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THE CENTER FOR THE STUDY  
OF INDUSTRIAL ORGANIZATION  
AT NORTHWESTERN UNIVERSITY

Working Paper #0027

Robust Predictions for Bilateral Contracting  
with Externalities<sup>\*</sup>

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## Abstract

The paper studies bilateral contracting between one principal and  $N$  agents when each agent's utility depends on the principal's unobservable contracts with other agents. By offering an agent a menu and then choosing from it, the principal can signal her trades with other agents, similar to signaling type in Maskin and Tirole's [1992] analysis of mechanism design by an informed principal. Requiring immunity to deviations to such menu contracts bounds equilibrium outcomes in a wide class of bilateral contracting games without imposing ad hoc restrictions on agents' beliefs. This bound yields competitive convergence as  $N \rightarrow \infty$  in some environments and other asymptotic predictions in other environments. We also examine the additional restrictions that arise in two commonly assumed bilateral contracting processes, the "offer game" in which the principal makes simultaneous offers to the agents (e.g., Hart and Tirole [1990], Segal [1999]) and the "bidding game" in which the agents make simultaneous offers to the principal (e.g., Bernheim and Whinston [1986]).

# 1 Introduction

This paper studies bilateral contracting between one principal and  $N$  agents in the presence of externalities among agents. Examples of such situations include (i) vertical contracting where sales by an upstream firm to a downstream firm reduce the downstream price received by other downstream firms (Hart-Tirole [1990], McAfee-Schwartz [1994], Rey-Tirole [1996]), (ii) nonexclusive insurance, where the contract between an insured and an insurer affects the level of care taken by the insured and hence the profits of other insurers (Pauly [1974], Bernheim and Whinston [1986]), Kahn and Mookherjee [1998]), Bisin and Gottardi [1998], Bisin et al. [1999]), (iii) lending with default, where the debt owed by the principal to one agent affects the expected repayment to other agents (Bizer and deMarzo [1992], Dubey et al. [1996], Rajan and Parlour [forth.]), and many others (see Segal [1999]).

The existing literature on such contracting situations is divided into two branches. One branch studies specific noncooperative bilateral contracting games. In the two most-studied games, one of the two sides makes take-it-or-leave-it offers to the other. The game in which the principal makes offers to the agents, which we refer to below as the *offer game*, has been studied, for example, by Hart and Tirole [1990], McAfee and Schwartz [1994], Rey and Tirole [1996], and Segal [1999]. The game in which the agents make offers to the principal, which we refer to below as the *bidding game*, appears in models of common agency, including Bernheim and Whinston [1986a,b, 1998], O'Brien and Shaffer [1997], and Martimort and Stole [1998, 2000]. As well, some specific dynamic contracting games have been considered (deMarzo and Bizer [1993], Kahn and Mookherjee [1998]).

While this literature has generated important insights about the nature of the inefficiency caused by contracting externalities, the games studied in this literature generate a wide range of equilibrium outcomes, both within a given game and across different games. This diversity of equilibrium outcomes can be traced to differences in agents' beliefs about the principal's contracts with other agents. For example, in the offer game an agent who observes an out-of-equilibrium offer by the principal can hold arbitrary beliefs

about the principal's offers to other agents, and these beliefs can sustain a large set of equilibrium outcomes. In contrast, in the bidding game, an agent knows (in equilibrium) the offers that will be made to the principal by other agents when contemplating his own offer.

The other branch of the literature studies the case with a large number of agents and postulates competitive equilibrium (Dubey et al. [1996], Bisin and Gottardi [1998]). For example, in the lending and nonexclusive insurance applications, the competitive concept demands that each lender/insurer take the principal's total borrowing/insurance, and hence her moral hazard action, as given. One problem with this approach, however, is that it has not been justified as a limit of outcomes of noncooperative contracting games with  $N$  agents as  $N \rightarrow \infty$ , unlike competitive equilibrium in economies without externalities which has been justified by Cournot-style competitive limit results.<sup>1</sup>

This paper considers a family of noncooperative games of contracting with externalities, which we call *bilateral contracting games*, and which includes the offer and bidding games as special cases. We address the concern we have raised about the first branch of the literature by identifying properties of equilibrium outcomes that are *robust* in the sense that they must be satisfied by *all* equilibria of *all* bilateral contracting games. Moreover, among the properties we establish is the necessity of convergence, in certain environments, to the competitive outcomes postulated by the second branch.

A second sense in which we identify outcomes that are robust is that, in contrast to the existing noncooperative contracting branch of the literature, we allow for fully general contracts between the principal and an agent. Indeed, a key idea behind our results is that the set of equilibrium outcomes is dramatically affected when the parties are allowed to offer each other menus from which the principal can then choose, rather than simple point contracts. From an agent's viewpoint, a menu can separate the different "types" of the principal corresponding to different trades with other agents. For example, in the offer game, the principal can offer an agent a menu to signal her trade with

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<sup>1</sup>An exception is Bisin et al. [1999].

other agents, which is similar to Maskin and Tirole’s [1992] analysis of signaling by an informed principal. According to Myerson’s [1983] “inscrutability principle” and Maskin and Tirole’s analysis, there exists a menu giving all types of the principal their maximum payoff among all menus that the agent must accept regardless of his beliefs. This menu is called the Rothschild-Stiglitz-Wilson (RSW) menu by Maskin and Tirole. In bilateral contracting games in which an agent makes an offer to the principal (such as the bidding game), the agent can use the same RSW menu (plus a lump-sum payment) to screen the principals of different types.

The requirement that deviations in which the parties offer each other an RSW menu be unprofitable imposes a significant bound on the set of equilibrium outcomes. Namely, if the equilibrium bilateral surplus of the principal and an agent  $i$  were too low, there would exist a bilateral contract (consisting of the RSW menu and a lump-sum transfer) that would guarantee each of them a higher payoff. Indeed, the consideration of such menus is central to our ability to develop meaningful robust restrictions on equilibrium outcomes.

We begin in Section 2 by considering a simple example, based on a setting of vertical contracting between a manufacturer and  $N$  retailers, in which we illustrate many of the paper’s themes.

In Section 3 we develop our general approach to contracting with externalities, defining a general family of bilateral contracting games and characterizing their outcomes using the notion of an RSW menu. To illustrate the additional restrictions that may arise in particular bilateral contracting games, we also characterize the equilibria of the offer and bidding games. Here the notion of an RSW menu again proves useful, as our characterization of equilibrium outcomes in offer games relies heavily upon it.

In the rest of the paper we apply our general results to three settings with specialized payoffs. In Section 4, we focus on a setting in which a well-defined notion of competitive equilibrium exists. We show that in this environment, all bilateral contracting outcomes must converge to the competitive outcome as  $N \rightarrow \infty$ . This result is implied by the possibility of deviations to the “competitive menu,” which allows the principal to buy an

arbitrary quantity at the competitive equilibrium price. We show, in addition, that under some additional assumptions this competitive menu is an RSW menu, so that deviations to it provide us with the tightest bound on equilibrium outcomes possible from our general results in Section 3. We also use this result to fully characterize equilibrium trade profiles in offer games. Finally, we develop a characterization of equilibrium trade profiles in bidding games and see that these are a strict subset of those in the offer game.

In Sections 5 and 6 we consider instead two cases in which competitive equilibrium does not exist: the case in which there is “decreasing marginal cost” (Section 5), and the case in which only a “partial competitive equilibrium” exists (Section 6). As in Section 4, we identify RSW menus, use them to derive the implications of our necessary condition from Section 3, and derive characterizations of equilibrium outcomes in offer and bidding games. Our results for the case of a partial competitive equilibrium relate closely to the “noncompetitive outcomes” observed in work by Kahn and Mookerjee [1998], Bisin and Gottardi [1998], and Rajan and Parlour [1998].

Section 7 concludes.

## 2 A Simple Example

In this section we consider a simple example of contracting with externalities. To be specific, we focus on the setting of vertical contracting (Hart-Tirole [1990], McAfee-Schwartz [1994]), where one manufacturer (the “principal”) sells her output to  $N \geq 2$  retailers (“agents”). The retailers then resell their purchases in the downstream market.<sup>2</sup>

For simplicity, we assume that the retail technology converts each unit of the manufacturer’s product into a unit of the final good at a zero marginal cost. We let the manufacturer’s cost function be  $c(X) = \alpha X + \frac{1}{2}\beta X^2$ , and the downstream inverse demand function be  $P(X) = \max\{a - bX, 0\}$ . We assume that  $a > \alpha > 0$ ,  $b > 0$ , and  $b + \beta > 0$ .

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<sup>2</sup>For simplicity we do not allow retailers to withhold a portion of the good from the market.

The efficient outcome for the vertical structure is to sell the monopoly quantity:

$$X^* = \underset{X \geq 0}{\text{Arg max}} P(X)X - c(X) = \frac{a - \alpha}{2b + \beta}.$$

On the other hand, if the vertical structure were to behave as a price-taker,<sup>3</sup> it would sell the “competitive” quantity  $X^c$  at which marginal cost equals price,

$$c'(X^c) = P(X^c) \Rightarrow X^c = \frac{a - \alpha}{b + \beta},$$

provided that its profit from doing so is nonnegative. This is so (and so a competitive equilibrium exists) if and only if  $\beta \geq 0$ . Figure 1 illustrates this case. The efficient quantity  $X^*$  and the competitive quantity  $X^c > X^*$  will serve as benchmarks against which to compare the outcomes of bilateral contracting.

### Figure 1 Here

We will consider bilateral contracting games in which each retailer contracts with the manufacturer without being able to observe other retailers’ contracts with or purchases from the manufacturer.<sup>4</sup> A retailer’s behavior in such a game depends on his beliefs about other retailers’ contracts and purchases.

Here we illustrate the role of beliefs in the *offer game*, in which the manufacturer makes simultaneous offers to the downstream firms, who then accept or reject. Suppose initially that the manufacturer offers each retailer  $i$  a “point” contract  $(x_i, t_i)$ , where  $x_i \geq 0$  is the delivery of the product and  $t_i$  is the retailer’s payment.<sup>5</sup> After observing the offer, retailer  $i$  forms beliefs about other retailers’ contracts.

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<sup>3</sup>Specifically, suppose that all the firms are merged and naively take the retail price as given. Alternatively, we can consider a notion of competitive equilibrium in which the manufacturer takes the wholesale price as given, while the each retailer takes both the wholesale and retail prices as given. This latter notion is formally defined in Definition 3 in Section 4.

<sup>4</sup>Likewise, we assume that the contract with a retailer cannot be conditioned on the retail price, which would provide an indirect way to condition a contract on the manufacturer’s sales to rivals; see Hart and Tirole [1990].

<sup>5</sup>Contracting outcomes would not change if the manufacturer could offer a menu from which the retailer would choose without observing other contracts, as in Hart and Tirole [1990].



In the literature on vertical contracting, and contracting with externalities more generally, it is common to restrict attention to *passive beliefs* (see, e.g., Hart and Tirole [1990], Segal [1999]) in which, after observing a deviation, each retailer continues to believe that other retailers receive their equilibrium offers. Let  $(\hat{x}_1, \dots, \hat{x}_N, \hat{t}_1, \dots, \hat{t}_N)$  denote the equilibrium outcome. With passive beliefs, if retailer  $i$  receives offer  $(x_i, t_i) \neq (\hat{x}_i, \hat{t}_i)$ , he still believes that other retailers make their equilibrium purchases  $\hat{x}_{-i}$ , and he will accept the offer if and only if  $P(\hat{X}_{-i} + x_i)x_i \geq t_i$ , where  $\hat{X}_{-i} = \sum_{j \neq i} \hat{x}_j$ . Given this, the manufacturer's equilibrium sale  $\hat{x}_i$  to retailer  $i$  must be *pairwise stable* in the sense that

$$\hat{x}_i \in \text{Arg max}_{x_i \geq 0} P(\hat{X}_{-i} + x_i)x_i - c(\hat{X}_{-i} + x_i). \quad (1)$$

Condition (1) says that it is impossible to increase the *bilateral surplus* between the manufacturer and any retailer  $i$  (the sum of retailer  $i$ 's revenue and the manufacturer's costs) given the purchases of all other retailers  $j \neq i$ . In the present parameterized example, the unique profile of pairwise-stable trades is the symmetric profile  $x^p = (\hat{X}_N^p/N, \dots, \hat{X}_N^p/N)$ , where

$$\hat{X}_N^p = \frac{a - \alpha}{(1 + \frac{1}{N})b + \beta}.^6$$

Note that  $\hat{X}_1^p = X^*$  (the efficient quantity),  $\hat{X}_N^p < X^c$  (the competitive quantity) for all  $N$ , and  $\hat{X}_N^p \rightarrow X^c$  as  $N \rightarrow \infty$  (so there is competitive convergence).

While the restriction to passive beliefs provides a simple story for the inefficiency of bilateral contracting and competitive convergence, it has at least two problems. First, a passive-beliefs equilibrium need not exist. For instance, in the present example any passive-beliefs equilibrium involves an aggregate trade that converges to the quantity  $X^c$  as  $N \rightarrow \infty$ . But we have already seen that if  $\beta < 0$  this would involve negative profits for the manufacturer, and so cannot be an equilibrium outcome.<sup>7</sup>

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<sup>6</sup>This outcome coincides with the Cournot outcome for  $N$  retail firms facing marginal cost  $c'(\hat{X}_N^p)$ .

<sup>7</sup>That is, even when a pairwise-stable trade profile exists, there may be no passive-beliefs equilibrium (see Segal [1999]). The cause of this non-existence can be traced to the possibility of deviations by the principal to several agents at once (here, she will prefer to offer no contracts at all). See Rey and Vergé [1997] for another example of nonexistence of a passive-beliefs equilibrium due to multilateral deviations.

Second, and more fundamentally, the *ad hoc* restriction to passive beliefs may not be compelling in many circumstances. To take an extreme example, if the manufacturer has only  $\bar{X}$  units for sale (i.e.,  $c(X) = \infty$  for  $X > \bar{X}$ ), then a retailer who is offered  $\bar{X}$  units can be sure that other retailers are not getting any of the good, regardless of what the equilibrium allocation was supposed to be. More generally, for our payoff specifications with  $\beta \neq 0$ , retailers should be aware that the manufacturer's optimal contract offer to one retailer depends on her contracts with other retailers.

Once one allows retailers to hold arbitrary beliefs after observing out-of-equilibrium offers, a large set of outcomes can be sustained in a weak perfect Bayesian equilibrium.<sup>8</sup> For example, as noted by McAfee and Schwartz [1994], the efficient outcome  $X^*$  can be sustained for any  $N$  by endowing retailers with *symmetry beliefs*, under which each retailer believes that the manufacturer offers all retailers the same contract. Therefore, with symmetry beliefs, competitive convergence does *not* obtain. More generally, for any  $N$ , we can support any outcome in which all parties receive nonnegative payoffs, and thus any aggregate quantity  $\hat{X} \geq 0$  with nonnegative total surplus  $P(\hat{X})\hat{X} - c(\hat{X}) \geq 0$ , by letting each retailer  $i$  believe that  $X_{-i} > a/b$  following any observed deviation.

A key idea of this paper is that the manufacturer may be able to avoid the problem of negative inferences by retailers by using contracts that specify a *menu* of possible trades from which the manufacturer can choose. For example, suppose that  $\beta > 0$  and that the manufacturer deviates from an equilibrium by offering retailer  $i$  a contract that gives the manufacturer the right to choose from the *competitive menu*

$$C = \{(x, t) : x \in [0, X^c], t = p^c x\}.$$

Observe that retailer  $i$  is *guaranteed a zero payoff if he accepts the offer, regardless of his belief about the aggregate quantity  $X_{-i}$  sold to other retailers*. Indeed, if  $X_{-i} < X^c$ , then the manufacturer will optimally choose  $x_i = X^c - X_{-i}$  from the menu, and the retailer's payoff is  $P(X^c)x_i - p^c x_i = 0$ . On the other hand, if  $X_{-i} > X^c$ , then the manufacturer will choose  $x_i = 0$  from the menu, and the retailer's payoff is again zero. By paying in

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<sup>8</sup>This is also true for stronger solution concepts such as sequential equilibrium.

addition an arbitrarily small amount  $\varepsilon > 0$  to the retailer, the manufacturer can convince him to accept the menu regardless of his beliefs.

Allowing the manufacturer to offer menus dramatically constrains weak perfect Bayesian equilibrium outcomes since any equilibrium outcome must now be immune to the manufacturer's deviations using such menus.<sup>9</sup> To see the implications of this in our example, suppose that  $(\hat{x}_1, \dots, \hat{x}_N, \hat{t}_1, \dots, \hat{t}_N)$  with  $\hat{X} = \sum_i \hat{x}_i \leq X^c$  is an equilibrium outcome of the offer game in which the manufacturer can offer menus (equilibria with aggregate trade above  $X^c$  can be ruled out with similar arguments). Consider the manufacturer's deviation to retailer  $i$  with the competitive menu (plus a small fixed payment  $\varepsilon > 0$ ), while keeping her offers to other agents unchanged. As argued above, retailer  $i$  will accept this deviation, and the manufacturer will sell him  $x_i = X^c - \hat{X}_{-i}$ . Note that the manufacturer's equilibrium revenue from retailer  $i$  was *at most*  $P(\hat{X})\hat{x}_i$  (otherwise the retailer would have rejected the equilibrium contract), and therefore the revenue loss on existing sales due to the deviation is at most  $[P(\hat{X}) - p^c]\hat{x}_i$ . On the other hand, the manufacturer's revenue on new sales is  $p^c[X^c - \hat{X}]$ . Finally, she incurs the extra production cost  $c(X^c) - c(\hat{X})$ . Adding up, we see that for the deviation not to be profitable we must have

$$p^c[X^c - \hat{X}] - [P(\hat{X}) - p^c]\hat{x}_i \leq c(X^c) - c(\hat{X}).$$

Since there is always an agent  $i$  with  $\hat{x}_i \leq \hat{X}/N$ , a necessary condition for equilibrium is

$$p^c[X^c - \hat{X}] - [P(\hat{X}) - p^c]\frac{\hat{X}}{N} \leq c(X^c) - c(\hat{X})$$

or equivalently

$$[p^c X^c - c(X^c)] - [p^c \hat{X} - c(\hat{X})] \leq [P(\hat{X}) - p^c]\frac{\hat{X}}{N}.$$

This inequality implies competitive convergence as  $N \rightarrow \infty$ . Intuitively, when  $N$  is large, the manufacturer can find a retailer who buys very little in equilibrium, and

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<sup>9</sup>Note that in equilibrium the retailers have correct beliefs, hence any equilibrium outcome can be sustained with the manufacturer offering a point contract  $(x_i, t_i)$  to each retailer  $i$  on the equilibrium path.

deviate to this retailer with the competitive menu. If  $\widehat{X}$  were substantially below  $X^c$ , this deviation would be profitable, since the revenue loss on existing sales to the retailer is negligible, and the manufacturer sells him an extra  $X^c - \widehat{X}$  to achieve her optimal output  $X^c$  at price  $p^c$ . Indeed, in our parameterized example the above inequality is quadratic in  $\widehat{X}$ , and its solutions can be calculated to be

$$\widehat{X} \in [\underline{X}_N, X^c], \text{ where } \underline{X}_N = \frac{a - \alpha}{\left(\frac{2b}{\beta N} + 1\right)(b + \beta)} \rightarrow X^c \text{ as } N \rightarrow \infty.$$

In particular, the efficient (monopoly) aggregate quantity  $X^*$ , which obtained with symmetry beliefs when the manufacturer could not offer menus, cannot be sustained now when  $N > 2(1 + b/\beta)$ . On the other hand, the symmetric pairwise-stable aggregate trade  $\widehat{X}_N^p$  satisfies the above condition for any  $N$ .

To foreshadow the developments of the next section, note that the competitive menu is not the only menu the manufacturer can offer that assures a retailer a nonnegative payoff. The degenerate menu offering only the null trade  $(0, 0)$  is of course one such menu. For another, consider the class of linear-price menus giving the manufacturer the right to sell to the retailer any quantity at a price  $p$  (we continue to assume that  $\beta > 0$ ). The manufacturer's optimal choice given trades  $X_{-i}$  with other retailers can be stated with reference to her supply function  $S(p) = \max\{(p - \alpha)/\beta, 0\}$ : If  $X_{-i} \geq S(p)$ , the manufacturer will sell nothing to retailer  $i$ , and the retailer will receive zero profit. If instead  $X_{-i} < S(p)$ , the manufacturer will sell  $x_i = S(p) - X_{-i} > 0$ , which brings the aggregate sale to  $S(p)$ , and the retail price to  $P(S(p))$ . The retailer's profit will then be  $[P(S(p)) - p]x_i$ . This profit is nonnegative if and only if  $P(S(p)) \geq p$ , which in turn holds if and only if  $p \leq p^c$ ; see Figure 2. Thus, any linear-price menu with price  $p \leq p^c$  guarantees retailer  $i$  a nonnegative profit for any belief he could hold about  $X_{-i}$ . Conversely, any linear-price menu with price above  $p^c$  would entail a loss to the retailer for some belief about  $X_{-i}$ . Therefore, the best linear-price menu for the manufacturer among those that are acceptable to the retailer regardless of his beliefs is the competitive menu considered above.

**Figure 2 Here**

In fact, in Section 4, we show that the competitive menu is the best menu for the principal among *all* (nonlinear) menus that guarantee a retailer a nonnegative payoff regardless of his belief. This result uses the ideas of Maskin and Tirole’s [1992] analysis of signaling by an informed principal. By choosing from a menu, the manufacturer (principal) can signal to retailer  $i$  her “type”, which in our case consists of her sales  $X_{-i}$  to other retailers. The only difference from Maskin and Tirole’s analysis is that the principal’s type is chosen by her rather than by nature. Maskin and Tirole show that there exists one menu, which they call the *RSW menu*, that maximizes the payoff of all of the principal’s types at once among those that are acceptable to the agent regardless of his beliefs. In the above example, the competitive menu *is* an RSW menu.

While we have so far focused on the offer game, the main insight is more general. For example, an alternative bargaining process that has been studied in the literature on vertical contracting is a *bidding (or common agency) game*, in which retailers simultaneously make offers to the manufacturer. By offering a menu, retailer  $i$  can “screen” the different “types”  $X_{-i}$  of the manufacturer. (The idea that an agent’s contract with other principals in the common agency game constitutes his “type” has been developed by Martimort and Stole [1998, 1999].) Immunity to deviations that screen the principal using an RSW menu couple with a lump-sum transfer implies the same bound on equilibrium trades as in the offer game. In the next section, we apply our approach in a more general setting to a large family of contracting games in which the manufacturer can make offers to some retailers and other retailers can make offers to the manufacturer, perhaps in the course of many periods.

### 3 Characterization of Contracting Outcomes

We consider bilateral contracting games between one principal and  $N$  agents ( $N$  will also denote the set of agents).<sup>10</sup> The set of possible trades between the principal and agent  $i$  is

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<sup>10</sup>We adopt this terminology for the sake of consistency across different bargaining processes among these parties. Note that this terminology reverses the standard terminology in the literature on common

$\mathcal{X}_i \subset \mathbb{R}$ , and the typical trade will be represented by  $x_i \in \mathcal{X}_i$ . In addition, the agent can make a monetary transfer  $t_i$  to the principal. Let  $0 \in \mathcal{X}_i$  denote the default trade between the principal and agent  $i$ . The default outcome between them is then  $(x_i, t_i) = (0, 0)$ .

The parties' utilities are quasilinear in money: the principal's payoff is  $\sum_i t_i - c(x)$ , and each agent  $i$ 's payoff is  $u_i(x) - t_i$ , where  $x = (x_1, \dots, x_N) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N$  is the agents' trade profile. We assume that the principal's cost  $c(\cdot)$  is lower semicontinuous, which allows, e.g., for a fixed cost.<sup>11</sup> Externalities among agents arise because each agent's utility can depend on all agents' trades. We assume, however, that agents' *reservation utilities* do not depend on others' trades:  $u_i(0, x_{-i}) = 0$  for all  $x_{-i} \in \mathcal{X}_{-i}$ . (In the terminology of Segal [1999], there are no externalities on non-traders.)

A bilateral contract between the principal and each agent  $i$  takes the form of a menu, i.e., a subset  $M_i \subset \mathcal{X}_i \times \mathbb{R}$ . After the contract is signed, the principal chooses a bundle  $(x_i, t_i) \in M_i$ . Her optimal choice in general depends on her contracts with other agents, which are not observed by agent  $i$ .<sup>12</sup> We restrict menus to be compact sets, to ensure that for any collection of menus  $(M_1, \dots, M_N)$ , the principal has an optimal choice:  $\arg \max_{(x,t) \in M_1 \times \dots \times M_N} [\sum_i t_i - c(x)] \neq \emptyset$ .<sup>13</sup>

We consider the following class of *bilateral contracting games*: A game lasts for  $K$  periods. In each period  $k = 1, \dots, K$ , a subset  $A_k \subset N$  of agents simultaneously offer (compact) menus to the principal, and simultaneously the principal offers (compact) menus to a subset  $P_k \subset N$  of agents (with  $P_k \cap A_k = \emptyset$ ). Then the principal and agents simultaneously decide whether to accept contracts offered to them. The game

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agency (bidding) games where the single party receiving offers is called the "agent" and the parties making offers are "principals".

<sup>11</sup>The function  $c(\cdot)$  is lower semi-continuous if  $x^n \rightarrow x$  implies that  $\liminf_{n \rightarrow \infty} c(x^n) \geq c(x)$ .

<sup>12</sup>We could allow more complicated contracts where agent  $i$  as well as the principal send messages determining  $(x_i, t_i)$ . However, since agents possess no private information, this would not affect the set of equilibrium outcomes.

<sup>13</sup>If a profile of menus from which the principal has no optimal choice could be offered in a bilateral contracting game, then there would be no continuation equilibrium following acceptance of these menus, hence the game would not have a weak perfect Bayesian equilibrium.

then proceeds to the next period. We assume that  $\cup_{k=1}^K (P_k \cup A_k) = N$ , i.e., each agent has at least one chance to contract with the principal. The principal observes all history, while each agent observes only menus offered to him and the principal's accept/reject decision from menus that he offers the principal. At the end of the game, for each agent  $i$  the principal chooses  $(x_i, t_i)$  from the last contract accepted with this agent (which supercedes all previous contracts), and  $(x_i, t_i) = (0, 0)$  if no contract has been agreed to. The parties then receive payoffs on the basis of this outcome (there is no discounting).<sup>14</sup> Two examples of bilateral contracting games are given by the offer and bidding games discussed in Section 2. The solution concept we adopt is that of (pure-strategy) weak Perfect Bayesian equilibrium (henceforth, "weak PBE").<sup>15</sup>

### 3.1 Acceptable Menus and Necessary Conditions

We will examine some properties that all pure-strategy weak PBE outcomes of any bilateral contracting game must satisfy. To fix ideas, consider first the offer game. It is clear that any weak PBE outcome  $(\hat{x}, \hat{t})$  of the game must satisfy agents' participation constraints:

$$u_i(\hat{x}) - \hat{t}_i \geq 0 \text{ for all } i \in N. \quad (2)$$

Indeed, if the inequality did not hold for some agent  $i$ , he could profitably deviate by rejecting all of the principal's offers and always offering her the null contract  $\{(0, 0)\}$ .

Another necessary condition on weak PBE outcomes of the offer game follows from considering the principal's deviations to menus that must be accepted by an agent  $i$  regardless of his belief about the principal's trades other agents, given that the agent expects the principal to choose optimally from this menu. Formally,

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<sup>14</sup>Our results would not be affected if the principal could choose from menus immediately upon their acceptance rather than at the end of the game.

<sup>15</sup>See Mas-Colell et al. [1995]. Our results would be unaffected by adopting instead stronger notions of perfect Bayesian equilibrium.

**Definition 1** A menu  $M_i \subset \mathcal{X}_i \times \mathbb{R}$  is acceptable to agent  $i$  if for any  $\bar{x}_{-i} \in \mathcal{X}_{-i}$  and any  $(\bar{x}_i, \bar{t}_i) \in \arg \max_{(x_i, t_i) \in M_i} [t_i - c(x_i, \bar{x}_{-i})]$  we have  $u_i(\bar{x}_i, \bar{x}_{-i}) - \bar{t}_i \geq 0$ . The set of compact menus that are acceptable to agent  $i$  is denoted by  $\mathcal{A}_i \subset 2^{\mathcal{X}_i \times \mathbb{R}}$ .

Consider the principal's deviation in the offer game in which she offers menu  $M_i \in \mathcal{A}_i$  to agent  $i$  (plus an arbitrarily small payment  $\varepsilon > 0$  to ensure acceptance), while following her equilibrium contracting with trades  $\hat{x}_{-i}$  with other agents. The principal's maximum profit from this deviation is<sup>16</sup>

$$\Pi_i^{M_i}(\hat{x}_{-i}) \equiv \max_{(x_i, t_i) \in M_i} [t_i - c(x_i, \hat{x}_{-i})].$$

Since the deviation must not be profitable, we must have

$$\hat{t}_i - c(\hat{x}) \geq \Pi_i^{M_i}(\hat{x}_{-i}).$$

Summing this inequality with agent  $i$ 's participation constraint (2), we see that any weak PBE outcome must satisfy

$$u_i(\hat{x}) - c(\hat{x}) \geq \Pi_i^{M_i}(\hat{x}_{-i}) \text{ for all } i \in N.$$

This inequality bounds from below the bilateral surplus  $u_i(\hat{x}) - c(\hat{x})$  of the principal and each agent  $i$  in any weak PBE of the offer game. Moreover, this condition must hold in *any* bilateral contracting game since if it did not either the principal or agent  $i$  could gain by offering the menu  $M_i$  to the other with an appropriately chosen lump-sum transfer. Indeed, if this inequality does not hold, then menu  $M_i$  would increase bilateral surplus since it gives the agent a payoff of at least 0, and gives the principal a payoff of at least  $\Pi_i^{M_i}(\hat{x}_{-i})$  (since she can always leave her trades with other agents unchanged). Thus, with an appropriately chosen lump-sum transfer, both parties would be made better off. A formalization of this idea yields

**Proposition 1** If  $M_i \in \mathcal{A}_i$ , then any pure-strategy weak PBE outcome  $(\hat{x}, \hat{t})$  of a bilateral contracting game must satisfy

$$u_i(\hat{x}) - c(\hat{x}) \geq \sup_{M_i \in \mathcal{A}_i} \Pi_i^{M_i}(\hat{x}_{-i}) \text{ for all } i \in N. \quad (3)$$

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<sup>16</sup>We define this profit not including the (fixed) transfer from agents  $j \neq i$ ,  $\sum_{j \neq i} \hat{t}_j$ .



**Proof.** Suppose in negation that there exists an equilibrium with outcome  $(\hat{x}, \hat{t})$  in which for some  $i \in N$  and some  $M_i \in \mathcal{A}_i$ ,

$$u_i(\hat{x}) - c(\hat{x}) > \Pi_i^{M_i}(\hat{x}_{-i}). \quad (4)$$

Let  $\bar{k}$  be the largest  $k$  for which  $i \in P_k \cup A_k$  (i.e., the last period in which the principal and agent  $i$  contract).

Suppose first that  $i \in P_{\bar{k}}$ . Consider a deviation by the principal in contracting with agent  $i$  in which she rejects all of agent  $i$ 's offers, offers the null contract  $\{(0, 0)\}$  to the agent in periods  $k < \bar{k}$  and offers him  $M_i$  plus a small payment  $\varepsilon > 0$  in period  $k = \bar{k}$ . Agent  $i$  will accept this contract for any beliefs about the principal's contracting with other agents. Suppose that the principal uses her equilibrium strategy in contracting with other agents. Then her payoff from the deviation is

$$\sum_{j \neq i} \hat{t}_j + \Pi_i^{M_i}(\hat{x}_{-i}) - \varepsilon > \sum_{j \neq i} \hat{t}_j + u_i(\hat{x}) - c(\hat{x}) \geq \sum_j \hat{t}_j - c(\hat{x}),$$

where the first inequality is by (4), and the second by agent  $i$ 's participation constraint (2). Thus, the deviation makes the principal better off.

Suppose instead that  $i \in A_{\bar{k}}$ . Consider a deviation by the agent in which he uses the equilibrium strategy in periods  $k < \bar{k}$  and offers  $M_i$  in period  $\bar{k}$  minus a payment

$$\Delta = \Pi_i^{M_i}(\hat{x}_{-i}) - [\hat{t}_i - c(\hat{x})] - \varepsilon,$$

where  $\varepsilon > 0$  is small. By construction, this deviation will make the principal better off than in equilibrium even if she continues to use her equilibrium strategy with other agents. (She might be able to do even better by changing her strategies with other agents.) Since rejecting the agent's deviation cannot make the principal better off than in equilibrium (this option was available to her in equilibrium), she will strictly prefer to accept this deviating offer. Since  $M_i$  is acceptable, the agent's payoff from this deviation will be at least  $\Delta$  for any contracts ultimately signed by the principal with other agents. Using the agent's equilibrium participation constraint (2) and (4), we see that

$$\Delta \geq \Pi_i^{M_i}(\hat{x}_{-i}) - [u_i(\hat{x}) - c(\hat{x})] - \varepsilon > 0$$

when  $\varepsilon$  is sufficiently small. Thus, the agent's deviation is profitable. ■

In principle, identifying the right-hand side of (3) involves identifying a distinct profit-maximizing compact acceptable menu for each type  $x_{-i}$  (assuming that such an optimal menu exists – i.e., that the sup is actually attained). It turns out, however, that just as in Maskin and Tirole [1992] (see also Myerson [1983]) there often exists a *single* compact acceptable menu that maximizes the principal's profit for *every* type  $x_{-i}$  that she may have. Formally, we call such a menu an *RSW menu*, in accord with the terminology introduced by Maskin and Tirole:

**Definition 2** *A menu  $R_i \in \mathcal{A}_i$  is an RSW menu for agent  $i$  if  $\Pi_i^{R_i}(x_{-i}) = \sup_{M_i \in \mathcal{A}_i} \Pi_i^{M_i}(x_{-i})$  for all  $x_{-i} \in \mathcal{X}_{-i}$ .*

At first it may appear that the existence of an RSW menu would be an unlikely coincidence. However, the following observation suggests that this existence is quite natural:

**Lemma 1** *The union of acceptable menus is an acceptable menu.*

**Proof.** Suppose  $M = \cup_{s \in S} M_s$ , where  $M_s$  is an acceptable menu to agent  $i$  for each  $s \in S$ . Take any  $\bar{x}_{-i} \in \mathcal{X}_{-i}$ , and any  $(\bar{x}_i, \bar{t}_i) \in \arg \max_{(x_i, t_i) \in M} [t_i - c(x_i, \bar{x}_{-i})]$ . By construction,  $(\bar{x}_i, \bar{t}_i) \in M_s$  for some  $s \in S$ , and  $(\bar{x}_i, \bar{t}_i) \in \arg \max_{(x_i, t_i) \in M_s} [t_i - c(x_i, \bar{x}_{-i})]$  since  $M_s \subset M$ . Since  $M_s$  is acceptable, we must have  $u_i(\bar{x}_i, \bar{x}_{-i}) - \bar{t}_i \geq 0$ . ■

Lemma 1 suggests that a natural candidate for an RSW menu is the union  $\Omega_i = \cup \mathcal{A}_i \equiv \cup_{M_i \in \mathcal{A}_i} M_i$  of all compact acceptable menus to agent  $i$ . Indeed, if  $\Omega_i$  is compact, then by definition we have

$$\Pi_i^{\Omega_i}(x_{-i}) = \max_{(x_i, t_i) \in \cup \mathcal{A}_i} [t_i - c(x_i, x_{-i})] = \sup_{M_i \in \mathcal{A}_i} \max_{(x_i, t_i) \in M_i} [t_i - c(x_i, x_{-i})] = \sup_{M_i \in \mathcal{A}_i} \Pi_i^{M_i}(x_{-i}).$$

A technical complication may arise when the menu  $\Omega_i$  is not compact, in which case the principal may not have an optimal choice from it. Indeed, there is usually an infinite number of compact acceptable menus, and the union of an infinite number of compact

menus need not be compact. In our cases of interest, however, there will always exist a compact RSW menu  $R_i \subset \Omega_i$ .<sup>17</sup> (We will find it convenient to use the smallest RSW menu, obtained by eliminating from  $\Omega_i$  all the bundles that are never optimal for the principal.)

In much of the rest of the paper we shall be interested in the implications of Proposition 1. Although we shall focus on special cases of the payoffs considered here in the sections that follow, a few more general observations about the implications of Proposition 1 are nevertheless possible. Consider, for example, the case without externalities, where  $u_i(x_i, x_{-i}) \equiv u_i(x_i)$ . In this case, agent  $i$  will accept a menu  $M_i$  if and only if the principal's optimal choices  $(x_i, t_i)$  from this menu for any  $x_{-i}$  satisfy  $u_i(x_i) - t_i \geq 0$ . Eliminating the principal's suboptimal choices, we see that  $\{(x_i, u_i(x_i)) : x_i \in \mathcal{X}_i\}$  is an RSW menu. This menu, in which agent  $i$  "sells out" to the principal, is similar to "truthful" menus considered by Bernheim and Whinston [1986b].<sup>18</sup> The necessary condition in (3) is then equivalent to the condition that  $u_i(\hat{x}_i) - c(\hat{x}) \geq \max_{x_i \in \mathcal{X}_i} u_i(x_i) - c(x_i, \hat{x}_{-i})$  in any bilateral contracting game: the equilibrium trade profile must maximize the bilateral surplus of each principal-agent pair given the principal's trades with other agents. (Note that this need not imply efficiency.)

More generally, we say that a trade profile  $\hat{x}$  is *pairwise stable* if it maximizes the bilateral surplus of each principal-agent pair; that is,

$$\hat{x}_i \in \text{Arg} \max_{x_i \in \mathcal{X}_i} u_i(x_i, \hat{x}_{-i}) - c(x_i, \hat{x}_{-i}) \text{ for each } i \in N.$$

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<sup>17</sup>Were this not the case, we would proceed by approximating the RSW profit arbitrarily closely using compact acceptable subsets of  $\Omega_i$ . Maskin and Tirole [1992] indeed perform such approximation in a model with finite domains  $\mathcal{X}_i$ . The RSW profit in their model is achieved by a menu on the boundary of  $\Omega_i$ , which is not acceptable because not all of the principal's optimal choices satisfy the agent's participation constraint. Maskin and Tirole use the existence of an arbitrarily close compact acceptable menu in which the principal's optimal choices are unique.

<sup>18</sup>We can relate our model formally to Bernheim and Whinston's [1986b] by letting each  $x_i$  represent the principal's promise of the whole public decision, and assuming that  $c(x)$  is prohibitively high unless  $x_1 = \dots = x_N$ . Note the absence of externalities in this model.

While without externalities condition (3) is equivalent to pairwise stability, we can show that pairwise stable outcomes are always consistent with condition (3) even when externalities are present:

**Proposition 2** *A pairwise-stable trade profile  $\hat{x}$  satisfies (3).*

**Proof.** Suppose in negation that there exist  $i \in N$  and  $M_i \in \mathcal{A}_i$  such that

$$\Pi_i^{M_i}(\hat{x}_{-i}) > u_i(\hat{x}) - c(\hat{x}).$$

Taking  $(\tilde{x}_i, \tilde{t}_i) \in \arg \max_{(x_i, t_i) \in M_i} [t_i - c(x_i, \hat{x}_{-i})]$ , the inequality means

$$\tilde{t}_i - c(\tilde{x}_i, \hat{x}_{-i}) > u_i(\hat{x}) - c(\hat{x}).$$

Since  $M_i$  is acceptable,  $u_i(\tilde{x}_i, \hat{x}_{-i}) - \tilde{t}_i \geq 0$ . Adding to the above inequality, we obtain

$$u_i(\tilde{x}_i, \hat{x}_{-i}) - c(\tilde{x}_i, \hat{x}_{-i}) > u_i(\hat{x}) - c(\hat{x}),$$

which contradicts pairwise stability. ■

Intuitively, the result follows because, as stated earlier, necessary condition (3) essentially requires that the bilateral surplus between any principal-agent pair cannot be increased by a deviation to a compact acceptable menu, which is necessarily true in any pairwise stable trade profile. Thus, although we may be critical of the assumption of passive beliefs, the pairwise stability condition that follows from passive beliefs is in fact compatible with our necessary condition. When externalities are present, however, trade profiles that are not pairwise stable may also be consistent with condition (3): this reflects the fact that in this case deviations that increase bilateral surplus may be prevented by adverse beliefs about the principal's trades with other agents.

It is natural to ask whether conditions (2) and (3) are sufficient for a trade outcome to emerge in a weak PBE of a given bilateral contracting game. The answer in general is no: the set of equilibrium trade profiles in a bilateral contracting game may be a proper subset of those satisfying (2) and (3). The reasons for additional restrictions on outcomes differ from game to game. We illustrate this point in the next two subsections

by characterizing weak PBE outcomes of two particular bilateral contracting games: the offer game and the bidding game.<sup>19</sup>

### 3.2 Equilibria of the Offer Game

In the offer game, even when the principal cannot raise profit by deviating to an RSW menu to one agent, she may be able to deviate profitably by offering RSW menus to several agents at once. To consider the profitability of such deviations, we need to define the supremum profit function from compact acceptable menus to a set  $D \subset N$  of agents:

$$\bar{\Pi}_D(x_{-D}) \equiv \sup_{M_i \in \mathcal{A}_i \text{ for } i \in D} \max_{(x_D, t_D) \in \prod_{i \in D} M_i} \left[ \sum_{i \in D} t_i - c(x_D, \hat{x}_{-D}) \right].$$

Note that when an RSW menu  $R_i$  exists for each agent  $i$ , this function can be obtained as

$$\bar{\Pi}_D(x_{-D}) = \max_{(x_D, t_D) \in \prod_{i \in D} R_i} \left[ \sum_{i \in D} t_i - c(x_D, \hat{x}_{-D}) \right]. \quad (5)$$

Consideration of such multilateral deviations gives rise to the following characterization of weak PBE outcomes:

**Proposition 3** *Suppose that  $u_i(\cdot)$  is bounded below for all  $i$ . Then  $(\hat{x}, \hat{t})$  is a weak PBE outcome of the offer game if and only if agents' participation constraints (2) hold, and*

$$\sum_{i \in D} \hat{t}_i - c(\hat{x}) \geq \bar{\Pi}_D(\hat{x}_{-D}) \text{ for all } D \subset N. \quad (6)$$

**Proof.** Necessity: If (2) did not hold, an agent would profitably deviate by rejecting the principal's offer. If (6) did not hold, the principal could profitably deviate to a set of agents  $D \subset N$  by offering each agent  $i \in D$  a menu  $M_i \in \mathcal{A}_i$  minus a small payment  $\varepsilon > 0$ . All agents from  $D$  would accept, and the principal would guarantee herself a payoff arbitrarily close to  $\sum_{i \notin D} \hat{t}_i + \bar{\Pi}_D(x_{-D}) > \sum_i \hat{t}_i - c(\hat{x})$ .

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<sup>19</sup>One can in fact derive some additional necessary conditions that must hold in any bilateral contracting game. In particular, the principal must not be able to gain from a deviation to null contracts with any set of agents. For example, this rules out any trade profile with negative surplus, such as the pairwise stable trade profile in the example of Section 2 with  $\beta < 0$  and  $N$  large.

Sufficiency: A weak PBE sustaining  $(\widehat{x}, \widehat{t})$  is described by the following strategies and beliefs. The principal's strategy is to offer the point contract  $\widehat{M}_i = \{(\widehat{x}_i, \widehat{t}_i)\}$  to each agent  $i$ . Following any accepted menu profile  $(M_1, \dots, M_N)$  in which only one agent, say  $j$ , accepts a non-degenerate menu, the principal chooses from among those elements of  $(x, t) \in \arg \max_{(x,t) \in \prod_i M_i} [\sum_i t_i - c(x)]$  that have the lowest value of  $u_j(x) - t_j$ .<sup>20</sup> Each agent  $i$ 's strategy is to accept  $\widehat{M}_i$  and all menus from  $\mathcal{A}_i$ , and to reject all other menus. As for agent  $i$ 's beliefs following an offer  $M_i \neq \widehat{M}_i$ , for any menu  $M_i \notin \mathcal{A}_i$  we can find  $\bar{x}_{-i}(M_i) \in \mathcal{X}_{-i}$  such that there exists  $(x'_i, t'_i) \in \arg \max_{(x_i, t_i) \in M_i} [t_i - c(x_i, \bar{x}_{-i}(M_i))]$  with  $u_i(x'_i, \bar{x}_{-i}(M_i)) - t'_i < 0$ . Following any menu  $M_i \neq \widehat{M}_i$  such that  $M_i \notin \mathcal{A}_i$  let agent  $i$  believe that the principal's offer to each other agent  $j \neq i$  is  $M_j = \{(\bar{x}_j(M_i), \bar{t}_j)\}$ , where  $\bar{t}_j < \inf_{x \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N} u_j(x)$  for all  $j$ . (Observe that  $M_j \in \mathcal{A}_j$ , so that such an offer will be accepted by agent  $j$ .)

Agents' participation constraints (2) ensure that agents have no profitable deviations. The principal's deviation to any menu  $M_i \notin \mathcal{A}_i$  will be rejected by agent  $i$  given his beliefs and the principal's strategy. The principal's deviation to any menus  $M_i \in \mathcal{A}_i$  (including the null contract) to agents  $i \in D$  cannot give her a higher payoff than  $\sum_{i \notin D} \widehat{t}_i + \overline{\Pi}_D(\widehat{x}_{-D})$ , which by (6) does not exceed her equilibrium payoff. ■

### 3.3 Equilibria of the Bidding Game

Recall that in the bidding game, agents simultaneously offer menus to the principal, who then chooses which menus to accept and makes choices from the accepted menus. Note that in contrast to the offer game, in a bidding game we need to consider only unilateral deviations, just as we did in Proposition 1. The additional constraints on equilibrium outcomes relative to Proposition 1 come from a different source: since an agent in the bidding game knows the equilibrium menus offered by other agents, he can predict the principal's choices from them following his deviation. Thus, his beliefs about the principal's trades with other agents are not arbitrary, which creates the possibility

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<sup>20</sup>We can specify any optimal choices by the principal for all other profiles of accepted contracts.

for additional profitable deviations, and hence reduces the set of sustainable outcomes.

One issue that arises in characterizing equilibrium outcomes in bidding games is that in situations in which the principal is indifferent among a number of trade profiles she may be able to prevent an agent from reducing his transfer by (effectively) threatening to choose a less desirable outcome from among these profiles.<sup>21</sup> This possibility is ruled out if the principal's choice is invariant to additive transformations in the menus she chooses from; that is, if the addition of lump-sum payments (of either sign) to a set of menus does not alter the principal's optimal choices from these menus.

**Proposition 4** *Take a strategy profile  $M = (M_1, \dots, M_N)$ . For  $S \subset N$  and  $x_{-S} \in \prod_{j \in N \setminus S} \mathcal{X}_j$ , define*

$$\pi_S(x_{-S}) = \max_{(x_S, t_S) \in \prod_{j \in S} [M_j \cup (0,0)]} \left[ \sum_{j \in S} t_j - c(x_S, x_{-S}) \right].$$

*If the outcome  $(\hat{x}, \hat{t})$  arises in a subgame perfect Nash equilibrium<sup>22</sup> in which agents use strategies  $M$ , then*

(i)

$$\sum_j \hat{t}_j - c(\hat{x}) = \pi_N;$$

(ii) *agents' participation constraints (2) hold;*

(iii) *for all  $i \in N$  and  $x_i \in \mathcal{X}_i$ , there exists*

$$(x_{-i}, t_{-i}) \in \arg \max_{(\tilde{x}_{-i}, \tilde{t}_{-i}) \in \prod_{j \in N \setminus i} [M_j \cup (0,0)]} \left[ \sum_{j \neq i} \tilde{t}_j - c(x_i, \tilde{x}_{-i}) \right]$$

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<sup>21</sup>For example, suppose that the principal's cost function is separable:  $c(x) = \sum_i c_i(x_i)$  and negative externalities are present. Then there is an equilibrium in which each agent  $i$  offers the menu  $M_i = \{(x_i, t_i) : t_i = c_i(x_i) + K\}$  where  $K = u_i(x^*)$ , agent  $i$ 's gross payoff at the first-best trade profile  $x^*$ , the principal chooses  $x^*$  given these offers, and punishes any agent for deviating with a very large choice of  $x_{-i}$ . The agents earn zero in this equilibrium and the principal earns the entire first-best surplus.

<sup>22</sup>Since the bidding game is one of perfect information, the concept of weak PBE reduces to that of subgame perfect Nash equilibrium.

such that

$$u_i(x_i, x_{-i}) + \pi_{-i}(x_i) \leq u_i(\hat{x}) + \sum_{j \neq i} \hat{t}_j - c(\hat{x}).$$

If, in addition, the principal's choice is invariant to additive transformations in the menus she chooses from, then<sup>23</sup>

(iv) For all  $i \in N$ ,

$$\sum_j \hat{t}_j - c(\hat{x}) = \pi_{-i}(0).$$

Moreover, if conditions (i)-(iv) hold, then  $(\hat{x}, \hat{t})$  can be supported as a subgame perfect Nash equilibrium outcome in which the agents use strategies  $M$ .

**Proof.** Necessity: (i) says that the principal's choice from agents' equilibrium menus is optimal, which is necessary for subgame perfection. If condition (ii) did not hold for agent  $i$ , then he could profitably deviate to  $M'_i = \{(0, 0)\}$ . If (iii) did not hold for some  $i$  and  $x_i$ , agent  $i$  could deviate to  $M'_i = \{(x_i, t_i)\}$ , where  $t_i = \pi_{-i}(0) - \pi_{-i}(x_i) + \varepsilon$  for a small  $\varepsilon > 0$ . The principal will accept this deviation, and letting  $(x_{-i}, t_{-i})$  denote her ensuing choice from other agents' menus, agent  $i$ 's payoff will be

$$\begin{aligned} u_i(x_i, x_{-i}) - t_i &= u_i(x_i, x_{-i}) - \pi_{-i}(0) + \pi_{-i}(x_i) - \varepsilon \\ &> u_i(\hat{x}) + \sum_{j \neq i} \hat{t}_j - c(\hat{x}) - \pi_{-i}(0) - \varepsilon \geq u_i(\hat{x}) - \hat{t}_i - \varepsilon, \end{aligned}$$

where the strict inequality is by the violation of (iii), and the weak inequality is by (i) and the fact that  $\pi_N \geq \pi_{-i}(0)$ . Therefore, for small enough  $\varepsilon$ , the deviation would make agent  $i$  better off. Finally, if the principal's choice is invariant to additive transformations of menus, then if (iv) did not hold, agent  $i$  would deviate to offer menu  $M_i$  minus a small  $\varepsilon > 0$ . The principal would accept the deviation and choose  $(x_i, t_i) = (\hat{x}_i, \hat{t}_i - \varepsilon)$  and  $(x_{-i}, t_{-i}) = (\hat{x}_{-i}, \hat{t}_{-i})$ , making agent  $i$  better off.

Sufficiency: (i) ensures that the principal does not want to deviate following  $(M_1, \dots, M_N)$ . Let the principal's strategy in response to a deviation  $M'_i$  by agent  $i$  be to choose some

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<sup>23</sup>Note that condition (iv) together with condition (iii) with  $x_i = 0$  implies condition (ii).



$(x_i, t_i) \in \arg \max_{M'_i} [t_i + \pi_{-i}(x_i)]$  and

$$(x_{-i}, t_{-i}) \in \arg \min_{(\bar{x}_{-i}, \bar{t}_{-i}) \in \arg \max_{(\bar{x}_{-i}, \bar{t}_{-i}) \in \prod_{j \in N \setminus i} [M_j \cup (0,0)]} [\sum_{j \neq i} \tilde{t}_j - c(x_i, \bar{x}_{-i})]} u_i(x_i, \bar{x}_{-i}).$$

If agent  $i$ 's deviation  $M'_i$  is accepted by the principal (and she chooses  $(x_i, t_i) \in M'_i$ ), we must have  $\pi_{-i}(x_i) + t_i \geq \pi_{-i}(0)$ . But then, agent  $i$ 's payoff from the deviation is

$$u_i(x_i, x_{-i}) - t_i \leq u_i(x_i, x_{-i}) + \pi_{-i}(x_i) - \pi_{-i}(0) \leq u_i(\hat{x}) + \sum_{j \neq i} \hat{t}_j - c(\hat{x}) - \pi_{-i}(0) = u_i(\hat{x}) - \hat{t}_i,$$

where the second inequality uses (iii) and the last equality uses (iv). Therefore, the deviation cannot be profitable. ■

Condition (iii) of Proposition 4 captures the fact that an agent  $i$  can predict the principal's reaction to his deviation based on his knowledge of the menus offered in equilibrium by other agents. Observe that an agent  $i$  who contemplates deviating with a menu from which the principal will choose  $(x_i, t_i)$  can achieve the same result by offering the point menu  $(x_i, t_i)$ . Condition (iii) says that for any such deviation there must exist an optimal response by the principal that lowers the bilateral surplus of agent  $i$  and the principal (which includes transfers earned from agents  $j \neq i$ ), given the menus offered by the other agents. It can be verified that condition (iii), along with (ii), implies the necessary condition (3). We will see in Section 6, however, that it can lead to further restrictions on the set of sustainable trade profiles.

## 4 A (Strict) Competitive Equilibrium Exists

In the remainder of the paper we apply the results of Section 3 to a particular class of payoff functions with externalities. In particular, we suppose that there are  $N \geq 2$  agents,  $\mathcal{X}_i = \mathbb{R}_+$  for each agent  $i$ , and his payoff is  $u_i(x_i, x_{-i}) - t_i = \alpha(X)x_i - t_i$ , where  $X = \sum_{i \in N} x_i$ . The principal's payoff is  $\sum_{i \in N} t_i - c(X)$ .<sup>24</sup> We assume that  $\alpha(\cdot)$  is bounded below and that  $c(\cdot)$  is lower semicontinuous. We also normalize  $c(0) = 0$ .

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<sup>24</sup>These payoffs corresponds to the case of Condition L with no externalities on nontraders in Segal [1999].

Note that this setting includes as a special case the vertical contracting example discussed in Section 2, studied by Hart and Tirole [1990] and McAfee and Schwartz [1994]. Another model that belongs to this class is the insurance with moral hazard model studied by Pauly [1974], Kahn and Mookerjee [1998], and Bisin et al. [1998, 1999] (see Segal [1999] for other examples). In the insurance model, a single risk-averse individual with constant absolute risk aversion  $r \geq 0$  and initial wealth  $W$  (the principal in our terminology) may contract with any of  $N$  risk-neutral firms. The individual chooses the unobservable probability  $\pi \in \Pi$  of a loss  $L > 0$  at a private cost  $\gamma(\pi)$ . Letting  $x_i \geq 0$  denote the payment promised by agent  $i$  in the event of a loss, the certainty equivalent of the principal's payoff is

$$W + \sum_i t_i - \gamma(\pi) - \frac{1}{r} \ln[1 + \pi \cdot (\exp\{-r(X - L)\} - 1)].$$

The principal chooses the probability of loss  $\pi(X)$  to maximize this expression. By the Monotone Selection Theorem of Milgrom and Shannon [1994],  $\pi(X)$  is nondecreasing in  $X$ . The payoff of each agent  $i$  is then  $-\pi(X)x_i - t_i$ .

We use the following notion of competitive equilibrium:

**Definition 3** *A price-quantity pair  $(p^c, X^c) \in \mathbb{R} \times \mathbb{R}_+$  is a competitive equilibrium if (i)  $X^c \in \arg \max_{X \in \mathbb{R}_+} p^c X - c(X)$ , and (ii)  $p^c = \alpha(X^c)$ . It is a strict competitive equilibrium if, in addition,  $\arg \max_{X \in \mathbb{R}_+} [p^c X - c(X)] = \{X^c\}$ .*

Condition (i) says that  $X^c$  is the principal's optimal supply given the market price  $p^c$ , while condition (ii) ensures that agents are willing to demand an aggregate quantity of  $X^c$  if each agent takes price  $p^c$  and the aggregate trade  $X^c$  as given. Figure 1 in Section 2 depicts a competitive equilibrium in the vertical contracting example. Figure 3 depicts a competitive equilibrium in the insurance with moral hazard model when  $\Pi = \{\underline{\pi}, \bar{\pi}\}$ . Observe that the shaded area  $A$  must be at least as large as the cross-hatched area  $B$  for  $(p^c, X^c)$  to be a competitive equilibrium. A competitive equilibrium  $(p^c, X^c)$  is *strict* if  $X^c$  is the principal's *unique* optimal trade when facing price  $p^c$ . In this section we assume that a strict competitive equilibrium exists, while in subsequent sections we consider some cases in which it does not.

Figure 3 Here

## 4.1 Necessary Conditions and Competitive Convergence

We begin by using Proposition 1, which requires that contracting outcomes be immune to unilateral deviations to acceptable menus, to establish a “competitive limit” result as the number of agents grows infinitely large. Specifically, we shall focus on one acceptable menu - the *competitive menu*  $C = \{(x, p^c x) : x \in [0, X^c]\}$ . Faced with this menu, the principal whose aggregate trade with other agents is  $X_{-i} \leq X^c$  will choose  $x_i = X^c - X_{-i}$ . We will also ensure that the principal with  $X_{-i} > X^c$  chooses  $x_i = 0$ , by assuming that  $p^c X - c(X)$  is strictly decreasing for  $X \geq X^c$ . Then, in both cases, agent  $i$  receives a zero payoff, and so  $C$  is an acceptable menu. Given this fact, condition (3) allows us to bound from below the bilateral surplus of the principal and agent  $i$  in any weak PBE outcome  $(\hat{x}, \hat{t})$  of a bilateral contracting game by the principal’s profit from the competitive menu:

$$\alpha(\hat{X})\hat{x}_i - c(\hat{X}) \geq \Pi^C(\hat{X}_{-i}) = \begin{cases} p^c(X^c - \hat{X}_{-i}) - c(X^c) & \text{when } \hat{X}_{-i} \leq X^c, \\ -c(\hat{X}_{-i}) & \text{otherwise.} \end{cases} \quad (7)$$

In the example in Section 2 we observed that this bound imposes a significant restriction on the set of equilibrium outcomes. Here we show more generally that when a strict competitive equilibrium exists, under mild additional assumptions, this condition *determines completely* the equilibrium aggregate trade as the number of agents  $N$  grows large:

**Proposition 5** *Suppose that  $(p^c, X^c)$  is a strict competitive equilibrium,  $p^c X - c(X)$  is strictly decreasing for  $X \geq X^c$ , and  $\alpha(X) \leq p^c$  for all  $X \geq X^c$ . Let  $\{\hat{X}^N\}_{N=1}^\infty$  be a sequence of weak PBE aggregate trades of a bilateral contracting game with  $N$  agents. Then*

(a)  $\hat{X}^N \leq X^c$  for all  $N$ ,

(b) if, moreover, the aggregate surplus  $W(X) = \alpha(X)X - c(X)$  is bounded above on  $X \in \mathbb{R}_+$ , then  $\hat{X}^N \rightarrow X^c$  as  $N \rightarrow \infty$ .

**Proof.** Suppose in negation that (a) does not hold. Consider an agent  $i$  with  $\hat{x}_i^N > 0$ . Since by assumption  $\alpha(X) \leq p^c$  for all  $X \geq X^c$ , we can write

$$\alpha(\hat{X}^N)\hat{x}_i^N - c(\hat{X}^N) \leq p^c\hat{x}_i^N - c(\hat{X}^N).$$

If  $\hat{X}_{-i}^N > X^c$ , then since  $p^c X - c(X)$  is strictly decreasing on  $X \geq X^c$ , the right-hand side is less than  $-c(\hat{X}_{-i}^N)$ , which contradicts (7). If instead  $\hat{X}_{-i}^N \leq X^c$ , then, since  $(p^c, X^c)$  is a strict competitive equilibrium, the right-hand side is less than  $p^c(X^c - \hat{X}_{-i}^N) - c(X^c)$ , which again contradicts (7). Therefore, (a) must hold.<sup>25</sup>

Part (a) implies that for each  $i$ ,  $\hat{X}_{-i}^N \leq \hat{X}^N \leq X^c$  and consequently  $\Pi^C(\hat{X}_{-i}^N) = p^c(X^c - \hat{X}_{-i}^N) - c(X^c)$ . Adding up (7) over  $i \in N$  yields:

$$\alpha(\hat{X}^N)\hat{X}^N - Nc(\hat{X}^N) \geq p^c \left( NX^c - (N-1)\hat{X}^N \right) - Nc(X^c). \quad (8)$$

Dividing by  $N-1$ , the inequality can be rewritten as

$$\begin{aligned} [p^c \hat{X}^N - c(\hat{X}^N)] &\geq \frac{N}{N-1} [p^c X^c - c(X^c)] - \frac{1}{N-1} W(\hat{X}^N) \\ &\geq \frac{N}{N-1} [p^c X^c - c(X^c)] - \frac{1}{N-1} \sup_{X \in \mathbb{R}_+} W(X) \\ &\rightarrow p^c X^c - c(X^c) \text{ as } N \rightarrow \infty. \end{aligned}$$

Since  $c(\cdot)$  is lower semi-continuous, this implies that  $\hat{X}^N \rightarrow \arg \max_{X \in \mathbb{R}_+} [p^c X - c(X)] = X^c$  as  $N \rightarrow \infty$ . ■

## 4.2 RSW menus and Equilibria of the Offer Game

One might wonder whether consideration of deviations to acceptable menus other than the competitive menu  $C$  might lead to tighter restrictions on the set of equilibrium trades. In fact, it turns out that, under some additional assumptions,  $C$  is an RSW menu, and so (7) represents the tightest bound obtainable using Proposition 1:

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<sup>25</sup>If  $\alpha(X) > \alpha(X^c)$  for  $\hat{X} > X^c$ , equilibrium outcomes with  $\hat{X} > X^c$  are possible. For example, when  $\alpha(\cdot)$  is increasing, the efficient aggregate trade  $X^*$  exceeds  $X^c$  and is sustainable by trading  $X^*$  with one agent and zero with all others (because of the nonnegativity of trades).

**Lemma 2** *Suppose that  $(p^c, X^c)$  is a strict competitive equilibrium, and*

*(i)  $c'(\cdot)$  exists and is bounded,  $c'(X) > p^c$  for  $X > X^c$ , and there exists  $\bar{X} < X^c$  such that  $c'(X) < p^c$  for  $X \in [\bar{X}, X^c)$ ,*

*(ii)  $\alpha(X) \leq p^c$  for  $X \geq X^c$ .*

*Then the competitive menu  $C$  is an RSW menu.*

**Proof.** See Appendix A. ■

The idea of the proof is easiest to see in the case in which  $c'(\cdot)$  is strictly increasing. Suppose, in negation, that there is a compact acceptable menu  $M$  under which the principal can achieve a higher profit than she does under menu  $C$  for some aggregate trade with agents other than  $i$ ,  $X'_{-i}$ ; that is, for which  $\Pi^M(X'_{-i}) > \Pi^C(X'_{-i})$ . One can first show that this is impossible at any  $X'_{-i} \geq X^c$ . To see why, observe that under menu  $C$  the principal receives all of the bilateral surplus of the principal-agent  $i$  pair (agent  $i$  receives a payoff of zero). However, when  $X'_{-i} \geq X^c$ , the bilateral surplus between the principal and agent  $i$  is maximized by setting  $x_i = 0$  (see Lemma 6 in Appendix A), resulting in a bilateral surplus of  $-c(X'_{-i})$  – precisely the profit level earned by the principal under menu  $C$ .

Suppose, instead, that  $\Pi^M(X'_{-i}) > \Pi^C(X'_{-i})$  at some  $X'_{-i} < X^c$ , as illustrated in Figure 4 which graphs the functions  $\Pi^M(\cdot)$  and  $\Pi^C(\cdot)$ . Let  $X^0_{-i}$  denote the smallest  $X_{-i} > X'_{-i}$  at which  $\Pi^M(X_{-i}) = \Pi^C(X_{-i})$  (note that  $X^0_{-i} \leq X^c$ ). Then there must exist some  $X''_{-i} \in (X'_{-i}, X^0_{-i})$  at which  $d\Pi^M(X''_{-i})/dX_{-i} < d\Pi^C(X''_{-i})/dX_{-i} < 0$  and  $\Pi^M(X''_{-i}) > \Pi^C(X''_{-i})$  (see again Figure 4). But by the Envelope Theorem, the derivative of the profit functions  $\Pi^M(\cdot)$  and  $\Pi^C(\cdot)$  can be evaluated holding the optimal choice fixed, thus  $c'(X^M(X''_{-i})) = d\Pi^M(X''_{-i})/dX_{-i} < d\Pi^C(X''_{-i})/dX_{-i} = c'(X^c)$  where  $X^M(X''_{-i})$  is an optimal choice for the principal from menu  $M$  given  $X''_{-i}$ . Then assumption (i) implies that  $X^M(X''_{-i}) > X^c$ . But then, using assumption (ii), it can be seen that bilateral surplus at  $X^M(X''_{-i})$  is lower than  $\Pi^C(X''_{-i})$ , since bilateral surplus falls if the trade between the principal and agent  $i$  pushes the aggregate trade beyond  $X^c$  (see again Lemma 6 in the

appendix). Since the agent must earn at least 0 if  $M$  is an acceptable menu, it must in fact be that  $\Pi^M(X''_{-i}) \leq \Pi^C(X''_{-i})$ , which yields a contradiction.

### Figure 4 Here

Lemma 2 also allows us to fully characterize the weak PBE outcomes of the offer game using Proposition 3. While Proposition 3 considers the principal's multilateral deviations, it turns out that in this setting it suffices to consider unilateral deviations to the RSW (competitive) menu – any outcome that is immune to them will also be immune to multilateral deviations. Thus we obtain:

**Proposition 6** *Suppose that the assumptions of Lemma 2 hold. Then  $\hat{x} \in \mathbb{R}_+^N$  is sustainable as a weak PBE trade profile in the offer game if and only if it satisfies condition (7).*

**Proof.** The necessity of (7) follows immediately from the necessity part of Proposition 3. For sufficiency, note first that condition (7) implies  $\hat{X} \leq X^c$  by Proposition 5(a) (whose assumptions follow from those of Lemma 2). Next, observe that since  $C$  is an RSW menu, by (5)

$$\bar{\Pi}_D(\hat{x}_{-D}) = \pi^c - p^c(\hat{X} - \hat{X}_D), \quad (9)$$

where  $\pi^c = p^c X^c - c(X^c)$ . Adding (7) over all agents in  $D$  yields

$$\begin{aligned} \sum_{i \in D} \alpha(\hat{X}) \hat{x}_i - |D| c(\hat{X}) &\geq |D| [\pi^c - p^c \hat{X}] + p^c \hat{X}_D \\ &= [\pi^c - p^c(\hat{X} - \hat{X}_D)] + |D - 1| (\pi^c - p^c \hat{X}) \end{aligned}$$

Since  $p^c \hat{X} - c(\hat{X}) \leq \pi^c$ , this implies

$$\sum_{i \in D} \alpha(\hat{X}) \hat{x}_i - c(\hat{X}) \geq \pi^c - p^c(\hat{X} - \hat{X}_D) = \bar{\Pi}_D(x_{-D}).$$

Therefore, letting  $\hat{t}_i = \alpha(\hat{X}) \hat{x}_i$  for each  $i$ , the outcome  $(\hat{x}, \hat{t})$  satisfies conditions (2) and (6), and Proposition 3 implies that it is sustainable in a weak PBE of the offer game. ■

Intuitively, the result holds because once the principal has access to a choice from menu  $C$  with one agent, having choices from menu  $C$  with other agents does not have any incremental value to the principal.

Recall that in the particular example considered in Section 2, the set of aggregate trades satisfying condition (7) consisted of an interval  $[\underline{X}_N, X^c]$ , which contained both the pairwise-stable aggregate trade  $\widehat{X}_N^p$  and the competitive aggregate trade  $X^c$ . More generally, since (7) is always satisfied by both pairwise-stable and competitive trade profiles (for the former, recall Proposition 2), we see that:

**Corollary 1** *Under the assumptions of Lemma 2, any pairwise-stable trade profile  $\widehat{x} \in \mathbb{R}_+^N$  is sustainable as a weak PBE trade profile in the offer game.*

**Corollary 2** *Under the assumptions of Lemma 2, any trade profile  $\widehat{x} \in \mathbb{R}_+^N$  such that  $\sum_i \widehat{x}_i = X^c$  is sustainable as a weak PBE trade profile in the offer game.*

### 4.3 Equilibria of the Bidding Game

Here we illuminate the additional restrictions that equilibrium outcomes of the bidding games must satisfy. These restrictions arise because an agent who deviates knows that all the other menus offered to the principal are equilibrium menus. The usefulness of this knowledge in planning a deviation can be seen from the following lemma, which establishes an additional necessary condition for equilibrium outcomes in bidding games when  $c'(\cdot)$  is either increasing or decreasing:

**Lemma 3** *Suppose that  $\alpha(\cdot)$  is nonincreasing and that  $c'(\cdot)$  is strictly increasing (resp. strictly decreasing). Then any subgame-perfect Nash equilibrium trade profile  $\widehat{x} \in \mathbb{R}_+^N$  in the bidding game satisfies*

$$\widehat{x}_i \in \underset{x_i \geq (\text{resp. } \leq) \widehat{x}_i}{\text{Arg max}} \alpha(x_i + \widehat{X}_{-i})x_i - c(x_i + \widehat{X}_{-i}) \text{ for each } i \in N. \quad (10)$$

**Proof.** Suppose in negation that  $(\widehat{x}, \widehat{t})$  is the outcome in an equilibrium in which the agents offer menus  $(M_1, \dots, M_N)$ , but that (10) is violated: i.e., there exists an  $i$  and

$x_i > \hat{x}_i$  (resp.  $x_i < \hat{x}_i$ ) such that  $\alpha(x_i + \hat{X}_{-i})x_i - c(x_i + \hat{X}_{-i}) > \alpha(\hat{X})\hat{x}_i - c(\hat{X})$ . Let  $(x_j, t_j)$  for  $j \neq i$  be an optimal choice for the principal from menus  $M_j$ ,  $j \neq i$ , given that she trades  $x_i$  with agent  $i$ . By the Monotone Selection Theorem (Milgrom and Shannon [1994]), if  $c'(\cdot)$  is strictly increasing (resp. strictly decreasing) and  $x_i > \hat{x}_i$  (resp.  $x_i < \hat{x}_i$ ), we must have  $X_{-i} \leq \hat{X}_{-i}$ . Then

$$\begin{aligned} \alpha(\hat{x}_i + \hat{X}_{-i})\hat{x}_i + \sum_{j \neq i} \hat{t}_j - c(\hat{x}_i + \hat{X}_{-i}) &< \alpha(x_i + \hat{X}_{-i})x_i + \sum_{j \neq i} \hat{t}_j - c(x_i + \hat{X}_{-i}) \\ &\leq \alpha(x_i + X_{-i})x_i + \sum_{j \neq i} t_j - c(x_i + X_{-i}) \end{aligned}$$

where the second inequality follows  $\alpha(\cdot)$  being nonincreasing and  $\sum_{j \neq i} t_j - c(x_i + X_{-i}) \geq \sum_{j \neq i} \hat{t}_j - c(x_i + \hat{X}_{-i})$ . But this violates condition (iii) of Proposition 4 – a contradiction to  $(\hat{x}, \hat{t})$  being an equilibrium. ■

When  $c'(\cdot)$  is strictly increasing, Lemma 3 says that at any equilibrium trade profile of the bidding game, it is impossible to raise bilateral surplus for any principal-agent pair by increasing the principal's trade with that agent. Informally, the trade profile must be pairwise stable to upward deviations. This condition follows from the fact that each agent knows that the menus offered by other agents will not change in response to his deviation. When  $c'(\cdot)$  is strictly increasing, an agent who offers a point contract with a trade level  $x_i$  above his equilibrium trade level  $\hat{x}_i$  can be assured that the trades of all the other agents will not increase if the principal accepts his offer. If this deviation increased bilateral surplus, then the agent could be assured of a higher payoff by deviating to a point contract that offered trade level  $x_i$  and gave the principal a payoff equal to her supposed equilibrium payoff if she did not alter her trade with other agents. In the example of Section 2, condition (10) implies that in every symmetric equilibrium each agent's trade cannot be lower than the pairwise-stable trade  $\hat{X}_N^p/N$ .

For the case where  $c'(\cdot)$  is strictly increasing and  $\alpha(\cdot)$  is nonincreasing we have been able to fully characterize the equilibrium trade profiles of the bidding game as follows:

**Proposition 7** *If  $c'(\cdot)$  is strictly increasing and  $\alpha(\cdot)$  is nonincreasing, then  $\hat{x} \in \mathbb{R}_+^N$  is sustainable as a subgame-perfect Nash equilibrium trade profile in the bidding game if and*



only if (7) holds and

$$\widehat{x}_i \in \underset{x_i \geq \widehat{x}_i}{\text{Arg max}} \alpha(x_i + \widehat{X}_{-i})x_i - c(x_i + \widehat{X}_{-i}) \text{ for each } i \in N. \quad (11)$$

**Proof.** Necessity follows from Proposition 1 and Lemma 3. Sufficiency can be shown by constructing an equilibrium in which each agent  $i$  offers the principal a menu  $\widehat{M}_i \equiv \{(x_i, t_i) : x_i \in [0, X^c]\}$  and

$$t_i \in \left\{ \begin{array}{ll} p^c x_i & \text{for } x_i \leq \widehat{x}_i \\ p^c x_i + [c(X^c) - c(\widehat{X})] - p^c(X^c - \widehat{X}) & \text{for } x_i \geq \widehat{x}_i. \end{array} \right\}.$$

Combined with the principal's choice of  $(\widehat{x}, \widehat{t})$  from the menu profile  $(\widehat{M}_1, \dots, \widehat{M}_N)$  and any optimal choices from other menu profiles, this constitutes a subgame perfect Nash equilibrium of the bidding game. The formal verification of this fact can be found in Appendix B.<sup>26</sup> ■

Thus, when  $c'(\cdot)$  is strictly increasing and  $\alpha(\cdot)$  is nonincreasing, not only is condition (11) necessary, but when combined with our previous necessary condition (7), it is sufficient as well. For example, a pairwise stable trade profile satisfies (11) by definition and satisfies (7) by Proposition 2; hence, it is sustainable in an equilibrium of the bidding game. Also, any competitive equilibrium trade profile (i.e., with  $\sum_i \widehat{x}_i = X^c$ ) is sustainable: it satisfies (7) trivially, and (11) by virtue of the fact that bilateral surplus is decreasing at aggregate trade levels above  $X^c$  (see Lemma 6). Thus, we have:

**Corollary 3** *Under the assumptions of Proposition 7, any pairwise-stable trade profile  $\widehat{x} \in \mathbb{R}_+^N$  is sustainable as a weak PBE trade profile in the bidding game.*

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<sup>26</sup>The idea of this construction is that the equilibrium menu profile makes the principal indifferent about expanding her trade by  $X^c - \widehat{X}$  units. However, if any agent reduces his trade by  $\varepsilon$ , the principal strictly prefers to increase the total trade by  $X^c - \widehat{X}$  units; then, for each additional unit that the agent further reduces his trade, the principal replaces it with an additional unit from another agent at the same price of  $p^c$ . Note that the principal's profit is therefore unaffected by having a zero trade with any one agent.

**Corollary 4** *Under the assumptions of Proposition 7, any trade profile  $\hat{x} \in \mathbb{R}_+^N$  such that  $\sum_i \hat{x}_i = X^c$  is sustainable as a weak PBE trade profile in the bidding game.*

In the example of Section 2, Proposition 7 in fact implies that the set of aggregate trade levels  $\hat{X}$  arising in symmetric equilibria of the bidding game is precisely the set  $[X_N^p, X^c]$ . Martimort and Stole [1999] independently obtain a similar result, except in the context of “intrinsic” common agency (where the principal in our terminology cannot accept offers from only a subset of agents).

## 5 Decreasing Marginal Cost

We now consider a situation in which the marginal cost  $c'(\cdot)$  is strictly decreasing, and thus a competitive equilibrium does not exist. Other than this change, we maintain the assumptions of Section 4. We show that in this case the null contract is an RSW menu:

**Lemma 4** *Suppose that*

(i)  *$c'(\cdot)$  exists and is strictly decreasing,*

(ii) *there exists  $\bar{X} \geq 0$  such that for  $X \geq \bar{X}$ ,  $\alpha(X)$  is nonincreasing and  $\alpha(X) \leq c'(X)$ .*

*Then the null contract  $N = \{(0, 0)\}$  is an RSW menu.*

**Proof.** See Appendix A. ■

Given this result, our necessary condition (3) reduces to the condition that  $\alpha(\hat{X})\hat{x}_i \geq c(\hat{X}) - c(\hat{X}_{-i})$ : each agent’s surplus must at least equal the incremental cost of providing  $x_i$  units to him. Turning to the offer game, observe that since in this case the RSW menu is a point menu, allowing menus rather than simple point offers  $(x_i, t_i)$  does not reduce the set of weak PBE outcomes. Using Proposition 3, the trade profiles in these outcomes are characterized as follows:

**Proposition 8** *Suppose the assumptions of Lemma 4 hold. Then  $\hat{x} \in \mathbb{R}_+^N$  is sustainable as a weak PBE trade profile in the offer game if and only if*

$$\alpha(\hat{X})\hat{X}_D \geq c(\hat{X}) - c(\hat{X}_{-D}) \text{ for all } D \subset N. \quad (12)$$

**Proof.** For necessity note that  $\bar{\Pi}_D(\hat{x}_{-D}) = -c(\hat{X}_{-D})$ , and so condition (6) of Proposition 3 can be written as

$$\sum_{i \in D} \hat{t}_i \geq c(\hat{X}) - c(\hat{X}_{-D}) \text{ for all sets } D \subset N. \quad (13)$$

Combining this with agents' participation constraints (2) yields (12). For sufficiency, let  $\hat{t}_i = \alpha(\hat{X})\hat{x}_i$  for all  $i$ , which yields (13) and also satisfies (2); Proposition 3 then yields the result. ■

Observe that (13) is exactly the condition that  $(\hat{x}, \hat{t})$  is “subsidy-free” for all sets  $D$  from the subsidy-free pricing literature: each subset of agents must pay at least its cost of service to the principal, otherwise the principal would drop them (by offering them the null contract  $(0, 0)$ ). The proposition reflects the fact that a necessary and sufficient condition for such a transfer scheme to exist is that each group's gross surplus  $\alpha(\hat{X})\hat{X}_D$  is at least its cost of service.

In fact, with decreasing marginal cost, an even simpler characterization applies: taking  $D = N$ , we see that the total surplus must be nonnegative:

$$\alpha(\hat{X})\hat{X} \geq c(\hat{X}). \quad (14)$$

Moreover, (14) is sufficient for (12): Since decreasing marginal cost implies decreasing average cost, (14) implies

$$\frac{c(\hat{X}) - c(\hat{X}_{-D})}{\hat{X}_D} \leq \frac{c(\hat{X})}{\hat{X}} \leq \alpha(\hat{X}).$$

Hence, we have:

**Corollary 5** *Suppose the assumptions of Lemma 4 hold. Then  $\hat{x} \in \mathbb{R}_+^N$  is sustainable as a weak PBE trade profile in the offer game if and only if  $\alpha(\hat{X})\hat{X} \geq c(\hat{X})$ .*

Finally, although we do not have a full characterization of bidding game equilibria, from Lemma 3 we see that the equilibrium trade profiles of the bidding game are once again a subset of those in the offer game: along with the nonnegative surplus condition (14), bidding game equilibria must also have the property that there is no downward deviation in the trade with an agent that increases bilateral surplus.

## 6 A Partial Competitive Equilibrium Exists

In this section, we consider situations in which a competitive equilibrium does not exist due to nonconvexities, but the following modification of competitive equilibrium does exist:

**Definition 4** A price-quantity pair  $(p^c, X^c) \in \mathbb{R} \times \mathbb{R}_+$  is a strict partial competitive equilibrium relative to  $\tilde{X}$  if (i)  $p^c = \alpha(X^c)$  and (ii)  $p^c X - c(X) < p^c \tilde{X} - c(\tilde{X}) = p^c X^c - c(X^c)$  for all  $X \in (\tilde{X}, X^c)$ .

Part (ii) of the definition means that when the principal faces price  $p^c$ , she will choose aggregate quantity  $X^c$  if forced to have  $X > \tilde{X}$ , and she is indifferent between  $\tilde{X}$  and  $X^c$ . Figures 5(a) and (b) depict cases in which  $(p^c, X^c)$  is a partial competitive equilibrium in the vertical contracting example and the insurance example where the principal has a binary moral hazard choice  $\pi \in \{\underline{\pi}, \bar{\pi}\}$ . By the definition of  $\tilde{X}$ , in both figures  $area(A) = area(B)$ . In the insurance example in Figure 5(b), for example, the strict partial competitive equilibrium price is actuarially fair given the low level of care when the individual has insurance level  $X^c$ , but at this price the individual would prefer not buying any insurance.

### Figure 5 Here

First we show that the competitive menu is still an acceptable menu, under the additional assumption that  $p^c X - c(X)$  is strictly decreasing on  $[0, \tilde{X}] \cup [X^c, +\infty)$ .<sup>27</sup>

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<sup>27</sup>The condition can be replaced with the assumptions that  $p^c X - c(X)$  is strictly decreasing for  $X > X^c$  and  $\alpha(X) \geq \alpha(X^c)$  for all  $X \in [0, \tilde{X})$ . In this case, if the principal does not choose  $X \in \{X^c, X_{-i}\}$ , she

This assumption ensures that when  $X_{-i} < \tilde{X}$  or  $X_{-i} > X^c$ , the principal faced with menu  $C$  will choose  $x_i = 0$ . On the other hand, when  $X_{-i} \in (\tilde{X}, X^c]$ , the principal will choose  $x_i = X^c - X_{-i}$ . (When  $X_{-i} = \tilde{X}$ , the principal is indifferent between  $x_i = 0$  and  $x_i = X^c - \tilde{X}$ ). In all these cases, the agent receives a zero payoff, and so  $C$  is an acceptable menu. Proposition 1 from Section 3 then implies that any weak PBE of a bilateral contracting game must satisfy,

$$\alpha(\hat{X})\hat{x}_i - c(\hat{X}) \geq \Pi^C(\hat{X}_{-i}) = \begin{cases} p^c(X^c - X_{-i}) - c(X^c) & \text{when } X_{-i} \in (\tilde{X}, X^c), \\ -c(X_{-i}) & \text{when } X_{-i} \leq \tilde{X} \text{ or } X_{-i} \geq X^c. \end{cases} \quad (15)$$

We can now bound the limiting behavior of aggregate trades in cases in which a competitive equilibrium does *not* exist, but a partial competitive equilibrium exists:

**Proposition 9** *Suppose that  $(p^c, X^c)$  is a strict partial competitive equilibrium relative to  $\tilde{X} > 0$ . Suppose also that  $p^c X - c(X)$  is strictly decreasing on  $[0, \tilde{X}] \cup [X^c, +\infty)$ , and that  $\alpha(X) \leq p^c$  for  $X \geq X^c$ . Suppose that, in addition,  $\alpha(X)$  is continuous at  $X = X^c$ , and the aggregate surplus  $W(X) = \alpha(X)X - c(X)$  is bounded above. Let  $\{\hat{X}^N\}_{N=1}^\infty$  be a sequence of weak PBE aggregate trades. Then  $\limsup_{N \rightarrow \infty} \hat{X}^N \leq \tilde{X}$ .<sup>28</sup>*

**Proof.** Suppose in negation that there exists a subsequence  $\{\hat{X}^K\}_{K=1}^\infty \subset \{\hat{X}^N\}_{N=1}^\infty$  such that  $\hat{X}^K \rightarrow X^0 > \tilde{X}$  as  $K \rightarrow \infty$ . By the same arguments as those in the proof of Proposition 5, (15) implies that

$$p^c \hat{X}^K - c(\hat{X}^K) \rightarrow p^c X^c - c(X^c) \text{ as } K \rightarrow \infty.$$

Since  $c(\cdot)$  is lower semicontinuous,  $(p^c, X^c)$  is a strict partial competitive equilibrium relative to  $\tilde{X}$ , and  $X^0 = \lim_{K \rightarrow \infty} \hat{X}^K > \tilde{X}$ , we must then have  $X^0 = X^c$ . Then, by continuity of  $\alpha(\cdot)$  at  $X^c$ , we should also have  $\alpha(\hat{X}^K) \rightarrow \alpha(X^c) = p^c$  as  $K \rightarrow \infty$ , and therefore

$$\lim_{K \rightarrow \infty} \left[ \alpha(\hat{X}^K)\hat{X}^K - c(\hat{X}^K) \right] = \lim_{K \rightarrow \infty} \left[ p^c \hat{X}^K - c(\hat{X}^K) \right] = p^c X^c - c(X^c).$$

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must choose  $X < \tilde{X}$ , and the agent is again assured a nonnegative payoff.

<sup>28</sup>This means that for any  $X' > \tilde{X}$  there exists  $N'$  such that  $\hat{X}^N < X'$  for all  $N \geq N'$ .

However, our assumptions imply that  $p^c X^c - c(X^c) = p^c \tilde{X} - c(\tilde{X}) < 0$ . But then the equilibrium total surplus  $\alpha(\hat{X}^K)\hat{X}^K - c(\hat{X}^K)$  must be negative for  $K$  large enough, which contradicts the fact that all parties must have nonnegative payoffs in an equilibrium. ■

A related result has been obtained in the insurance model by Kahn and Mookherjee [1998], who call the threshold  $\tilde{X}$  the “third-best” outcome. (Similar noncompetitive outcomes have been observed by Bisin and Gottardi [1998] in the insurance model and Rajan and Parlour [1998] in the lending model.) In the bilateral contracting game they consider, in equilibrium the principal obtains precisely the aggregate amount  $\tilde{X}$  of insurance. At this aggregate trade level, the principal is exactly indifferent between buying an additional amount  $X^c - \tilde{X}$  of insurance at the given price  $p^c$  and not buying any additional insurance. Proposition 9 establishes that under the appropriate assumptions, the equilibrium aggregate trade in a bilateral contracting game cannot exceed  $\tilde{X}$  when the number of agents is sufficiently large. Intuitively, if  $\hat{X} \in (\tilde{X}, X^c)$ , then there will exist a profitable bilateral deviation between the principal and an agent  $i$  whose trade with the principal is sufficiently small. In this deviation the parties will sign the competitive contract  $C$ , from which the principal then chooses quantity  $X^c - \hat{X}_{-i}$ , so that the aggregate trade becomes  $X^c$ . On the other hand, the aggregate trade  $X^c$  itself cannot be sustained in equilibrium, for this would result in negative total surplus. Therefore, asymptotically we must have  $\hat{X} \leq \tilde{X}$ .

This prediction of our model should be contrasted with the “convexification” approach used in general equilibrium theory. The convexification approach considers the case of a continuum of identical principals, which ensures that their aggregate supply correspondence is convex and thus a “competitive equilibrium” at which aggregate supply equals aggregate demand exists. In economies without externalities this approach is justified, since a seller is indifferent between, say, selling 5 apples to each buyer A and buyer B, and selling all 10 apples to buyer A. However, the use of convexification in economies with externalities, such as the lending model of Dubey et al. [1996], is incorrect. Intuitively, a lender is usually not indifferent between lending \$5 to each borrower A and borrower

B, and lending all \$10 to borrower A. (For example, imagine that a borrower who borrows more than \$7 might default.) For this reason, lenders will not treat borrowers as anonymous, as the convexification approach assumes, but rather contract with each borrower individually, as in our noncooperative bargaining model. Our analysis establishes that the aggregate trade in noncooperative contracting with a large number of agents must be below the “third-best” aggregate trade, and therefore quite different from the “convexified competitive equilibrium,” as depicted in Figure 6.

**Figure 6 Here**

## 6.1 RSW Menus and Equilibria of the Offer Game

Under some additional assumptions, we can show that the competitive menu  $C$  is in fact an RSW menu:

**Lemma 5** *Suppose that  $(p^c, X^c)$  is a strict partial competitive equilibrium relative to  $\tilde{X}$ , and*

(i)  *$c'(\cdot)$  exists and is bounded,  $c'(X) > p^c$  for  $X > X^c$ , there exists  $\bar{X} < X^c$  such that  $c'(X) < p^c$  for  $X \in [\bar{X}, X^c)$ , and  $c'(X) > p^c$  and is nonincreasing for  $X \leq \tilde{X}$ .*

(ii)  *$\alpha(X) \leq p^c$  for  $X \geq X^c$ .*

*Then the competitive menu  $C$  is an RSW menu.*

**Proof.** See Appendix A. ■

This lemma allows us to characterize the equilibrium outcomes of the offer game as follows:

**Proposition 10** *Suppose that the assumptions of Lemma 5 hold. Then  $\hat{x} \in \mathbb{R}_+^N$  is sustainable as a weak PBE trade profile in the offer game if and only if (15) holds for all  $i \in N$  such that  $\hat{X}_{-i} \geq \tilde{X}$ , and*

$$\alpha(\hat{X})\hat{X}_D \geq c(\hat{X}) - c(\hat{X}_{-D}) \text{ for all sets } D \subset N \text{ such that } \hat{X}_{-D} < \tilde{X}. \quad (16)$$

**Proof.** The proof of the necessity part follows in a similar manner to previous arguments. For sufficiency, (15) holding for all  $i \in N$  such that  $\hat{X}_{-i} \geq \tilde{X}$  implies that  $\hat{X} \leq X^c$  (as in the proof of Proposition 5). Observe next that for all sets  $D \subset N$  such that  $\hat{X}_{-D} \in [\tilde{X}, X^c]$ ,  $\bar{\Pi}_D(\hat{x}_{-D})$  is given by (9), while for sets  $D \subset N$  such that  $\hat{X}_{-D} < \tilde{X}$ , we have  $\bar{\Pi}_D(\hat{x}_{-D}) = -c(\hat{X}_{-D})$ . Let  $\hat{t}_i = \alpha(\hat{X})\hat{x}_i$  for all  $i$ . Then the proof that condition (6) in Proposition 3 is satisfied for sets  $D \subset N$  such that  $\hat{X}_{-D} \geq \tilde{X}$  is the same as in Proposition 6. For sets  $D \subset N$  such that  $\hat{X}_{-D} < \tilde{X}$ , (16) ensures that if  $\hat{t}_i = \alpha(\hat{X})\hat{x}_i$  for all  $i$  then condition (6) is satisfied for these sets as well. ■

The characterization in Proposition 10 can be viewed as a combination of the results for the cases in which a strict competitive equilibrium exists and in which marginal costs are strictly decreasing. For symmetric equilibria, Proposition 10 implies that  $\hat{X}$  can be sustained as the aggregate trade in a PBE of an offer game if and only if either  $(\frac{N-1}{N})\hat{X} < \tilde{X}$  and  $\alpha(\hat{X})\hat{X} - c(\hat{X}) \geq 0$ , or  $(\frac{N-1}{N})\hat{X} \geq \tilde{X}$  and  $(\frac{1}{N})\alpha(\hat{X})\hat{X} + (\frac{N-1}{N})p^c\hat{X} - c(\hat{X}) \geq p^cX^c - c(X^c)$ .

## 7 Conclusion

In this paper we have studied bilateral contracting between one principal and  $N$  agents when each agent's utility depends on the principal's unobservable contracts with other agents. In such settings, by offering an agent a menu and then choosing from it, the principal can signal her trades with other agents, similar to signaling type in Maskin and Tirole's [1992] analysis of mechanism design by an informed principal. We have seen that requiring immunity to deviations to such menu contracts often provides significant bounds on equilibrium outcomes in a wide class of bilateral contracting games without imposing ad hoc restrictions on agents' beliefs. Indeed, in settings in which a notion of competitive equilibrium can be said to exist, this bound yields, under certain mild assumptions, competitive convergence as  $N \rightarrow \infty$ . To highlight how specific contracting processes lead to further restrictions on equilibrium outcomes, we have also examined the additional restrictions that arise in two commonly assumed bilateral contracting



processes, the “offer game” in which the principal makes simultaneous offers to the agents (e.g., Hart and Tirole [1990], Segal [1999]) and the “bidding game” in which the agents make simultaneous offers to the principal (e.g., Bernheim and Whinston [1986]).

## 8 Appendix A

**The Bilateral Surplus Function:** For the proofs in this appendix it will prove useful to have defined the function

$$B(X, X_{-i}) = \alpha(X)(X - X_{-i}) - c(X),$$

giving the bilateral surplus when the aggregate trade with agents  $j \neq i$  is  $X_{-i}$  and the aggregate trade is  $X$  (and so the trade with agent  $i$  is  $x_i = X - X_{-i}$ ). Note that if the principal of type  $X_{-i}$  offers agent  $i$  an acceptable menu  $M$  and chooses  $x_i = X - X_{-i}$  from it, the agent’s participation constraint (2) can be written as

$$\Pi^M(X_{-i}) \leq B(X, X_{-i}). \quad (17)$$

The proofs of several results utilize the following properties of bilateral surplus:

**Lemma 6** *Suppose that  $(p^c, X^c)$  is a competitive equilibrium and  $\alpha(X) \leq p^c$ .*

(a) *If  $X \geq X^c \geq X_{-i}$ , then  $B(X, X_{-i}) \leq B(X^c, X_{-i})$ .*

(b) *If  $X \geq X_{-i} \geq X^c$  and  $p^c X - c(X)$  is also nonincreasing for  $X \geq X^c$ , then  $B(X, X_{-i}) \leq B(X_{-i}, X_{-i})$ .*

**Proof.** For part (a), observe that

$$\begin{aligned} B(X, X_{-i}) &= \alpha(X)(X - X_{-i}) - c(X) \\ &= [p^c X - c(X)] + [\alpha(X) - p^c](X - X_{-i}) - p^c X_{-i} \\ &\leq [p^c X^c - c(X^c)] - p^c X_{-i} = B(X^c, X_{-i}), \end{aligned}$$

where the inequality uses the fact that  $(p^c, X^c)$  is a competitive equilibrium. For part (b), we can write

$$B(X, X_{-i}) = [p^c X - c(X)] + [\alpha(X) - p^c](X - X_{-i}) - p^c X_{-i} \leq [p^c X_{-i} - c(X_{-i})] - p^c X_{-i} = B(X_{-i}, X_{-i}),$$

where the inequality follows from the fact that  $p^c X - c(X)$  is nonincreasing for  $X \geq X^c$ . ■

**Lemma 7** *Suppose that  $\alpha(\cdot)$  is nonincreasing,  $c(\cdot)$  is differentiable, and  $\alpha(X) \leq c'(X)$  for all  $X > X_{-i}$ . Then  $B(X, X_{-i}) \leq B(X_{-i}, X_{-i})$  for all  $X > X_{-i}$ .*

**Proof.**

$$\begin{aligned} B(X, X_{-i}) - B(X_{-i}, X_{-i}) &= \alpha(X)(X - X_{-i}) - c(X) + c(X_{-i}) \\ &= \int_{X_{-i}}^X [\alpha(X) - c'(Y)] dY. \end{aligned}$$

The result follows since  $c'(Y) \geq \alpha(Y) \geq \alpha(X)$  for all  $Y \in (X_{-i}, X)$ . ■

**Proof of Lemma 2.** The assumptions of the lemma imply the assumptions of Proposition 5 under which  $C$  is an acceptable menu, with the associated profit function  $\Pi^C(\cdot)$  described in (7). We will show that for an arbitrary compact acceptable menu  $M$ ,  $\Pi^M(X_{-i}) \leq \Pi^C(X_{-i})$  for all  $X_{-i} \in \mathbb{R}_+$ . For  $X_{-i} \geq X^c$ , this follows from the agent's participation constraint (17) and Lemma 6(b):

$$\Pi^M(X_{-i}) \leq B(X, X_{-i}) \leq B(X_{-i}, X_{-i}) = \Pi^C(X_{-i}).$$

Now suppose in negation that there exists  $X'_{-i} < X^c$  and an acceptable menu  $M$  such that  $\Pi^M(X'_{-i}) > \Pi^C(X'_{-i})$ . By Lemma 1, the union menu  $U = M \cup C$  is also acceptable. Moreover, it is compact. Its associated profit function is  $\Pi^U(X_{-i}) = \max\{\Pi^M(X_{-i}), \Pi^C(X_{-i})\}$ . Let

$$X_{-i}^0 = \min \{X_{-i} \in [X'_{-i}, X^c] : \Pi^U(X_{-i}) = \Pi^C(X_{-i})\}. \quad (18)$$

(The minimum is achieved because  $\Pi^U(\cdot)$  and  $\Pi^C(\cdot)$  are absolutely continuous under our assumptions – see, e.g., Milgrom and Segal [forth., Theorem 2].) By construction,  $X_{-i}^0 > X'_{-i}$ .

Take  $\varepsilon = \min \{X_{-i}^0 - X'_{-i}, X^c - \bar{X}\}$ . Let  $(x^M(X_{-i}), t^M(X_{-i}))$  be an optimal choice from a menu  $M$  for the principal of type  $X_{-i}$ , and let  $X^M(X_{-i}) = x^M(X_{-i}) + X_{-i}$ . Standard envelope theorem arguments of mechanism design (formalized by Milgrom and Segal [2000, Corollary 1] imply that for any acceptable menu  $M$ ,

$$\Pi^M(X_{-i}^0) = \Pi^M(X_{-i}^0 - \varepsilon) - \int_{X_{-i}^0 - \varepsilon}^{X_{-i}^0} c'(X^M(X_{-i})) dX_{-i}$$

Therefore,

$$\Pi^U(X_{-i}^0 - \varepsilon) - \Pi^C(X_{-i}^0 - \varepsilon) = \int_{X_{-i}^0 - \varepsilon}^{X_{-i}^0} [c'(X^U(X_{-i})) - c'(X^C(X_{-i}))] dX_{-i}. \quad (19)$$

Since  $\Pi^U(X_{-i}^0 - \varepsilon) > \Pi^C(X_{-i}^0 - \varepsilon)$ , we must have  $c'(X^U(X''_{-i})) > c'(X^C(X''_{-i})) = c'(X^c)$  for some  $X''_{-i} \in (X_{-i}^0 - \varepsilon, X_{-i}^0)$ . Therefore, we must have either (a)  $X^U(X''_{-i}) > X^c$  or (b)  $X^U(X''_{-i}) < \bar{X}$ . At the same time, by (18), we must have  $\Pi^U(X''_{-i}) > \Pi^C(X''_{-i})$ .

Case (a) can be ruled out because then the agent's participation constraint (17) and Lemma 6 would imply

$$\Pi^U(X''_{-i}) \leq B(X^U(X''_{-i}), X''_{-i}) \leq B(X^c, X''_{-i}) = \Pi^C(X''_{-i}).$$

Thus, we can focus on case (b), where  $X''_{-i} \leq X^U(X''_{-i}) < \bar{X}$ . Let  $(x'', t'') \equiv (x(X''_{-i}), t(X''_{-i}))$ . By (18), we have

$$t'' - c(X''_{-i} + x'') = \Pi^U(X''_{-i}) > \Pi^C(X''_{-i}) \geq p^c x'' - c(X''_{-i} + x'').$$

Therefore,  $t'' > p^c x''$ . But then, since  $(x'', t'') \in U$ , we have

$$\Pi^U(X^c - x'') \geq t'' - c(X^c - x'' + x'') > p^c x'' - c(X^c) = \Pi^C(X^c - x'').$$

Since

$$X^c - x'' = X^c - X^U(X''_{-i}) + X''_{-i} > X^c - X'' + X''_{-i} > X^c - X'' + X_{-i}^0 - \varepsilon > X_{-i}^0,$$

this contradicts (18). ■

**Proof of Lemma 4.**  $N$  is trivially an acceptable menu, with the associated profit function  $\Pi^N(X_{-i}) = -c(X_{-i})$ . To see that  $N$  is an RSW menu, take an arbitrary acceptable menu  $M$ . By Lemma 1, the union menu  $U = M \cup N$  is also acceptable, with

$\Pi^U(X_{-i}) = \max\{\Pi^M(X_{-i}), \Pi^N(X_{-i})\}$ . We will show that  $\Pi^U(X_{-i}) = \Pi^N(X_{-i})$ , and therefore  $\Pi^M(X_{-i}) \leq \Pi^N(X_{-i})$ , for all  $X_{-i}$ .

First, note that if  $(x, t) \in U$  is the principal's optimal choice when  $X_{-i} \geq \bar{X}$ , the agent's participation constraint (17) and Lemma 7 imply that

$$\Pi^U(X_{-i}) \leq B(X_{-i} + x, X_{-i}) \leq B(X_{-i}, X_{-i}) = \Pi^N(X_{-i}).$$

Therefore,  $(0, 0)$  is an optimal choice from  $U$  for the principal when  $X_{-i} \geq \bar{X}$ .

Note that with  $c'(\cdot)$  strictly decreasing, the principal's payoff has strictly increasing differences in  $(x_i, X_{-i})$ . Therefore, by the Monotone Selection Theorem of Milgrom and Shannon [1994],  $(0, 0)$  must be an optimal choice from  $U$  for any  $X_{-i} \leq \bar{X}$ , which implies that  $\Pi^U(X_{-i}) = \Pi^N(X_{-i})$  for all such  $X_{-i}$ . ■

**Proof of Lemma 5.** Follows that of Lemma 2, with one modification:  $X^C(X''_i) = X''_i$  for  $X''_i < \tilde{X}$ . Our assumptions imply that it is again not possible to have  $X^U(X''_i) \in [X''_i, X^c]$  such that  $c'(X^U(X''_i)) > c'(X^C(X''_i))$ . ■

## 9 Appendix B

**Verification of the sufficiency claim in Proposition 7:** Here we verify that each agent  $i$  offering a menu  $\widehat{M}_i$  in which the principal can choose any  $x_i \in [0, X^c]$  with accompanying payments of

$$t_i \in \left\{ \begin{array}{ll} p^c x_i & \text{for } x_i \leq \widehat{x}_i \\ p^c x_i + \{[c(X^c) - c(\widehat{X})] - p^c(X^c - \widehat{X})\} & \text{for } x_i \geq \widehat{x}_i \end{array} \right\},$$

combined with the choice by the principal of  $(\widehat{x}, \widehat{t})$  from these menus and any optimal choices when facing any other profile of offered menus, constitutes a subgame perfect Nash equilibrium of the bidding game under the assumptions of the proposition.

The verification proceeds through a series of lemmas. Note, first, that (7) implies that  $\widehat{X} < X^c$  (see the proof of Proposition 5).

**Lemma 8** Suppose we fix  $x_i$ . Any solution to

$$\max_{(x_j, t_j) \in \widehat{M}_i \forall j \neq i} \sum_{j \neq i} t_j - c(x_i + \sum_{j \neq i} x_j) \quad (20)$$

has at most 1 element  $k$  with  $x_k > \widehat{x}_k$ .

**Proof.** Observe first that any solution to (20) must have  $X \leq X^c$ : Were this not true, we would have an  $x_j > \widehat{x}_j$  (since  $\widehat{X} \leq X^c$ ). A reduction in  $x_j$  to  $x_j - \varepsilon$  where  $\varepsilon < \min\{X - X^c, x_j - \widehat{x}_j\}$  would then increase the objective function in (20) by at least  $[c'(x_j - \varepsilon) - p^c]\varepsilon > 0$ .

Suppose now that there is a solution to (20) in which  $(x_j, x_k) \gg (\widehat{x}_j, \widehat{x}_k)$ . Consider a change in which only  $j$  and  $k$ 's trades are changed, and these are changed to  $[\widehat{x}_j, x_k + (x_j - \widehat{x}_j)]$  (note that this is feasible since  $x_k + (x_j - \widehat{x}_j) < X \leq X^c$ ). Given the maximal transfers associated with these trades, the change in profit is

$$\begin{aligned} & p^c(x_j + x_k) + \{[c(X^c) - c(\widehat{X})] - p^c(X^c - \widehat{X})\} \\ & - p^c(x_j + x_k) - 2\{[c(X^c) - c(\widehat{X})] - p^c(X^c - \widehat{X})\} \\ = & -\{[c(X^c) - c(\widehat{X})] - p^c(X^c - \widehat{X})\} \\ > & 0 \end{aligned}$$

since  $\widehat{X} \leq X^c$  and  $c(X^c) - c(\widehat{X}) \leq c'(X^c)(X^c - \widehat{X}) = p^c(X^c - \widehat{X})$ . ■

**Lemma 9** Suppose we fix  $x_i$ . The solution and value of the solution [denoted  $\pi_{-i}(x_i)$ ] to problem (20) take the following form:

1. If  $x_i < \widehat{x}_i$ : Any solution has  $\sum_{j \neq i} x_j = X^c - x_i$  and has  $\pi_{-i}(x_i) = p^c(\widehat{X} - x_i) - c(\widehat{X})$ .
2. If  $x_i > \widehat{x}_i$  but  $x_i < X^c - \widehat{X}_{-i}$ : Any solution has  $\sum_{j \neq i} x_j = \widehat{X}_{-i}$  and has  $\pi_{-i}(x_i) = p^c \widehat{X}_{-i} - c(x_i + \widehat{X}_{-i})$ .
3. If  $x_i > \widehat{x}_i$  and  $x_i \in [X^c - \widehat{X}_{-i}, X^c]$ : Any solution has  $\sum_{j \neq i} x_j = X^c - x_i$  and has  $\pi_{-i}(x_i) = p^c(X^c - x_i) - c(X^c)$ .

4. If  $x_i > \hat{x}_i$  and  $x_i > X^c$ : Any solution has  $x_j = 0$  for all  $j \neq i$  and has  $\pi_{-i}(x_i) = -c(x_i)$ .

**Proof.** Given  $x_i$ , call a configuration with  $x_j \leq \hat{x}_j$  for all  $j \neq i$  a type A configuration. Its value is  $p^c \sum_{j \neq i} x_j - c(x_i + \sum_{j \neq i} x_j)$ . When  $x_i < X^c - \hat{X}_{-i}$ , the best configuration within this class has  $x_j = \hat{x}_j$  for all  $j$  and has value  $p^c \hat{X}_{-i} - c(x_i + \hat{X}_{-i})$ . When  $x_i \in [X^c - \hat{X}_{-i}, X^c]$ , the best configuration within this class has  $x_j \leq \hat{x}_j$  for all  $j$  and  $\sum_{j \neq i} x_j = X^c - x_i$ , and has value  $p^c(X^c - x_i) - c(X^c)$ . When  $x_i > X^c$ , the best solution in this class has  $x_j = 0$  for all  $j \neq i$  and has value  $-c(x_i)$ .

Call a configuration with  $x_k > \hat{x}_k$  for some  $k$  a type B configuration. Its value is

$$p^c \sum_{j \neq i} x_j - c(x_i + \sum_{j \neq i} x_j) + \{[c(X^c) - c(\hat{X})] - p^c(X^c - \hat{X})\}.$$

If  $x_i \leq X^c - \hat{x}_k$ , this is maximized with  $\sum_{j \neq i} x_j = X^c - x_i$  and yields a value of  $p^c(\hat{X} - x_i) - c(\hat{X})$ . When  $x_i \geq X^c - \hat{x}_k$ , this would be uniquely maximized by setting  $x_k = \hat{x}_k$  and  $x_j = 0$  for all  $j \neq i, k$ , but this violates  $x_k > \hat{x}_k$ ; hence, such type B configurations yield a value strictly less than

$$\begin{aligned} p^c \hat{x}_k - c(x_i + \hat{x}_k) + \{[c(X^c) - c(\hat{X})] - p^c(X^c - \hat{X})\} &< p^c(X^c - x_i) - c(X^c) + \{[c(X^c) - c(\hat{X})] - p^c(X^c - \hat{X})\} \\ &= p^c \hat{X} - c(\hat{X}). \end{aligned}$$

Now consider the optimal choices in the four cases of the lemma:

1. If  $x_i < \hat{x}_i$  ( $\leq X^c - \hat{X}_{-i} \leq X^c - \hat{x}_k$  for all  $k$ ): The best type A configuration has value  $p^c \hat{X}_{-i} - c(x_i + \hat{X}_{-i})$  while the best type B configuration has value  $p^c(\hat{X} - x_i) - c(\hat{X})$ . Since  $\hat{X}_{-i} < \hat{X} - x_i \leq X^c$ , a type B configuration is better and any optimal solution to problem (20) has  $\sum_{j \neq i} x_j = X^c - x_i$ .
2. If  $x_i > \hat{x}_i$  but  $x_i < X^c - \hat{X}_{-i}$  ( $\leq X^c - \hat{x}_k$  for all  $k$ ): The best type A configuration has value  $p^c \hat{X}_{-i} - c(x_i + \hat{X}_{-i})$ , while the best type B configuration has value  $p^c(\hat{X} - x_i) - c(\hat{X})$ . Since  $\hat{X} - x_i < \hat{X}_{-i} \leq X^c$ , a type A configuration is better and any optimal solution to problem (20) has  $\sum_{j \neq i} x_j = \hat{X}_{-i}$ .

3. If  $x_i > \hat{x}_i$  and  $x_i \in [X^c - \hat{X}_{-i}, X^c]$ : The best type A configuration has value  $p^c(X^c - x_i) - c(X^c)$ . The best type B configuration has value no greater than  $p^c(\hat{X} - x_i) - c(\hat{X})$ . Hence, a type A configuration is optimal and any optimal solution to problem (20) has  $\sum_{j \neq i} x_j = X^c - x_i$  (note that if a type B configuration is also optimal, it must also have  $\sum_{j \neq i} x_j = X^c - x_i$ ).
4. If  $x_i > \hat{x}_i$  and  $x_i > X^c$ : The optimal type A configuration has value  $-c(x_i)$ . Since  $x_i > X^c - \max_{k \neq i} \{\hat{x}_k\}$ , the best type B configuration has value strictly less than  $\max_k p^c \hat{x}_k - c(x_i + \hat{x}_k)$ . Since  $x_i > X^c$ , a type A configuration is better and any optimal solution to problem (20) has  $x_j = 0$  for all  $j \neq i$ . ■

We now prove the result by verifying that conditions (i)-(iv) of Proposition 4 are satisfied:

To check condition (i) we show that the principal does not improve her payoff by deviating to some  $x \neq \hat{x}$ . Suppose first that she deviates to an  $x$  with  $X \leq \hat{X}$ . Then  $x_i \leq \hat{x}_i$  for some  $i$ . So the principal can earn at most  $p^c x_i + p^c(\hat{X} - x_i) - c(\hat{X}) = p^c \hat{X} - c(\hat{X})$ . Suppose that she deviates to an  $x$  with  $X > \hat{X}$ . Then there is an  $i$  with  $x_i > \hat{x}_i$ . If  $x_i < X^c - \hat{X}_{-i}$ , then the principal earns at most

$$p^c x_i + \{[c(X^c) - c(\hat{X})] - p^c(X^c - \hat{X})\} + p^c \hat{X}_{-i} - c(x_i + \hat{X}_{-i})$$

which equals

$$p^c(x_i + \hat{X}_{-i}) - c(x_i + \hat{X}_{-i}) + \{[c(X^c) - c(\hat{X})] - p^c(X^c - \hat{X})\}$$

which is bounded above by

$$p^c X^c - c(X^c) + \{[c(X^c) - c(\hat{X})] - p^c(X^c - \hat{X})\}$$

which equals  $p^c \hat{X} - c(\hat{X})$ .

If instead  $x_i \in [X^c - \hat{X}_{-i}, X^c]$ , then the principal earns no more than

$$\begin{aligned} & p^c x_i + \{[c(X^c) - c(\hat{X})] - p^c(X^c - \hat{X})\} + p^c(X^c - x_i) - c(X^c) \\ &= p^c \hat{X} - c(\hat{X}). \end{aligned}$$

Finally, if  $x_i > X^c$ , then the principal earns no more than

$$\begin{aligned}
& p^c x_i + \{[c(X^c) - c(\widehat{X})] - p^c(X^c - \widehat{X})\} - c(x_i) \\
& \leq p^c X^c - c(X^c) + \{[c(X^c) - c(\widehat{X})] - p^c(X^c - \widehat{X})\} \\
& = p^c \widehat{X} - c(\widehat{X}).
\end{aligned}$$

Condition (ii) is satisfied since agent  $i$  earns  $\alpha(\widehat{X})\widehat{x}_i - p^c \widehat{x}_i \geq 0$ .

To see that condition (iii) holds, note that

$$\alpha(\widehat{X})\widehat{x}_i + \sum_{j \neq i} t_j(\widehat{x}_j) - c(\widehat{X}) = \alpha(\widehat{X})\widehat{x}_i + p^c \widehat{X}_{-i} - c(\widehat{X}). \quad (21)$$

Consider four cases:

1. If  $x_i < \widehat{x}_i$ : For any optimal choice  $x_{-i}$  by the principal, we have

$$\begin{aligned}
\alpha(x_i + \sum_{j \neq i} x_j)x_i + \pi_{-i}(x_i) &= \alpha(x_i + (X^c - x_i))x_i + p^c(\widehat{X} - x_i) - c(\widehat{X}) \\
&\leq \alpha(x_i + (X^c - x_i))x_i + p^c(X^c - x_i) - c(X^c) \\
&= p^c X^c - c(X^c) \\
&= [p^c(X^c - \widehat{X}_{-i}) - c(X^c)] + p^c \widehat{X}_{-i} \\
&\leq [\alpha(\widehat{X})\widehat{x}_i - c(\widehat{X})] + p^c \widehat{X}_{-i}
\end{aligned}$$

where the last inequality follows from condition (7).

2. If  $x_i > \widehat{x}_i$  but  $x_i < X^c - \widehat{X}_{-i}$ : For any optimal choice by the principal, we have

$$\alpha(x_i + \sum_{j \neq i} x_j)x_i + \pi_{-i}(x_i) = \alpha(x_i + \widehat{X}_{-i})x_i + p^c \widehat{X}_{-i} - c(x_i + \widehat{X}_{-i}) \leq \alpha(\widehat{x}_i + \widehat{X}_{-i})\widehat{x}_i + p^c \widehat{X}_{-i} - c(\widehat{x}_i + \widehat{X}_{-i})$$

by condition (11).

3. If  $x_i > \widehat{x}_i$  and  $x_i \in [X^c - \widehat{X}_{-i}, X^c]$ : For any optimal choice by the principal, we have

$$\begin{aligned}
\alpha(x_i + \sum_{j \neq i} x_j)x_i + \pi_{-i}(x_i) &= \alpha(x_i + (X^c - x_i))x_i + p^c(X^c - x_i) - c(X^c) \\
&= p^c X^c - c(X^c) \\
&= [p^c(X^c - \widehat{X}_{-i}) - c(X^c)] + p^c \widehat{X}_{-i} \\
&\leq [\alpha(\widehat{X})\widehat{x}_i - c(\widehat{X})] + p^c \widehat{X}_{-i}
\end{aligned}$$



by condition (7).

4. If  $x_i > \hat{x}_i$  and  $x_i > X^c$ : Now

$$\begin{aligned}\alpha(x_i + \sum_{j \neq i} x_j)x_i + \pi_{-i}(x_i) &= \alpha(x_i)x_i - c(x_i) \\ &\leq p^c x_i - c(x_i) \\ &< p^c X^c - c(X^c) \\ &= [p^c(X^c - \hat{X}_{-i}) - c(X^c)] + p^c \hat{X}_{-i} \\ &\leq [\alpha(\hat{X})\hat{x}_i - c(\hat{X})] + p^c \hat{X}_{-i}\end{aligned}$$

by condition (7).

For condition (iv), note that our analysis above tells us that the principal's optimal payoff when she does not trade with one agent  $i$  (i.e. when  $x_i = 0$ ) is  $p^c \hat{X} - c(\hat{X})$ , which exactly equals her equilibrium payoff. ■

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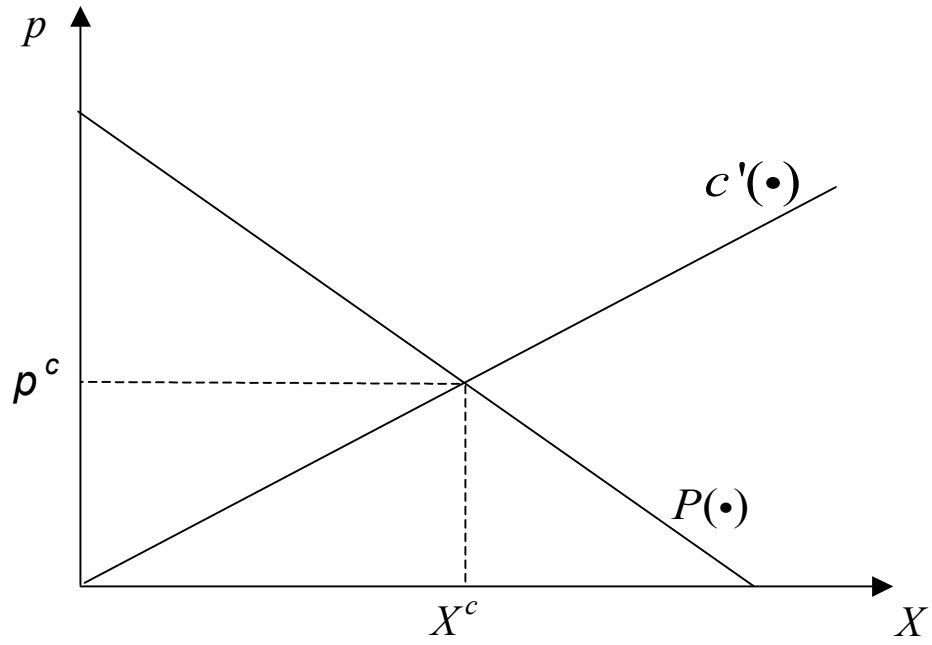


Figure 1.

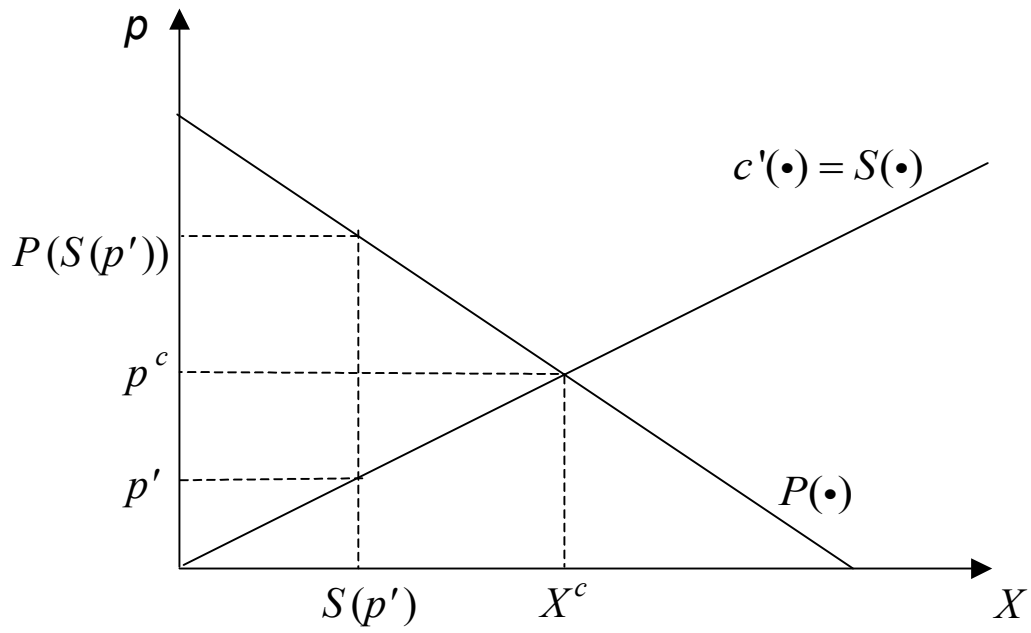


Figure 2.

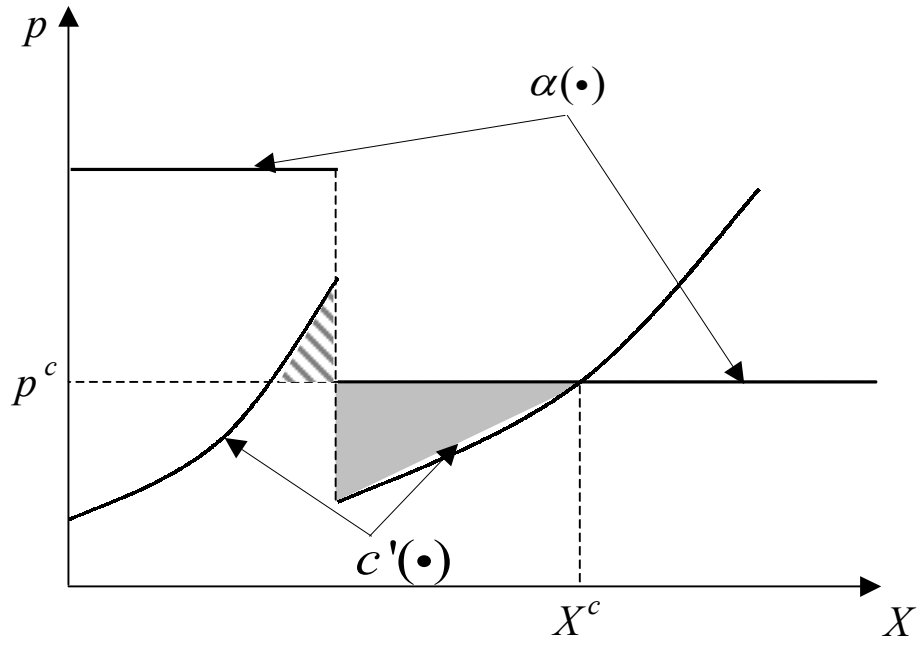


Figure 3.

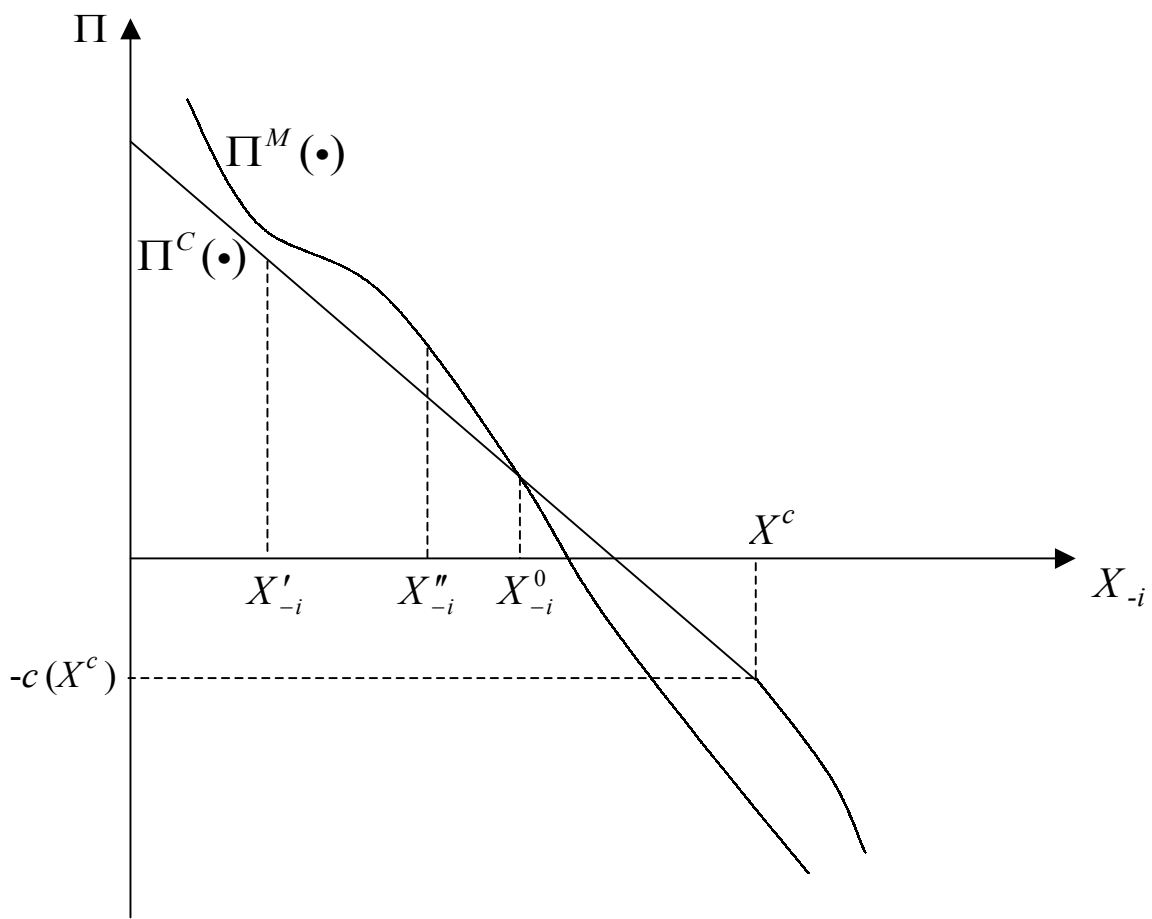


Figure 4.

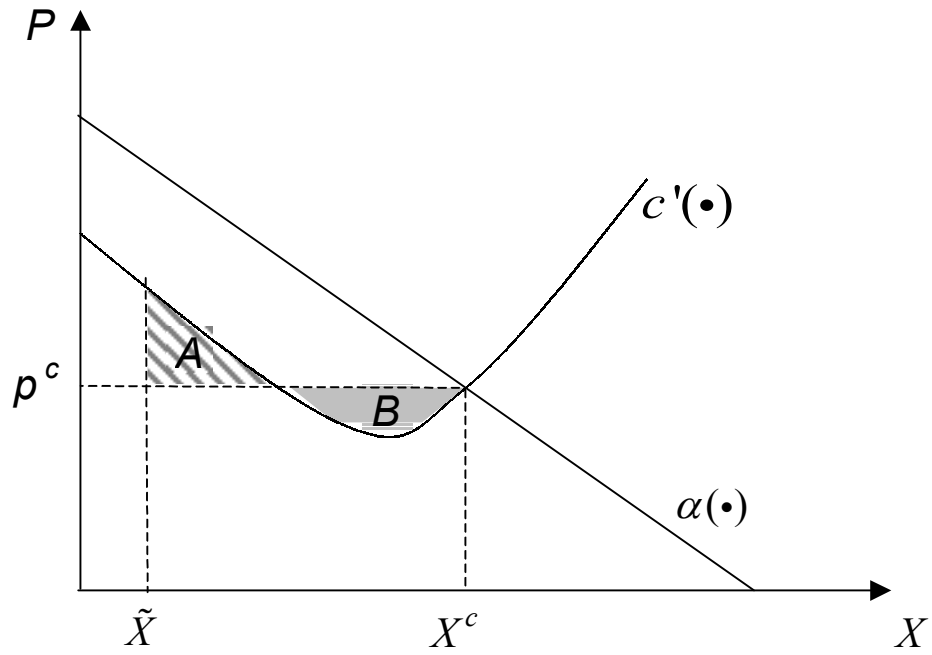


Figure 5(a).

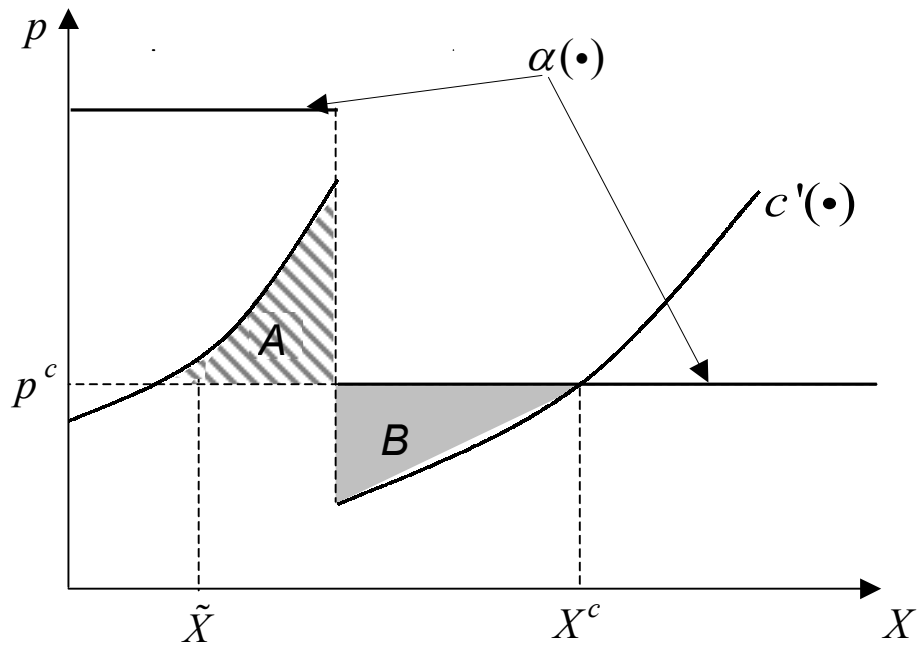


Figure 5(b).



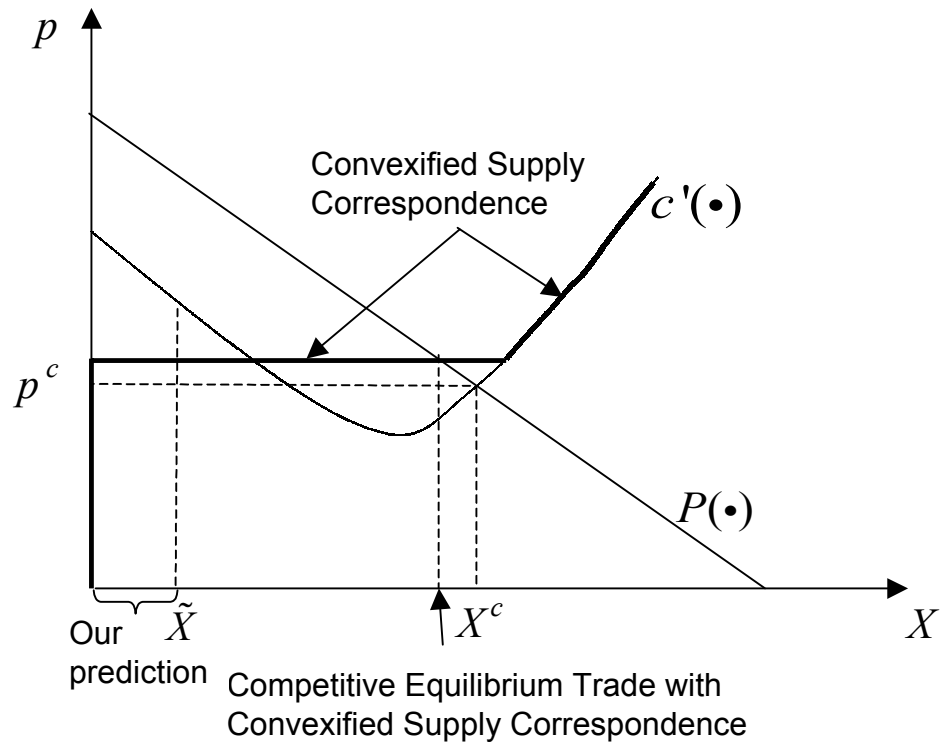


Figure 6.