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**Conference Paper**

## The Theory of Incentive Mechanisms and the Samuelson Critique of a Contractarian Approach to Public-Good Provision

Beiträge zur Jahrestagung des Vereins für Socialpolitik 2010: Ökonomie der Familie - Session: Optimal Monitoring and Mechanism Design, No. B17-V3

**Provided in Cooperation with:**

Verein für Socialpolitik / German Economic Association

*Suggested Citation:* Bierbrauer, Felix; Hellwig, Martin (2010) : The Theory of Incentive Mechanisms and the Samuelson Critique of a Contractarian Approach to Public-Good Provision, Beiträge zur Jahrestagung des Vereins für Socialpolitik 2010: Ökonomie der Familie - Session: Optimal Monitoring and Mechanism Design, No. B17-V3, Verein für Socialpolitik, Frankfurt a. M.

This Version is available at:

<https://hdl.handle.net/10419/37420>

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# The Theory of Incentive Mechanisms and the Samuelson Critique of a Contractarian Approach to Public-Good Provision\*

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December 2009. Preliminary Version. Please do not circulate.

## Abstract

The assessment that the implementation of efficient outcomes by means of “decentralized spontaneous solutions” is to be expected for private goods, but not for public goods has become part of the conventional wisdom of our discipline. This paper uses a mechanism design approach to clarify under which conditions this proposition is indeed correct. The main result is that the following has to be satisfied: There is a large number of individuals and only mechanisms which are robust, in the sense that they do not exploit assumptions about individuals’ probabilistic beliefs, are considered.

*Keywords:* Public Goods, Private Goods, Robust Mechanism Design, Large Economy

*JEL:* D60, D70, D82, H41

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\*We benefited from conversations with Alia Gizatulina.

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# 1 Introduction

The concept of a public good as one that involves nonrivalry in consumption was introduced by Lindahl (1919) in an attempt to provide a contractarian foundation for studying the allocative role of the state. When Samuelson (1954, 1955) revived the concept, he went out of his way to criticize the contractarian approach, and emphasized that efficient provision of a public good is not to be expected by means of “decentralized spontaneous solutions”. According to Samuelson, incentive and information problems for public goods are quite different from those for private goods. Because of the jointness of the benefits that different people obtain from a public good, the criteria for provision cannot be decomposed in a manner that lends itself to decentralization. Moreover, the requisite elicitation of preferences is impeded by the free-rider problem. Whereas, for private goods, such problems disappear when there are many participants so that markets may be deemed to be competitive, for public goods, a larger number of participants does not help at all.

The assessment that the implementation of efficient outcomes by means of “decentralized spontaneous solutions” is to be expected for private goods, but not for public goods has become part of the conventional wisdom of our discipline.<sup>1</sup> In the following, we shall refer to this assessment as the *Samuelson Critique*.

The argument that Samuelson gave in support of his critique was entirely informal. Since then, the theory of incentive mechanisms has provided precise methods for studying the information and incentive problems that stand in the way of efficiency. This literature, however, tends to look at different specifications in isolation. There is a bewildering variety of possibility theorems and impossibility theorems for *both*, public *and* private goods. There does not seem to be a systematic overall account of what the theory of incentive mechanisms has to say about the Samuelson Critique - and what the Samuelson Critique can teach us about the theory of incentive mechanisms.

This paper proposes to fill the gap. Taking stock of the various possibility and impossibility theorems, we ask to what extent and in what sense precisely the theory of incentive mechanisms confirms the view that “decentralized spontaneous solutions” are available for the efficient provision of private goods, but *not* for the efficient provision of public goods? We will not provide any really new theorems. Our purpose is only to put some order into the array of existing theorems and to deepen our understanding of why precisely the difference between goods that exhibit nonrivalry in consumption and goods that do not have this property should be important for

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<sup>1</sup>See, e.g., Musgrave (1959), Atkinson-Stiglitz (1980), or Hindriks and Myles (2006).

efficient implementation.

In the absence of participation constraints, efficient implementation is possible regardless of whether we are concerned about public or private goods.<sup>2</sup> However, regimes that permit efficient implementation may violate participation constraints, i.e., people may have to be coerced into participating. Mechanisms that rely on such coercion can hardly be interpreted as “decentralized spontaneous solutions” to the allocation problem. Any contractarian approach to the solution of allocation problems must involve an element of voluntariness - for a given assignment of initial property rights. We therefore restrict our analysis to mechanisms that people would voluntarily submit to.

If participation constraints are imposed, the Bayesian approach to mechanism design tells us that, whether efficient implementation is possible or not, depends on the specification of the information variables, rather than the nature of the goods involved. In a model with  $n$  participants, with private information about their preferences, efficient implementation is typically impossible if the different agents’ preferences are stochastically independent, and efficient implementation is typically possible if the different agents’ preferences are correlated. The distinction has nothing to do with the distinction between public and private goods.<sup>3</sup>

To confirm the Samuelson Critique in the Bayesian approach, one has to assume independent private values and to consider models with a large number of participants. In a model with  $n$  participants having independent private values, efficient implementation is typically impossible for private as well as public goods. However, for private goods, there are mechanisms ensuring that efficiency losses are small if  $n$  is large. For public goods, there are no such mechanisms; if  $n$  is large and the provision cost of the public good is roughly proportional to  $n$ , it is not even possible to have a significant level of public-good provision at all without violating participation constraints.

The difference between the large-numbers results for private and for public goods reflects the different roles of the number of participants in the two settings. For private goods, with rivalry in consumption, an increase in  $n$  means that there is more competition. The market power

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<sup>2</sup>For Bayesian implementation, this has been shown for independent private value models by d’Aspremont and Gérard-Varet (1979) and d’Aspremont et al. (2004) and for models with correlated values by Crémer and McLean (1988) and Kosenok and Severinov (2008). For implementation in dominant strategies, it follows from Clarke (1971) and Groves (1973) that efficient decision rules can be implemented. While this may necessitate a budget surplus, this surplus can be made arbitrarily small with a large number of participants, see Green and Laffont (1977, 1979).

<sup>3</sup>Well-known impossibility results are due to Myerson and Satterthwaite (1983) for private goods, and Güth and Hellwig (1986) and Mailath and Postlewaite (1990) for public goods.

effects that impede efficient provision of private goods in finite economies are therefore the less important the more participants there are. In contrast, for public goods, with nonrivalry in consumption, there is no competition effect of an increase in  $n$ , only a strengthening of the free-rider effect because, the more participants there are, the less likely each participant considers the prospect of his having an effect on the level of public-good provision. This strengthening of the free-rider effect can be neutralized if the fact that many people benefit from the public good provides sufficient scale economies. However, if there are no scale economies in public-good provision, in particular, if the *per capita* costs of public-good provision are independent of  $n$ , the scope for public-good provision without coercion becomes ever smaller as  $n$  increases. The assumption of sufficient scale economies is met if we think of the public good in question as a research finding; the assumption is not met if we think of the public good as the reliability of the legal system. In this case, the Samuelson Critique of the contractarian approach to public-good provision is fully supported by a Bayesian analysis of models with many participants with independent private values.

In such models, however, the question of how much of the public good should be provided is moot. For the limiting case where  $n$  goes out of bounds, the law of large numbers implies that the cross-section distribution of preferences in the economy is (converges to) a nonrandom constant, which is common knowledge. Knowing this cross-section distribution, the mechanism designer can infer the efficient level of public-good provision, and the only problem left is to determine which participants value the public good highly and should be made to contribute a lot.

For the question of how a society finds out about the efficient level of public-good provision, one must either look at models with finitely many participants, where it is hard to assess the difference between public and private goods, or one must look at models with correlated values. For instance, we may think of a common underlying parameter so that the participants' preferences are conditionally independent given this parameter, but this parameter itself affects the entire distribution of preferences in the economy. In such a setting, the efficient level of public-good provision would depend on the realization of the underlying parameter and would not be common knowledge. For such specifications with correlated private values, however, the literature provides us with various possibility theorems showing that, even if participation constraints are imposed, efficient implementation is possible regardless of the number of participants and regardless of whether we are concerned with public or private goods.<sup>4</sup>

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<sup>4</sup>A possibility result for a private goods model is obtained by Crémer and McLean (1988), and extended to a public goods model by Kosenok and Severinov (2008).

Thus, in the Bayesian approach to mechanism design, the Samuelson Critique is valid if and only if the analysis is limited to models with many participants with independent private values, where the question of how much of the public good should be provided is moot.

As a main result, the paper shows that the Samuelson Critique is universally valid in models with large numbers of participants if we insist on dominant-strategy implementation or, equivalently, if we require incentive mechanisms to be robust in the sense of Bergemann and Morris (2005) or Ledyard (1979). For robust Bayesian implementation or for dominant-strategy implementation with participation constraints, we get qualitatively the same results as for Bayesian implementation with independent private values, namely (i) with finitely many participants, efficient implementation is impossible regardless of whether we are dealing with public or with private goods; (ii) if the number of participant is large, then, with private goods, one can achieve approximate efficiency, and, with public goods, with provision costs proportional to the number of participants, it is not possible have a significant level of public-good provision at all. Hence, if we assess a formal framework for the analysis of public goods by asking whether it delivers the propositions that most economists would expect from a pure theory of public goods, our analysis gives rise to the conclusion that robust mechanism design is the “right” approach.

More specifically, the paper proceeds as follows.

In a first step we revisit the independent private values model. We first show that, with finitely many individuals, allocation problems involving public goods and allocation problems involving private goods have the same qualitative properties: depending on the properties of the underlying probability distribution, efficiency may or may not be compatible with incentive, participation and resource constraints. However, as we let the number of individuals become large the probability of an efficient outcome converges to 0 with a public goods problem and to 1 with a private goods problem. Hence, with an independent private values model, the Samuelson critique is justified, provided that the number of individuals is large.<sup>5</sup>

What tells us that independent private values are the appropriate stochastic specification? This is an important question because, with correlated private values, efficient outcomes are always possible. We state, in a second step, that this holds true for public as well as private goods, and does not depend on whether the number of individuals is large or not.<sup>6</sup>

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<sup>5</sup>Our analysis in this part reproduces some results that are already known, in particular an impossibility result due to Mailath and Postlewaite (1990), and also contains some extensions. Its main purpose is to set the stage so that the results from our robust mechanism design approach are more easily interpreted.

<sup>6</sup>This part of the analysis briefly recalls results from the literature on the possibility of surplus extraction

The main part of the analysis is concerned with a model of robust mechanism design.<sup>7</sup> For this part, we do not impose any assumption about the beliefs of individuals. They may or may not be derived from a common prior, be consistent with an independent values model, or display a correlation between individuals' privately held information. We require a mechanism to be robust in the sense that it satisfies incentive and participation constraints, whatever the beliefs of individuals are.

We evaluate the performance of such a mechanism from the perspective of a mechanism designer who has subjective beliefs about the data generating process. We impose a mild assumption of “conditional independence” on these beliefs which makes it possible to evaluate the performance of mechanisms as we let the number of individuals go to infinity. In particular, if the mechanism designer's beliefs were derived from a common prior, then “conditional independence” contains both independent and correlated value environments as special cases. Hence, this assumption is innocuous with respect to the possibility and impossibility theorems that are available in independent, or correlated value models, respectively.

Our main result is as follows: Robust mechanism design gives the same conclusions as the independent private values model. The analysis whether or not efficiency is possible is inconclusive with finitely many individuals. This holds true both for an allocation problem involving public goods and for one with private goods. However, as the number of individuals becomes large we get efficiency in a private goods model, but not in a public goods model. Thus, the findings in the independent private values model are robust, even if the stochastic specification is “special”. By contrast, the findings based on correlated values are as “special” as the stochastic specification.<sup>8</sup>

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(in particular, Crémer and McLean (1988), and Kosenok and Severinov (2008)) and adapts them to the given framework.

<sup>7</sup>See Ledyard (1978) and Bergemann and Morris (2005).

<sup>8</sup>By “special” we simply mean that assumptions about the stochastic specification of the model are crucial for the results. An alternative approach would be to use a mathematical notion of genericity to shed light on the question which stochastic specifications are “general” and which ones are “special”. Kosenok and Severinov (2008) employ a notion of genericity so that the independent private values model turns out to be “special”, whereas the correlated values model is “general.” Heifetz and Neeman (2006) have an extended description of an individual's characteristics, which lends itself to a notion of genericity so that the stochastic specification of Crémer and McLean (1988) and Kosenok and Severinov (2008) looks “special”. Using a stochastic specification that is generic in the sense of Heifetz and Neeman (2006), Neeman (2004) obtains similar conclusion as we do for a public goods model. This suggests that a robust mechanism design approach and a Bayesian mechanism design approach with priors that are generic in the sense of Heifetz and Neeman (2006), may give rise to the same conclusions. A clarification of this point is left to future research.

As a final step, we show that the inefficiency result in the public goods model is really driven by the requirement of voluntary participation. We characterize an optimal robust mechanism that has to satisfy only incentive and resource constraints and show that it approximates efficient outcomes if the number of individuals is large. Hence, the impossibility result in the public goods model is really due to the requirement that no individual is made worse off as compared to a status quo situation with no public goods provision. If coercion is possible, then an efficient allocation of public goods can be reached in a robust way.<sup>9</sup>

The remainder of the paper is organized as follows. Section specifies the model. In Section 3 we study public and private goods provision in an independent private values environment. Section 4 states results for the case of correlated values. The robust mechanism design approach is analyzed in Section 5.

## 2 Model

### 2.1 Preferences and Beliefs

There is a set of  $n$  individuals, denoted by  $I^n = \{1, \dots, n\}$ . Individual  $i$  has preferences  $u_i = \theta q_i - t_i$ , where  $t_i$  is a monetary payment, and  $q_i$  is  $i$ 's consumption of a private or public good. The taste parameter takes values in a finite ordered set  $\Theta = \{\theta^0, \theta^1, \dots, \theta^m\}$ , with  $\theta^0 = 0$ ,  $\theta^1 = 1$ , etc. Individuals have private information on their taste parameter.

We denote the vector of all taste parameters by  $\theta := (\theta_1, \dots, \theta_n)$ , and write  $\theta_{-i}$  for a vector that lists all taste parameters except  $\theta_i$ . Individual  $i$  has probabilistic beliefs about  $\theta_{-i}$ . Possibly these beliefs vary with an individual's taste parameter. We denote by  $p_i^k(\theta_{-i})$  the probability that individual  $i$  assigns to  $\theta_{-i}$ , given that  $\theta_i = \theta^k$ .<sup>10</sup> Formally, we can represent the beliefs of type  $k$  of individual  $i$  by a vector  $p_i^k$  with length  $(\#\Theta)^{n-1} = m^{n-1}$  whose entries lie between 0 and 1 and add up to 1. We collect the beliefs of  $i$ 's different types in the  $(m \times m^{n-1})$ -matrix  $P^i = (p_i^1, \dots, p_i^m)'$ .

We assume that there is mechanism designer who assigns a probability of  $p(\theta)$  to the vector  $\theta$ . We use the expectations operator  $E$  to indicate that expectations are taken with respect to

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<sup>9</sup>Similar results were previously obtained by Green and Laffont (1979), albeit with a different approach. Green and Laffont showed this by constructing a mechanism that relies on sampling, i.e., a mechanism so that only a subset of the population is used for preference elicitation, and the remainder of the population is used to balance the budget. Our approach is different in that we characterize the optimal robust mechanism directly.

<sup>10</sup>In Section 5 on robust mechanism design we use a more flexible framework that does not impose any a priori restriction on the relation between beliefs and preferences.



the mechanism designer's probability measure  $p$ . We say that the common prior assumption holds if an individual's beliefs can be derived from  $p$  by conditioning on the individual's private information; i.e., for every  $i$ , every  $k$ , and every  $\theta_{-i}$ ,

$$p_i^k(\theta_{-i}) = p(\theta \mid \theta_i = \theta^k),$$

where the right hand side is the probability that the vector of taste parameters equals  $\theta$  conditional on  $i$ 's taste parameter being equal to  $\theta^k$ ; i.e.,

$$p(\theta \mid \theta_i = \theta^k) := \frac{p(\theta)}{\sum_{\{\theta' \mid \theta'_i = \theta^k\}} p(\theta')}.$$

## 2.2 Social choice functions

A social choice function consists of a decision rule  $\hat{q}_i^n : \theta \mapsto \hat{q}_i^n(\theta)$  and a payment rule  $\hat{t}_i^n : \theta \mapsto \hat{t}_i^n(\theta)$ ,  $i \in I^n$ , which prescribes a decision and a payment for each individual  $i$  as a function of the vector of preference parameters  $\theta$ .

A social choice function is said to be efficient if, for all  $\theta$ , the budget condition holds as an equality,

$$\sum_{i=1}^n \hat{t}_i^n(\theta) = K^n(\hat{q}_1^n(\theta), \dots, \hat{q}_n^n(\theta)), \quad (1)$$

and the decision rule is surplus maximizing, i.e.,

$$(\hat{q}_1^n(\theta), \dots, \hat{q}_n^n(\theta)) \in \operatorname{argmax}_{(x_1, \dots, x_n)} \sum_{i=1}^n \theta_i x_i - K^n(x_1, \dots, x_n), \quad (2)$$

where  $K^n(q_1^n(\theta), \dots, q_n^n(\theta))$  is the resource cost associated with decision  $(q_1^n(\theta), \dots, q_n^n(\theta))$ . We assume that  $K^n(0, \dots, 0) = 0$ .

## 2.3 Mechanisms

We seek to implement a social choice function by means of a mechanism. We appeal to the revelation principle and limit attention to direct mechanisms and to truth-telling equilibria. A direct mechanism consists of a decision rule  $q_i^n : \theta \mapsto q_i^n(\theta)$  and a payment rule  $t_i^n : \theta \mapsto t_i^n(\theta)$ ,  $i \in I^n$ , which specify outcomes as a function of a profile of announced preferences.

We require that a mechanism satisfies incentive, participation and resource constraints. The incentive constraints ensure that truth-telling constitutes a Bayes-Nash equilibrium: For all  $i$ , for all  $l$  and for all  $k$ ,

$$V_i^n(\theta^l, \theta^l) \geq V_i^n(\theta^l, \theta^k),$$

where

$$V_i^n(\theta^l, \theta^k) := \sum_{\theta_{-i} \in \Theta^{n-1}} p_i^l(\theta_{-i}) \left( \theta^l q_i^n(\theta_{-i}, \theta^k) - t_i^n(\theta_{-i}, \theta^k) \right) .$$

The participation constraints are satisfied provided that an individual's expected payoff does not fall short off a default utility level that would be reached if there was no mechanism at all. This default utility level is normalized to 0. Consequently, the participation constraints require that, for all  $i$ , and for all  $\theta_i$ ,

$$V_i^n(\theta_i, \theta_i) \geq 0 .$$

A mechanism is said to be weakly feasible if, from the mechanism designer's perspective, expected payments suffice to cover the expected costs. Formally,

$$E \left[ \sum_{i=1}^n t_i^n(\theta) \right] \geq E[K^n(q_1^n(\theta), \dots, q_n^n(\theta))] , \quad (3)$$

A social choice function  $(\hat{q}_i^n, \hat{t}_i^n)_{i=1}^n$  is said to be implementable if there is a mechanism  $(q_i^n, t_i^n)_{i=1}^n$  that is incentive compatible, weakly feasible, satisfies the participation constraints, and achieves the given social choice function, i.e., for all  $\theta$ , and all  $i$ ,

$$\hat{q}_i^n(\theta) = q_i^n(\theta) \quad \text{and} \quad \hat{t}_i^n(\theta) = t_i^n(\theta) .$$

A mechanism that achieves an efficient social choice function is henceforth called an efficient mechanism. A mechanism that is weakly feasible and achieves a surplus maximizing social choice function is called weakly efficient.

Following the literature, our analysis focusses on the question whether weakly efficient outcomes can be achieved subject to incentive and participation constraints. We will, however, repeatedly comment on the implications that the results have for the possibility to reach full efficiency.

## 2.4 Public Goods

In the public goods model, a decision decision means that an indivisible, non-excludable public good is either provided or not. Formally, for each  $\theta$ , there is a number  $q^n(\theta) \in \{0, 1\}$  so that  $q_i^n(\theta) = q^n(\theta)$ , for all  $i$ .

We assume that the cost function exhibits constant returns to scale and that per capita costs do not depend on the number of individuals. Formally, there exists  $k$  so that

$$\frac{1}{n} K^n(q_1^n(\theta), \dots, q_n^n(\theta)) = k q^n(\theta) ,$$

for all  $n$ .

A surplus maximizing decision rule is as follows:  $q^n(\theta) = 1$  if and only if  $\frac{1}{n} \sum_{i=1}^n \theta_i \geq k$ . We assume that  $0 < k < \theta^m$ . Hence, public goods provision is desirable if all individuals have the highest possible valuation and undesirable if all individuals have the lowest possible valuation.

## 2.5 Private Goods

In the private goods model, a decision determines (i) whether or not an indivisible private good is produced, and (ii) which individual gets the good if production takes place. Formally,  $q_i^n(\theta) \in [0, 1]$ , for all  $i$ , and  $\sum_{i=1}^n q_i^n(\theta) \in [0, 1]$ , where  $q_i^n(\theta)$  is interpreted as the probability that individual  $i$  gets the good if the vector of taste parameters equals  $\theta$ .

We assume a constant returns to scale technology with marginal costs  $c$  that do not depend on the number of individuals. Formally, there exists  $c$  so that

$$K^n(q_1^n(\theta), \dots, q_n^n(\theta)) = c \sum_{i=1}^n q_i^n(\theta).$$

Surplus maximization then requires that  $q_i^n(\theta) = 1$ , for some  $i$ , if and only if  $\max_{i \in I^n} \theta_i \geq c$ , and  $q_i^n(\theta) = 1$  only if  $\theta_i \geq \theta_j$ , for all  $j \neq i$ . We assume that  $0 < c < \theta^m$ .

## 3 Independent private values

An independent private values model is a special case of the above environment which satisfies the following properties: There is a common prior, and the random variables  $(\theta_i)_{i=1}^n$  are independent and identically distributed according to a distribution with density  $f = (f(\theta^0), f(\theta^1), \dots, f(\theta^m))$ , and cumulative distribution function  $F$ , with  $F(\theta^k) = \sum_{l=0}^k f(\theta^l)$ . The independence assumption implies, in particular, that all types of individual  $i$  have the same probabilistic beliefs about  $\theta_{-i}$ , i.e., for all  $l$  and  $k$ ,  $p_i^l(\theta_{-i}) = p_i^k(\theta_{-i}) =: p_i(\theta_{-i})$ .

Our results below focus on the possibility to achieve weakly efficient outcomes.<sup>11</sup> We will first derive necessary and sufficient conditions that apply both to a public and a private goods model. Subsequently, we show that, with many individuals, these conditions are violated in a public goods models but not in a private goods model. By contrast, with a small number of individuals, efficiency may or may not be possible in either setting.

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<sup>11</sup>With an independent private values model, if weak efficiency is compatible with incentive and participation constraints, then the same holds for full efficiency, see Börgers and Norman (2009).

**Proposition 1** *In the public goods model and in the private goods model, weak efficiency is compatible with incentive compatibility and voluntary participation if and only if*

$$E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) q_i^{n*}(\theta) - K^n(q_1^{n*}(\theta), \dots, q_n^{n*}(\theta)) \right] \geq 0, \quad (4)$$

where  $(q_i^{n*})_{i=1}^n$  is a surplus maximizing decision rule.

**Proof** See Appendix. ■

The proof uses arguments that are familiar from the analysis of non-linear pricing problems. We first derive an upper bound on the expected payments of individuals by considering the following relaxed revenue maximization problem: For an arbitrary, but given decision rule  $(q_i^n)_{i=1}^n$  maximize expected per capita revenues  $E[\frac{1}{n} \sum_{i=1}^n t_i^n(\theta)]$  subject to local downward incentive compatibility constraints which prevent an individual with a  $\theta^k$ -preference to declare a  $\theta^{k-1}$ -preference and the participation constraints for individuals with a  $\theta^0$ -preference. Since these constraints are a subset of all incentive and participation constraints the resulting per capita revenue is an upper bound on the per capita revenue that can be raised if all constraints are taken into account. This upper bound is shown to be equal to  $E \left[ \frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) q_i^n(\theta) \right]$ . Second, we exploit the single crossing condition in order to show that if the given decision rule satisfies a monotonicity condition, then this upper bound can actually be achieved. Finally, we verify that this monotonicity condition is satisfied by a surplus maximizing decision rule, both in the public goods model and in the private goods model. Consequently, expected budget balance is possible if and only if the maximal payments  $E \left[ \frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) q_i^{n*}(\theta) \right]$  exceed the per capita cost of efficient public goods provision  $E \left[ \frac{1}{n} K^n(q_1^{n*}(\theta), \dots, q_n^{n*}(\theta)) \right]$ .

### 3.1 Public Goods

The following Proposition shows that, for the public goods model with a fixed number of individuals, the question whether weak efficiency is possible or not is inconclusive in the sense that there are both specifications of the probability distribution  $f$  so that the condition (4) holds, and others such that it is violated. Hence, unless one is prepared to make assumptions about the distribution  $f$ , one does not know whether efficiency is possible.<sup>12</sup>

**Proposition 2** *Given the public goods model and fixed number  $n$  of individuals. There exists a probability distribution  $f$  such that condition (4) holds and a probability distribution  $f'$  such*

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<sup>12</sup>For a proof that efficiency is out of reach if  $f$  is an atomless distribution, see Hellwig (2001).

that condition (4) fails.

**Proof** In a public goods model, condition (4) can be written as

$$E \left[ \left( \frac{1}{n} \sum_{i=1}^n \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - k \right) q^{n^*}(\theta) \right] \geq 0, \quad (5)$$

Fix the number of individuals  $n$  and observe that  $\theta^m - \frac{1 - F(\theta^m)}{f(\theta^m)} = \theta^m > k$ . Hence, as  $f(\theta^m)$  converges to 1,  $E \left[ \left( \frac{1}{n} \sum_{i=1}^n \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - k \right) q^{n^*}(\theta) \right]$  converges to  $\theta^m - k$ , so that condition (4) is fulfilled.

Alternatively, consider  $f$  such that there is  $\theta^l > k$  with  $\theta^l - \frac{1 - F(\theta^l)}{f(\theta^l)} < k$ . Consequently, if  $f(\theta^l)$  is sufficiently large  $E \left[ \left( \frac{1}{n} \sum_{i=1}^n \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - k \right) q^{n^*}(\theta) \right]$  is negative, implying that weak efficiency is not possible. ■

By contrast, as we let the number of individuals go out of bounds, we obtain a conclusive answer: The probability of public goods provision goes to zero, irrespective of whether or not provision would be efficient.

**Proposition 3** *Let  $((t_i^n)_{i=1}^n, q^n)_{n=1}^\infty$  be a sequence of mechanisms that are incentive compatible, weakly feasible and satisfy participation constraints, with the property that the associated sequence of provision probabilities  $(\rho^n)_{n=1}^\infty$  converges to a limit probability  $\rho^\infty$ . For any such sequence,  $\rho^\infty = 0$ .*

**Proof** See Appendix. ■

The key step in the proof is the observation that  $\left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right)_{i=1}^n$  is a sequence of independent and identically distributed random variables. Hence, the law of large numbers implies that  $\frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right)$  converges in probability to  $E \left[ \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right]$ . Moreover, after some rearrangements, one finds that  $E \left[ \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right] = 0$ . This implies that that, with many individuals, the maximal per capita revenue that can be extracted, for an arbitrary public goods provision rule, is equal to 0. This is insufficient to finance a positive supply of the public good.

This proposition shows that the inconclusiveness of the analysis based on a fixed number of individuals may be overcome by letting the number of individuals go to  $\infty$ . The intuition is that an individual's influence on public goods provision vanishes as  $n$  becomes large. But if no individual can influence how likely it is that the public good is provided, incentive compatibility can only be maintained if each individual is asked to make the same payment as an individual

with a  $\theta^0$  preference. Consequently, every individual's expected payment converges to zero as the number of individuals goes out of bounds.

Proposition 3 weakens a result Mailath and Postlewaite (1990). They arrive at the same conclusion under a monotone hazard rate assumption which ensures that, for every  $l$ ,  $\theta^l - \frac{1-F(\theta^l)}{f(\theta^l)} \geq \theta^{l-1} - \frac{1-F(\theta^{l-1})}{f(\theta^{l-1})}$ . Our result above shows that this assumption is not needed. Irrespective of whether or not the monotone hazard rate assumption holds, the probability of public goods provision has to go to zero, as the number of individuals goes out of bounds.

### 3.2 Private Goods

Proposition 2 has an analog in the private goods model. To see this note that, in a private goods model, condition (4) can be written as

$$E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) q_i^{n*}(\theta) - c \sum_{i=1}^n q_i^{n*}(\theta) \right] \geq 0, \quad (6)$$

If  $f$  is such that there is sufficient probability mass on some  $\theta^k$  so that  $\theta^k - \frac{1-F(\theta^k)}{f(\theta^k)} > c$ , then weak efficiency is possible. By contrast, if there is sufficient mass on  $\theta^l$  so that  $\theta^l > c > \theta^l - \frac{1-F(\theta^l)}{f(\theta^l)}$ , then efficiency fails. Again, this inconclusiveness can be overcome by letting the number of individuals go out of bounds.

**Proposition 4** *Let  $f$  be such that  $f(\theta^m) > 0$ . There exists  $N$  so that for all  $n > N$ , condition (4) holds.*

**Proof** As  $n \rightarrow \infty$ , the probability that at least one individual has  $\theta_i = \theta^m$  converges to 1. Moreover  $\theta^m - \frac{1-F(\theta^m)}{f(\theta^m)} = \theta^m$ . Consequently, the probability of the event

$$\sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) q_i^{n*}(\theta) = \theta^m$$

converges to 1, which implies that

$$\lim_{n \rightarrow \infty} E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) q_i^{n*}(\theta) - c \sum_{i=1}^n q_i^{n*}(\theta) \right] = \theta^m - c > 0.$$

■

The Proposition exploits the fact that  $\theta^m - \frac{1-F(\theta^m)}{f(\theta^m)} = \theta^m$ , i.e., the valuation of those with the highest valuation of the public good is not distorted downwards by the presence of information rents. Moreover, in a private goods model with many individuals, the individual with the

maximal preference is almost surely a  $\theta^m$ -individual. Hence, information rents do not impede surplus maximization.

The comparison of Propositions 3 and 4 reveals a major difference between private and public goods. In the public goods model, the probability of being decisive for public goods provision vanishes so that individuals reveal their public goods preferences only if they need not contribute to the cost of provision. In the private goods model, by contrast, there are many individuals who compete for the private good. Competition among those with the highest valuation implies that information rents play no role, so that a surplus maximizing outcome can be ensured.

We can therefore conclude that, in an independent private values environment, the Samuelson critique is vindicated: If there are many individuals, an efficient allocation of private goods can be reached, whereas an efficient allocation of public goods is impossible. However, we will see in the following section that this result is itself an artefact of the independent private values assumption.

## 4 Correlated values/ Beliefs determine preferences

In this section, we briefly demonstrate that the result of the previous section – with many individuals, public goods and private goods can be distinguished by the possibility to achieve weakly efficient outcomes subject to incentive and participation constraints – crucially relies on the independence private values assumption. We show that, both for the public goods as well as the private goods model, weak efficiency is possible irrespective of the number of individuals, if the individuals' beliefs satisfy a certain condition. This observation is a more or less straightforward modification of a result due to Crémer and McLean (1988).<sup>13</sup> Our reason to state it is not its novelty. Rather that, in concert with the results from the independent private values models, it conveys a major lesson: The details of the stochastic specification determine whether or not efficient outcomes can be reached, and, in particular, whether there is a qualitative difference between public and private goods.

We say that beliefs determine preferences if, for all  $i$ , the vectors  $p_i^1, p_i^2, \dots, p_i^m$  are linearly independent, i.e., the matrix  $P^i$  has full rank. Note that if these beliefs are derived from a common prior, then the full rank assumption implies that  $(\theta_i)_{i=1}^n$  cannot be collection of independent random variables. Otherwise the marginal distributions  $p_i^1, p_i^2, \dots, p_i^m$  would have to be identical, for all  $i$ . Moreover, with the full rank condition, there exists a bijection between

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<sup>13</sup>Again, the results in this section can be strengthened so as to refer to efficient outcomes and not just to weakly efficient outcomes, see Kosenok and Severinov (2008).

payoffs and beliefs: Knowing that individual  $i$  has type  $k$  implies both that  $\theta_i = \theta^k$  and that  $p_i = p_i^k$ . Hence, beliefs determine preferences in the sense that knowledge of an individual's probabilistic beliefs implies knowledge of the individual's utility function.<sup>14</sup>

**Proposition 5** *Suppose that there is a common prior and that beliefs determine preferences. Then, for all  $n$ , there exists a mechanism that is weakly efficient and satisfies all incentive and participation constraints.*

**Proof** See Appendix. ■

## 5 A model of robust mechanism design

In the following we use a robust mechanism design approach, so that the working of the mechanism must not depend on assumptions about the probabilistic beliefs of individuals. We embed this mechanism design problem into an environment that contains both the independent private values model and the beliefs determine preferences model as special cases.

We will show that the results from the independent private values model turn out to be robust: With finitely many individuals, a private and a public goods model have the same qualitative properties. However, as we let the number of individuals go out of bounds, we get efficiency almost surely in the private goods model and get no public goods provision at all in the public goods model.

### The mechanism designer's beliefs

We assume that there is a mechanism designer who has subjective beliefs about the data generating process, i.e., about the joint probability distribution of  $(\theta_i)_{i=1}^n$ .<sup>15</sup>

We make the following assumptions about the mechanism designer's beliefs: He assumes that there is a set of possible states  $S = \{s_1, \dots, s_r\}$  that occur with probabilities  $\alpha = (\alpha_1, \dots, \alpha_r)$ . Each state  $s_x \in S$  is identified with a probability distribution  $f_x = (f_x(\theta^0), \dots, f_x(\theta^m))$  with full support, so that, conditional on  $s = s_x$ , the random variables  $(\theta_i)_{i=1}^n$  are *i.i.d.* according to

<sup>14</sup>Neeman (2004) argues that the possibility to achieve efficiency depends on the possibility to infer an individual's preferences from its beliefs, i.e., that there is no possibility that individuals with different preferences have the same beliefs; see also Heifetz and Neeman (2006).

<sup>15</sup>There is no need to identify these beliefs with a common prior probability distribution. However, there is also no need to exclude this possibility.



distribution  $f_x$ . Consequently, the mechanism designer assigns probability

$$p(\theta) := \sum_{x=1}^r \alpha_x \prod_{i=1}^n f_x(\theta_i)$$

to the event  $\theta$ . If we use the expectations operator  $E$  in the following, this indicates that expectations are taken with respect to this probability distribution.

The assumption of “conditional independence” is made for reasons of tractability. It implies that we can use the Law of Large Numbers when studying the properties of mechanisms as we let  $n$  go to  $\infty$ . This assumption says that the mechanism designer’s beliefs satisfy an anonymity condition: Whenever two profiles  $\theta$  and  $\theta'$  are such that, for each  $\theta^k$ ,  $\#\{i \mid \theta_i = \theta^k\} = \#\{i \mid \theta'_i = \theta^k\}$ , then  $p(\theta) = p(\theta')$ ; i.e., the mechanism designer assigns the same probability to profiles that give rise to the same cross-section distribution of preferences. Otherwise, the specification is flexible. In particular, it is both consistent with an independent values specification and with a beliefs determine preferences specification. To see this, suppose first that the set  $S$  is a singleton, and there is a common prior which determines both the mechanism designer’s beliefs and the beliefs of individuals. These assumptions give rise to an independent private values environment. By contrast, if we stick to the common prior assumption but assume that the set  $S$  has more than one entry we may generate an environment that satisfies the beliefs determine preferences assumption, as is illustrated by the following example.

**Example 1** *Suppose that individuals either have a high or a low public goods preference, so that  $\Theta = \{\theta^0, \theta^1\}$ . Also suppose that there are two states of the economy,  $S = \{s_1, s_2\}$  with associated probability distributions  $f_1 = (\frac{2}{3}, \frac{1}{3})$ , and  $f_2 = (\frac{1}{3}, \frac{2}{3})$ , respectively. Assume that there is a common prior according to which the two states are equally likely,  $\alpha = (\frac{1}{2}, \frac{1}{2})$ .*

*Under these assumptions, an application of Bayes’ rule reveals that individual  $i$  with type  $\theta^0$  has different beliefs about the current state than individual  $i$  with type  $\theta^1$ ,*

$$pr(s = s_1 \mid \theta_i = \theta^0) = \frac{2}{3} \quad \text{and} \quad pr(s = s_1 \mid \theta_i = \theta^1) = \frac{1}{3}.$$

*This implies that the beliefs  $p_i^0$  and  $p_i^1$  are linearly independent. To see this, note that, e.g.,*

$$\frac{p_i^0(\theta_{-i} = (\theta^0, \dots, \theta^0))}{p_i^1(\theta_{-i} = (\theta^0, \dots, \theta^0))} = \frac{(\frac{2}{3})^n + (\frac{1}{3})^n}{\frac{1}{3}(\frac{2}{3})^{n-1} + \frac{2}{3}(\frac{1}{3})^{n-1}},$$

*whereas*

$$\frac{p_i^0(\theta_{-i} = (\theta^1, \dots, \theta^1))}{p_i^1(\theta_{-i} = (\theta^1, \dots, \theta^1))} = \frac{\frac{1}{3}(\frac{2}{3})^{n-1} + \frac{2}{3}(\frac{1}{3})^{n-1}}{(\frac{2}{3})^n + (\frac{1}{3})^n}.$$

Hence, if we impose a common prior assumption, then the analysis in section 3 is relevant with independent values, and the analysis of section 4 is relevant otherwise.

## 5.1 Robust Mechanism Design

In the following we do not exploit a common prior assumption. Instead, we require that a mechanism works – in the sense of being incentive compatible and satisfying participation constraints – whatever the beliefs of individuals are; that is, we use a mechanism design approach that is robust in the sense of Bergemann and Morris (2005) and Ledyard (1978).

Formally, we introduce the notion of an individual's extended type. For simplicity, we assume that, for each individual  $i$ , there is a finite set of possible extended types, denoted by  $\Gamma_i$  with typical entry  $\gamma_i$ . An individual's extended type determines both the individual's preferences and the individual's probabilistic beliefs with respect to the extended types of other individuals. We refer to the former as the individual's payoff type which is determined by a function  $v_i : \Gamma_i \rightarrow \Theta$ , and to the latter as the individual's belief type which is determined by  $\beta_i : \Gamma_i \rightarrow \mathcal{M}(\Gamma_{-i})$ , where  $\mathcal{M}(\Gamma_{-i})$  is the set of probability distributions on  $\prod_{j \neq i} \Gamma_j$ . Hence, if individual  $i$  has type  $\gamma_i$ , then the individual's preference parameter equals  $v_i(\gamma_i)$ , and the individual assigns a probability  $\beta_i(\gamma_{-i} | \gamma_i)$  to the event that the types of all individuals different from  $i$  are given by  $\gamma_{-i}$ .

A collection  $\mathcal{T} := (\Gamma_i, v_i, \beta_i)_{i=1}^n$ , is henceforth called a type space. Without loss of generality, we limit attention to type spaces so that, for each  $i$ , the function  $v_i$  is surjective.

**Example 2** *Such a type space is compatible with the existence of a common prior if there exists a probability measure  $P$  on  $\Gamma := \prod_{i=1}^n \Gamma_i$  so that, for each  $i$ , and each  $\gamma = (\gamma_i, \gamma_{-i})$ ,  $\beta_i(\gamma_{-i} | \gamma_i) = P(\gamma_{-i} | \gamma_i)$ ; i.e., an individual's beliefs can be derived from conditioning the common prior on the individual's privately observed extended type.*

*The independent private values model is a special case of such a common prior type space so that, for any  $i$  and any pair  $\gamma_i$  and  $\gamma'_i$ ,  $\beta_i(\gamma_i) = \beta_i(\gamma'_i)$ ; i.e., all extended types give rise to the same belief type. Two extended types differ only in the payoff type dimension.*

*The beliefs determine preferences model is the special case of a common prior type space so that, for any  $i$  and any pair  $\gamma_i$  and  $\gamma'_i$ ,  $v(\gamma_i) \neq v(\gamma'_i)$ , and  $(\beta_i(\gamma_i))_{\gamma_i \in \Gamma_i}$  is a collection of linearly independent vectors.*

We seek to characterize the set of social choice functions that can be implemented without relying on assumptions about the functions  $\beta_i$ , that determine the beliefs of individuals. As a preliminary step, we first define what it means to implement a social choice function on a given type space  $\mathcal{T}$ , and in particular, for a given profile of beliefs.

## Implementation of a social choice function on a given type space

Given that an individual is now described by an extended type, a direct mechanism asks each individual to declare his extended type and specifies outcomes as a function of the extended type profile  $\gamma$ . In the following we represent such a mechanism by a collection of functions  $q_i^{ne} : \gamma \mapsto q_i^{ne}(\gamma)$  and  $t_i^{ne} : \gamma \mapsto t_i^{ne}(\gamma)$ , for each  $i \in I^n$ . Such a mechanism is incentive compatible on a given type space  $\mathcal{T}$ , if for each  $i$ ,  $\gamma_i$ , and  $\gamma'_i$ ,

$$\begin{aligned} & \sum_{\Gamma_{-i}} \{v_i(\gamma_i)q_i^{ne}(\gamma_i, \gamma_{-i}) - t_i^{ne}(\gamma_i, \gamma_{-i})\} \beta_i(\gamma_{-i} | \gamma_i) \\ & \geq \sum_{\Gamma_{-i}} \{v_i(\gamma_i)q_i^{ne}(\gamma'_i, \gamma_{-i}) - t_i^{ne}(\gamma'_i, \gamma_{-i})\} \beta_i(\gamma_{-i} | \gamma_i). \end{aligned} \quad (7)$$

It satisfies the participation constraints, if for each  $i$ , and each  $\gamma_i$ ,

$$\sum_{\Gamma_{-i}} \{v_i(\gamma_i)q_i^{ne}(\gamma_i, \gamma_{-i}) - t_i^{ne}(\gamma_i, \gamma_{-i})\} \beta_i(\gamma_{-i} | \gamma_i) \geq 0. \quad (8)$$

Remember that a social choice function  $(\hat{q}_i^n, \hat{t}_i^n)_{i=1}^n$  specifies for each individual  $i$ , a public or private goods consumption level, and a payment as a function of the profile of *payoff types*, i.e., the social choice function prescribes the same outcome for two type profiles  $\gamma$  and  $\gamma'$  such that  $v(\gamma) = v(\gamma')$ , where  $v(\gamma) := (v_1(\gamma_1), \dots, v_n(\gamma_n))$ . For instance, the decision rule for individual  $i$   $\hat{q}_i^n : \theta \mapsto \hat{q}_i^n(\theta)$  requires that for all extended type profiles  $\gamma$  such that  $v(\gamma) = \theta$  the private or public goods consumption of individual  $i$ , equals  $\hat{q}_i^n(\theta)$ .

We say that the social choice function  $(\hat{q}_i^n, \hat{t}_i^n)_{i=1}^n$  is implementable subject to participation constraints, on a given type space, if there is a mechanism  $(q_i^{ne}, t_i^{ne})_{i=1}^n$  that satisfies the incentive and participation constraints in (7) and (8), respectively, and moreover, achieves the social choice function, i.e., for each  $i$  and  $\gamma$ ,

$$q_i^{ne}(\gamma) = \hat{q}_i^n(v(\gamma)) \quad \text{and} \quad t_i^{ne}(\gamma) = \hat{t}_i^n(v(\gamma)). \quad (9)$$

## Robust implementation of a social choice function

We say that the social choice function  $(\hat{q}_i^n, \hat{t}_i^n)_{i=1}^n$  is *robustly implementable subject to participation constraints* if, given  $\Gamma = (\Gamma_i)_{i=1}^n$  and  $v = (v_i)_{i=1}^n$ , there is a mechanism  $(q_i^{ne}, t_i^{ne})_{i=1}^n$  that achieves it and, moreover satisfies the incentive and participation constraints in (7) and (8) for every type space,  $\mathcal{T} = (\Gamma_i, v_i, \beta_i)_{i=1}^n$ .

**Proposition 6** *A social choice function  $(\hat{q}_i^n, \hat{t}_i^n)_{i=1}^n$  is robustly implementable subject to participation constraints if and only if for all  $i$ , for all  $\theta_i \in \Theta$ , all  $\theta'_i \in \Theta$ , and all  $\theta_{-i} \in \Theta^{n-1}$ ,*

$$\theta_i \hat{q}_i^n(\theta_i, \theta_{-i}) - \hat{t}_i^n(\theta_i, \theta_{-i}) \geq \theta_i \hat{q}_i^n(\theta'_i, \theta_{-i}) - \hat{t}_i^n(\theta'_i, \theta_{-i}), \quad (10)$$

and, for all  $i$ , for all  $\theta_i \in \Theta$ , and all  $\theta_{-i} \in \Theta^{n-1}$ ,

$$\theta_i \hat{q}_i^n(\theta_i, \theta_{-i}) - \hat{t}_n^i(\theta_i, \theta_{-i}) \geq 0. \quad (11)$$

**Proof** See Appendix. ■

The proposition follows from an adaptation of arguments due to Bergemann and Morris (2005) and Ledyard (1978) to a setting that involves participation constraints. It can be interpreted as follows: A social choice function is robustly implementable subject to participation constraints if and only if it is achieved by a mechanism with the following properties: (i) The mechanism is “very direct”: it asks individuals to communicate only their payoff types, and remains ignorant with respect to beliefs types; i.e., we may without loss of generality limit attention to the class of direct mechanisms that were introduced in Section 2. (ii) Truth-telling must be a best response from an ex post perspective; i.e., after the payoff types of all other individuals have been revealed, no individual regrets to have revealed the own payoff type.<sup>16</sup> (iii) The participation constraints are also satisfied from the ex post perspective: after all payoff types are revealed, no individual is worse off as compared to a situation with no public or private goods provision at all.

In the following we refer to the inequalities in (10) as ex post incentive compatibility conditions and the inequalities in (11) as ex post participation constraints.

We will now revisit the question whether efficient outcomes are possible using a robust mechanism design approach. Specifically, we study whether there is a surplus maximizing public goods provision rule that satisfies simultaneously the ex post incentive compatibility constraints, the ex post participation constraints, and the weak feasibility constraint in (3), with the expectations operator referring to the mechanism designer’s beliefs.

The following Proposition is the analog to Proposition 1 for a model of robust mechanism design.

**Proposition 7** *In the public goods model and in the private goods model, weak efficiency is compatible with incentive compatibility and voluntary participation if and only if*

$$E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) q_i^{n*}(\theta) - K^n(q_1^{n*}(\theta), \dots, q_n^{n*}(\theta)) \right] \geq 0, \quad (12)$$

where  $(q_i^{n*})_{i=1}^n$  is a surplus maximizing decision rule.

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<sup>16</sup>In the given private values environment, this is equivalent to the postulate that revealing one’s payoff type must be a dominant strategy.

**Proof** See Appendix. ■

The proof of this Proposition follows the same logic as the proof of Proposition 1; i.e., we first study a relaxed revenue maximization problem that takes only a subset of all constraints into account, namely the local downward ex post incentive compatibility constraints and the ex post participation constraints for those individuals with the lowest valuation of  $\theta^0$ . Second, we observe that if the decision rule satisfies a monotonicity condition, then only local downward ex post incentive constraints and the ex post participation constraints for individuals with the lowest payoff type need to be taken into account. In the given setting, the relevant monotonicity constraint is also an ex post constraint – given the payoff types of individuals different from  $i$ , the public goods provision level must not go down if individual  $i$ 's payoff type goes up – as opposed to the expected monotonicity condition in the independent private values model. Given that a surplus maximizing decision rule satisfies also this stronger monotonicity requirement, we can extend Proposition 1 immediately to the present setting.

The proposition is of interest because it shows that, if we evaluate the possibility of reaching efficiency from the mechanism designer's ex ante perspective, we get to same expression as with an independent private values model. Replacing the Bayes-Nash incentive and participation constraints from the independent private values model by the more stringent ex post incentive and participation constraint does not affect the criterion for whether or not (weakly) efficient outcomes can be obtained.

## 5.2 Public Goods

The results that we obtain for the public goods model parallel those from the independent private values environment in Section 3.1.

In the public goods model, condition (12) can be written as

$$E \left[ \left( \frac{1}{n} \sum_{i=1}^n \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - k \right) q^{n*}(\theta) \right] \geq 0, \quad (13)$$

where  $q^{n*}(\theta) = 1$  if  $\frac{1}{n} \sum_{i=1}^n \theta_i > k$ , and  $q^{n*}(\theta) = 0$ , otherwise. Using the law of iterated expectations we can rewrite this once more as

$$\sum_{x=1}^r \alpha_x E \left[ \left( \frac{1}{n} \sum_{i=1}^n \theta_i - \frac{1 - F_x(\theta_i)}{f_x(\theta_i)} - k \right) q^{n*}(\theta) \mid s_x \right] \geq 0. \quad (14)$$

This decomposition makes it easy to reproduce Proposition 2 according to which, with finitely many individuals, the question whether or not efficiency is possible can not be answered in

general, but depends on the details of the probability distribution. As Proposition 2 shows, there exist specifications of  $f_x$  so that

$$E \left[ \left( \frac{1}{n} \sum_{i=1}^n \theta_i - \frac{1 - F_x(\theta_i)}{f_x(\theta_i)} - k \right) q^{n^*}(\theta) \mid s_x \right] \geq 0; \quad (15)$$

and other specifications so that this inequality fails. As the inequality in (14) reveals, if there is sufficient weight on states so that (15) holds as a strict inequality, then (13) will be satisfied. By contrast, if there is sufficient weight on states so that (15) fails, then efficiency will be impossible.

As in the independent private values this inconclusiveness can be overcome by letting the number of individuals go to infinity. For every distribution  $f_x$ ,  $\left( \theta_i - \frac{1 - F_x(\theta_i)}{f_x(\theta_i)} \right)_{i=1}^n$  is a sequence of *i.i.d.* random variables with mean 0, so that  $\frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1 - F_x(\theta_i)}{f_x(\theta_i)} \right)$  converges in probability to 0. This implies that, conditional on any state  $s_x$ , the maximal per capita revenue that can be extracted is equal to 0, and does not suffice to finance a positive supply of the public good. Trivially, this conclusion also holds if we compute an average of various such states. This proves the following Proposition.

**Proposition 8** *Let  $((t_i^n)_{i=1}^n, q^n)_{n=1}^\infty$  be a sequence of mechanisms in the public goods model that are ex post incentive compatible and satisfy the ex post participation constraints. Suppose that the associated sequence of provision probabilities  $(\rho^n)_{n=1}^\infty$  converges to a limit probability  $\rho^\infty$ . For any such sequence,  $\rho^\infty = 0$ .*

The analysis so far has been based on a notion of weak efficiency. Proposition 8 shows that, with many individuals, the second best mechanism that satisfies ex post incentive compatibility, ex post participation constraints and expected budget balance is, from the mechanism designer's perspective, equivalent to a trivial mechanism that never collects any payment and never provides the public good. This second-best mechanism satisfies not only expected budget balance, but also budget balance from an ex post perspective. Consequently, with many individuals, whether we work with expected or ex post budget balance is not essential.

### 5.3 Private Goods

Similarly as for the public goods model, with finitely many individuals, an efficient allocation of private goods may or may not be possible depending on the details of the probability distributions. Once more, as we let the number of individuals go out of bounds, the results change. Again, in the private goods model, we get efficiency almost surely.

To see this formally, it is helpful to look at the analog of condition (14) for the private goods model. Accordingly, weak efficiency can be reached if and only if

$$\sum_{x=1}^r \alpha_x E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F_x(\theta_i)}{f_x(\theta_i)} \right) q_i^{n*}(\theta) - c \sum_{i=1}^n q_i^{n*}(\theta) \mid s_x \right] \geq 0. \quad (16)$$

The arguments in the proof of Proposition 4 can now be applied for each state  $s_x$  separately: Conditional on  $s = s_x$ , the individual with the highest valuation of the public good is almost surely one with a  $\theta^m$ -preference and whose valuation is not distorted by the presence of information rents,  $\theta^m - \frac{1 - F_x(\theta^m)}{f_x(\theta^m)} = \theta^m$ . Consequently, efficiency is possible. The fact that for each  $s_x$ ,

$$\lim_{n \rightarrow \infty} E \left[ \frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1 - F_x(\theta_i)}{f_x(\theta_i)} \right) q_i^{n*}(\theta) - c \sum_{i=1}^n q_i^{n*}(\theta) \mid s_x \right] = \theta^m - c > 0,$$

implies that condition (16) is satisfied if  $n$  is sufficiently large. The following Proposition summarizes.

**Proposition 9** *There exists  $N$  so that for all  $n > N$ , condition (16) holds.*

This possibility result has been derived relying on an expected budget balance condition. However, with many individuals, we may specify the mechanism so that the probability that the budget is not balanced from an ex post perspective – i.e., the probability of the event  $\sum_{i=1}^n t_i^n(\theta) \neq c \sum_{i=1}^n q_i^n(\theta)$  – converges to zero. To see this, consider a mechanism so that the public good is produced with probability 1 and each individual receives a lump sum payment (that is, a payment that does not depend on the announced type) of  $\frac{1}{n}(\theta^m - c)$ . In addition, an individual gets the good only if it declares a type of  $\theta^m$ . In case of getting the good an individual has to pay  $\theta^m$  to the mechanism designer. This mechanism is ex post incentive compatible and satisfies the ex post participation constraints. As  $n \rightarrow \infty$ , the probability that at least one individual has a  $\theta^m$ -preference goes to 1. Consequently, the probability of an outcome that is surplus maximizing and budget balanced in an ex post sense, also converges to 1.

## 5.4 On Participation constraints

The analysis so far has established one major result: In a model of robust mechanism design (which is both consistent with an independent private values environment as well as one in which beliefs determine preferences), there is a qualitative difference between public and private goods: With many individuals, and given that incentive, participation and resource constraints have to be respected, efficiency is possible in a private goods model but not in a public goods model.

We show in the following that this result really has to be attributed to the presence of participation constraints, i.e., if there are no participation constraints, efficiency is possible in either environment. Hence, it is really the postulate that participation must be voluntary which implies that there is no way to overcome the free-rider problem in public goods provision.

To establish this claim we study the problem of a mechanism designer who chooses a public goods provision rule  $q^n$  and individual payments  $(t_i^n)_{i=1}^n$  in order to maximize expected welfare

$$W := E \left[ \frac{1}{N} \sum_{i=1}^n (\theta_i q^n(\theta) - t_i^n(\theta)) \right]$$

subject to ex post incentive compatibility and an ex post resource constraints: for all  $\theta$ ,

$$\frac{1}{n} \sum_{i=1}^n t_i^n(\theta) \geq k q^n(\theta) .$$

**Proposition 10** *Let  $(q^{nr}, t^{nr})_{n=1}^\infty$ , with  $t^{nr} = (t_i^{nr})_{i=1}^n$ , be a sequence of solutions of the above robust mechanism design problem. Any such sequence has the following properties.*

1. *The sequence  $\{q^{nr}\}_{n=1}^\infty$  converges in probability to a surplus maximizing provision rule.*
2. *The sequence  $\{t^{nr}\}_{n=1}^\infty$  converges in probability to a payment rule that is ex post budget balanced.*
3. *The sequence  $\{t^{nr}\}_{n=1}^\infty$  converges in probability to a lump sum payment rule, i.e., a payment rule so that all individuals make the same payment, irrespective of their announcement to the mechanism designer. This lump sum payment rule prescribes equal cost sharing, i.e. all individuals pay  $k$  if the public good is provided, and pay 0 otherwise.*

The proof of the first two claims in Proposition 10 exploits the following observations: (i) A Clarke-Groves mechanism is ex post incentive compatible, and ex post resource feasible, and therefore satisfies the constraints of the robust mechanism design problem. Consequently, for any given  $n$ , the welfare generated by a Clarke-Groves mechanism is a lower bound on the welfare generated by the optimal robust mechanism. (ii) The welfare generated by the optimal robust mechanism is bounded from above by the welfare induced by a first-best mechanism, i.e., a mechanism that prescribes surplus-maximization and ex post budget balance. (iii) A Clarke-Groves mechanism converges to a first best mechanism. The reason is that a Clarke-Groves mechanism differs from a first-best mechanism only in those circumstances where at least one individual is pivotal for surplus maximizing public goods provision, i.e., if this individuals has a low preference, the public good is not provided and it is provided if the individual in question



has a high preference. However, the probability that at least one individual is pivotal goes to 0 as  $n$  goes to  $\infty$ . Hence, the probability that a Clarke-Groves mechanism differs from a first best mechanism also converges to zero.

The third claim in Proposition 10 is also based on the observation, that no individual is pivotal if the economy is sufficiently large. This implies that ex post incentive compatibility can be satisfied only if all individuals are requested to make the same contribution to the cost of public goods provision. Hence, public goods provision will be financed by lump sum contributions, so that  $i$ 's payment does no longer depend on  $i$ 's announcement to the mechanism designer. Moreover, the only way to have simultaneously lump sum payments and first best outcomes is to have public goods finance via equal cost sharing.

## 6 Concluding Remarks

This paper has used a mechanism design approach to assess the validity of Samuelson's critique of "the voluntary exchange theory of public finance" which claimed that, for public as well as private goods, efficient outcomes can be obtained without government intervention. We show that Samuelson's critique is vindicated only if we insist on robustness in the sense that possibility and impossibility results for efficient outcomes must not depend on assumptions about the individuals' probabilistic beliefs about the preferences and beliefs of other individuals.

Since Samuelson's view has become the profession's conventional wisdom – as, for instance, expressed in textbooks on public finance – our conclusion is that robust mechanism design is the "right" formal framework for the analysis of public goods. Of course, this assessment is based on the choice of particular standard: A formal framework for the analysis of public goods should give us the possibility to generate those propositions that are uncontroversial. Only if this criterion is met, we can have the confidence that the model is useful as a basis for extensions, generalizations, thought experiments etc.

This standard supports the use of robust mechanisms also in other settings. For instance, as shown in Bierbrauer (2009a), a robust mechanism design approach to a problem of incentive compatible redistribution and public goods provision gives rise to a mechanism design problem that is formally equivalent to the optimal income tax problem that is studied in the literature building on Mirrlees (1971).

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## A Appendix

### Proof of Proposition 1

The independent private values assumption implies that an individual's beliefs do not vary with an individual's preference parameter, so that the incentive compatibility constraints of individual  $i$  can be written as follows: for all  $k$  and all  $l$ ,

$$\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) \geq \theta^k Q_i^n(\theta^l) - T_i^n(\theta^l),$$

where

$$Q_i^n(\theta^k) := \sum_{\Theta_{-i}} p_i(\theta_{-i}) q_i^n(\theta^k, \theta_{-i}) \quad \text{and} \quad T_i^n(\theta_i) := \sum_{\Theta_{-i}} p_i(\theta_{-i}) t_i^n(\theta^k, \theta_{-i}).$$

The participation constraints require that for all  $i$  and  $k$ ,

$$\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) \geq 0;$$

The Proposition now follows from the series of Lemmas below.

**Lemma 1** *Let  $(q_i^n)_{i=1}^n$  be a given decision rule. Then the per capita revenue,  $E \left[ \frac{1}{n} \sum_{i=1}^n t_i^n(\theta) \right]$ , is bounded from above by  $E \left[ \frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) q_i^n(\theta) \right]$ .*

**Proof** *Step 1.* We first establish the following Claim: Let a provision rule  $q^n$  for the public good be given and let the expected payments  $(T_i^n(\theta^0))_{i=1}^n$  of individuals with  $\theta_i = \theta^0$  be given. If all local downward incentive constraints, i.e., all constraints of the form

$$\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) \geq \theta^k Q_i^n(\theta^{k-1}) - T_i^n(\theta^{k-1}),$$

are binding, then

$$E \left[ \sum_{i=1}^n t_i^n(\theta) \right] = \sum_{i=1}^n T_i^n(\theta^0) + E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta)}{f(\theta)} \right) q_i^n(\theta) \right].$$

To see this, note that since all local downward incentive constraints are binding, we have that for any given  $i$  and any  $k \geq 1$ ,

$$T_i(\theta^k) = T_i(\theta^0) + \sum_{l=1}^k \theta^l (Q_i^n(\theta^l) - Q_i^n(\theta^{l-1})).$$

Rearranging terms and using that  $\theta^0 = 0$ ,  $\theta^1 = 1$ , etc., shows that this can be equivalently written as

$$T_i(\theta^k) = T_i(\theta^0) + \theta^k Q_i^n(\theta^k) - \sum_{l=0}^k Q_i^n(\theta^l).$$

By the law of iterated expectations

$$E[t_i^n(\theta)] = T_i^n(\theta^0) + \sum_{k=0}^m f(\theta^k) Q_i^n(\theta^k) - \sum_{k=0}^m f(\theta^k) \sum_{l=0}^k Q_i^n(\theta^l).$$

Straightforward manipulations show that

$$\sum_{k=0}^m f(\theta^k) \sum_{l=0}^k Q_i^n(\theta^l) = \sum_{k=0}^m f(\theta^k) \frac{1 - F(\theta^k)}{f(\theta^k)} Q_i^n(\theta^k).$$

Using the law of iterated expectations once more implies that  $E[t_i^n(\theta)]$  can be written as

$$E[t_i^n(\theta)] = T_i^n(\theta^0) + E \left[ \left( \theta_i - \frac{1 - F(\theta)}{f(\theta)} \right) q_i^n(\theta) \right].$$

*Step 2.* To complete the proof of the Lemma, take an arbitrary decision rule  $(q_i^n)_{i=1}^n$  as given and consider the problem of maximizing aggregate revenues  $E[\sum_{i=1}^n t_i^n(\theta)]$  subject to the local downward incentive compatibility constraints of all individuals, and the participation constraints for individuals with the lowest valuation,  $T_i^n(\theta^0) \leq 0$ , for all  $i$ . Obviously, at a solution to the relaxed problem all constraints have to be binding. Otherwise, for some types of some individuals expected payments could be increased, without violating any one of the constraints. By the reasoning in *Step 1.*, this implies that the maximal aggregate revenue equals  $E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta)}{f(\theta)} \right) q_i^n(\theta) \right]$ .

Since the above problem takes only a subset of all incentive compatibility and participation constraints into account  $E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta)}{f(\theta)} \right) q_i^n(\theta) \right]$  is an upper bound on the revenues that can be generated if decision rule  $(q_i^n)_{i=1}^n$  is to be implemented and all incentive and participation constraints are taken into account. ■

**Lemma 2** *Consider a mechanism such that all local downward incentive compatibility constraints are binding, and a provision rule so that, for all  $i$ , and all  $k$  the following monotonicity constraints are satisfied,  $Q_i^n(\theta^k) \leq Q_i^n(\theta^{k+1})$ . Then, this mechanism satisfies all incentive compatibility constraints.*

**Proof** We first show that if the monotonicity constraints hold, and all local downward incentive constraints are satisfied, then all downward incentive constraints (i.e., constraints of the form  $\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) \geq \theta^k Q_i^n(\theta^l) - T_i^n(\theta^l)$ , for some  $l \leq k$ ) are satisfied. To see this, note that the two local constraints,

$$\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) \geq \theta^k Q_i^n(\theta^{k-1}) - T_i^n(\theta^{k-1}),$$

and

$$\theta^{k-1} Q_i^n(\theta^{k-1}) - T_i^n(\theta^{k-1}) \geq \theta^{k-1} Q_i^n(\theta^{k-2}) - T_i^n(\theta^{k-2}),$$

imply that

$$\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) \geq \theta^k Q_i^n(\theta^{k-2}) - T_i^n(\theta^{k-2}) + (\theta^k - \theta^{k-1})(Q_i^n(\theta^k) - Q_i^n(\theta^{k-1})).$$

The monotonicity constraint implies that the right hand side of this inequality is larger than  $\theta^k Q_i^n(\theta^{k-2}) - T_i^n(\theta^{k-2})$ . Hence, the downward incentive constraint,

$$\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) \geq \theta^k Q_i^n(\theta^{k-2}) - T_i^n(\theta^{k-2}),$$

is satisfied. Iterating this argument further shows that all downward incentive constraints hold.

In a similar way, one can show that, if the monotonicity constraints holds, then local upward incentive compatibility,  $\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) \geq \theta^k Q_i^n(\theta^{k+1}) - T_i^n(\theta^{k+1})$  for all  $i$  and  $k$ , is sufficient to ensure that all upward incentive constraints,  $\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) \geq \theta^k Q_i^n(\theta^l) - T_i^n(\theta^l)$  for all  $i$ ,  $k$  and  $l > k$ , are satisfied.

To complete the proof we show that if all local downward incentive compatibility constraints are binding, and the monotonicity constraint holds, then all local upward incentive compatibility constraints are satisfied. To see this, suppose that

$$\theta^k Q_i^n(\theta^k) - T_i^n(\theta^k) = \theta^k Q_i^n(\theta^{k-1}) - T_i^n(\theta^{k-1}),$$

or, equivalently,

$$T_i^n(\theta^k) - T_i^n(\theta^{k-1}) = \theta^k (Q_i^n(\theta^k) - Q_i^n(\theta^{k-1})). \quad (17)$$

The local upward incentive compatibility constraint,

$$\theta^{k-1} Q_i^n(\theta^{k-1}) - T_i^n(\theta^{k-1}) \geq \theta^{k-1} Q_i^n(\theta^k) - T_i^n(\theta^k),$$

can be equivalently written as

$$T_i^n(\theta^k) - T_i^n(\theta^{k-1}) \geq \theta^{k-1} \left( Q_i^n(\theta^k) - Q_i^n(\theta^{k-1}) \right). \quad (18)$$

If the monotonicity constraint  $Q_i^n(\theta^k) \geq Q_i^n(\theta^{k-1})$  holds, then (17) implies (18). ■

**Lemma 3** *Consider a mechanism that satisfies local downward incentive compatibility. Suppose that for all  $i$ ,  $\theta^0 Q_i^n(\theta^0) - T_i^n(\theta^0) \geq 0$ . Then, this mechanism satisfies all participation constraints.*

**Proof** By assumption the mechanism satisfies the participation constraints for  $\theta^0$ -individuals. The local downward incentive compatibility constraint for a  $\theta^l$  individual implies that

$$\theta^l Q_i^n(\theta^l) - T_i^n(\theta^l) \geq \theta^l Q_i^n(\theta^{l-1}) - T_i^n(\theta^{l-1}) \geq \theta^{l-1} Q_i^n(\theta^{l-1}) - T_i^n(\theta^{l-1}).$$

Consequently, if the participation constraint is satisfied for a  $\theta^{l-1}$  individual, then it is also satisfied for a  $\theta^l$  individual. ■

**Lemma 4** *Let  $(q_i^n)_{i=1}^n$  be a decision rule so that for all  $i$ , and all  $k$  the following monotonicity constraints are satisfied,  $Q_i^n(\theta^k) \leq Q_i^n(\theta^{k+1})$ . Then the maximal per capita revenue that can be extracted under all incentive and participation constraints equals  $E \left[ \frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) q_i^n(\theta) \right]$ .*

**Proof** Immediately from Lemmas 1-3. ■

**Lemma 5** *Both in the public goods and the private goods model, the surplus maximizing decision rule satisfies  $Q_i^n(\theta^k) \leq Q_i^n(\theta^{k+1})$ , for all  $i$  and  $k$ ,*

**Proof** Given our assumptions on the cost function  $K^n$  and the preferences of individuals, both in the public goods and the private goods model, a surplus maximizing decision rule is such that, for all  $i$ , and all  $\theta_{-i}$ ,  $q_i^{n*}(\theta_{-i}, \theta^k) \leq q_i^{n*}(\theta_{-i}, \theta^{k+1})$ . Consequently, it also satisfies the monotonicity constraint  $Q_i^{n*}(\theta^k) \leq Q_i^{n*}(\theta^{k+1})$ . By Lemma 4 this implies that the maximal per capita payment that can be extracted from individuals equals  $E \left[ \frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) q_i^{n*}(\theta) \right]$ . Expected budget balance is possible if only if this maximal payment is larger or equal than the expected per capita cost  $E [K^n(q_1^{n*}(\theta), \dots, q_n^{n*}(\theta))]$ . ■

### Proof of Proposition 3

*Step 1.* We first argue that the probability of the event  $\frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) \geq k$ , goes to zero if  $n$  goes to  $\infty$ . This is a consequence of the following two observations: (i) Since  $\left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right)_{i=1}^n$  is a sequence of independent and identically distributed random variables, the law of large numbers, implies that  $\frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right)$  converges in probability to  $E \left[ \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right]$ . (ii) Moreover,  $E \left[ \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right] = 0$ . To see this note that

$$\begin{aligned} E \left[ \frac{1-F(\theta_i)}{f(\theta_i)} \right] &= \sum_{k=0}^m f(\theta^k) \frac{1-F(\theta^k)}{f(\theta^k)} = \sum_{k=0}^m \sum_{k+1}^m f(\theta^k) \\ &= 0f(\theta^0) + 1f(\theta^1) + 2f(\theta^2) + \dots + mf(\theta^k) = \sum_{k=0}^m \theta^k f(\theta^k) \\ &= E[\theta_i]. \end{aligned}$$

*Step 2.* Let  $((\bar{t}_i^n)_{i=1}^n, \bar{q}^n)_{n=1}^\infty$  be a sequence of mechanisms with the property that the associated sequence of provision probabilities  $(\rho^n)_{n=1}^\infty$  converges to a limit probability  $\rho^\infty$ . Denote the associated sequence of per capita budget surpluses by  $(S^n)_{n=1}^\infty$  with limit surplus  $S^\infty$ . We show that  $\rho^\infty > 0$  implies  $S^\infty < 0$ .

By Proposition 1, for any  $n$ ,

$$S^n \leq E \left[ \frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) \bar{q}^n(\theta) - k \bar{q}^n(\theta) \right]. \quad (19)$$

Suppose that  $\rho^\infty > 0$ . Then, by *Step 1*, it must be the case that

$$\text{plim}_{n \rightarrow \infty} \left( \bar{q}^n(\theta) = 1 \text{ and } \frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) < k \right) > 0,$$

and,

$$\text{plim}_{n \rightarrow \infty} \left( \bar{q}^n(\theta) = 1 \text{ and } \frac{1}{n} \sum_{i=1}^n \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) \geq k \right) = 0.$$

Hence, as  $n \rightarrow \infty$ , the right hand side of (19) converges to a negative number, which implies that  $\lim_{n \rightarrow \infty} S^n < 0$ . ■

### Proof of Proposition 5

The proof proceeds in two steps. We first show that, under the full rank condition, we can simultaneously achieve surplus maximization, incentive compatibility, and binding participation constraints for all types of all individuals. We then show that, under the common prior assumption, there exists a mechanism that share this property and is such that the weak feasibility constraint is binding.



*Step 1.* Let  $((t_i^{n*})_{i=1}^n, (q_i^{n*})_{i=1}^n)$  be an incentive compatible mechanism such that  $(q_i^{n*})_{i=1}^n$  is a surplus maximizing decision rule.<sup>17</sup> Define,

$$h_i^k := \sum_{\theta_{-i} \in \Theta^{n-1}} p_i^k(\theta_{-i}) \left( \theta^k q_i^{n*}(\theta_{-i}, \theta^k) - t_i^{n*}(\theta_{-i}, \theta^k) \right) .$$

In words,  $h_i^k$  is the expected utility that type  $k$  of individual  $i$  realizes in equilibrium. We denote by  $h_i = (h_i^1, \dots, h_i^m)'$  the vector that lists these expected utility levels. The full rank condition implies that there exists a vector  $g_i = (g_i(\theta_{-i}))_{\theta_{-i} \in \Theta^{n-1}}$  so that

$$P^i \times g_i = h_i ,$$

or, equivalently, for each  $k$ ,

$$\sum_{\theta_{-i} \in \Theta^{n-1}} p_i^k(\theta_{-i}) g_i(\theta_{-i}) = h_i^k .$$

Consider the mechanism  $((t_i^{n'})_{i=1}^n, q_i^{n*})$ , with the surplus maximizing decision rule as above, but a transfer scheme that satisfies, for all  $\theta$ , and for all  $i$ ,

$$t_i^{n'}(\theta) = t_i^{n*}(\theta) + g_i(\theta_{-i}) - \gamma ,$$

where  $\gamma$  is a scalar. By construction, this mechanism is also incentive compatible: It adds components to  $i$ 's payment that do not depend on  $i$ 's behavior, which implies that incentive compatibility remains intact. We claim that  $\gamma \geq 0$  implies that for each type of each individual the participation constraint is satisfied. To see this note that

$$\begin{aligned} & \sum_{\Theta^{n-1}} p_i^k(\theta_{-i}) \left( \theta^k q_i^{n*}(\theta_{-i}, \theta^k) - t_i^{n'}(\theta_{-i}, \theta^k) \right) \\ &= \sum_{\Theta^{n-1}} p_i^k(\theta_{-i}) \left( \theta^k q_i^{n*}(\theta_{-i}, \theta^k) - t_i^{n*}(\theta_{-i}, \theta^k) \right) - \sum_{\Theta^{n-1}} p_i^k(\theta_{-i}) g_i(\theta_{-i}) + \gamma \\ &= \sum_{\Theta^{n-1}} p_i^k(\theta_{-i}) \left( \theta^k q_i^{n*}(\theta_{-i}, \theta^k) - t_i^{n*}(\theta_{-i}, \theta^k) \right) - h_i^k + \gamma \\ &= \gamma . \end{aligned}$$

*Step 2.* Under the common prior assumption, we can make use of the law of iterated expectations so that

$$E \left[ \sum_{i=1}^n t_i^{n'}(\theta) \right] = \sum_{i=1}^n E \left[ E[t_i^{n'}(\theta) \mid \theta_i] \right] .$$

Since all individuals realize an expected payoff equal to  $\gamma$ , we have that

$$E[t_i^{n'}(\theta) \mid \theta_i] = E[\theta_i q_i^{n*}(\theta) \mid \theta_i] - \gamma .$$

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<sup>17</sup>In the public goods environment this could be a Groves mechanism, in the private goods environment this could be a second price auction.

Hence, using the law of iterated expectations once more,

$$E \left[ \sum_{i=1}^n t_i^{n'}(\theta) \right] = E \left[ \sum_{i=1}^n \theta_i q_i^{n*}(\theta) \right] - n\gamma .$$

The fact that  $q_i^{n*}$  is an efficient provision rule implies that

$$E \left[ \sum_{i=1}^n \theta_i q_i^{n*}(\theta) \right] \geq E[K^n(q_1^{n*}(\theta), \dots, q_n^{n*}(\theta))] .$$

Consequently, budget balance holds if we choose

$$\gamma = \frac{1}{n} \left( E \left[ \sum_{i=1}^n \theta_i q_i^{n*}(\theta) \right] - E[K^n(q_1^{n*}(\theta), \dots, q_n^{n*}(\theta))] \right) .$$

■

### Proof of Proposition 6

“ $\Leftarrow$ ”. We first show that robust implementability subject to participation constraints implies that the constraints (10) and (11) are satisfied.

Consider a social choice function  $(\hat{q}_i^n, \hat{t}_n^i)_{i=1}^n$  that is robustly implementable subject to participation constraints, and let  $(q_i^{ne}, t_i^{ne})_{i=1}^n$  be the corresponding mechanism.

Fix  $\gamma$  and let  $(\beta_i)_{i=1}^n$  be such that, for all  $i$ ,  $\beta_i(\gamma_{-i} \mid \gamma_i) = 1$ . On such a “complete information type space” the constraints (10) and (11), respectively, can be written as follows: for each  $i$  and  $\gamma'_i$

$$v_i(\gamma_i) q_i^{ne}(\gamma_i, \gamma_{-i}) - t_i^{ne}(\gamma_i, \gamma_{-i}) \geq v_i(\gamma_i) q_i^{ne}(\gamma'_i, \gamma_{-i}) - t_i^{ne}(\gamma'_i, \gamma_{-i}) , \quad (20)$$

and, for each  $i$ ,

$$v_i(\gamma_i) q_i^{ne}(\gamma_i, \gamma_{-i}) - t_i^{ne}(\gamma_i, \gamma_{-i}) \geq 0 . \quad (21)$$

Since  $\gamma$  was arbitrary these constraints have to be satisfied for all  $\gamma$ .

By assumption, the mechanism  $(q_i^{ne}, t_i^{ne})_{i=1}^n$  achieves the social choice function  $(\hat{q}_i^n, \hat{t}_n^i)_{i=1}^n$ . Using (9), the inequalities in (20) and (21) can therefore be rewritten as: For all  $i$ , for all  $\gamma_i$ , for all  $\gamma'_i$ , and for all  $\gamma_{-i}$ ,

$$\begin{aligned} & v_i(\gamma_i) \hat{q}_i^n(v_i(\gamma_i), v_{-i}(\gamma_{-i})) - \hat{t}_n^i(v_i(\gamma_i), v_{-i}(\gamma_{-i})) \\ & \geq v_i(\gamma_i) \hat{q}_i^n(v_i(\gamma'_i), v_{-i}(\gamma_{-i})) - \hat{t}_n^i(v_i(\gamma'_i), v_{-i}(\gamma_{-i})) , \end{aligned} \quad (22)$$

and, for all  $i$ , for all  $\gamma_i$ , and for all  $\gamma_{-i}$ ,

$$v_i(\gamma_i) \hat{q}_i^n(v_i(\gamma_i), v_{-i}(\gamma_{-i})) - \hat{t}_n^i(v_i(\gamma_i), v_{-i}(\gamma_{-i})) \geq 0 . \quad (23)$$

Using that, for each  $i$ ,  $v_i$  is surjective, this implies that (10) and (11) have to be satisfied.

“ $\implies$ ”. Let  $(\hat{q}_i^n, \hat{t}_i^n)_{i=1}^n$  be a social choice function that satisfies (10) and (11). If, for each  $i$ ,  $v_i$  is surjective, this implies that the inequalities in (22) and (23) are satisfied. Let  $(q_i^{ne}, t_i^{ne})_{i=1}^n$  be a mechanism that achieves this social choice function. Consequently, the constraints (20) and (21) have to be satisfied. Obviously, this implies that the incentive constraints in (7) and the participation constraints in (8) hold for every collection of beliefs  $(\beta_i)_{i=1}^n$ .  $\blacksquare$

### Proof of Proposition 7

The proof follows from the series of Lemmas below.

**Lemma 6** *Let a decision rule  $(q_i^n)_{i=1}^n$  be given. If all local downward ex post incentive constraints, i.e., all constraints of the form*

$$\theta^k q_i^n(\theta^k, \theta_{-i}) - t_i^n(\theta^k, \theta_{-i}) \geq \theta^k q_i^n(\theta^{k-1}, \theta_{-i}) - t_i^n(\theta^{k-1}, \theta_{-i}),$$

are binding, then

$$E \left[ \sum_{i=1}^n t_i^n(\theta) \right] = \sum_{i=1}^n E[t_i^n(\theta) \mid \theta_i = \theta^0] + E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta)}{f(\theta)} \right) q_i^n(\theta) \right].$$

**Proof** The expected payment of individual  $i$  equals

$$\begin{aligned} E[t_i^n(\theta)] &= \sum_{x=1}^r \alpha_x E[t_i^n(\theta) \mid s_x] \\ &= \sum_{x=1}^r \alpha_x \rho_x(\theta_{-i}) \left( \sum_{k=0}^m f_x(\theta^k) t_i^n(\theta^k, \theta_{-i}) \right). \end{aligned} \quad (24)$$

where  $\rho_x(\theta_{-i}) := \prod_{j \neq i} f_x(\theta_j)$ . If all local downward ex post incentive constraints are binding, we have that for any given  $i$ , and any  $\theta_{-i}$ , and any  $k \geq 1$ ,

$$t_i^n(\theta^k, \theta_{-i}) = t_i^n(\theta^0, \theta_{-i}) + \theta^k q_i^n(\theta^k, \theta_{-i}) - \sum_{l=0}^{k-1} q_i^n(\theta^l, \theta_{-i}).$$

Using similar arguments as in the proof of Lemma 1, this implies that  $\sum_{k=0}^m f_x(\theta^k) t_i^n(\theta^k, \theta_{-i})$  can be written as

$$\sum_{k=0}^m f_x(\theta^k) t_i^n(\theta^k, \theta_{-i}) = t_i^n(\theta^0, \theta_{-i}) + E \left[ \left( \theta_i - \frac{1 - F_x(\theta_i)}{f_x(\theta_i)} \right) q_i^n(\theta) \mid \theta_{-i}, s_x \right]. \quad (25)$$

Upon substituting (25) into (24), we obtain

$$\begin{aligned} E[t_i^n(\theta)] &= \sum_{x=1}^r \alpha_x E[t_i^n(\theta) \mid s_x] \\ &= \sum_{x=1}^r \alpha_x \left( E[t_i^n(\theta_i, \theta_{-i}) \mid \theta_i = \theta^0, s_x] + E \left[ \left( \theta_i - \frac{1 - F_x(\theta_i)}{f_x(\theta_i)} \right) q_i^n(\theta) \mid s_x \right] \right) \\ &= E[t_i^n(\theta_i, \theta_{-i}) \mid \theta_i = \theta^0] + E \left[ \left( \theta_i - \frac{1 - F_x(\theta_i)}{f_x(\theta_i)} \right) q_i^n(\theta) \right]. \end{aligned} \quad (26)$$

Consequently,

$$E \left[ \sum_{i=1}^n t_i^n(\theta) \right] = \sum_{i=1}^n E[t_i^n(\theta) \mid \theta_i = \theta^0] + E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta)}{f(\theta)} \right) q_i^n(\theta) \right].$$

The Lemma now follows from the fact that, in a public goods model,  $q_i^n(\theta) = q^n(\theta)$ , for all  $i$ , and  $\theta$ . ■

Taking an arbitrary decision rule  $(q_i^n)_{i=1}^n$  as given consider the problem of maximizing aggregate revenues  $E[\sum_{i=1}^n t_i^n(\theta)]$  subject to the local ex post downward incentive compatibility constraints of all individuals, and the ex post participation constraints for individuals with the lowest valuation,  $t_i^n(\theta^0, \theta_{-i}) \leq 0$ , for all  $i$ , and  $\theta_{-i}$ . Obviously, at a solution to the relaxed problem all constraints have to be binding. Otherwise, for some types of some individuals expected payments could be increased, without violating any one of the constraints. By Lemma 6 this implies that the maximal aggregate revenue equals  $E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta)}{f(\theta)} \right) q_i^n(\theta) \right]$ .

Since the above problem takes only a subset of all incentive compatibility and participation constraints into account  $E \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta)}{f(\theta)} \right) q_i^n(\theta) \right]$  is an upper bound on the revenues that can be generated if provision rule  $q^n$  is to be implemented and all ex post incentive and participation constraints are taken into account.

The following three Lemmas extend Lemmas 2, 3, and 5 to the present setting. The extension is straightforward so that we omit a formal proof.

**Lemma 7** *Consider a mechanism such that all local downward ex post incentive compatibility constraints are binding, and a provision rule so that, for all  $i$ , all  $k$ , and all  $\theta_{-i}$ ,  $q_i^n(\theta^k, \theta_{-i}) \leq q_i^n(\theta^{k+1}, \theta_{-i})$ . Then, this mechanism satisfies all ex post incentive compatibility constraints.*

**Lemma 8** *Consider a mechanism that satisfies local downward ex post incentive compatibility. Suppose that for all  $i$ , and all  $\theta_{-i}$ ,  $\theta^0 q_i^n(\theta^0, \theta_{-i}) - t_i^n(\theta^0, \theta_{-i}) \geq 0$ . Then, this mechanism satisfies all participation constraints.*

**Lemma 9** *Both in the public goods and the private goods model, the surplus maximizing decision rule satisfies  $q_i^n(\theta^k, \theta_{-i}) \leq q_i^n(\theta^{k+1}, \theta_{-i})$ , for all  $i$ ,  $k$ , and  $\theta_{-i}$ .*

## Proof of Proposition 10

The proof follows from Lemmas 10 to 14 and Corollary 1 below.

A Clarke-Groves Mechanism is a mechanism consisting of a surplus maximizing public goods provision rule and a payment rule  $t^{nCG}$  that is defined as follows:

- i) Let  $\frac{1}{n} \sum_{i=1}^n \theta_i \geq k$  and  $\frac{1}{N} \sum_{j \neq i} \theta_j \geq k$ . Then  $t_i^{nCG}(\theta) = k$ .
- ii) Let  $\frac{1}{n} \sum_{i=1}^n \theta_i \geq k$  and  $\frac{1}{N} \sum_{j \neq i} \theta_j < k$ . Then  $t_i^{nCG}(\theta) = Nk - \sum_{j \neq i} \theta_j$ .
- iii) Let  $\frac{1}{n} \sum_{i=1}^n \theta_i < k$ . Then  $t_i^{nCG}(\theta) = 0$ .

The following property of Clarke-Groves mechanisms is well known.<sup>18</sup>

**Lemma 10** *The Clarke-Groves mechanism is ex post incentive compatible and ex post resource feasible.*

Under a surplus maximizing provision rule, the probability that at least one individual  $i$  is pivotal – i.e.  $\frac{1}{n} \sum_{i=1}^n \theta_i \geq k$  and  $\frac{1}{n} \sum_{j \neq i} \theta_j < k$ , for at least one  $i$  – can be written as  $1 - \Omega^n$ , where

$$\Omega^n := \text{prob}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \frac{1}{n} \sum_{j \neq 2} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k\right) + \text{prob}\left(\frac{1}{n} \sum_{i=1}^n \theta_i < k\right)$$

**Lemma 11** *Consider a sequence of surplus maximizing provision rules. The probability that at least one individual is pivotal converges to 0 as  $n \rightarrow \infty$ .*

**Proof** The probability that at least one individual is pivotal can be written as

$$1 - \Omega^n = 1 - \sum_{x=1}^r \alpha_x \Omega_x^n,$$

where

$$\begin{aligned} \Omega_x^n := & \text{prob}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \frac{1}{n} \sum_{j \neq 2} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k \mid s = s_x\right) \\ & + \text{prob}\left(\frac{1}{n} \sum_{i=1}^n \theta_i < k \mid s = s_x\right). \end{aligned}$$

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<sup>18</sup>For a proof see, for instance, Mas-Colell et al. (1995).

Hence, it suffices to show that, for every  $x$ ,  $\Omega_x^n \rightarrow 1$  as  $n \rightarrow \infty$ . To establish this claim we will show that

$$\begin{aligned} & \text{plim}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \frac{1}{n} \sum_{j \neq 2} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k \mid s = s_x\right) \\ &= \begin{cases} 1, & \text{if } E[\theta_i \mid s = s_x] > k, \\ 0, & \text{if } E[\theta_i \mid s = s_x] < k, \end{cases} \end{aligned}$$

and that

$$\text{plim}\left(\frac{1}{n} \sum_{i=1}^N \theta_i < k \mid s = s_x\right) = \begin{cases} 1, & \text{if } E[\theta_i \mid s = s_x] < k, \\ 0, & \text{if } E[\theta_i \mid s = s_x] > k. \end{cases}$$

*Step 1.* We first show that  $E[\theta_i \mid s = s_x] > k$  implies that

$$\text{plim}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k \mid s = s_x\right) = 1.$$

Note that

$$\begin{aligned} & \text{prob}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k \mid s = s_x\right) \\ &= \text{prob}\left(\frac{1}{n} \sum_{i=1}^n \theta_i \geq k + \frac{\theta_1}{n}, \dots, \frac{1}{n} \sum_{i=1}^n \theta_i \geq k + \frac{\theta_n}{n} \mid s = s_x\right) \\ &\geq \text{prob}\left(\frac{1}{n} \sum_{i=1}^n \theta_i \geq k + \frac{\theta^m}{n} \mid s = s_x\right). \end{aligned}$$

Obviously,  $(\frac{\theta^m}{n})_{n=1}^\infty$  is a deterministic sequence that converges to 0. Conditional on  $s = s_x$ ,  $(\frac{1}{n} \sum_{i=1}^n \theta_i)_{n=1}^\infty$  is a sequence of random variables that converges in probability to  $E[\theta_i \mid s = s_x]$ , by the law of large numbers. Consequently,  $(\frac{1}{n} \sum_{i=1}^n \theta_i - k - \frac{\theta^m}{n})_{n=1}^\infty$  is a sequence of random variables that converges in probability to  $E[\theta_i \mid s = s_x] - k$ . Hence, if  $E[\theta_i \mid s = s_x] > k$ , then

$$\text{prob}\left(\frac{1}{n} \sum_{i=1}^n \theta_i \geq k + \frac{\theta^m}{n} \mid s = s_x\right) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

*Step 2.* We now show that  $E[\theta_i \mid s = s_x] < k$  implies that

$$\text{plim}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k \mid s = s_x\right) = 0.$$

Note that

$$\text{prob}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k \mid s = s_x\right) \leq \text{prob}\left(\frac{1}{n} \sum_{i=1}^n \theta_i \geq k_n \mid s = s_x\right).$$

Conditional on  $s = s_x$ ,  $(\frac{1}{n} \sum_{i=1}^n \theta_i - k)_{n=1}^\infty$  is a sequence of random variables that converges in

probability to  $E[\theta_i | s = s_x] - k$ . Hence, if  $E[\theta_i | s = s_x] < k$ , then

$$\text{prob}\left(\frac{1}{n} \sum_{i=1}^n \theta_i \geq k \mid s = s_x\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

*Step 3.* Analogously, if  $E[\theta_i | s = s_x] > k$ , then

$$\text{prob}\left(\frac{1}{n} \sum_{i=1}^n \theta_i < k \mid \bar{s} = s\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

■

**Lemma 12** *Denoty  $\mu_x := E[\theta_i | s = s_x]$ . The expected per capita surplus that is realized under a Clarke-Groves mechanisms converges to*

$$\sum_{\{s_x | \mu_x \geq k\}} \alpha_x (\mu_x - k) ,$$

*i.e., to the welfare level realized with the first best provision rule and a first best payment rule.*

**Proof** The expected per capita surplus under a Clarke-Groves mechanism can be written as

$$\text{prob}\left(\frac{1}{n} \sum_{i=1}^n \theta_i < k\right) \cdot 0 + (1 - \Omega^n) \cdot \Lambda^n + \text{prob}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k\right) \cdot \Gamma^n$$

where

$$\Lambda^n := E \left[ \frac{1}{n} \sum_{i=1}^n [\theta_i - t_i^{CG}(\theta)] \mid \exists i : \frac{1}{n} \sum_{i=1}^n \theta_i \geq k \text{ and } \frac{1}{n} \sum_{j \neq i} \theta_j < k \right] ,$$

and

$$\Gamma^n := E \left[ \left( \frac{1}{n} \sum_{i=1}^n \theta_i \right) - k \mid \forall i : \frac{1}{n} \sum_{j \neq i} \theta_j \geq k \right] .$$

*Step 1.* Note that  $\Lambda^n$  is bounded from above by  $\theta^m$ . Hence, for every  $n$ ,  $(1 - \Omega^n) \cdot \Lambda^n \leq (1 - \Omega_N) \theta^m$ .

By Lemma 11,  $\Omega^n \rightarrow 1$  and  $(\Lambda^n)_{n=1}^\infty$  bounded imply that

$$(1 - \Omega^n) \cdot \Lambda^n \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

*Step 2.* We have that

$$\begin{aligned} & \text{prob}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k\right) \cdot \Gamma^n \\ &= \sum_{x=1}^r \alpha_x \text{prob}\left(\frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k \mid s = s_x\right) \cdot \Gamma_x^n \end{aligned}$$

where

$$\Gamma_x^n = E \left[ \left( \frac{1}{n} \sum_{i=1}^n \theta_i \right) - k \mid \forall i : \frac{1}{n} \sum_{j \neq i} \theta_j \geq k, s = s_x \right]$$

The arguments in the proof of Lemma 11 imply that

$$\sum_{x=1}^r \alpha_x \text{prob} \left( \frac{1}{n} \sum_{j \neq 1} \theta_j \geq k, \dots, \frac{1}{n} \sum_{j \neq n} \theta_j \geq k \mid s = s_x \right) \cdot \Gamma_x^n$$

converges to  $\sum_{\{s_x \mid \mu_x \geq k\}} \alpha_x (\mu_x - k)$ . ■

**Lemma 13** *The expected per capita surplus that is realized with the optimal robust mechanism,  $(q^{nr}, t^{nr})_{n=1}^\infty$ , converges to  $\sum_{\{s_x \mid \mu_x \geq k\}} \alpha_x (\mu_x - k)$ .*

**Proof** For any  $n$ , the per capita surplus that can be realized by the optimal robust mechanism is bounded from above by the welfare level that is realized with a surplus maximizing provision rule and budget balance. By Lemma 10, a Clarke-Groves mechanism is both ex post feasible and ex post incentive compatible. Hence, for any  $n$ , the per capita surplus that can be realized by the optimal robust mechanism is bounded from below by the one attained by the Clarke-Groves mechanism. By Lemma 12, the surplus under a Clarke Groves mechanism converges to the surplus under an efficient. Hence, the surplus under the optimal robust mechanism converges as well to the first best surplus. ■

**Corollary 1**  *$(q^{nr})_{n=1}^\infty$  converges in probability to the first best provision rule and  $(t^{nr})_{n=1}^\infty$  converges in probability to a payment rule that is ex post budget balanced.*

**Lemma 14**  *$(t^{nr})_{n=1}^\infty$  converges in probability to a lump sum payment rule.*

**Proof** Consider a sequence  $\{i^n\}_{n=1}^\infty$  with  $i^n \in I^n$  for every  $n$ . We show that, for every pair  $(\theta^l, \theta^k)$ , and every  $\varepsilon > 0$ ,

$$\text{prob}(|t_{i^n}^{nr}(\theta^l, \theta_{-i^n}) - t_{i^n}^{nr}(\theta^k, \theta_{-i^n})| \geq \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We assume without loss of generality that  $\theta^k > \theta^l$ . By ex post incentive compatibility, for every



$\theta_{-i}$ ,

$$\theta^k [q^{nr}(\theta^k, \theta_{-i}) - q^{nr}(\theta^l, \theta_{-i})] \geq t_i^{nr}(\theta^k, \theta_{-i}) - t_i^{nr}(\theta^l, \theta_{-i}) \geq \theta^l [q^{nr}(\theta^k, \theta_{-i}) - q^{nr}(\theta^l, \theta_{-i})].$$

This implies in particular that

$$t_i^{nr}(\theta^k, \theta_{-i}) - t_i^{nr}(\theta^l, \theta_{-i}) \geq 0 \tag{27}$$

for every  $\theta_{-i}$ , and that

$$\begin{aligned} & \theta^k E[q^{nr}(\theta^k, \theta_{-i}) - q^{nr}(\theta^l, \theta_{-i})] \\ & \geq E[t_i^{nr}(\theta^k, \theta_{-i}) - t_i^{nr}(\theta^l, \theta_{-i})] \geq \theta^l E[q^{nr}(\theta^k, \theta_{-i}) - q^{nr}(\theta^l, \theta_{-i})]. \end{aligned} \tag{28}$$

By Lemma 11, the probability that any one individual is pivotal converges to 0. Hence,

$$E[q^{nr}(\theta^k, \theta_{-i}) - q^{nr}(\theta^l, \theta_{-i})] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Consequently, the inequalities (28) imply that

$$E[t_i^{nr}(\theta^k, \theta_{-i}) - t_i^{nr}(\theta^l, \theta_{-i})] \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Using (27) this implies that  $t_i^{nr}(\theta^k, \theta_{-i}) - t_i^{nr}(\theta^l, \theta_{-i})$  converges in probability to 0 as  $n$  grows without limit. ■