation equilibrium (as in Proposition 1) during the decisive moment triggered by the dropout. That is the intuition to extend the predictions in the no-cross-bidding case to the cross-bidding case.

A continuation equilibrium is called *exposure-free* if the global bidder knows the total price for both items before buying any of them and he wins all or none of the items.

**Lemma 8** *Suppose there is only one active local bidder and he is cross-bidding against the global bidder for both items. If this cross-bidder quits only one item now and if the posteriors so far are common knowledge, then the continuation game given this dropout has an exposure-free equilibrium.*

**Proof** Without loss of generality, let  $\beta$  be the lone active local bidder, who has been cross-bidding until now when he quits only one item. To avoid triviality, suppose this item is A, with current price  $p_A > 0$ . One can calculate the remaining bidders' interim values  $w_i$  during the decisive moment triggered by this dropout, depending on  $\beta$ 's dropout action:

$$\beta\text{'s dropout action is "stop"} \implies [w_\beta = t_\beta - p_A; w_\gamma = t_\gamma - p_A]; \quad (14)$$

$$\beta\text{'s dropout action is "withdraw"} \implies [w_\beta = t_\beta - p_A/2; w_\gamma = t_\gamma - p_A/2]. \quad (15)$$

For instance, suppose  $\beta$ 's dropout action (from A) is "withdraw". By the calculation in Lemma 3, global bidder  $\gamma$ 's interim value  $w_\gamma = t_\gamma - p_A/2$ . To calculate  $\beta$ 's interim value: If  $\beta$  gets to buy B at its current price  $p_B$ , then  $\gamma$  must have withdrawn from A (since  $\beta$  has withdrawn from A,  $\alpha$  would still have to pay for A at full price  $p_A$  if  $\alpha$ 's dropout action is "stop"), hence  $\beta$ 's payoff is  $t_\beta - p_B - p_A/2$  since  $\beta$  shares the withdrawal penalty  $p_A$  with bidder  $\gamma$ . If  $\beta$  quits from B, then both items are sold to bidder  $\gamma$  and hence  $\beta$ 's withdrawal penalty is zero and his payoff is zero. Thus,  $\beta$ 's interim value for B is  $w_\beta = t_\beta - p_A/2$ .

Then this lemma follows from Proposition 1 applied to the above interim values. The proposition is applicable since posteriors are assumed to be common knowledge here. ■

If the above exposure-free continuation equilibrium is expected, cross-bidding does not help local bidders: Free from exposure problems, global bidder  $\gamma$  will bid for both items up to  $t_\gamma$ , whether there is cross-bidding or not. To expand his winning event, a local bidder wants

to couple his bid with the other local bidder's; to do so, he quits one item and jump-bids for the other item immediately after the other local bidder becomes the first dropout:

**Lemma 9** *If a local bidder quits both items simultaneously and if, up to this point, the other local bidder has been cross-bidding against the global bidder for both items and the posteriors are common knowledge, then the continuation game from now on has an exposure-free equilibrium, where the local bidder immediately withdraws from his unvalued item and jump-bids for the other item with the global bidder.*

**Proof** Let  $\alpha$  be the first dropout who quit both items simultaneously, with current price  $p_A$  for item A. For the continuation game, we construct an equilibrium:

- a. Local bidder  $\beta$  immediately withdraws from A and then bidders  $\beta$  and  $\gamma$  play the continuation equilibrium in Proposition 1 with interim values  $(w_\beta, w_\gamma)$  calculated below.
- b. If  $\beta$  immediately stops (not withdrawing) from A, then  $\beta$  and  $\gamma$  play the continuation equilibrium in Proposition 1 with interim values different from (a), calculated below.
- c. If  $\beta$  does not immediately quit A, then:
  - i. if neither  $\beta$  nor  $\gamma$  has quit up to current prices  $(p_A, p_B)$ , then bidder  $i$ 's ( $i \in \{\beta, \gamma\}$ ) type-distribution is updated conditional on the event  $t_i > p_A + p_B$ ;
  - ii. if  $\beta$  stays cross-bidding,  $\gamma$  stays for both items if  $p_A + p_B < t_\gamma$  and else quits both;
  - iii. if subsequently bidder  $\beta$  quits only one item, then bidders  $\beta$ 's type-distribution is updated conditional on  $p_B < t_\beta$ , and  $\gamma$ 's distribution is updated as in c.i; then bidders  $\beta$  and  $\gamma$  play the exposure-free continuation equilibrium in Lemma 8.

If this construction constitutes an equilibrium, then it is exposure-free due to its on-path action (a) and the conclusion of Proposition 1.

We shall verify the equilibrium conditions by backward induction. First, consider plan c. Plan c.iii constitutes a continuation equilibrium given the event that bidder  $\beta$  subsequently quits only one item; that follows from Lemma 8, which is applicable because the

posteriors are common knowledge due to the updating rule c.i. Expecting this continuation equilibrium, bidder  $\gamma$  finds strategy c.ii optimal.

Second, expecting plan c, bidder  $\beta$  knows that, if he continues cross-bidding after  $\alpha$ 's dropout, the game ends with the *Vickrey outcome* with values  $(\mathbf{t}_\beta, \mathbf{t}_\gamma)$ , i.e.,  $\beta$ 's winning event is  $\{t_\gamma : t_\gamma \leq t_\beta\}$  and his payment conditional on winning is in expectation equal to  $t_\gamma$ . This is obvious if  $\beta$  keeps cross-bidding until he quits both items. Suppose  $\beta$  quits only one item. Then the jump-bidding continuation equilibrium is played (plan c.iii); by Proposition 1, the game ends with the Vickrey outcome with interim values  $(\mathbf{w}_\beta, \mathbf{w}_\gamma)$  determined by (14) and (15); one can show that this outcome is equivalent to the Vickrey outcome with values  $(\mathbf{t}_\beta, \mathbf{t}_\gamma)$ . For instance, suppose  $\beta$  withdraws from item A and stays for B. Then his winning event for item B is “ $w_\gamma \leq w_\beta$ ”, which, by (15), is equivalent to “ $t_\gamma \leq t_\beta$ ”. If  $\beta$  wins B, then  $\gamma$  must have withdrawn from A, hence  $\beta$  needs to pay half the withdrawal penalty, so his payment is equal to  $p_B + p_A/2 = w_\gamma + p_A/2 = t_\gamma - p_A/2 + p_A/2 = t_\gamma$ .

Third, consider plan a. Now that  $\beta$  withdraws from A and stays for B, one can calculate the remaining bidders' interim values  $w_i$  during the decisive moment (triggered by  $\alpha$ 's and  $\beta$ 's dropouts), depending on  $\alpha$ 's dropout action for item A:

$$\alpha\text{'s dropout action for A is “stop”} \implies [w_\beta = t_\beta; w_\gamma = t_\gamma - p_A]; \quad (16)$$

$$\alpha\text{'s dropout action for A is “withdraw”} \implies [w_\beta = t_\beta - p_A/3; w_\gamma = t_\gamma - 2p_A/3]. \quad (17)$$

To prove (17), suppose that bidder  $\alpha$ 's dropout action is “withdraw”. Suppose bidder  $\gamma$  gets to buy B at some price  $p_B$ . If he buys it, he also buys A at its current price  $p_A$ , hence his payoff is  $t_\gamma - p_A - p_B$ ; if he does not buy B,  $\gamma$  needs to withdraw from A (as in Lemma 3), then his payoff is  $-p_A/3$ , since he needs to pay 1/3 of the bid  $p_A$  withdrawn by all three bidders. Thus, bidder  $\gamma$  buys B in the decisive moment if and only if its price is less than  $t_\gamma - 2p_A/3$  as in (17). To calculate  $\beta$ 's interim value, suppose he gets to buy B. Then bidder  $\gamma$  must have withdrawn from A, so if  $\beta$  buys B then  $\beta$ 's payoff is  $t_\beta - p_B - p_A/3$ , since all three bidders have withdrawn the bid  $p_A$ ; alternatively,  $\beta$  can quit item B, then bidder  $\gamma$  wins both items and bidder  $\beta$  gets zero payoff. Thus, bidder  $\beta$  buys B during the decisive moment if and only if its price is less than  $t_\beta - p_A/3$ , as in (17). Hence (17) is true. The proof for (16) is similar and simpler. With the interim values determined by (16) and (17), the jump-bidding equilibrium in Proposition 1 is a continuation equilibrium given  $\beta$ 's withdrawal from A. The

proposition is applicable since posteriors are assumed common knowledge in this lemma.

Fourth, consider plan b. Given that bidder  $\beta$  immediately stops from item A, one can mimic the previous paragraph to calculate the interim values and then show that the jump-bidding equilibrium in Proposition 1 with such interim values is a continuation equilibrium.

We now complete the proof by showing that it is optimal for bidder  $\beta$  to follow plan a, i.e., to withdraw from A immediately after  $\alpha$ 's dropout. It is weakly dominated for  $\beta$  to stop instead of withdrawing from A: They make no difference unless bidder  $\gamma$  withdraws from both during the decisive moment. Suppose we are in that event; if  $\alpha$ 's dropout action is "withdraw", bidder  $\beta$  has to pay the entire  $p_A$  for the zero-value item A if he merely stops, and he pays only  $p_A/3$  for A if he withdraws; if  $\alpha$ 's action is "stop",  $\beta$  pays the entire  $p_A$  with a positive probability if he merely stops (since A is randomly allocated between him and  $\alpha$ ), and he pays zero penalty if he withdraws. Hence bidder  $\beta$  prefers plans a to b.

Thus, other than withdrawing from A, the only alternative left for  $\beta$  is to continue cross-bidding, i.e., plan c. On this path, as shown in the second step, bidder  $\beta$ 's winning event is  $\{t_\gamma : t_\beta \geq t_\gamma\}$  and his payment if he wins is in expectation equal to  $t_\gamma$ . In contrast, under plan a, one can prove, from the Vickrey outcome prediction of Proposition 1 and (16) and (17), that  $\beta$ 's winning event is bigger and his payment is smaller: if  $\alpha$  stopped from A,  $\beta$ 's winning event is " $t_\gamma - p_A \leq t_\beta$ " and  $\beta$ 's winning payment is in expectation equal to  $t_\gamma - p_A$ ; if  $\alpha$  withdrew from A,  $\beta$ 's winning event is " $t_\gamma - p_A/3 \leq t_\beta$ " and  $\beta$ 's winning payment is in expectation equal to  $t_\gamma - p_A/3$ . Thus, bidder  $\beta$  prefers plans a to c. ■

To extend the over-concentration prediction, we embed the continuation equilibrium of Lemma 9 into the path where local bidders cross-bid. If they do not cross-bid, let them play the equilibrium of Proposition 2. As each path is exposure-free, the only source of inefficiency is the threshold problem, which leads to probable over-concentration.

**Proposition 3** *Proposition 2 is true in the simultaneous ascending auctions that allow both jump- and cross-bidding.*

**Proof** Pick any equilibrium of the package auction, which corresponds to a pair  $(s_\alpha, s_\beta)$  of local bidders' dropout strategies that best reply each other, with global bidder  $\gamma$  straight-

forward. We construct an equilibrium for the simultaneous auctions that replicates  $(s_\alpha, s_\beta)$ . The provisions in the construction are indexed by letters as in Proposition 2 with possibly different accents: a provision  $\hat{x}$  here is identical to the provision  $\hat{x}$  in Proposition 2; a provision  $\acute{x}$  here corresponds to, but is different from, the provision  $\hat{x}$  in Proposition 2.

- â. If for each item at least one local bidder is active, then, whether local bidders cross-bid or not, global bidder  $\gamma$  stays for both items if  $p_A + p_B < t_\gamma$  and quits both if otherwise.
- á. A local bidder  $i$  does not cross-bid, and he keeps bidding for his valued item until someone else quits or his dropout price prescribed by  $s_i$  has been reached.
- ê. If a local bidder quits, his dropout action is “stop” rather than “withdraw”.
- đ. If a local bidder  $i$  becomes the first dropout with the other local bidder  $j$  still active:
  - i. if bidder  $j$  is not cross-bidding, then bidders  $j$  and  $\gamma$  play the jump-bidding continuation equilibrium in Proposition 1;
  - ii. if bidder  $j$  has been cross-bidding up to  $i$ 's dropout, then  $j$  and  $\gamma$  play the continuation equilibrium in Lemma 9.
- ê. Given any (commonly known) current history  $h$  (including current prices  $(p_A, p_B)$ ):
  - i. if no one has quit yet, then local bidder  $\alpha$ 's type-distribution is updated conditional on  $s_\alpha(t_\alpha, h) > p_A$  and likewise for local bidder  $\beta$ , and global bidder  $\gamma$ 's distribution is updated conditional on  $t_\gamma > p_A + p_B$ ;
  - ii. if a local bidder is the first to quit, updating follows the continuation equilibria in plan đ; if the global bidder is the first to quit, the game ends.

As in the proof of Proposition 2, the following claim is true: Given any pair of dropout prices, the outcome in the simultaneous auctions according to plans  $\hat{a}$ ,  $\hat{c}$ , and  $\acute{d}$  is identical to the outcome in the package auction. It follows that the simultaneous auctions according to plans  $\hat{a}$ – $\hat{e}$  replicate the package auction according to dropout strategies  $(s_\alpha, s_\beta)$ .

Hence we complete the proof by showing that plans  $\hat{a}$ – $\hat{e}$  constitute an equilibrium of the simultaneous auctions. First, consider plan  $\acute{d}$ . The case of  $\acute{d}$ .ii follows from Lemma 9.

For case  $\acute{d}.i$ , as the remaining local bidder who is not cross-bidding cannot switch to cross-bidding, he is equivalently living in the no-cross-bidding world, hence plan  $\acute{d}.i$  follows from Proposition 1, which is applicable due to the commonly known posteriors ensured by updating rule  $\hat{e}$ . Second, as recommended in plan  $\acute{b}$ , a local bidder does not cross-bid: By plans  $\acute{d}.i$  and  $\acute{d}.ii$ , a local bidder never gains from cross-bidding; moreover, he may have to pay a withdrawal penalty for the unvalued item for which he cross-bids, by plan  $\acute{d}.ii$  and Lemma 9. Third, with plan  $\acute{d}$  expected, the exposure problem vanishes, hence strategy  $\hat{a}$  is optimal for the global bidder and strategy  $\hat{c}$  is optimal for each local bidder (Corollary 1). The only remaining step is to check the dropout strategies in plan  $\acute{b}$ . This is done by the same step in the proof of Proposition 2, using the aforementioned claim about dropout prices and the fact that the dropout strategies  $(s_\alpha, s_\beta)$  best reply each other in the package auction. ■

## 6 Extension to resale

We have seen so far that equilibrium allocations of simultaneous auctions are inefficient with a positive probability. Thus, with the same type of arguments in Zhèng [18], we know that resale after the simultaneous auctions, if allowed, must occur with a positive probability. Even if resale is declared illegal, the strict incentive for resale makes the prohibition costly.

Let us extend the model to incorporate resale. The main idea is to allow winners in an auction to select any selling mechanism for possible resale and give winners in his mechanism the same option. That might sound odd at first glance, because it treats the initial auctions exogenously and resale auctions endogenously. But this formulation is actually natural. Recall that the goal is to understand the performance of a given mechanism, simultaneous ascending auctions. Hence it is appropriate to hold this initial auction mechanism as exogenous. Assuming exogenous resale mechanisms, in contrast, would be inappropriate, because renegotiation can take many forms, and we do not know a priori which specific format will prevail. It is therefore natural to let resale mechanisms emerge as optimal actions chosen by some players. To reflect the friction in bargaining, we use the standard mechanism-design formulation: allow one player to select a mechanism and commit to it by having it operated by a neutral trustworthy mediator and, to be even-handed, preserve the privacy of the other



players' types unless they are willingly revealed by the players themselves.

## 6.1 The auction-resale game

There are  $N$  periods, with no discounting, and  $N$  is an exogenous large number. (The exogenous  $N$  is to ensure that the equilibrium concept is well formed, as explained in Zhèng [18]). In period one, the items are auctioned off via simultaneous ascending auctions, which may allow or ban cross-bidding or jump-bidding. In period two, resale among bidders is allowed. If a bidder has won all items in period one, he can pick any mechanism (defined in the next paragraph) and commit to it for possible resale. (“No sale no matter what” is counted as one such mechanism.) If items are sold to different bidders, one of the winners is randomly selected to pick a resale mechanism; if no other winner vetoes it, the mechanism is implemented; else the mechanism is not implemented and every winner commits to a resale price for the item he currently owns. The probability with which a winner is selected to pick a resale mechanism is proportional to the number of items he currently owns. If a resale mechanism results in no-sale or if period  $N$  is reached, the game ends; else in the next period a winner is chosen to pick a resale mechanism, as in the current period.

A selling *mechanism* for player  $i$  (who is the current seller) is a mapping from the profile of types across players other than  $i$  to a payment arrangement and a lottery that assigns the items to the players (including  $i$ ). The lottery is called *allocation outcome* from  $i$ 's viewpoint.

## 6.2 Myerson auctions

If a seller could costlessly prohibit resale after the operation of his mechanism, his mechanism becomes the endgame and his mechanism is *incentive feasible* if truth-telling is a Bayesian Nash equilibrium in this endgame. The *Myerson auction for player  $i$*  maximizes player  $i$ 's expected profit among all incentive feasible selling mechanisms for  $i$  under the assumption that player  $i$  can costlessly prohibit resale after the operation of his mechanism (Myerson [16]). Given a type-profile, the *virtual utility* of an allocation outcome *from player  $i$ 's standpoint*

is defined to be the ex post gain of trade generated by this outcome minus

$$\sum_{j \neq i} \frac{1 - F_j(t_j)}{f_j(t_j)},$$

where index  $j$  ranges through all players but  $i$  who are involved in the trade specified by the allocation outcome, and  $t_j$  is the realized type of such a player  $j$ . For instance, if player  $\gamma$  sells item A to player  $\alpha$  and B to  $\beta$ , the virtual utility from  $\gamma$ 's viewpoint is equal to

$$t_\alpha + t_\beta - t_\gamma - \frac{1 - F_\alpha(t_\alpha)}{f_\alpha(t_\alpha)} - \frac{1 - F_\beta(t_\beta)}{f_\beta(t_\beta)}. \quad (18)$$

The next lemma, due to Levin [14], characterizes Myerson auctions in our multiple-object environment. It is proved by extending the standard optimal auctions technique and using the assumption that each bidder's type is one-dimensional.

**Lemma 10** *Suppose that the hazard rate  $f_i/(1 - F_i)$  for every player  $i$  ( $i = \alpha, \beta, \gamma$ ) is weakly increasing. Then in any Myerson auction for player  $i$ , for almost every type-profile, the allocation outcome maximizes the virtual utility from  $i$ 's standpoint among all allocation outcomes from  $i$ 's standpoint.*

**Lemma 11** *There is no loss of generality to assume that a Myerson auction for the global player  $\gamma$  has the property that item A never goes to bidder  $\beta$  and B never goes to  $\alpha$ .*

**Proof** Selling A to  $\beta$ , global player  $\gamma$  receives at most zero revenue. Instead, keeping A to himself,  $\gamma$ 's payoff is either zero (if  $\gamma$  does not keep item B) or almost surely positive (if  $\gamma$  also keeps B). Hence  $\gamma$  cannot decrease his payoff by keeping item A to himself instead of selling it to bidder  $\beta$ . Symmetrically,  $\gamma$  cannot decrease his payoff by keeping B instead of selling it to  $\alpha$ . The lemma then follows from the definition of Myerson auction. ■

### 6.3 Endogenous separation between primary and resale markets

Since a bidder's action in period one will be used to update information about him, it is obvious that a bidder who expects a positive probability of buying an item at resale has an

incentive to conceal his true value by shading his bids in period one. One type of bid shading that facilitates tractability is not to bid in period one at all. This strategy is a best reply if the bidder expects some other bidder to bid for all items no matter how high the prices become. Although such bidding behavior also constitutes an equilibrium in the case where resale is prohibited, it is weakly dominated there. In contrast, when resale is allowed, such bidding behavior is not weakly dominated. That is because a high bidder can consistently believe that his resale revenue can cover his payments for the items, and the bidders who shy away in period one can consistently believe that entering a bid in period one can only result in being charged a higher price at resale. (This is similar to a point already made by Garratt and Tröger [8] for a single-good model.) Footnote 6 in the following comments on robustness relative to certain trembles.

Next we construct an equilibrium of the auction-resale game where the global bidder wins all items in period one and then offers them for resale. Although there may be other equilibria where players other than the global bidder act as resellers, it seems more plausible that the global bidder assumes the middleman role, as the goods are over-concentrated to the global bidder before bidders learn to exploit resale opportunities.

**Proposition 4** *If the hazard rate  $f_i/(1 - F_i)$  for every local bidder  $i \in \{\alpha, \beta\}$  is weakly increasing, then there is an equilibrium where the global bidder  $\gamma$  wins both items in period one and offers resale to local bidders via the Myerson auction from his standpoint, and there is no further resale after the operation of  $\gamma$ 's mechanism.*

**Proof** We shall show that the following constitutes an equilibrium:

- a. In period one, the global bidder continues bidding until he wins both items, and local bidders do not participate in the auctions.
- b. If no one deviates in period one, the global bidder in period two offers the items for possible resale via the Myerson auction from his viewpoint, based on the prior beliefs and subject to the property in Lemma 11; if a local bidder wins an item at resale, he chooses not to resell it.
- c. If a local bidder deviates to bidding in period one and quits item  $k$  at price  $p_k$ , the

global bidder's resale mechanism in period two is the Myerson auction based on the posterior that the deviant bidder's type is drawn from the prior distribution conditional on its being greater than  $p_k$  (and again subject to the property in Lemma 11).

To verify this equilibrium, we need only to prove four claims: First, a local bidder who wins in the Myerson auction finds it optimal to not offer the item for further resale. Second, expecting no further resale, a local bidder finds it optimal to be truthful in the Myerson auction. Third, the Myerson auction is optimal for the global player conditional on his winning both items in period one. Fourth, given the resale mechanisms specified in (b) and (c), a local bidder finds it optimal not to bid in period one.

The first claim implies the second claim: as the winners in the Myerson auction choose no-resale, the Myerson auction is the last stage of the auction-resale game, so truth-telling in the Myerson auction is optimal for each local bidder, as this auction is incentive feasible when it is the endgame. If the first and second claims are true, the third one follows, since the Myerson auction by definition maximizes the global player's expected profits when the global player can choose to ban resale, and the he cannot do better when he loses that option. The fourth claim is obvious: by definition of the Myerson auction, if the infimum of the support of a bidder's type is higher, he cannot pay less.<sup>6</sup>

Thus, it suffices to prove the first claim. After the Myerson auction, due to its property found in Lemma 11, local bidder  $\alpha$  does not get item B and  $\beta$  does not get A. Suppose bidder  $\alpha$  gets item A. He cannot profit from selling A to bidder  $\beta$ , who does not value A. Nor can he profit from selling back to player  $\gamma$ , since the fact that  $\gamma$  sells A to  $\alpha$  via the Myerson auction implies that the virtual utility  $t_\alpha - (1 - F_\alpha(t_\alpha))/f_\alpha(t_\alpha) - t_\gamma \geq 0$  (Lemma 10; note that the formula for virtual utilities is the same in cases (b) and (c)); as  $\frac{1 - F_i(t_i)}{f_i(t_i)} \geq 0$ , we have  $t_\alpha \geq t_\gamma$ . Can  $\alpha$  profit from coordinating with bidder  $\beta$ , in case that  $\beta$  wins B, to resell both items back to player  $\gamma$ ? No because this trade leads to no surplus: the fact that the

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<sup>6</sup> A local bidder with some sufficiently low types may have zero probability of buying any item at resale, so he is indifferent between bidding and not bidding in period one, since he cannot get the good in either way. Even for such a bidder, the equilibrium prescription of no-bidding-in-period-1 is a robust best reply relative to certain trembles: for example, conditional on global player  $\gamma$ 's making an error in period one (so that the local bidder may win if he bids in period one), there is a higher probability with which  $\gamma$  makes an error in period two that sells the good to the local bidder.

Myerson auction gives the two items to the local bidders implies that the virtual utility (18) is nonnegative and hence  $t_\alpha + t_\beta \geq t_\gamma$ . Hence bidder  $\alpha$  chooses to not resell A. The other local bidder's no-resale incentive is symmetric. This proves the desired claim. ■

**Corollary 2** *At the equilibrium in Proposition 4, the final allocation is over-concentrated with a positive probability and is never over-diffused.*

**Proof** Since that equilibrium implements the Myerson auction from the global bidder's viewpoint, the equilibrium final allocation is determined by the virtual-utility algorithm characterized in Lemma 10. Since  $\frac{1-F_i(t_i)}{f_i(t_i)} > 0$  for almost all types, the virtual utility (18) of selling A to  $\alpha$  and B to  $\beta$  is less than the social surplus  $t_\alpha + t_\beta - t_\gamma$  of this trade. Thus, if this trade eventually takes place, then its social surplus is positive; but even if its social surplus is positive, the virtual utility of the trade might still be negative and so the trade need not take place. Hence the final allocation is never over-diffused and is probably over-concentrated. ■

The conclusion of Corollary 2 is true even without the optimal-resale-mechanism assumption, i.e., even if not all resale mechanisms are available for a reseller. As long as his mechanism is individually rational for the reseller from his standpoints both before and after the operation of his mechanism, the global player  $\gamma$  never resells item A to bidder  $\beta$  and never B to  $\alpha$  (as in Lemma 11), and he never resells both items while the total revenue is less than his own value. Thus, as in the last paragraph in the proof of Proposition 4, there will be no further resale after  $\gamma$ 's resale mechanism. As a monopolist at resale,  $\gamma$  would under-sell the goods if he has some policy instruments such as reserve prices. What the optimal-resale-mechanism assumption offers is a prediction of the final outcome of the game that can be directly calculated from the prior distributions.

## 6.4 Extension to more bidders

Let us extend Proposition 4 to the case where there are  $n_k$  copies of the  $k$ -bidder, with  $k = \alpha, \beta, \gamma$ . For each  $k \in \{\alpha, \beta, \gamma\}$ , assume that all the  $k$ -bidders value the same item and their values are independently drawn from the same distribution  $F_k$ .

**Corollary 3** *Assume monotone hazard rate as in Proposition 4 and assume that, for each  $k \in \{\alpha, \beta, \gamma\}$ , the values of all the  $k$ -bidders are independently drawn from the same distribution  $F_k$ . There is an equilibrium where all global bidders participate in period-one auctions and all other bidders do not, the global bidder with the highest type wins both items in period one and offers resale to local bidders via the Myerson auction from his standpoint, and there is no further resale after the operation of his mechanism.*

**Proof** In period one, when another global bidder is staying, a global bidder's maximum bid is equal to his realized type plus the maximum expected profit that he can obtain during the resale stage; he keeps bidding for both items until  $p_A + p_B$  reaches his maximum bid. If all but one global bidder have quit, the remaining global bidder will continue bidding forever should there be remaining local bidders. One can prove that the maximum bid is strictly increasing in the global bidder's type by mimicking the envelope theorem argument in the proof of Proposition 2 in Zhèng [18] (from the start of that proof down to its second displayed equation). Hence the winner in period-1 is the one whose type is highest among all global bidders, so he suffers no loss to not include the other global bidders ( $\gamma$ -bidders) as potential buyers in his resale mechanism.

As in Proposition 4, we complete the proof by showing that a local bidder who wins in the global player's Myerson auction finds it optimal to not offer the item for further resale. For each  $k \in \{\alpha, \beta, \gamma\}$ , let  $k^*$  denote the bidder who has the highest type among the  $k$ -bidders. Since bidders of the same kind have the same distribution and hence the same strictly increasing virtual utility function, only bidders  $\alpha^*$  and  $\beta^*$  have chances to win in the Myerson auction. By Lemma 10, we may assume without loss of generality that  $\alpha^*$  does not get item B and  $\beta^*$  does not get A. Suppose bidder  $\alpha^*$  gets item A. He cannot profit from selling A to other  $\alpha$ -bidders, who value A less, or to  $\beta$ - or  $\gamma$ -bidders with  $\gamma \neq \gamma^*$ , who do not value A alone. As in the proof of Proposition 4, nor can  $\alpha^*$  profit from selling A back to its previous owner  $\gamma^*$ . The only case left is where the two winning local bidders,  $\alpha^*$  and  $\beta^*$ , could resell both items to a single  $\gamma$ -bidder. But this trade generates no surplus either, because  $t_{\alpha^*} + t_{\beta^*} \geq t_{\gamma^*}$  as in the proof of Proposition 4, and  $t_{\gamma^*} \geq t_{\gamma'}$  for any global bidder  $\gamma'$  who lost the period-one auctions to bidder  $\gamma^*$ . The case of  $\beta^*$  is symmetric. ■

Most results in the paper can be extended to this model. For example, the jump-bidding equilibrium of Proposition 1 becomes: when the last  $\alpha$ -bidder is quitting A, all the remaining  $\beta$ -bidders jump-bid for B and each remaining  $\gamma$ -bidder either responds with a jump bid or quits. Item B is won by a  $\beta$ - or  $\gamma$ -bidder with the highest value of his kind. (Bidders of the same kind do not free ride each other: a  $\beta$ -bidder who stops bidding will never win the good if some other  $\beta$ -bidder remains active.)

## 7 Discussions

### 7.1 The complication when there are more than two items

In the two-item case, once a local bidder quits, the other local bidder cannot free ride others. In contrast, when there are more than two items, after one local bidder quits, there are still multiple active local bidders and they may free ride one another in the continuation game: One may want to slow down the price ascension for his valued item so that the other active local bidders pay bigger shares of the total bid that tops the global bidder. Hence the jump-bidding arrangement may be undermined.

However, this free-riding problem among remaining local bidders does not necessarily overturn the over-concentration prediction. Recall that the prediction is driven by two forces: First, local bidders tend to underbid because they suffer the free-riding (threshold) problem in competing against the global bidder. Second, the global bidder does not underbid, owing to the jump-bidding continuation equilibrium: when one item is available for him to buy, the auctions of its complements proceed fast enough for the global bidder to know the total price before buying any item. Obviously, the first force can only be strengthened by the additional free-riding problem caused by having more than two items. The second force does not seem to be weakened either: Although a local bidder may want to slow down the auction in which he participates, he cannot do so unilaterally, because the global bidder can always speed up the auction by jump-bidding successively.

The global bidder's jump-bidding decision is complicated, since in each remaining auc-

tion he needs to decide how much and how fast to jump-bid. In the two-item case, he only needs to bid fast enough to finish the remaining auction within the decisive moment. With more than two items, the relative speeds of price ascension for the various items affect local bidders' equilibrium dropout strategies. Hence the global bidder needs to find an optimal list of speeds, one for each auction. Calculation or merely existence of such optimum would require existence of equilibrium in the continuation game given the choice of such speeds, where local bidders may choose dropout strategies as in a package auction. A proof of existence of such equilibria needs to overcome the complexity mentioned in footnote 4.

## 7.2 Evolution of mechanisms through self-emergence

The various mechanisms analyzed in this paper are linked by a chain of self-emerging attempts to reform mechanisms. To capture the decentralized nature of markets, we start with the basic mechanism of simultaneous ascending auctions. Conditional on its equilibrium, we find two possible directions of mechanism reform: First, a local bidder wishes to jump-bid to scare off the global bidder and the global bidder benefits from the information inferred from the jump-bids. Second, when a local bidder is the first to quit, the other local bidder wishes to cross-bid in order to keep the global bidder from becoming more competitive after winning an item, and then the global bidder does not have to buy an item right now without knowing the price of its complement. Along the first direction, the basic mechanism may be reformed to allow jump-bidding; along the second direction, cross-bidding may emerge. Conditional on the equilibrium in the mechanism that allows jump- and cross-bidding, the goods are overly concentrated to the global bidder. That triggers attempts to further reform the mechanism into one that allows resale from global to local bidders.

These mechanism-reform attempts are *self-emerging* in the sense that, if players follow the equilibrium of a given mechanism up to a certain event and if, at that event, all active players are given a chance to reform the given mechanism in a certain way, then they would unanimously vote for the reform. Conceivably, if a mechanism is operated repeatedly among different clones of the same kind of players, then a self-emerging reform attempt would erode the rules of the mechanism and may eventually turn the mechanism into a new one.



If one takes the viewpoint that mechanisms evolve through such self-emerging attempts, two interesting properties are possible: First, an evolution process is path-dependent. For instance, if the basic mechanism first turns into one that allows cross-bidding, jump-bidding will emerge, as argued in §5.1; in contrast, if the basic mechanism allows jump-bidding first, then cross-bidding does not emerge (Proposition 3). Second and somewhat paradoxically, a new mechanism may work against the intention for which it was created. For instance, the over-concentration problem creates an incentive for the global bidder to resell the goods occasionally; however, once he is expected to do so, other players adjust their bids so that this bidder becomes a middleman more often; being a monopolist at resale, he chooses to under-sell the goods; hence resale does not alleviate the over-concentration problem.

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