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**Working Paper**

## When is price discrimination profitable?

CSIO Working Paper, No. 0072

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*Suggested Citation:* Anderson, Eric T.; Dana, James D. (2005) : When is price discrimination profitable?, CSIO Working Paper, No. 0072, Northwestern University, Center for the Study of Industrial Organization (CSIO), Evanston, IL

This Version is available at:

<https://hdl.handle.net/10419/38645>

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THE CENTER FOR THE STUDY  
OF INDUSTRIAL ORGANIZATION  
AT NORTHWESTERN UNIVERSITY

Working Paper #0072

When is Price Discrimination Profitable? <sup>†</sup>

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This Draft: October 5, 2005

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<sup>†</sup> We would like to acknowledge Justin Johnson, Marco Ottoviani, Kathryn Spier, Duncan Simester, Lars Stole, Birger Wernerfelt, and participants in the Northwestern and University of Chicago Joint Conference on Marketing and Economics for their helpful comments.

## Abstract

We analyze a model of a quality-constrained monopolist's product line decision that encompasses a variety of important examples of second-degree price discrimination, including intertemporal price discrimination, coupons, advance purchase discounts, versioning of information goods, and damaged goods. We derive necessary and sufficient conditions for price discrimination to be profitable that generalize existing results in the literature. Specifically, we show that when a continuum of product qualities are feasible, price discrimination is profitable if and only if the ratio of the marginal social value from an increase in quality to the total social value of the good is increasing in consumers' willingness to pay. Unlike third-degree price discrimination, we show that second-degree price discrimination may result in a Pareto improvement. However, in general the welfare effects are ambiguous.

## 1. Introduction

Sellers often price discriminate by allowing consumers to select among multiple product qualities at different prices (i.e., second-degree price discrimination). Examples of this type of price discrimination are numerous, yet there are also many instances in which firms choose not to price discriminate: Saturn sells automobiles at a single, no-hassle price; iTunes charges 99 cents for each song; retailers often charge the same price for color and size variants. A fundamental question for a firm is “When to price discriminate?” or conversely, “When not to price discriminate?”

In this paper, we present a model that yields a single necessary and sufficient condition for a monopolist to price discriminate. The fact that we derive conditions under which price discrimination is not profitable isn’t terribly surprising. Several papers, which we will discuss later, have previously made this point. What is surprising is that a single, elegant and intuitive condition characterizes when price discrimination is optimal. Specifically, we show that when a continuum of product qualities, up to some constraint, is feasible, a monopolist price discriminates if and only if the ratio of the marginal social value from an increase in product quality to the total social value of the good is increasing in consumers’ willingness to pay. We show that this condition, which we call an *increasing percentage differences* condition, integrates an extensive number of existing applications of second-degree price discrimination, including intertemporal pricing, damaged goods, advance purchase discounts, coupons (rebates), and versioning information goods.

We are not the first to tackle this issue. Our framework generalizes work by Salant (1989), who sought to explain apparently contradictory findings in Mussa and Rosen (1978) and Stokey (1979). Mussa and Rosen's analysis yields the prediction that price discrimination by a monopolist is always optimal, which starkly contrasts with Stokey's analysis of intertemporal pricing that shows a monopolist may forgo price discrimination. Salant proves that these seemingly contradictory findings can be reconciled in a single model.

Two key insights from Salant's analysis are also readily apparent in our model. First, quality constraints are a necessary condition for firms to forgo price discrimination. In his analysis, Salant bounds quality between 0 and 1 and we make a similar assumption. In the absence of quality constraints, a monopolist always price discriminates. Quality constraints are often implicit in price discrimination models because they are rather natural in many applications for at least two reasons. First, firms are endowed with a given product technology, which bounds the maximum level of quality. Second, and perhaps more importantly, the technologies available for lowering product quality (e.g., coupons, travel restrictions, disabling product features, and delaying delivery times) are often much richer and more diverse than the technologies available for raising quality.

A second key insight from Salant's work is that convexity in the cost function is necessary for price discrimination; if utility and cost are both linear in quality, price discrimination is never optimal. Salant shows this and separately presents a sufficient condition for price discrimination with linear utility and convex cost. Our analysis generalizes Salant's intuition by showing that the optimality of price discrimination depends on the shape of both the utility and cost functions. Further, we derive a single

condition that is both necessary and sufficient for this more general environment; we show that price discrimination is profitable if and only if the percentage change in the social value function (utility less cost) is increasing in consumers' willingness to pay.

While applications of second-degree price discrimination have much in common, few papers have attempted to unify them in a single framework. This paper does exactly that. As we shall demonstrate, the necessary and sufficient conditions that we derive are implicit in many models. Our paper unifies these literatures by explicitly recognizing their common elements.

Note that Johnson and Myatt (2003) explicitly state our *increasing percentage differences* condition in their analysis of product line pricing. However they described it only as a necessary condition, not as a necessary and sufficient condition. Moreover, the condition's relationship to other applications of second-degree price discrimination has not been previously recognized.

Our analysis also shows that the welfare implications of second and third-degree price discrimination are quite different. In particular, we focus on whether price discrimination leads to a Pareto improvement. Under third-degree price discrimination, a Pareto improvement is never feasible since some customers always pay a higher price. In contrast, we show that there always exists a distribution of consumers such that a Pareto improvement is possible under second-degree price discrimination.

Before we proceed, we recognize that there are multiple views on what constitutes price discrimination. We take the position that selling different products to different consumers when it would have been more efficient to sell them the same product constitutes price discrimination. This definition of price discrimination is appealing

because it corresponds to asking whether the solution to the monopolist's problem is a separating or a pooling solution. However in some of our analysis it is efficient to sell different products to different consumers. In this case we say that the firm is price discriminating when the monopolist is distorting the quality of some of its products away from the efficient level in order to increase its profits.

The remainder of the paper is organized as follows. In Section 2, we offer a simple example with two types of consumers and two exogenously given product qualities to illustrate the intuition of our increasing percentage differences condition. We extend the example to a monopolist selecting two product qualities subject to a quality constraint. Section 3 considers the more general problem in which a monopolist sells to continuum of consumers. Section 4 analyzes the welfare properties of price discrimination. Section 5 examines the relationship of our results to the existing literature on intertemporal price discrimination, damaged goods, coupons, advance purchase discounts, and information goods, and provides useful intuition about the conditions under which price discrimination is profitable.

## 2. An Example with Two Consumer Types

Consider a monopolist who can sell either or both of two products, one with exogenously given quality  $\underline{q}$  and another with exogenously given quality  $\bar{q}$ , to two distinct groups of consumers,  $n_H$  high types, denoted  $\theta_H$ , and  $n_L$  low types, denoted  $\theta_L$ . The monopolist cannot directly distinguish between consumer types, but can sell a different product to each type as long as the purchase decision is individually rational and incentive compatible. Consumers have unit demands and maximize their consumer

surplus,  $V(q, \theta_L) - p(q)$  and  $V(q, \theta_H) - p(q)$  respectively. The firm has unit costs of production,  $c(q)$ , that vary with product quality. For convenience, we assume

$$V(q, \theta_L) > c(q) \text{ for all } q.$$

Figure 1 gives a graphical depiction of the problem. Note that in the Figure, the willingness to pay is depicted as decreasing in  $q$  and the marginal cost is depicted as increasing in  $q$ , however our example is more general. We assume

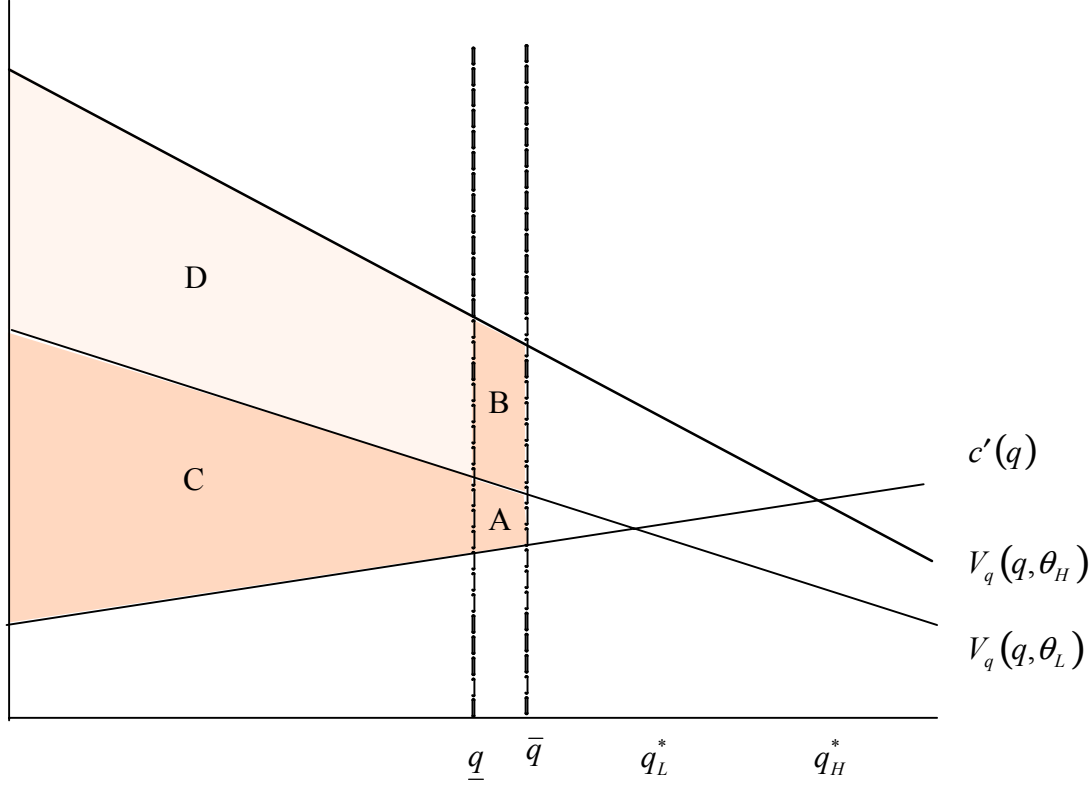
$$V(q, \theta_H) > V(q, \theta_L), \forall q, \text{ and } V(\bar{q}, \theta_H) - V(\bar{q}, \theta_L) > V(\underline{q}, \theta_H) - V(\underline{q}, \theta_L),$$

or equivalently that the consumers that are willing to pay the most for a low quality product are also the consumers that are willing to pay the most to increase the quality from low to high. This is the well-known sorting condition. Graphically, our assumptions are equivalent to assuming that the areas of the regions A, B, C, and D depicted in the figure are all positive.

If the firm served all consumers with a single quality  $\bar{q}$  at a single price, its profit on each sale would be  $A + C$ . The high type would capture surplus  $D + B$  while the low type captures 0 surplus. If, instead, the firm chose to offer both a high and a low quality product,  $\underline{q}$ , it would lose  $A$  (capture only  $C$ ) on each sale to a low type but gain an additional amount  $B$  (capture  $C + A + B$ ) on each sale to a high type. So the firm's profits

would increase as long as  $Bn_H > An_L$ , or  $\frac{A}{A+B} < \frac{n_H}{n_L + n_H}$ .





**Figure 1**

If the firm served only the high type consumers it would be able to capture producer surplus  $A + B + C + D$ . If instead the firm chose to offer both high and low quality, its profits would increase only if the profit earned on the new low-type consumers covered the lower margin on high-type customers. That is  $Cn_L > Dn_H$ , or

$$\frac{C}{C+D} > \frac{n_H}{n_L + n_H}.$$

Hence the firm is willing to offer both product qualities if and only if

$$\frac{C}{C+D} > \frac{n_H}{n_L + n_H} > \frac{A}{A+B} \text{ which can only hold if } \frac{A+B+C+D}{C+D} > \frac{A+C}{C}, \text{ or}$$

$$\frac{V(\bar{q}, \theta_H) - c(\bar{q})}{V(\underline{q}, \theta_H) - c(\underline{q})} > \frac{V(\bar{q}, \theta_L) - c(\bar{q})}{V(\underline{q}, \theta_L) - c(\underline{q})}. \quad (1)$$

This condition implies that both products are offered only if the ratio of the high type's total surplus to the low type's total surplus is increasing in quality. Equivalently both products are offered only if the marginal surplus from an increase in quality as a percentage of the total surplus is increasing in the consumer type. We call this condition *increasing percentage differences*. There exist values of  $n_L$  and  $n_H$  such that offering both products is optimal only if this condition is met.

This example is easily generalized to allow the firm to choose its product quality optimally. Suppose there are  $n_L$  buyers of type  $\theta_L$  and  $n_H$  buyers of type  $\theta_H$ . Buyer  $\theta$ 's consumer surplus from purchasing a product of quality  $q$  at price  $t$  is  $V(q, \theta) - t$ , and buyers purchase the product that gives them the greater consumer surplus. Assume the firm's cost for selling  $n$  units of quality  $q$  is  $nc(q)$ . Quality is constrained to be less than or equal to one. We assume  $V(0, \theta) - c(0) < 0, \forall \theta$ , which implies that quality will always be strictly positive. We assume that  $V$  and  $c$  are continuously differentiable with respect to  $q$ , that  $V_q > 0$  and  $V_q(q, \theta) - c'(q) > 0$ , and that  $V(q, \theta)$  and  $V_q(q, \theta)$  are increasing in  $\theta$ . Finally, we assume  $q^*(\theta) \equiv \arg \max_{q \leq 1} V(q, \theta) - c(q) = 1, \forall \theta$ , that is, that the quality constraint binds on both types.

**Proposition 1**

Let  $N^*$  denote the open interval  $\left( \frac{V_q(1, \underline{\theta}) - c'(1)}{V_q(1, \bar{\theta}) - c'(1)}, \max_{\phi} \frac{V(\phi, \underline{\theta}) - c(\phi)}{V(\phi, \bar{\theta}) - c(\phi)} \right)$ .

- a) The firm will offer multiple qualities only if  $V(q, \theta) - c(q)$  is log supermodular at  $q = 1$ .
- b) If  $V(q, \theta) - c(q)$  is log supermodular at  $q = 1$ , then  $N^*$  is non-empty and the firm will offer multiple qualities if and only if  $\frac{n_H}{n_H + n_L} \in N^*$ .

**Proof:** All proofs are in the Appendix.

Proposition 1 shows that when quality is constrained, a monopolist will price discriminate only if the ratio of the marginal social value of quality to the total social value of quality is increasing in the consumer's type,  $\theta$ . Proposition 1 also characterizes the distributional conditions that, along with log supermodularity of the surplus function, are sufficient for price discrimination to be profitable.

An implication of Proposition 1 is the following.

**Corollary A**

If  $V(\theta, q) = \theta q$  and if  $c'(q) > c(q)/q$  for all  $q \in [0, 1]$ , then offering multiple products is optimal for all  $n_L$  and  $n_H$  satisfying

$$\max_q \frac{\theta_L q - c(q)}{\theta_H q - c(q)} > \frac{n_H}{n_H + n_L} > \frac{\theta_L - c'(1)}{\theta_H - c'(1)}. \quad (2)$$

We will relate Corollary A to previous applications of second-degree price discrimination in §5.

### 3. The General Model

In this section we analyze a general model in which there are a continuum of buyers of type  $\theta \in [\underline{\theta}, \bar{\theta}]$ , with probability distribution  $f(\theta)$  and cumulative distribution  $F(\theta)$ , and the firm can produce any number of products of any quality,  $q$ , subject to the constraint that  $q \in [0, 1]$ . The unit cost of production is  $c(q)$ . Consumers maximize their consumer surplus, equal to their strictly positive utility,  $V(q, \theta)$ , less the price,  $p(q)$ . We assume that  $V$  and  $c$  are twice continuously differentiable and the  $V$  satisfies  $V_q > 0$ ,  $V_\theta > 0$ , and  $V_{q\theta} > 0$ . Letting  $S(q, \theta)$  denote the surplus function  $V(q, \theta) - c(q)$ , we also assume that  $S_{qq} \leq 0$  and  $S_{qq\theta} \geq 0$ .

Finally, we assume that  $V(1, \bar{\theta}) - c(1) > 0$  and

$$(V(1, \underline{\theta}) - c(1))f(\underline{\theta}) - V_\theta(1, \underline{\theta}) < 0. \quad (3)$$

These assumptions guarantee that a monopolist selling a single product of quality 1 will serve some, but not all, consumers.

The monopolist's product quality decision depends on whether or not the surplus function is log supermodular, that is, on whether or not  $S(q_1, \theta)/S(q_2, \theta)$  is increasing in  $\theta$  for all  $q_1 > q_2$ . When  $S(q, \theta)$  is twice continuously differentiable, log supermodularity of  $S(q, \theta)$  is also equivalent to  $\partial^2 \ln S / \partial q \partial \theta = [S_{\theta q} S - S_\theta S_q] / S^2 > 0$ . It is worth noting that any function that is multiplicatively separable in  $\theta$  and  $q$  is not log supermodular because  $S_{\theta q} S - S_\theta S_q = 0$ . However it is easy to see that the functions

$a + f(q)g(\theta)$  and  $a + h(q) + f(q)g(\theta)$  are log supermodular as long as  $a > 0$  and  $f, g,$  and  $h$  are positive, increasing functions.

The seller's problem is to choose the menu of prices and qualities  $(p(\theta), q(\theta))$  on  $[\theta_L, \bar{\theta}]$  which maximizes

$$\max_{\theta_L, p(\theta), q(\theta)} \int_{\theta_L}^{\bar{\theta}} [p(\theta) - c(q(\theta))] f(\theta) d\theta, \quad (4)$$

subject to  $q(\theta) \leq 1$  as well as incentive compatibility and individual rationality constraints. Without loss of generality we assume the seller chooses to serve a range of consumers that includes the highest type,  $\bar{\theta}$ .

This is a standard mechanism design problem.<sup>1</sup> The solution to (4) satisfies

$$H(\theta, q(\theta)) \equiv \frac{\partial V(q(\theta), \theta)}{\partial q} - c'(q(\theta)) - \frac{(1 - F(\theta)) \partial^2 V(q(\theta), \theta)}{f(\theta) \partial \theta \partial q} = 0 \quad (5)$$

where  $0 < q(\theta) < 1$  and satisfies  $H(\theta, q(\theta)) \geq 0$  where  $q(\theta) = 1$  and  $H(\theta, q(\theta)) \leq 0$  where  $q(\theta) = 0$ .

We assume  $H(\theta, q)$  is increasing in  $\theta$  and decreasing in  $q$ . Given our assumptions on  $V$  and  $c$ , this holds if  $F(\theta)$  satisfies a monotone likelihood ratio property.

Finally, the lowest type buyer that the firm chooses to serve,  $\theta_L$ , must satisfy either

$$J(\theta_L, q(\theta_L)) \equiv -V(q(\theta_L), \theta_L) + c(q(\theta_L)) + \left[ \frac{1 - F(\theta_L)}{f(\theta_L)} \right] \frac{\partial V(q(\theta_L), \theta_L)}{\partial \theta} = 0, \quad (6)$$

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<sup>1</sup> Note that by appropriately rescaling the quality measure this problem can be written with linear costs, but for ease of application we chose not to make this simplification.

or  $\theta_L = \underline{\theta}$  and  $J(\underline{\theta}, q(\underline{\theta})) \leq 0$ .

Let  $q^*(\theta) \equiv \arg \max_{q \leq 1} V(q, \theta) - c(q)$  denote the optimal quality for each consumer type, subject to the constraint that quality be less than or equal to one. Without loss of generality we assume  $q^*(\bar{\theta}) = 1$ .

Under our assumptions,  $q^*(\theta)$  is weakly increasing in  $\theta$ . We analyze three separate cases. First, we consider the case in which  $q^*(\theta)$  is strictly increasing. Second we consider the case in which  $q^*(\theta) = 1$  for all  $\theta$ . And finally, we consider the case in which  $q^*(\theta) = 1$  for only some  $\theta$ .

If the quality constraint does not bind, that is if  $q^*(\theta)$  is strictly increasing, then the firm offers multiple qualities and the surplus function is always log supermodular.

**Proposition 2**

*If  $q^*(\theta)$  is strictly increasing, then  $V(q, \theta) - c(q)$  is log supermodular for all  $\theta$  and  $q = 1$ , then  $q(\theta)$  is strictly increasing for all  $\theta$ , that is, the firm strictly prefers to sell multiple product qualities.*

If the quality constraint does bind, then it may not be optimal for the firm to offer multiple qualities; it no longer follows that  $V(q, \theta) - c(q)$  is necessarily log supermodular, and it no longer follows that the firm will necessarily offer multiple product qualities. These two properties are related.

We now consider the case in which the quality constraint binds everywhere, or  $q^*(\theta) = 1$  for all  $\theta$ . In this case, price discrimination may no longer be optimal. We

show that log supermodularity of the surplus function is both necessary and sufficient for offering multiple products to be optimal.

The following is our main result.

**Proposition 3**

*If  $q^*(\theta) = 1, \forall \theta$ , then*

- a) if  $V(q, \theta) - c(q)$  is log submodular then the firm sells a single quality, and*
- b) if  $V(q, \theta) - c(q)$  is log supermodular then the firm sells multiple qualities.*

As we show in the proof of Proposition 3, it isn't necessary that  $V(q, \theta) - c(q)$  be log supermodular everywhere for multiple products to be optimal, but only that  $V(1, \theta) - c(1)$  be locally log supermodular. Also, if  $S_{qq} < 0$  and  $S_{qq\theta} > 0$ , then if  $V(1, \theta) - c(1)$  is locally log submodular, the firm will sell a single product.

An immediate implication of Proposition 3 is that when consumers have utility  $V(q, \theta) = \theta q$ , the firm will offer multiple products as long as the marginal cost of quality is greater than the average cost for all quality.

**Corollary B**

*If  $V(q, \theta) = \theta q$ , then offering multiple products is optimal if  $c'(q) > c(q)/q$  for all  $q \in [\hat{q}, 1]$ ,  $0 \leq \hat{q} < 1$ , and offering multiple products is not optimal if  $c'(q) \leq c(q)/q$  for all  $q \in [\hat{q}, 1]$ ,  $0 \leq \hat{q} < 1$ .*

Another implication of Proposition 3 is that for multiplicatively separable utility and strictly positive costs, multiple products are more likely when the marginal value of quality is lower than the average value of quality.

### Corollary C

If  $V(q, \theta) = g(q)h(\theta)$  and  $c(q) > 0$ , then multiple products are optimal if

$$\frac{c'(q)}{c(q)/q} > \frac{g'(q)}{g(q)/q} \text{ for all } q \in [\hat{q}, 1], \quad 0 \leq \hat{q} < 1, \text{ and multiple products is not optimal if}$$

$$\frac{c'(q)}{c(q)/q} \leq \frac{g'(q)}{g(q)/q} \text{ for all } q \in [\hat{q}, 1], \quad 0 \leq \hat{q} < 1.$$

This clearly implies that if cost is strictly positive and independent of quality,  $c(q) = c > 0$ , then multiple products are optimal if  $g'(q) < g(q)/q$  and not optimal if  $g'(q) > g(q)/q$ . It also implies that for any multiplicatively separable utility function,  $g'(q) < g(q)/q$  and  $c'(q) > c(q)/q$  are sufficient for multiple products to be optimal. Some of the intuition for these results can be seen in Figure 1 which depicts utility functions and cost functions that satisfy both these conditions.

Finally, a third implication of Proposition 3 is that if the firm is only able to produce a finite number of products, log supermodularity of  $V(q, \theta) - c(q)$  is still a necessary condition for the firm to offer multiple products, however log supermodularity is no longer sufficient.<sup>2</sup>

Finally we consider the third case in which the efficient quality is only partially constrained. In this case a social planner would always offer multiple products, but it is still useful to compare the monopolist's behavior to the social planner's behavior. In this case, the standard Mussa and Rosen result still holds when the monopolist faces a quality constraint.

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<sup>2</sup> To see this, let  $V(\theta, q_l) = 1 + \theta$  and  $V(\theta, q_h) = 1000 + 1000\theta$  and  $\theta : U[0, 1]$ . It is easy to see that  $V(\theta, q_h)/V(\theta, q_l)$  is increasing in  $\theta$ , yet it is optimal for the firm to sell only product  $q_h$ .



**Proposition 4**

*If  $q^*(\theta)$  is weakly increasing, then the monopolist's quality,  $q(\theta)$ , satisfies*

$$q(\bar{\theta}) = q^*(\bar{\theta}), \quad q(\theta) \leq q^*(\theta), \quad \forall \theta, \quad \text{and} \quad q(\theta) < q^*(\theta) \quad \text{whenever} \quad 0 < q^*(\theta) < 1.$$

It follows that if  $V(q, \theta) - c(q)$  is everywhere log supermodular, then the firm always sells multiple qualities, so log supermodularity is again sufficient for price discrimination.

To summarize, when quality is unconstrained price discrimination is always optimal. When quality is partially or completely constrained then price discrimination is optimal if the surplus function,  $V(q, \theta) - c(q)$ , is everywhere log supermodular.

## 4. Welfare

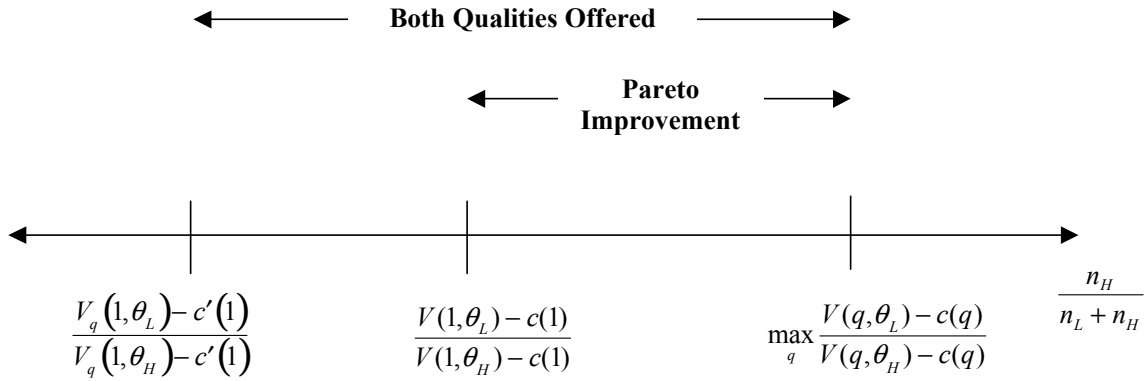
In this section, we focus on a specific aspect of welfare: does price discrimination lead to a Pareto improvement? Third-degree price discrimination can never lead to a Pareto improvement since buyers with relatively inelastic demands necessarily face higher prices. However, second-degree price discrimination can lead to a Pareto improvement (Deneckere and McAfee 1996, Anderson and Song 2004). A necessary condition for second-degree price discrimination to be Pareto improving is that the firm serves more buyers than it would have otherwise. In serving more buyers, the firm will need to sell at a lower price, but by distorting quality, the firm can lower its price without passing on the same price increase to the buyers it would have otherwise served. Nevertheless, incentive compatibility constraints may force the firm to lower its price to all of its buyers.

We first characterize necessary and sufficient conditions for a Pareto improvement to occur when there are two types of buyers. A Pareto improvement occurs when both types of buyers are served when price discrimination is allowed, but only the high type is served when price discrimination is banned. These conditions are summarized in Proposition 5 and illustrated graphically in Figure 2.

**Proposition 5**

If there are two types of consumers,  $V(q, \theta) - c(q)$  is log supermodular in a neighborhood of  $q = 1$ ,  $\frac{n_H}{n_H + n_L} \in N^*$ , and  $\frac{n_H}{n_H + n_L} > \frac{V(1, \underline{\theta}) - c(1)}{V(1, \overline{\theta}) - c(1)}$ , then offering multiple qualities results in a Pareto improvement.

If the fraction of high-type consumers is sufficiently high (i.e., greater than  $(V(1, \theta_L) - c(1)) / (V(1, \theta_H) - c(1))$ ), then in the absence of the ability to price discriminate the firm will sell to only the high types. But when the firm is able to price discriminate, the high-type consumers are better off because they capture some surplus due to the incentive compatibility constraint. In contrast, if the fraction of high-type consumers is small, then in the absence of the ability to price discriminate the firm will sell to both consumer types. In this case, when the firm is able to price discriminate, the high type consumer are worse off because they face a higher price for the same quality.



**Figure 2: Regions of Price Discrimination and Pareto Improvement**

When there are a continuum of buyer types, a similar intuition holds. However, the insight from the two-type case applies to the marginal consumer. The marginal consumer receives zero surplus when price discrimination is not allowed. But if price discrimination is allowed and more buyers are served then the marginal consumer receives strictly positive surplus<sup>3</sup>. While the marginal consumer and every consumer with a lower type are strictly better off when price discrimination is allowed, price discrimination does not result in a Pareto improvement unless every consumer with a higher type is also better off. Specifically, the highest-type buyer must also receive more surplus when the firm price discriminates. Since the firm serves more consumers, the incentive compatibility constraint becomes more difficult to satisfy. But the firm can satisfy the constraint either by lowering price or distorting quality. Lowering product quality sold to customers *who would have been served in the absence of price discrimination* allows the firm to raise the price to the highest-type buyer. So allowing price discrimination has two offsetting effects on the surplus of the highest-type consumer. However it is clear that for some distributions of consumer types this

<sup>3</sup> The market may not always expand; fewer buyers may be served when a firm price discriminates.

consumer is better off, and as a consequence, allowing price discrimination leads to a Pareto improvement.

### **Claim**

There exists a continuous distribution function  $f$  such that allowing price discrimination is a Pareto improvement.

The proof of Claim uses distributions of  $f$  that approximate the two-type distribution and so they do not satisfy the monotone likelihood ratio property (MLRP). We also considered a number of analytically tractable distributions functions that satisfied MLRP, but none of the examples we considered had the property that allowing price discrimination lead to a Pareto improvement. We conjecture that while a Pareto improvement is feasible for some distributions, it may only occur with irregular distributions such as the bi-modal distributions we used to prove the claim.

## **5. Applications**

While our results are developed in the context of product line pricing, they readily generalize to other types of second-degree price discrimination. In this section, we link our model to five common applications: intertemporal pricing, damaged goods, versioning, advance purchase discounts, and coupons (rebates). We show that specific results from these applications are both replicated and generalized by our results. We also use these applications to emphasize several key intuitions from our increasing percentage differences condition.

## A. Intertemporal Price Discrimination

Stokey (1979) considers a monopoly model of intertemporal price discrimination in which consumers' utility functions are  $U(\theta, t) = \theta \delta^t$  and the unit cost of production is  $k(t) = \kappa \delta^t$ . These assumptions imply that firm cost is independent of time except for the time value of money. A well-known result from this model is that intertemporal price discrimination is never optimal. We now show that this important result follows immediately from Proposition 3.

A monopolist chooses a menu of prices paid at time 0 and delivery times, subject to the constraint that  $t \geq 0$ , to maximize profits. Similar to Salant (1989), we use a change of variables,  $q = \delta^t$ , so that  $V(\theta, q) = \theta q$  and costs  $c(q) = cq$ . With this transformation, the firm's problem is to choose the profit-maximizing menu of prices and qualities subject to the constraint that  $q \leq 1$ . Clearly  $V(\theta, q) - c(q) = \theta q - cq$  is not log supermodular and  $q = 1$  is the efficient quality for all  $\theta$ . So by Proposition 3, intertemporal price discrimination is never optimal, even though it is clearly feasible.

Both Stokey (1979) and Salant (1989) demonstrate that intertemporal price discrimination is optimal with more general cost functions, such as  $k(t) = \kappa(t) \delta^t$ . After a change of variables this implies  $c(q) = \kappa \left( \frac{\log q}{\log \delta} \right) q$ . The surplus function,

$V(\theta, q) - c(q) = \theta q - c(q)$ , is log supermodular if  $\frac{(qc'(q) - c(q))}{(\theta q - c(q))^2} > 0$ , or

$c'(q) > c(q)/q$ . So if the marginal cost of quality is positive and greater than the average cost of quality then intertemporal price discrimination is profitable.<sup>4</sup>

Salant showed that  $c'(q) > c(q)/q$  was necessary and that

$$\frac{\theta_L - c'(0)}{\theta_H - c'(0)} > \frac{n_H}{n_H + n_L} > \frac{\theta_L - c'(1)}{\theta_H - c'(1)} \quad (7)$$

was sufficient for intertemporal price discrimination. The lower bound that we derive (see condition (2) in Corollary A) and Salant's lower bound are identical, but the upper bound that we derive is strictly greater than the upper bound in (7). Thus, we both replicate and generalize Salant's previous findings. Further, while (7) is implied by Salant's analysis, he does not formally state his sufficient condition in terms of the fraction of high types in the market.

Using Proposition 3 and Corollary B, we can also generalize Salant's results for discrete types to a market with a continuum of consumer types<sup>5</sup>. By Corollary B,  $c'(q) > c(q)/q$  is necessary and sufficient condition for price discrimination, and since

$c(q) = \kappa \left( \frac{\log q}{\log \delta} \right) q$ , it immediately follows that  $c'(q) > c(q)/q$  if and only if

$$\kappa \left( \frac{\log q}{\log \delta} \right) + \kappa' \left( \frac{\log q}{\log \delta} \right) \frac{1}{\log \delta} > \kappa \left( \frac{\log q}{\log \delta} \right). \quad (8)$$

Assuming  $\delta < 1$  and  $\log \delta < 0$ , this means that intertemporal price discrimination is profitable if and only if  $\kappa'(t) < 0$ . Thus, if the firm's production costs are declining over time the firm will offer declining prices and induce some consumers to delay their

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<sup>4</sup> Johnson and Myatt (2003) have a related result about the product range of a multiproduct monopolist.

<sup>5</sup> Salant claims that his results generalize to the n-type case, but does not consider a continuum of types.

purchases, while if the firm's production costs are rising over time the firm will offer a constant price over time and all consumers will purchase immediately.<sup>6</sup>

Finally, note that intertemporal price discrimination is also profitable when consumers have heterogeneous valuations and a common discount rate,  $r$ , that is greater than the firm's rate,  $r_f$  (see Landsberger and Meilijson, 1985). We can write consumers' utility,  $U(t, \theta) = \theta e^{-rt}$ , as  $V(\theta, q) = \theta q$ , where  $q = e^{-rt}$ , and the firm's cost function as  $c(q) = \kappa q^{r_f/r}$ . This implies  $c'(q) > c(q)/q$  and so by Corollary B intertemporal price discrimination is clearly profitable.

## B. Damaged Goods

A damaged good is one for which  $c'(q) \leq 0$ , that is, it is weakly more expensive to produce lower quality goods. It follows from our increasing percentage surplus condition that price discrimination is less likely to be profitable for damaged goods. For example, when consumers' utility is  $V(q, \theta) = \theta q$ , by Corollary B price discrimination is never optimal if  $c'(q) \leq 0$  but is optimal for all cost functions satisfying  $c'(q) > c(q)/q$ .

Deneckere and McAfee (1996) derive conditions for optimal price discrimination with damaged goods. They demonstrate that it can be both profitable and Pareto improving to offer a damaged good<sup>7</sup>. They assume a continuum of types with unit demands, and restrict attention to two product qualities,  $q_L$  and  $q_H$ . Consumers have quasi-linear utilities  $V(q_H, \theta) = \theta$  and  $V(q_L, \theta) = \lambda(\theta)$ .

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<sup>6</sup> As Stokey points out, when  $\kappa'(t) < r = -\log \delta$  for some  $t$  competitive markets will also exhibit this pattern of prices and purchases. But by Proposition 4, when competitive market exhibit such delay, the monopoly market exhibit weakly greater delay for all consumers and strictly greater delay for any consumers who don't purchase immediately.

<sup>7</sup> Here we discuss the second of the two models that Deneckere and McAfee (1996) analyze.

The necessary and sufficient condition derived by Deneckere and McAfee is a special case of our more general condition. Specifically, in Deneckere and McAfee's model,  $V(q, \theta) - c(q)$  is log supermodular if and only if

$$\frac{1}{\theta - c_H} > \frac{\lambda'(\theta)}{\lambda(\theta) - c_L},$$

or  $\lambda(\theta) - c_L - (\theta - c_H)\lambda'(\theta) > 0$ . To see how this is related to the condition derived by Deneckere and McAfee, note that the price a single product firm would charge is

$p = V(q_H, \underline{\theta}) = \underline{\theta}$  where  $\underline{\theta}$  is defined by  $\underline{\theta} - c_H - \frac{1 - F(\underline{\theta})}{f(\underline{\theta})} = 0$ . So  $V(q, \underline{\theta}) - c(q)$  is log

supermodular if and only if

$$\lambda(\hat{\theta}) - c_L - \left( \frac{1 - F(\hat{\theta})}{f(\hat{\theta})} \right) \lambda'(\hat{\theta}) > 0,$$

which is the necessary and sufficient condition for the provision of damaged goods derived by Deneckere and McAfee.

### C. Other Applications

Our results also integrate three other common applications of second-degree price discrimination: versioning, advance purchase discounts, and coupons (rebates). We use these applications to show additional intuitions from our increasing percentage differences condition.

A key intuition from the increasing percentage differences condition is that a decrease in marginal costs increases the likelihood of satisfying the increasing percentage differences condition. Thus, a firm that faces lower costs is more likely to price discriminate. This may in part explain why price discrimination with information goods



has received considerable attention. Information goods is a term used to describe goods like software, books, music, newspapers and magazines, which have high fixed costs of production and small or negligible variable costs. The practice of selling multiple versions of information goods has been described by informally by Shapiro and Varian (1998) and more formally by Varian (1995 & 2001) and Bhargava and Choudhary (2001b, 2004). It follows immediately from Corollary B that if costs are zero, a firm will not price discriminate when  $V(\theta, q) = \theta q$ , and this result is shown by Bhargava and Choudhary (2001a). Bhargava and Choudhary (2001b) consider a two good model with a uniform distribution of consumer types and show that the firm will produce both the high and low quality good only if  $V(q_H, \theta)/V(q_L, \theta)$  is increasing in  $\theta$ . This necessary condition is analogous to our increasing percentage differences condition.

A firm may also face lower production costs due to learning effects in the manufacturing process. In turn, such a firm is more likely to price discriminate. As an example of such behavior, consider the product strategy of Keurig, a small manufacturer of single-serve coffee brewing systems. In 2003, Keurig entered the U.S. consumer market with a single coffee brewer priced at \$249 and faced costs of over \$200 per unit. In 2004, Keurig significantly lowered its production costs via reengineering efforts and overseas manufacturing. By 2005, Keurig offered three versions of the consumer coffee brewer priced at \$199, \$149 and \$99. Our increasing percentage differences condition highlights that lower costs increase the likelihood of price discrimination. But we recognize that lower costs are only one of many factors that can explain an increase in product variants.

A second intuition from the increasing percentage differences condition is that an increase in product valuation increases the likelihood that a firm price discriminates. To illustrate this point, we use a stylized model of advance purchase requirements. Recent examples of the use of advance purchase requirements for price discrimination include Shugan and Xie (2000), Courty and Li (2000), and Gale and Holmes (1992, 1993).<sup>8,9</sup> Purchasing in advance requires consumers to give up flexibility in their purchase decision, departure time, or destination. Consider the following model, which is inspired by Courty and Li (2000). Assume the firm can set one price,  $p_0$ , for travel if the ticket is purchased at time 0 (e.g., 14-days in advance) and another price,  $p_1$ , for travel if the ticket is purchased at time 1 (e.g., one day in advance). Assume two types of consumers, business travelers and leisure travelers, who differ in their valuations for the product and in their cost of planning. Specifically, consumers value for travel is  $v_B = v + \varepsilon$  and  $v_L = v$  if they buy in the spot market and is  $v_B - x_B$  and  $v_L - x_L$  if they buy in advance.<sup>10</sup> We also assume  $x_B > x_L$  and zero marginal product cost.

The firm has three pricing options. It can sell to all the business travelers at price  $v_B$  ( $p_0 = p_1 = v_B$ ), or sell to all buyers at price  $v_L$  ( $p_0 = p_1 = v_L$ ), or sell to leisure travelers at price  $p_0 = (v_L - x_L)$  at time 0 and to sell to business travelers at price

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<sup>8</sup> Price discrimination can help the firm extract greater surplus from heterogeneous consumers (see Shugan and Xie 2001, Courty and Li 2000, and Dana, in progress) and also enable the firm to increase capacity utilization (see Gale and Holmes, 1992, 1993, and Dana, 1998, 1999).

<sup>9</sup> Advance purchase discounts can also benefit the firm in other ways. First, advance purchase discounts can be used to improve production efficiency of production by giving the firm better forecast of spot market demand (Tang et. al. 2004, and McCardle et. al. 2004). Also, firms may find it more profitable to sell in advance when consumers have an imperfect forecast of their spot market preferences (Shugan and Xie 2001, and Courty 2003).

<sup>10</sup> The literature on advance purchase discounts derives the value of flexibility explicitly from consumers' demands – consumers who buy in advance are either uncertain about their spot market valuations (Courty and Li, 2000, Dana 1998, and Shugan and Xie 2000) or about their departure time preferences (Gale and Holmes 1992, 1993 and Dana 1999).

$p_1 = p_0 + x_B$  at time 1. By Proposition 1, price discrimination option is the most profitable option if and only if

$$\frac{x_B}{v + \varepsilon - x_B} > \frac{x_L}{v - x_L}. \quad (9)$$

Clearly condition (9) is more likely to be satisfied if consumer valuations,  $v$ , increase. Thus, our increasing percentage differences condition is more likely to be satisfied in markets where the average product valuation is greater.

A common intuition for price discrimination is that consumers must have a positive correlation between product valuations and disutility of the inferior good. We use an application to coupons (Anderson and Song 2004, Nevo and Wolfram 2002, Gerstner and Hess 1991) to illustrate that this intuition is not sufficient. Models of coupons (or rebates) assume that the hassle cost of clipping and redeeming coupons is positively correlated with product valuations. We consider a model of coupon-based price discrimination based on Anderson and Song (2004). Assume that consumers are uniformly distributed on  $[\underline{\theta}, \bar{\theta}]$  the unit interval and that their utility is  $V(\theta, N) = \alpha + \theta\varphi$  if they do not use a coupon and  $V(\theta, C) = \alpha + \theta\varphi - H(\theta)$  if they do use a coupon. The function  $H(\theta)$  represents the cost of using a coupon and is assumed to be increasing in the consumer's type. The parameters  $\alpha$  and  $\varphi$  are positive scalars. The firm chooses,  $d$ , the face value of the coupon, and  $p$ , the shelf price. The constant marginal cost of the good is  $c$ , and the cost of printing the coupons is  $\lambda$  per coupon user.

From Proposition 3, coupons are profitable only if  $V(\theta, q) - c(q)$ ,  $q \in \{C, N\}$ , is log supermodular, and  $V(\theta, q) - c(q)$ ,  $q \in \{C, N\}$ , is log supermodular if

$$\frac{\varphi}{\alpha + \theta\varphi - c} > \frac{\varphi - H'(\theta)}{\alpha + \theta\varphi - H(\theta) - c - \lambda}, \quad (10)$$

or equivalently

$$\frac{\varphi}{\alpha + \theta\varphi - c} < \frac{H'(\theta)}{H(\theta) + \lambda}.$$

If  $H(\theta) = \theta H$  and  $\lambda = \alpha = 0$ , then there is perfect positive correlation between hassle cost and product valuation but price discrimination is not optimal. Thus, positive correlation is not sufficient for a firm to price discriminate. In contrast, our increasing percentage difference condition is both intuitive and sufficient. The parameter  $\lambda$  illustrates that if price discrimination is costly to implement, then it is less likely to be profitable.

The coupon model also yields a parallel result to Corollaries A and B. A necessary condition for price discrimination when  $\lambda = \alpha = 0$  is that  $H'(\theta) > H(\theta)/\theta$ , which one might loosely interpret as marginal hassle cost is greater than average hassle cost.

## 6. Conclusion

We offer a general theory of the optimality of price discrimination that is useful in analyzing product line decisions, intertemporal price discrimination, coupons, the versioning of information goods, the practice of crimping or selling intentionally damaged goods, and the use of advance purchase discounts. We derive a single, intuitive condition that is both necessary and sufficient for price discrimination to be profitable.

We also link common elements of many existing, but disparate, applications of price discrimination into a general theory.

Our paper studies when price discrimination will be profitable, but we found it was sometimes easier to ask when it would not be profitable. We began with a generalized Mussa and Rosen environment in which price discrimination is always profitable and looked at some modifications of that environment in which price discrimination does not occur. We found that with a continuum of consumer types a cap, or constraint, on quality implies that the firm offers a single product only if the surplus function is log submodular. With just two consumer types, price discrimination may fail to be profitable either because quality is constrained and the surplus function is log submodular, or because there are too few of one type of consumer.

## 7. Proofs

### Proof of Proposition 1

The seller selects quality levels,  $q_L$  and  $q_H$ , and transfers,  $t_L$  and  $t_H$ , subject to incentive compatibility and participation constraints:

$$\max_{q_L, q_H, t_L, t_H} I(V(q_L, \theta_L) - t_L) n_L (t_L - c(q_L)) + I(V(q_H, \theta_H) - t_H) n_H (t_H - c(q_H)) \quad (11)$$

subject to

$$V(q_H, \theta_H) - t_H \geq V(q_L, \theta_H) - t_L, \quad (\text{IC-1})$$

$$V(q_L, \theta_L) - t_L \geq V(q_H, \theta_L) - t_H, \quad (\text{IC-2})$$

and

$$q_L \leq q_H \leq 1$$

where  $I$  is the indicator function (consumers purchase only if their surplus is non-negative).

Clearly any solution to (11) satisfies  $q_H = 1$ . Hence, the solution to (11) takes on one of three possible forms. The first, which we label strategy S1, is to sell a single quality,  $q_H = 1$ , to only the high type buyers at  $t_H = V(1, \theta_H)$  and profit

$n_H (V(q_H, 1) - c)$ . The second, which we label strategy S2, is to sell a single quality,  $q_L = q_H = 1$ , to all buyer types at price  $t_L = V(1, \theta_L)$  and profit  $(n_L + n_H)(V(1, \theta_L) - c)$ .

The third, which we label strategy S3, is to offer multiple qualities and sell to both buyer types. The low-type buyer pays  $t_L = V(q_L, \theta_L)$  for quality  $q_L < 1$ , and the high

type buyer pays  $t_H = V(1, \theta_H) - (V(q_L, \theta_H) - V(q_L, \theta_L))$  for quality  $q_H = 1$  and the firm earns a profit  $n_L (V(q_L, \theta_L) - c(q_L)) + n_H (V(1, \theta_H) - c(1) - (V(q_L, \theta_H) - V(q_L, \theta_L)))$ . When the firm adopts strategy S3, the low quality level solves

$$\max_{\tilde{\phi}} n_L (V(\tilde{\phi}, \theta_L) - c(\tilde{\phi})) + n_H (V(1, \theta_H) - c(1) - (V(\tilde{\phi}, \theta_H) - V(\tilde{\phi}, \theta_L))). \quad (12)$$

The first order condition,

$$G(q) = n_L (V_q(q, \theta_L) - c'(q)) + n_H (V_q(q, \theta_L) - V_q(q, \theta_H)) = 0, \quad (13)$$

has a strictly interior solution,  $\tilde{\phi} \in (0, 1)$ , if and only if  $G(0) > 0$  and  $G(1) < 0$ . Under our assumptions the second order condition is satisfied.

Comparing the three solution strategies,  $\hat{q} < 1$ , or equivalently strategy S3 strictly dominates strategy S2, if and only if  $G(0) > 0$  and  $G(1) < 0$ , the later of which can be written as

$$n_L (V_q(1, \theta_L) - c'(1)) + n_H (V_q(1, \theta_L) - V_q(1, \theta_H)) < 0. \quad (14)$$

Strategy S3 strictly dominates strategy S1 if and only if

$$\begin{aligned} n_L (V(\tilde{\phi}, \theta_L) - c(\tilde{\phi})) + n_H (V(1, \theta_H) - c(1) - (V(\tilde{\phi}, \theta_H) - V(\tilde{\phi}, \theta_L))) \\ > n_H (V(1, \theta_H) - c(1)), \end{aligned} \quad (15)$$

or equivalently  $n_L (V(\tilde{\phi}, \theta_L) - c(\tilde{\phi})) - n_H (V(\tilde{\phi}, \theta_H) - V(\tilde{\phi}, \theta_L)) > 0$ , for some  $\tilde{\phi}$ . Note that (15) and  $V(0, \theta) - c(\theta) < 0, \forall \theta$  imply  $G(0) > 0$ , so S3 dominates both S1 and S2 if and only if (14) and (15) hold, or equivalently

$$\frac{V_q(1, \theta_L) - c'(1)}{V_q(1, \theta_H) - c'(1)} < \frac{n_H}{n_H + n_L}, \quad (16)$$

and

$$\frac{V(\hat{\phi}, \theta_L) - c(\hat{\phi})}{V(\hat{\phi}, \theta_H) - c(\hat{\phi})} > \frac{n_H}{n_H + n_L}, \quad (17)$$

for some  $\hat{q}$ . Clearly a necessary condition for (16) and (17) to hold simultaneously is

$$\frac{V_q(1, \theta_L) - c'(1)}{V(\hat{\phi}, \theta_L) - c(\hat{\phi})} < \frac{V_q(1, \theta_H) - c'(1)}{V(\hat{\phi}, \theta_H) - c(\hat{\phi})}, \quad (18)$$

for some  $\hat{\phi}$ .

Equation (18) defines the interval  $N^*$ . So, if  $V(q, \theta) - c(q)$  is everywhere log submodular then (18) cannot hold,  $N^*$  is empty, conditions (16) and (17) cannot both be satisfied, and either strategy S1 or S2 dominates strategy S3. That is, the firm produces only a high quality product.

If  $V(q, \theta) - c(q)$  is log supermodular on  $\{\theta_L, \theta_H\} \times \{\hat{q}, 1\}$  for some  $\hat{q} < 1$  then (18) holds,  $N^*$  is non-empty, and (16) and (17) both hold for all  $n_L$  and  $n_H$  such that

$\frac{n_H}{n_H + n_L} \in N^*$ , and strategy S3 dominates both strategies S1 and S2. That is, the firm

offers both a high and low quality product.

### **Proof of Proposition 2:**

When  $q^*(\theta)$  is increasing,  $V(\theta, q) - c(q)$  it follows that

$$S_q(\theta, 1) = V_q(1, \theta) - c'(1) < 0, \quad S_{\theta q} > 0, \quad \text{and} \quad S_\theta > 0, \quad \text{so}$$

$S_{\theta q}(\theta, 1)S(\theta, 1) - S_\theta(\theta, 1)S_q(\theta, 1) > 0$  and  $V(\theta, q) - c(q)$  is log supermodular in a neighborhood on 1.



**Lemma:** *If  $V(\theta, q) - c(q)$  is log supermodular in a neighborhood on 1, then the firm offers multiple products.*

Suppose the firm offers a single product quality, so  $q(\underline{\theta}) = 1$ . Recall that either (6) holds, that is  $J(\theta_L, q(\theta_L)) = 0$ , or  $\theta_L = \underline{\theta}$ .

First, consider the case in which  $\theta_L = \underline{\theta}$ . Equation (3) and  $q(\underline{\theta}) = 1$  imply  $J(\underline{\theta}, q(\underline{\theta})) > 0$ , which implies  $\theta_L > \underline{\theta}$ , so if  $\theta_L = \underline{\theta}$  then  $q(\underline{\theta}) < 1$  which is a contradiction.

Second, consider the case in which  $J(\theta_L, q(\theta_L)) = 0$ . Then

$$\left[ V(1, \underline{\theta}) - c(1) - \left[ \frac{1 - F(\underline{\theta})}{f(\underline{\theta})} \right] \frac{\partial V(1, \underline{\theta})}{\partial \theta} \right] = 0 \quad (19)$$

uniquely defines  $\underline{\theta}$ . Because  $H(\theta, q(\theta)) \geq 0$  where  $q(\theta) = 1$  we also have

$$\frac{\partial V(1, \underline{\theta})}{\partial q} - c'(1) - \left[ \frac{1 - F(\underline{\theta})}{f(\underline{\theta})} \right] \frac{\partial^2 V(1, \underline{\theta})}{\partial q \partial \theta} \geq 0. \quad (20)$$

Equations (19) and (20) imply

$$\frac{\partial(V(1, \underline{\theta}) - c(1))}{\partial \theta} \frac{\partial(V(1, \underline{\theta}) - c(1))}{\partial q} \geq (V(1, \underline{\theta}) - c(1)) \frac{\partial^2(V(1, \underline{\theta}) - c(1))}{\partial q \partial \theta}, \quad (21)$$

which implies  $\frac{\partial^2 \ln(V(1, \underline{\theta}) - c(1))}{\partial \theta \partial q} \leq 0$ . So if  $V(1, \underline{\theta}) - c(1)$  is log supermodular,

offering a single product cannot be optimal.

**Proof of Proposition 3:**

a) From (5), and our assumptions on  $H$ , the quality sold to the lowest-type buyer served is lower than the quality sold to the highest type buyer, i.e.,  $q(\theta_L) < q(\bar{\theta}) = 1$ , if and only if  $H(\theta_L, 1) < 0$ , or

$$\frac{\partial V(1, \theta_L)}{\partial q} - c'(1) - \left[ \frac{1 - F(\theta_L)}{f(\theta_L)} \right] \frac{\partial^2 V(1, \theta_L)}{\partial q \partial \theta} < 0. \quad (22)$$

The lowest-type buyer served is  $\theta_L$  only if  $J(\theta_L, q(\theta_L)) \leq 0$ . Together

$J(\theta_L, q(\theta_L)) \leq 0$  and (22) imply that  $q(\theta_L) < 1$  only if

$$\frac{\partial V(q(\theta_L), \theta_L)}{\partial \theta} \left[ \frac{\partial V(1, \theta_L)}{\partial q} - c'(1) \right] < [V(q(\theta_L), \theta_L) - c(q(\theta_L)))] \frac{\partial^2 V(1, \theta_L)}{\partial q \partial \theta}. \quad (23)$$

Inequality (23) can be re-written as

$$\begin{aligned} & \frac{\frac{\partial V(q(\theta_L), \theta_L)}{\partial \theta}}{\frac{\partial V(1, \theta_L)}{\partial \theta}} \left[ \left( \frac{\partial V(1, \theta_L)}{\partial q} - c'(1) \right) \frac{\partial V(1, \theta_L)}{\partial \theta} \right] \\ & < \frac{V(q(\theta_L), \theta_L) - c(q(\theta_L))}{V(1, \theta_L) - c(1)} \left[ (V(1, \theta_L) - c(1)) \frac{\partial^2 V(1, \theta_L)}{\partial q \partial \theta} \right], \end{aligned} \quad (24)$$

which implies either

$$\frac{\partial V(1, \theta_L)}{\partial \theta} \left( \frac{\partial V(1, \theta_L)}{\partial q} - c'(1) \right) < (V(1, \theta_L) - c(1)) \frac{\partial^2 V(1, \theta_L)}{\partial q \partial \theta} \quad (25)$$

or

$$\frac{\partial V(q(\theta_L), \theta_L)}{\partial \theta} / \frac{\partial V(1, \theta_L)}{\partial \theta} < \frac{V(q(\theta_L), \theta_L) - c(q(\theta_L))}{V(1, \theta_L) - c(1)}. \quad (26)$$

Equations (25) and (26) can be rewritten as

$$\begin{aligned} & \frac{\partial(V(1, \theta_L) - c(1))}{\partial \theta} \left( \frac{\partial(V(1, \theta_L) - c(1))}{\partial q} \right) \\ & < (V(1, \theta_L) - c(1)) \frac{\partial^2(V(1, \theta_L) - c(1))}{\partial q \partial \theta} \end{aligned} \quad (27)$$

and

$$\frac{\partial(V(q(\theta_L), \theta_L) - c(q(\theta_L)))}{\partial \theta} \bigg/ \frac{\partial(V(1, \theta_L) - c(1))}{\partial \theta} < \frac{V(q(\theta_L), \theta_L) - c(q(\theta_L))}{V(1, \theta_L) - c(1)}. \quad (28)$$

But we shall now see that neither equation (27) nor (28) holds unless  $V(\hat{q}, \theta_L) - c(\hat{q})$  is log supermodular for some  $\hat{q}$ . Clearly (27) holds if and only if  $V(\hat{q}, \theta_L) - c(\hat{q})$  is

log supermodular. Similarly (28) holds if only if  $\frac{V(1, \theta_L) - c(1)}{V(q(\theta_L), \theta_L) - c(q(\theta_L))}$  is

increasing in  $\theta$  at  $\theta = \theta_L$ . But if  $\frac{\partial^2 \ln(V(\hat{\phi}, \theta_L) - c(\hat{\phi}))}{\partial \theta \partial q} \leq 0$  for all  $\hat{q}$  then

$$\begin{aligned} \int_{\hat{\phi}}^1 \frac{\partial^2 \ln(V(q, \theta_L) - c(q))}{\partial \theta \partial q} dq &= \frac{\partial(\ln(V(1, \theta_L) - c(1)) - \ln(V(\hat{\phi}, \theta_L) - c(\hat{\phi})))}{\partial \theta} \\ &= \frac{\partial \ln \left( \frac{V(1, \theta_L) - c(1)}{V(\hat{\phi}, \theta_L) - c(\hat{\phi})} \right)}{\partial \theta} \leq 0 \end{aligned} \quad (29)$$

so (28) cannot hold. Therefore neither (27) nor (28) holds unless

$V(\hat{q}, \theta_L) - c(\hat{q})$  is log supermodularity for some  $\hat{q}$ . That is, log supermodularity of  $V(\hat{q}, \theta_L) - c(\hat{q})$  for some  $\hat{q}$  is a necessary condition for the firm to sell multiple products.

Finally, since  $S_{qq\theta}(q, \theta) > 0$  and  $S_{qq}(q, \theta) < 0$ ,

$S(1, \theta)S_{q\theta}(1, \theta) - S_q(1, \theta)S_\theta(1, \theta) < 0$  for all  $\theta$  implies that

$S(q, \theta)S_{q\theta}(q, \theta) - S_q(q, \theta)S_\theta(q, \theta)$  is increasing in  $q$ , and so

$S(q, \theta)S_{q\theta}(q, \theta) - S_q(q, \theta)S_\theta(q, \theta) < 0$  for all  $q \leq 1$ . In other words, if  $V(1, \theta) - c(1)$  is log submodular then  $V(q, \theta) - c(q)$  is log submodular for all  $\theta$  and all  $q \leq 1$ . So log submodularity of  $V(1, \theta_L) - c(1)$  implies that the firm will not sell multiple products.

b) See the lemma in the proof of Proposition 2.

**Proof of Proposition 4:**

The firm's optimal product line is described by  $H(\theta, q(\theta)) = 0$  where  $0 < q(\theta) < 1$ ,

$H(\theta, q(\theta)) \geq 0$  where  $q(\theta) = 1$ , and  $H(\theta, q(\theta)) \leq 0$  where  $q(\theta) = 0$ . First, it is clear these imply  $q(\bar{\theta}) = q^*(\bar{\theta}) = 1$ . Second,  $q(\theta) < q^*(\theta)$  whenever  $0 < q^*(\theta) < 1$  follows from  $H(\theta, q(\theta)) = 0$ . Finally,  $q(\theta) \leq q^*(\theta)$  whenever  $q^*(\theta) = 1$  follows from the constraint, so  $q(\theta) \leq q^*(\theta), \forall \theta$ .

**Proof of Proposition 5:**

If a seller is restricted to offering a single quality, it will sell only high quality. Also, it will sell exclusively to the high types if and only if

$$\bar{n} (V(1, \bar{\theta}) - c(1)) > (\bar{n} + \underline{n}) (V(1, \underline{\theta}) - c(1))$$

or

$$\frac{V(1, \underline{\theta}) - c(1)}{V(1, \bar{\theta}) - c(1)} < \frac{\bar{n}}{\bar{n} + \underline{n}}. \quad (30)$$

Log supermodularity implies

$$\frac{\partial V(1, \underline{\theta}) / \partial q - c'(1)}{\partial V(1, \bar{\theta}) / \partial q - c'(1)} < \frac{V(1, \underline{\theta}) - c(1)}{V(1, \bar{\theta}) - c(1)} < \frac{V(\underline{\phi}, \underline{\theta}) - c(\underline{\phi})}{V(\underline{\phi}, \bar{\theta}) - c(\underline{\phi})}, \quad (31)$$

so in the subinterval  $\left[ \frac{V(1, \underline{\theta}) - c(1)}{V(1, \bar{\theta}) - c(1)}, \frac{V(\underline{\phi}, \underline{\theta}) - c(\underline{\phi})}{V(\underline{\phi}, \bar{\theta}) - c(\underline{\phi})} \right]$  of  $N^*$  allowing price

discrimination results in a Pareto improvement. That is, it weakly increases seller profits by revealed preference, weakly increases type  $\underline{\theta}$  buyers' consumer surplus because they were not previously served, and strictly increases type  $\bar{\theta}$  buyers' consumer surplus from zero to something positive because their incentive compatibility constraint strictly binds. QED.

## 8. References

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