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Conference Paper

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Beiträge zur Jahrestagung des Vereins für Socialpolitik 2010: Ökonomie der Familie - Session: Income Risk, Savings and Intergenerational Links, No. A14-V4

Provided in Cooperation with:
Verein für Socialpolitik / German Economic Association

Suggested Citation: Hillebrand, Marten (2010) : Governmental Debt, Interest Policy, and Tax Stabilization in an Overlapping Generations Economy, Beiträge zur Jahrestagung des Vereins für Socialpolitik 2010: Ökonomie der Familie - Session: Income Risk, Savings and Intergenerational Links, No. A14-V4, Verein für Socialpolitik, Frankfurt a. M.

This Version is available at:
https://hdl.handle.net/10419/37409

Governmental Debt, Interest Policy, and Tax Stabilization in an Overlapping Generations Economy*

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August 14, 2010

Abstract

The paper analyzes the role of governmental debt in an overlapping generations economy with stochastic production and capital accumulation. In the absence of taxation, equilibria with positive debt generically converge to debtless equilibria with probability one which are dynamically inefficient. It is shown that this may be overcome by a tax on labor income which minimizes stabilization efforts by stabilizing the level of debt along a stable set of the underlying dynamical system. A long-run welfare criterion is formulated which measures consumer welfare at the stabilized equilibrium. Based on this criterion, the welfare effects of different levels of debt and different interest policies are investigated with the help of numerical simulations.

Keywords: OLG, governmental debt, interest policy, risk sharing, tax stabilization, stabilized equilibrium, long-run welfare.
JEL Classification: C62, E27, E62, H21, H63.

*Acknowledgment. This research was begun while the author was a visiting scholar at the W.P. Carey School of Business at Arizona State University and the Research Department at the Federal Reserve Bank of Minneapolis. I am grateful for the generous hospitality received there and to Ed Prescott for inspiring comments. I would also like to thank Martin Barbie and Veronica Block for fruitful discussions and Thorsten Pampel for valuable technical advice. Financial support from the German Research Foundation (DFG) under contract No. Hi 1381/2-1 is gratefully acknowledged.

Introduction

Most industrialized countries have large governmental debt. In the U.S., total outstanding debt amounted to a little less than 60% of GDP in 2002 and rose moderately to about 66% in 2007. Largely due to the gigantic fiscal stimuli in response to the recent economic crisis, the past two years have seen a dramatic increase of this ratio to more than 85% as of 2009. Similar figures apply for other countries suggesting that the sustainability of governmental debt is - or should be - a highly relevant issue for policy making.
From a theoretical perspective, it is well-known that an increase in governmental debt may stimulate aggregate demand in the short run but crowds out capital investment in the long run, cf. Elmendorf & Mankiw (1999). The latter effect is particularly important in overlapping generations (OLG) economies where the first welfare theorem need not hold and competitive equilibria may be inefficient due to an overaccumulation of capital. In such a situation, as first shown by Diamond (1965), introducing governmental debt leads to a welfare improvement by implementing a dynamically efficient allocation. Subsequent studies to investigate the role of governmental debt in deterministic OLG economies may be found, e.g., in de la Croix & Michel (2002, Ch.4), Farmer (1986), Rankin & Roffia (1999) and in Bullard & Russell (1999) for consumers with multiperiod lives.
There is a close relationship between the sustainability of governmental debt and the emergence of a bubble. The latter corresponds to an intrinsically worthless asset that is traded at a positive price such as fiat money or a private asset that does not pay dividends. The differences between debt and a bubble are thoroughly exhibited in de la Croix & Michel (2002, p.212). Starting with the work by Tirole (1985), a large body of the literature discusses the emergence of bubbles in deterministic OLG models. Examples may be found, e.g., in Bertocchi & Wang (1994), Kunieda (2008), or Michel & Wigniolle (2003). Due to the structural similarities between debt and a bubble, the results by Tirole (1985) are directly applicable to characterize sustainable levels of governmental debt in deterministic OLG models, cf. de la Croix & Michel (2002, Ch.4). It is shown there that steady states with positive debt are saddle-path stable which implies that sustainable levels of debt are required to lie on a lower-dimensional subset (the so-called *stable manifold*) of the state space. This determines a unique debt-to GDP ratio under which the economy converges to the golden-rule steady state with positive debt. Debt smaller than the critical level leads to an asymptotically debtless (and inefficient) situation while larger values imply an unsustainable situation with explosive debt.
Starting with the work of Wang (1993), the literature has increasingly focused on OLG economies with aggregate risk due to random production shocks. It is not yet known, however, how the previous findings carry over to a stochastic setting and under what conditions equilibria with positive debt exist. A first approach in this direction is put forward in Bertocchi (1994), who analyzes possible equilibrium scenarios in a stochastic OLG model with riskless debt. The present paper provides a refinement of her results.
If there is aggregate risk, another function of governmental debt is to provide a possibility of *risk-sharing* between generations. If interest payments on outstanding debt are

financed by issuing new debt to the next generation, the implied risk sharing is essentially determined by the interest on debt. This motivates the question of how different interest policies on debt affect intergenerational risk-sharing and consumer welfare.
The present paper extends the model of Wang (1993) to study the role of governmental debt in a stochastic OLG framework which has not been done in the literature. Following above's motivation, two issues are at the center of interest: 1.*Which levels of debt are sustainable in the presence of random shocks*? 2.*Which interest policy is favorable and induces optimal risk sharing between generations*? To quantify the welfare effects of different debt policies, the paper develops a long-run welfare concept on the basis of which the optimum quantity of debt and an optimal interest policy can be determined. The paper is organized as follows. Section 1 introduces the model. Section 2 analyzes equilibria with capital-equivalent debt where the return on debt coincides with the capital return. This structure is generalized in Section 3 which allows for general interest policies including riskless debt. Section 4 demonstrates how the level of debt can be stabilized against unfavorable shocks through a labor income tax. The welfare properties of stabilized equilibria under different debt policies are investigated in Section 5. Section 6 concludes, all proofs are placed in the Mathematical Appendix.

1 The Model

The framework to be introduced in this section generalizes the stochastic overlapping generations model in Wang (1993) to include governmental debt and a tax system.

Population. The consumption sector consists of overlapping generations of homogeneous consumers who live for two periods. The index $j \in \{y, o\}$ identifies the young and old generation in each period. Each young consumer is endowed with one unit of labor time supplied inelastically to the market. Old consumers are retired and do not supply labor. Abstracting from population growth, each generation consists of $N > 0$ consumers such that $L_t^S \equiv N$ denotes aggregate labor supply at time $t \geq 0$. Old consumers in period t own the existing stock of capital K_t which they supply to the production process.

Production. Labor and capital are employed by a single firm which produces Y_t units of a consumption good using labor L_t and capital K_t as inputs in period t. In addition, the production process in period t is subjected to random shocks corresponding to the random variable ε_t. The technology is represented by the linear homogeneous production function $F(\cdot;\varepsilon_t) : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ producing gross output (including depreciated capital) as

$$Y_t = F(K_t, L_t; \varepsilon_t). \tag{1}$$

The noise process $\{\varepsilon_t\}_{t\geq 0}$ consists of independent, identically distributed random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each ε_t is distributed according to the probability measure ν supported on $\mathcal{E} \subset [\varepsilon_{min}, \varepsilon_{max}] \subset \mathbb{R}_{++}$. The process is adapted to a suitable filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub σ-algebras of $\mathcal{F}$ such that each $\varepsilon_t : \Omega \longrightarrow \mathcal{E}$ is Borel-measurable with respect to $\mathcal{F}_t$.[1] Let $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ denote the

[1] The underlying probability space may be constructed by defining $\Omega := \mathcal{E}^{\mathbb{N}_0}$ which may be endowed with the product topology to become a topological space on which the Borel-σ-algebra $\mathcal{F} := \mathcal{B}(\Omega)$

expectations operator conditional on the information represented by $\mathcal{F}_t$. Throughout, the notion of an adapted stochastic process $\{\xi_t\}_{t\geq 0}$ taking values in some set $\Xi \subset \mathbb{R}^M$ refers to the probability space and the filtration defined. It implies that each random variable $\xi_t : \Omega \longrightarrow \Xi$ is Borel- measurable with respect to $\mathcal{F}_t$ and hence observable in period t. All equalities or inequalities involving random variables are assumed to hold $\mathbb{P}$-almost surely without further notice.
Define $y_t := \frac{Y_t}{N}$ and $k_t := \frac{K_t}{N}$ as output respectively capital per labor force. By the linear homogeneity of $F(\cdot;\varepsilon_t)$, the technology (1) may be written in intensive form as

$$y_t = f(k_t;\varepsilon_t) := F(k_t, 1;\varepsilon_t). \tag{2}$$

The function f is continuous and twice differentiable with respect to its first argument with continuous derivatives satisfying $f_{kk}(k;\varepsilon) < 0 < f_k(k;\varepsilon)$ for all $k > 0$ and $\varepsilon \in \mathcal{E}$ as well as the Inada conditions $\lim_{k\to 0} f_k(k;\varepsilon) = \infty$ and $\lim_{k\to\infty} f_k(k;\varepsilon) < 1$. Profit maximization and linear homogeneity of the technology (1) imply that market clearing prices for labor and capital in period $t \geq 0$ are given by their marginal products, i.e.,

$$w_t^g = \mathcal{W}(k_t;\varepsilon_t) := f(k_t;\varepsilon_t) - k_t f_k(k_t;\varepsilon_t) \tag{3}$$

$$r_t = \mathcal{R}(k_t;\varepsilon_t) := f_k(k_t;\varepsilon_t). \tag{4}$$

Government. The infinitely-lived government taxes consumers and issues debt to finance its deficit. For the purpose of this paper, debt may be thought of as a one-period lived bond which pays a (possibly random) return $r^\star_{t+1} > 0$ per unit invested at time $t \geq 0$. Negative debt will not be considered in this paper. Let $b_t \geq 0$ be the number of bonds per young consumer issued at time t and τ_t^y an τ_t^o be the taxes levied on a young consumer's labor income and old consumer's capital income, respectively. Negative taxes correspond to subsidies on the income of the respective group. It follows that debt evolves as

$$b_t = r_t^\star b_{t-1} - \tau_t^y - \tau_t^o, \quad t \geq 0. \tag{5}$$

Consumers. At time $t \geq 0$ a young consumer earns net labor income $w_t := w_t^g - \tau_t^y > 0$ that can be consumed and invested in bonds and capital. Let s_t and b_t be the investments in capital and bonds at time $t \geq 0$. These choices define current consumption as

$$c_t^y = w_t - b_t - s_t \tag{6}$$

while next period's consumption is given by the random variable

$$c_{t+1}^o = b_t\, r^\star_{t+1} + s_t\, r_{t+1} - \tau^o_{t+1}. \tag{7}$$

Here the randomness enters through the uncertain returns on both investments and uncertain tax payments which are treated as given random variables in the decision. Young consumers evaluate the expected utility of different consumption plans (c_t^y, c_{t+1}^o) defined by (6) and (7) according to the von-Neumann Morgenstern utility function

$$U(c^y, c^o) := u(c^y) + v(c^o). \tag{8}$$

may be defined. The measure $\mathbb{P}$ corresponds to the product measure $\mathbb{P} := \otimes_{t\geq 0}\nu$ while the sub-σ-algebra $\mathcal{F}_t$ is generated by the class of measurable rectangular sets $A = \prod_{n=0}^{\infty} A_n$ where each A_n is a Borel-measurable subset of $\mathcal{E}$ and $A_n = \mathcal{E}$ for $n > t$.

Both functions u and v are C^2 with derivatives $z''(c) < 0 < z'(c)$ for $c > 0$ and satisfy

$$\lim_{c \to 0} z'(c) = \infty \quad \text{for} \quad z \in \{u, v\}. \tag{9}$$

Each young consumer chooses investment to maximize her expected utility of lifetime consumption. The decision problem reads:

$$\max_{b,s} \Big\{ u(w_t - b - s) + \mathbb{E}_t \big[v\big(r^\star_{t+1}\, b + r_{t+1}\, s - \tau^o_{t+1} \big) \big] \,\Big|\, s \geq 0, b + s \leq w_t - \tau^y_t \Big\}. \tag{10}$$

Note that no short-selling constraints on b are imposed at the individual level. The investment in capital s_t determines next period's per-capita capital stock as

$$k_{t+1} = s_t. \tag{11}$$

Old consumers in period $t \geq 0$ consume the proceeds on their investments in bonds and capital during the previous period - net of taxes - as defined by (7).

Equilibrium. Combining the assumptions of market clearing, individual optimality and rational expectations the following definition of an equilibrium is straightforward.

Definition 1.1
Given initial values $b_0 \geq 0$, $k_0 > 0$, and $\varepsilon_0 \in \mathcal{E}$, an equilibrium is an adapted stochastic process $\big\{ w^g_t, r_t, r^\star_t, \tau^y_t, \tau^o_t, b_t, s_t, c^y_t, c^o_t, k_{t+1} \big\}_{t \geq 0}$ which satisfies the following for each $t \geq 0$:

(i) Wages $w^g_t > 0$ and returns $r_t > 0$ are determined by (3) and (4) and the returns on debt satisfy $r^\star_t > 0$.

(ii) Taxes satisfy $\tau^y_t < w^g_t$ and $\tau^o_t < b_t r^\star_t + k_t r_t$ while debt $b_t \geq 0$ evolves as in (5).

(iii) The pair (b_t, s_t) solves the decision problem (10) at the given wage, returns, and taxes while c^y_t, c^o_t, and k_{t+1} are determined by (6), (7), and (11).

Indeterminacy of fiscal policy. The following result shows that without further restrictions on taxes $\{\tau^y_t, \tau^o_t\}_{t \geq 0}$, any debt process is consistent with equilibrium. This is a straightforward generalization of the deterministic case in de la Croix & Michel (2002).

Lemma 1.1
Given $k_0 > 0$, let an interior allocation $\big\{ s_t, c^y_t, c^o_t, k_{t+1} \big\}_{t \geq 0}$, and prices $\{ w^g_t, r_t, r^\star_t \}_{t \geq 0}$ satisfy (3), (4), and (11), the feasibility condition $c^y_t + c^o_t + k_{t+1} = f(k_t, \varepsilon_t)$ for all $\varepsilon_t \in \mathcal{E}$ and the intertemporal efficiency conditions $u'(c^y_t) = \mathbb{E}_t[r_{t+1} v'(c^o_{t+1})] = \mathbb{E}_t[r^\star_{t+1} v'(c^o_{t+1})]$ for all $t \geq 0$. Then, for any non-negative debt process $\{b_t\}_{t \geq 0}$ there is a feasible tax process $\{\tau^y_t, \tau^o_t\}_{t \geq 0}$ such that $\big\{ w^g_t, r_t, r^\star_t, \tau^y_t, \tau^o_t, b_t, s_t, c^y_t, c^o_t, k_{t+1} \big\}_{t \geq 0}$ is an equilibrium.

In light of the last result, investigating the sustainability of governmental debt requires further restrictions on tax policies. As in Diamond (1965), the subsequent analysis therefore assumes that there is no taxation of capital income such that $\tau^o_t \equiv 0$ for all $t \geq 0$. Since the ultimate goal is to analyze the long-run welfare effects of debt policies, this restriction is also in line with the findings of Chari & Kehoe (1999), who show that optimal policies in OLG models are characterized by zero capital taxation in the long-run.

2 Equilibria with Capital-Equivalent Debt

The next two sections study dynamic equilibria in the absence of taxation ($\tau_t^y \equiv 0$) under different assumptions on the return on debt, i.e., on the process $\{r_t^\star\}_{t\geq 0}$. In the sequel we denote by $E_h(x) := \frac{xh'(x)}{h(x)}$ the elasticity of a differentiable function $h : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$. The following additional restrictions on f in (2) and v in (8) will be used frequently:

$$(P1)\ E_{v'}(c) \geq -1\ \forall c > 0 \quad (P2)\ \lim_{c\to\infty} c\, v'(c) = \infty \quad (P3)\ E_{f_k}(k;\varepsilon) \geq -1\ \forall k > 0, \varepsilon \in \mathcal{E}.$$

While (P1) and (P3) are standard, (cf. de la Croix & Michel (2002) and Wang (1993)), (P2) is more restrictive as it excludes several popular parameterizations such as log utility. Examples satisfying (P1) and (P2) are power utility $v(c) = \theta^{-1}c^\theta$, $0 < \theta < 1$, or CES utility $v(c) = [1 - \theta + \theta c^\beta]^{\frac{1}{\beta}}$, $0 < \theta < 1$, $\beta > 0$.

As a first scenario, suppose the government commits itself to paying the capital return on debt such that $r_t^\star \equiv r_t$ for each $t \geq 0$. This case will be called *capital-equivalent (CE) debt* and the remainder of this section will study the existence and properties of equilibria under this assumption. To unveil the recursive structure of equilibria for the economy, consider an arbitrary period t. Let current capital $k_t > 0$ and the shock $\varepsilon_t \in \mathcal{E}$ be given which determine the wage $w_t = w_t^g > 0$ and the return $r_t > 0$ on capital and debt according to (3) and (4). Then, current debt $b_t \geq 0$ corresponding to the supply of bonds follows from its previous value b_{t-1} and (5). Assume that $w_t > b_t$. Since investment in debt and capital are perfect substitutes and the number of bonds traded is determined by the supply side, the equilibrium problem is to determine next period's capital stock $0 < k_{t+1} < w_t - b_t$ in a way consistent with an optimal savings decision derived from (10) and rational, self-confirming expectations. Let $\mathbb{E}_\nu[\cdot]$ denote the expected value with respect to the distribution ν of next period's production shock. Using (4), (11), and the first order conditions of (10), define the map $H(\cdot; w, b) :]\,0, w - b\,[\longrightarrow \mathbb{R}$,

$$H(k; w, b) := u'(w - b - k) - \mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'\big(\mathcal{R}(k;\cdot)(b + k)\big)\right]. \tag{12}$$

Then, given $w_t > b_t \geq 0$, the expectations-consistent solution k_{t+1} is determined by $H(k_{t+1}; w_t, b_t) = 0$. Existence and uniqueness of such a zero are established next.

Lemma 2.1
Let v satisfy (P1). Then for each $w > 0$ there exists an upper bound $0 < b^{max}(w) \leq w$ such that $H(\cdot; w, b)$ has a zero (which is unique) in $]0, w - b[$ if and only if $b < b^{max}(w)$. If, in addition, (P2) holds, then $b^{max}(w) = w$.

In the sequel we assume that the hypotheses of Lemma 2.1 are satisfied permitting us to define the set $\mathbb{V} := \{(w, b) \in \mathbb{R}^2_+ \,|w > 0, b < b^{max}(w)\}$ and the mapping $\mathcal{K} : \mathbb{V} \longrightarrow \mathbb{R}_{++}$ which determines the unique zero of $H(\cdot; w, b)$. Invoking the Implicit Function Theorem (IFT), this map is continuously differentiable with derivatives satisfying the following.

Lemma 2.2
Let v satisfy (P1). Then, at each point $(w, b) \in \mathbb{V}$ (cf. Remark A.1) the partial derivatives of the map $\mathcal{K}$ are continuous and satisfy

$$\partial_b\mathcal{K}(w, b) < 0 \quad \text{and } 0 < \partial_w\mathcal{K}(w, b) < |\partial_b\mathcal{K}(w, b)| \leq 1.$$

Combining equations (3) to (5), and (11) defines a C^1 map $\Phi = (\Phi_w, \Phi_b) : \mathbb{V} \times \mathcal{E} \longrightarrow \mathbb{R}^2_+$ which determines the evolution of wages and debt under the exogenous noise process as

$$w_{t+1} = \Phi_w(w_t, b_t; \varepsilon_{t+1}) := \mathcal{W}(\mathcal{K}(w_t, b_t); \varepsilon_{t+1}) \tag{13a}$$

$$b_{t+1} = \Phi_b(w_t, b_t; \varepsilon_{t+1}) := \mathcal{R}(\mathcal{K}(w_t, b_t); \varepsilon_{t+1}) b_t. \tag{13b}$$

Given initial values $(w_0, b_0) \in \mathbb{V}$, the equilibrium process $\{w_t, b_t\}_{t \geq 0}$ is therefore generated by randomly mixing the family of mappings $\{\Phi(\cdot; \varepsilon)\}_{\varepsilon \in \mathcal{E}}$, i.e., the realization of next period's shock selects a map that determines the next state from the current one. Structurally, this corresponds to a two-dimensional version of the one-dimensional dynamics in Wang (1993). The endogenous state variables $\{w_t, b_t\}_{t \geq 0}$ together with the exogenous noise process $\{\varepsilon_t\}_{t \geq 0}$ completely determine the other equilibrium variables of the model. Therefore, existence of a dynamic equilibrium is equivalent to determining $(w_0, b_0) \in \mathbb{V}$ such that the process generated by (13a), (13b) satisfies $(w_t, b_t) \in \mathbb{V}$ for all $t \geq 0$ under $\mathbb{P}$-almost all paths of the noise processes. Since $b_0 = 0$ implies $b_t = 0$ for all $t > 0$, it is clear that a trivial equilibrium with no debt exists for all $w_0 > 0$. In this case, the dynamics reduce to the evolution of wages defined by the map $\phi_0 : \mathbb{R}_{++} \times \mathcal{E} \longrightarrow \mathbb{R}_{++}$

$$w_{t+1} = \phi_0(w_t; \varepsilon_{t+1}) := \mathcal{W}(\mathcal{K}(w_t, 0); \varepsilon_{t+1}). \tag{14}$$

The next assumption ensures existence and rules out multiplicity of steady states of ϕ_0.

Assumption 2.1
For each $\varepsilon \in \mathcal{E}$, the map $\phi_0(\cdot; \varepsilon)$ possesses a unique fixed point $\bar{w}^0_\varepsilon > 0$ which is stable.

In the sequel, let $\mathbb{V}_+ := \mathbb{V} \cap \mathbb{R}^2_{++}$ and $\Phi^t(\cdot; \varepsilon) := \Phi(\cdot; \varepsilon) \circ \ldots \circ \Phi(\cdot; \varepsilon)$ the t-fold composition of the map $\Phi(\cdot; \varepsilon)$ for $t > 0$. From above's structure, it is clear that the existence and properties of equilibrium depend crucially on the dynamic properties of the mappings $(\Phi(\cdot; \varepsilon))_{\varepsilon \in \mathcal{E}}$ and whether these exhibit contractive or expansive behavior. We therefore begin by fixing a value $\varepsilon \in \mathcal{E}$ to study the dynamic properties of the map $\Phi(\cdot; \varepsilon)$. The next result shows that the return $\mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon)$ at the trivial steady state $(\bar{w}^0_\varepsilon, 0)$ from Assumption 2.1 determines whether $\Phi(\cdot; \varepsilon)$ displays stable - along a certain direction - or explosive behavior. In anticipation of this result, let $\mathcal{E}_s := \{\varepsilon \in \mathcal{E} \,|\, \mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon) < 1\}$ and $\mathcal{E}_x := \{\varepsilon \in \mathcal{E} \,|\, \mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon) > 1\}$.[2] Since the case $\mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon) = 1$ is non-generic, $\mathcal{E}_0 := \mathcal{E} \backslash (\mathcal{E}_s \cup \mathcal{E}_x)$ is assumed to have measure zero, i.e., $\nu(\mathcal{E}_0) = 0$.

Lemma 2.3
Let (P1) and Assumption 2.1 be satisfied. Then, the following holds true:

(i) For $\varepsilon \in \mathcal{E}^s$ the map $\Phi(\cdot; \varepsilon)$ possesses a unique non-trivial fixed point $(\bar{w}_\varepsilon, \bar{b}_\varepsilon) \in \mathbb{V}_+$. This fixed point is saddle-path stable, i.e., the Eigenvalues of the Jacobian matrix $D\Phi(\bar{w}_\varepsilon, \bar{b}_\varepsilon; \varepsilon)$ are real and satisfy $0 < |\lambda_1| < 1 < |\lambda_2|$.

(ii) For $\varepsilon \in \mathcal{E}^x$ the map $\Phi(\cdot; \varepsilon)$ is explosive, i.e., for each $(w, b) \in \mathbb{V}_+$ there exists a $t_0 \in \mathbb{N}$ such that $(w_{t_0}, b_{t_0}) := \Phi^{t_0}(w, b; \varepsilon) \notin \mathbb{V}$, that is, $w_{t_0} \leq b_{t_0}$.

[2] If $\mathcal{E}$ is infinite, continuity of $\varepsilon \longmapsto \mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon)$ ensures (Borel-) measurability of $\mathcal{E}_s$, $\mathcal{E}_x$, and $\mathcal{E}_0$.

Result (i) implies that for $\varepsilon \in \mathcal{E}^s$ the dynamics generated by $\Phi(\cdot;\varepsilon)$ converge to a non-trivial steady state for certain initial values. These are defined by the *stable manifold*

$$\mathcal{M}_\varepsilon := \Big\{(w,b) \in \mathbb{V} \,|\Phi^n(w,b;\varepsilon) \in \mathbb{V}\ \forall n \geq 1 \wedge \lim_{n\to\infty} \Phi^n(w,b;\varepsilon) = (\bar{w}_\varepsilon, \bar{b}_\varepsilon)\Big\},\ \varepsilon \in \mathcal{E}^s. \quad (15)$$

The set $\mathcal{M}_\varepsilon$ will play a key-role throughout this paper. Note that $\mathcal{M}_\varepsilon$ is *self-supporting* under $\Phi(\cdot;\varepsilon)$, i.e., $\Phi(\mathcal{M}_\varepsilon;\varepsilon) \subset \mathcal{M}_\varepsilon$. For each $\varepsilon \in \mathcal{E}^s$, Theorem A.1 in the appendix establishes existence of a map $\psi_\varepsilon : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ which is strictly increasing such that $\mathcal{M}_\varepsilon = \text{graph}(\psi_\varepsilon)$. Based on this representation, the next result shows that $\mathcal{M}_\varepsilon$ separates initial points which diverge from those which converge to the trivial steady state.

Lemma 2.4
Let (P1) and Assumption 2.1 be satisfied and $w > 0$ be arbitrary. Then, for each $\varepsilon \in \mathcal{E}^s$:

(i) $b < \psi_\varepsilon(w) \ \Rightarrow\ \Phi^t(w,b;\varepsilon) \in \mathbb{V}_+\ \forall t > 0 \quad \wedge \quad \lim_{t\to\infty} \Phi^t(w,b;\varepsilon) = (\bar{w}^0_\varepsilon, 0).$

(ii) $b > \psi_\varepsilon(w) \ \Rightarrow\ \exists t_0 > 0$ *such that* $\Phi^{t_0}(w,b;\varepsilon) \notin \mathbb{V}.$

Geometrically, Lemma 2.4 implies that if (w,b) is below the curve $\mathcal{M}_\varepsilon$, then the sequence $\Phi^t(w,b;\varepsilon)$ stays below $\mathcal{M}_\varepsilon$ for all $t \geq 0$ and converges to the trivial steady state with zero debt. Conversely, any state above $\mathcal{M}_\varepsilon$ stays above and leaves $\mathbb{V}$ in finite time.
Based on the properties of the mappings $\Phi(\cdot;\varepsilon)$ $\varepsilon \in \mathcal{E}$, we are now in a position to state conditions for the existence of equilibrium with positive debt. Let $w_0 := \mathcal{W}(k_0;\varepsilon_0) > 0$ be given. First observe that if $\nu(\mathcal{E}^x) > 0$, any initial value in $\mathbb{V}_+$ will leave this set in finite time with positive probability. Hence, $\nu(\mathcal{E}^x) = 0$ is a necessary condition for non-trivial equilibria to exist. For $w > 0$, let $\psi^{min}(w) := \min_{\varepsilon\in\mathcal{E}^s}\{\psi_\varepsilon(w)\}$ which is well-defined, if either $\mathcal{E}^s$ is finite or compact and $\varepsilon \longmapsto \psi_\varepsilon(w)$ continuous. By Lemma 2.4, $b_0 \leq \psi^{min}(w_0)$ is necessary for an equilibrium to exist. Sufficiency requires the following assumption that initial states below $\text{graph}(\psi^{min})$ stay below this curve under all shocks.

Assumption 2.2
For all $w > 0$ one has $b \leq \psi^{min}(w) \ \Rightarrow\ \Phi_b(w,b;\varepsilon) \leq \psi^{min}(\Phi_w(w,b;\varepsilon)) \quad \forall \varepsilon \in \mathcal{E}_s$.

Combining the results from Lemma 2.3 and 2.4 leads to the following theorem which extends and, if the noise is degenerate recover the findings of Tirole (1985).

Theorem 2.1
Under (P1) and Assumptions 2.1 and 2.2, suppose $\nu(\mathcal{E}^x) = 0$. Then, any $b_0 \in]0, \psi^{min}(w_0)]$ defines an equilibrium with positive debt $b_t > 0\ \forall t > 0$.

While equilibria exist under the hypotheses of Theorem 2.1 for any initial value b_0 sufficiently small, typically the level of debt converges to zero with probability one. This is illustrated in Figure 1 for the case with only two shocks where $\mathcal{E} = \{\varepsilon, \varepsilon'\}$. The dotted arrow represents the case excluded by Assumption 2.2. For any $b_0 \leq \bar{b}_0 := \psi^{min}(w_0)$ the state remains in $\mathbb{V}$ and below the $\mathcal{M}_\varepsilon$-curve but b_t converges to zero $\mathbb{P}$-almost surely. Conversely, for any $b_0 > \bar{b}_0$ the state leaves $\mathbb{V}$ in finite time with positive probability implying no-existence of equilibrium in this case.
The following example, however, shows that equilibria with persistent debt may exist.

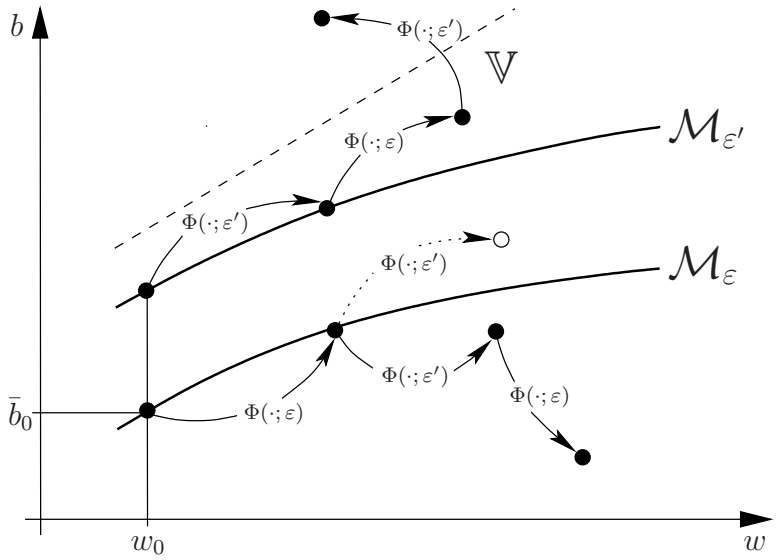


Figure 1: State dynamics under different initial values and noise paths.

Let $U(c^y, c^o) = \ln c^y + \gamma c^o$, $\gamma > 0$ and $f(k;\varepsilon) = \varepsilon k^\alpha$, $0 < \alpha < 1$. The bound from Lemma 2.1 computes $b^{max}(w) = \frac{\gamma}{1+\gamma} w$ such that $\mathbb{V} = \{(w,b) \in \mathbb{R}^2_+ | b < \frac{\gamma}{1+\gamma} w\}$. Moreover, $\Phi_w(w,b;\varepsilon) = \varepsilon(1-\alpha)(\frac{\gamma}{1+\gamma} w - b)^\alpha$ and $\Phi_b(w,b;\varepsilon) = \varepsilon\alpha(\frac{\gamma}{1+\gamma} w - b)^{\alpha-1} b$ implying $\mathcal{E}^s \neq \emptyset$, iff $\frac{\gamma}{1+\gamma} > \frac{\alpha}{1-\alpha}$. The next result shows that the stable sets in (15) are independent of ε.

Lemma 2.5
Under the previous parametrization, suppose $\frac{\gamma}{1+\gamma} > \frac{\alpha}{1-\alpha}$. *Then* $\mathcal{E}^s = \mathcal{E}$ *and*

$$\mathcal{M}_\varepsilon \equiv \mathcal{M} := \left\{(w,b) \in \mathbb{R}^2_{++} \,\middle|\, b = \left(\frac{\gamma}{1+\gamma} - \frac{\alpha}{1-\alpha}\right) w\right\} \quad \forall \varepsilon \in \mathcal{E}.$$

Since $\mathcal{M}$ is self-supporting for the family $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$, $(w_0, b_0) \in \mathcal{M}$ implies $(w_t, b_t) \in \mathcal{M}$ for $t \geq 0$ and debt is bounded away from zero under all possible sequences of shocks.

3 Equilibria with General Debt

To extend the previous analysis to the case with more general interest policies on debt, the remainder confines attention to the case with multiplicative shocks, i.e., $f(k;\varepsilon) = \varepsilon g(k)$ in (2) where $g := f(\cdot;1) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ inherits the properties of $f(\cdot;\varepsilon)$. While under the previous scenario the return on debt offered at time t would be $r^\star_{t+1} = \varepsilon_{t+1} g'(k_{t+1})$, the present section generalizes this structure by supposing that[3]

$$r^\star_{t+1} = \mathcal{R}^\star_\vartheta(z_t; \varepsilon_{t+1}) := z_t\, \vartheta(\varepsilon_{t+1}), \quad t \geq 0. \tag{16}$$

[3] The multiplicative form (16) seems natural under multiplicative shocks. Under more general technologies (2), one could generalize (16) to some continuous function $\mathcal{R}^\star : \mathbb{R}_{++} \times \mathcal{E} \longrightarrow \mathbb{R}_{++}$ where $z \longmapsto \mathcal{R}^\star(z;\varepsilon)$ is a bijection. While Lemma 3.1 and the dynamic equations (18a,b) below would still be valid under this extension, a theoretical characterization of the dynamics seems impossible.

The value $z_t > 0$ is determined in period t and $\vartheta : \mathcal{E} \longrightarrow \mathbb{R}_{++}$ is a time-invariant[4] *interest policy* that determines the risk to which debt investments are subjected. If $\vartheta \equiv \bar{\vartheta}$, debt is riskless while $\vartheta = \mathrm{id}_{\mathcal{E}}$ recovers the case with CE debt.
In the sequel we fix some interest policy ϑ and assume that in each period $t \geq 0$ the return on debt is of the form (16). To derive the recursive equilibrium structure of the economy, consider an arbitrary period t. Let current capital k_t and the shock $\varepsilon_t \in \mathcal{E}$ be given which determine the wage $w_t = w_t^g > 0$ according to (3). Given previous values $b_{t-1} \geq 0$ and $z_{t-1} > 0$, the current shock determines the return on debt $r_t^\star = z_{t-1}\vartheta(\varepsilon_t)$ and current debt/supply of bonds $b_t \geq 0$ according to (5). In addition to finding an expectations-consistent capital stock k_{t+1}, the equilibrium problem for period t is to determine the return offered on debt by fixing a value $z_t > 0$ such that consumers are willing to absorb the predetermined supply of bonds. Since there are no short-selling restrictions on debt, any solution $s > 0$ and $b \geq 0$ to (10) satisfies the corresponding first order conditions. Given $w > b \geq 0$, let $H_i^\vartheta(\cdot,\cdot; w, b) : \mathbb{R}_{++} \times]0, w-b[\longrightarrow \mathbb{R}$, $i \in \{1, 2\}$,

$$H_1^\vartheta(z,k;w,b) := u'(w-b-k) - \mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'\big(b\,\mathcal{R}_\vartheta^\star(z;\cdot) + k\mathcal{R}(k;\cdot)\big)\right] \tag{17a}$$

$$H_2^\vartheta(z,k;w,b) := u'(w-b-k) - \mathbb{E}_\nu\left[\mathcal{R}_\vartheta^\star(z;\cdot)v'\big(b\,\mathcal{R}_\vartheta^\star(z;\cdot) + k\mathcal{R}(k;\cdot)\big)\right]. \tag{17b}$$

Then, given $w_t > b_t \geq 0$ the previous problem reduces to solving $H_1^\vartheta(z_t, k_{t+1}; w_t, b_t) = H_2^\vartheta(z_t, k_{t+1}; w_t, b_t) = 0$. Existence and uniqueness of such a solution is established next.

Lemma 3.1
Let (P1)–(P3) be satisfied and $\vartheta : \mathcal{E} \longrightarrow \mathbb{R}_{++}$ be continuous. Then, for each $w > b \geq 0$ there exist unique $z > 0$, $0 < k < w - b$ to satisfy $H_1^\vartheta(z,k;w,b) = H_2^\vartheta(z,k;w,b) = 0$.

Based on this result, let $\mathbb{V} := \{(w,b) \in \mathbb{R}_+^2 \,|\, w > b\}$ denote the set of feasible wage-debt combinations. By Lemma 3.1 and the Implicit Function Theorem, there exist mappings $\mathcal{K}^\vartheta : \mathbb{V} \longrightarrow \mathbb{R}_{++}$ and $\mathcal{Z}^\vartheta : \mathbb{V} \longrightarrow \mathbb{R}_{++}$ which are C^1 on $\mathbb{V}$ (cf. Remark A.1) and which determine the unique zeros of (17a) and (17b) for each $(w,b) \in \mathbb{V}$. Before stating properties of these mappings in Lemma 3.2, we introduce additional restrictions on the elasticities of preferences and the production technology that will be used subsequently:

$$(P4)\ |E_{v'}(c)| = \theta\ \forall c > 0 \quad (P5)\ |E_{u'}(c)| \leq 1\ \forall c > 0 \quad (P6)\ E_g(k) + |E_{g'}(k)| \leq 1\ \forall k > 0.$$

Under (P4), second period utility v exhibits constant relative risk aversion. Property (P5) is automatically satisfied if (P1) holds and $v(c) = \beta u(c)$, $\beta > 0$. Finally, (P6) is necessary and sufficient for the elasticity $E_g(k)$ to be a non-decreasing function of k, which holds, e.g., if g is Cobb-Douglas or CES with elasticity of substitution $\sigma \geq 1$.

Lemma 3.2
Let (P1)–(P3) be satisfied and $\vartheta : \mathcal{E} \longrightarrow \mathbb{R}_{++}$ be continuous. Then, $\mathcal{K}^\vartheta$ and $\mathcal{Z}^\vartheta$ are continuously differentiable at each point $(w,b) \in \mathbb{V}$. Moreover, the following holds true:

(i) The derivatives of $\mathcal{K}^\vartheta$ satisfy $0 < \partial_w\mathcal{K}^\vartheta(w,b) < -\partial_b\mathcal{K}^\vartheta(w,b)$.

[4] A straightforward generalization would be to consider dynamic risk-sharing rules by making ϑ state dependent. Lemma 3.1 below would continue to hold under this extension.

(ii) If, in addition, (P4) holds, then $\mathcal{Z}^\vartheta$ satisfies $0 < -\partial_w \mathcal{Z}^\vartheta(w,b) < \partial_b \mathcal{Z}^\vartheta(w,b)$.[5]

Combining equations (3) to (5) and (11) defines a map $\Phi^\vartheta = (\Phi^\vartheta_w, \Phi^\vartheta_b) : \mathbb{V} \times \mathcal{E} \longrightarrow \mathbb{R}^2_+$ which determines the evolution wages and debt under the exogenous shocks as

$$w_{t+1} = \Phi^\vartheta_w(w_t, b_t; \varepsilon_{t+1}) := \mathcal{W}(\mathcal{K}^\vartheta(w_t, b_t), \varepsilon_{t+1}) \tag{18a}$$

$$b_{t+1} = \Phi^\vartheta_b(w_t, b_t; \varepsilon_{t+1}) := \vartheta(\varepsilon_{t+1}) \mathcal{Z}^\vartheta(w_t, b_t) b_t. \tag{18b}$$

As before, an equilibrium exist if and only if the process $\{w_t, b_t\}_{t \geq 0}$ generated by (18a,b) satisfies $(w_t, b_t) \in \mathbb{V}$ $\mathbb{P}$-almost surely for all $t \geq 0$. Since the equilibrium process is generated by randomly mixing the family $(\Phi^\vartheta(\cdot;\varepsilon))_{\varepsilon \in \mathcal{E}}$, we proceed as in the previous section and fix a value $\varepsilon \in \mathcal{E}$ to study the dynamic properties of the map $\Phi^\vartheta(\cdot;\varepsilon)$. Note that $\Phi^\vartheta(\cdot;\varepsilon)$ is independent of ϑ for $b = 0$ and identical to the map ϕ^0 in (14). In particular, there exists a trivial equilibrium and, under Assumption 2.1 each map $\Phi^\vartheta(\cdot;\varepsilon)$ possesses a unique stable trivial steady state $(\bar{w}^0_\varepsilon, 0)$ which is independent of ϑ. In the sequel, the following slightly stronger version of Assumption 2.1 will be necessary.

Assumption 3.1
For each $\varepsilon \in \mathcal{E}$, the map $\phi_0(\cdot;\varepsilon)$ from (14) possesses a unique fixed point $\bar{w}^0_\varepsilon > 0$ which is stable. Moreover, the corresponding capital stock $\bar{k}^0_\varepsilon := \mathcal{K}(\bar{w}^0_\varepsilon, 0)$ satisfies $E_g(\bar{k}^0_\varepsilon) \leq \frac{1}{2}$.

As in the previous section, the dynamic behavior of $\Phi^\vartheta(\cdot;\varepsilon)$ depends crucially on the return on debt at the trivial steady state. The latter is given by $\bar{z}^0_\varepsilon \vartheta(\varepsilon)$ where

$$\bar{z}^0_\varepsilon := \mathcal{Z}^\vartheta(\bar{w}^0_\varepsilon, 0) = \frac{\mathbb{E}_\nu[\mathcal{R}(\bar{k}^0_\varepsilon;\cdot) v'(\bar{k}^0_\varepsilon \mathcal{R}(\bar{k}^0_\varepsilon;\cdot))]}{\mathbb{E}_\nu[\vartheta(\cdot) v'(\bar{k}^0_\varepsilon \mathcal{R}(\bar{k}^0_\varepsilon;\cdot))]}. \tag{19}$$

Using (19), let $\mathcal{E}^\vartheta_s := \{\varepsilon \in \mathcal{E} \mid \bar{z}^0_\varepsilon \vartheta(\varepsilon) < 1\}$ and $\mathcal{E}^\vartheta_x := \{\varepsilon \in \mathcal{E} \mid \bar{z}^0_\varepsilon \vartheta(\varepsilon) > 1\}$. As before, the set $\mathcal{E}^\vartheta_0$ where $\bar{z}^0_\varepsilon \vartheta(\varepsilon) = 1$ is assumed to have measure zero, i.e., $\nu(\mathcal{E}^\vartheta_0) = 0$. The next result extends Lemma 2.3 to the present case with general interest policies. The proof draws heavily on ideas put forward in Galor (1992).

Lemma 3.3
Under Assumption 3.1 and properties (P1)– (P6), the following holds for any policy ϑ:

(i) For all $\varepsilon \in \mathcal{E}^\vartheta_s$ the mapping $\Phi^\vartheta(\cdot;\varepsilon)$ possesses a unique non-trivial steady state $(\bar{w}^\vartheta_\varepsilon, \bar{b}^\vartheta_\varepsilon) \in \mathbb{V}_+$. This steady state is saddle path-stable.

(ii) For all $\varepsilon \in \mathcal{E}^\vartheta_x$ the mapping $\Phi^\vartheta(\cdot;\varepsilon)$ is explosive.

Given policy ϑ, the previous result permits to define for each $\varepsilon \in \mathcal{E}^\vartheta_s$ the stable manifold

$$\mathcal{M}^\vartheta_\varepsilon := \Big\{(w,b) \in \mathbb{V} \,|\, (\Phi^\vartheta)^n(w,b;\varepsilon) \in \mathbb{V} \;\forall n \geq 1 \wedge \lim_{n\to\infty} (\Phi^\vartheta)^n(w,b;\varepsilon) = (\bar{w}^\vartheta_\varepsilon, \bar{b}^\vartheta_\varepsilon)\Big\}. \tag{20}$$

By Theorem A.1, $\mathcal{M}^\vartheta_\varepsilon$ can be represented as the graph of a map $\psi^\vartheta_\varepsilon : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ and, as shown in the appendix, Lemma 2.4 continues to hold in the present setup. It follows

[5] Numerical experiments with utility functions v not satisfying (P4) have throughout displayed the same properties of $\mathcal{Z}^\vartheta$ as in Lemma 3.2(ii) suggesting that this restriction could probably be relaxed.

that all findings from the previous section about existence of equilibria and the persistence of debt including Theorem 2.1 carry over to the case with general interest policies. Under the restriction imposed in Assumption 2.2 – which can be shown to hold automatically under riskless debt – equilibria exist if $\nu(\mathcal{E}_x^\vartheta) = 0$ and $b_0 \leq \min_{\varepsilon\in\mathcal{E}_s^\vartheta}\{\psi_\varepsilon^\vartheta(w_0)\}$ but are generically asymptotically debtless with probability one.
Lemma 3.3 also entails important insights concerning the discussion in Bertocchi (1994) about stable sets under safe debt. Referring to the cases discussed there, it shows that steady states which are asymptotically stable and give rise to stable sets with positive debt do not exist. In particular, exploiting that the shocks in (18a),(18b) enter in a multiplicative fashion, it is possible to show that a scenario as in the example of Section 2 where the set $\mathcal{M}_\varepsilon$ was independent of ε is not possible under safe debt. The reason is that for this to happen the stable manifold (20) would have to be a horizontal line in $\mathbb{V}_+$. This, however, is impossible since each $\psi_\varepsilon^\vartheta$ is strictly increasing by Theorem A.1(ii).

4 Debt Stabilization through Taxation

To analyze the long-run welfare effects of debt, it seems natural to formulate a criterion that measures consumer welfare at some stationary solution of the state dynamics. In the stochastic case, the latter corresponds to an invariant probability distribution on the state space $\mathbb{V}$ which extends the deterministic concept of a steady state. The previous analysis revealed, however, that even if they exist, equilibria are generically asymptotically debtless. Therefore, neither the optimum quantity of debt nor the risk-sharing effects of different interest policy can be analyzed because, asymptotically, equilibria are independent of ϑ. Structurally, the reason is that stable subsets of the state space $\mathbb{V}_+$ (which can be associated with invariant distributions, cf. Wang (1993)) fail to exist.
The present section investigates whether this may be overcome by a tax on labor income which stabilizes debt against unfavorable shocks. More specifically, given a subset of $\mathbb{V}_+$ to be stabilized, the idea is to design a tax policy to counteract shocks under which the state would leave the set. In this regard, the goal is to keep stabilization taxes as minimal as possible. In particular, taxes should be zero if no stabilization is required, as in the example of Section 2. The policy to be developed satisfies all these requirements.
We consider the scenario of Section 3 with multiplicative noise and some interest policy ϑ with $\mathcal{E}_s^\vartheta \neq \emptyset$. Note that we permit $\mathcal{E}_x^\vartheta \neq \emptyset$, i.e., some maps $\Phi^\vartheta(\cdot;\varepsilon)$ may be explosive. To motivate the approach, suppose for a moment that the shocks are degenerate, i.e., $\varepsilon_t \equiv \varepsilon$ or, equivalently, $\mathcal{E} = \{\varepsilon\}$. In this case, the deterministic case studied in Tirole (1985) is recovered and a stable set would be given by (a subset of) the stable manifold $\mathcal{M}_\varepsilon^\vartheta$ defined in (20). For any initial value $(w_0, b_0) \in \mathcal{M}_\varepsilon^\vartheta$ the system converges to the golden rule steady state $(\bar{w}_\varepsilon, \bar{b}_\varepsilon) \in \mathcal{M}_\varepsilon^\vartheta$. In the non-degenerate case, the stable manifold $\mathcal{M}_\varepsilon^\vartheta$ in (20) associated with some shock $\varepsilon \in \mathcal{E}_s^\vartheta$ is self-supporting under the map $\Phi^\vartheta(\cdot;\varepsilon)$, i.e., $\Phi^\vartheta(\mathcal{M}_\varepsilon^\vartheta;\varepsilon) \subset \mathcal{M}_\varepsilon^\vartheta$ but, in general, not under $\Phi^\vartheta(\cdot;\varepsilon')$ where $\varepsilon' \neq \varepsilon$. Nevertheless, if stabilization taxes are to be small, it seems natural to exploit the system's inherent stability forces by stabilizing the state along the set $\mathcal{M}_\varepsilon^\vartheta$ associated with some $\varepsilon \in \mathcal{E}$. In this case, taxes are zero whenever realization ε occurs and, by continuity, small for shocks

close to this value. Therefore, stabilization expenditures should remain small at least if the variance of shocks is not too large. As an example, suppose $\mathcal{E} = \{\varepsilon_{min}, \bar{\varepsilon}, \varepsilon_{max}\}$ where $\nu(\{\varepsilon_{min}\}) = \nu(\{\varepsilon_{max}\}) = .05$ and $\bar{\varepsilon} \in \mathcal{E}_s^\vartheta$. In this case, stabilizing the set $\mathcal{M}_{\bar{\varepsilon}}^\vartheta$ requires governmental intervention on average every ten periods only. By contrast, any other objective such as stabilizing the absolute level of debt ($b_t \equiv \bar{b}$, cf. Diamond (1965)) or the debt-to output ratio ($b_t/y_t \equiv \bar{\beta}$, cf. de la Croix & Michel (2002)) is essentially arbitrary and not related to the system's stability properties. It stands to reason that, in general, such a stabilization objective requires much higher stabilization taxes.

To formalize the previous ideas, let $w_t^g > 0$ denote the gross wage defined by (3) and $\tau_t := \tau_t^y < w_t^g$ the tax levied on labor income w_t^g in period $t \geq 0$. Then, $w_t := w_t^g - \tau_t$ is the net wage and $b_t = r_t^\star b_{t-1} - \tau_t$ is the debt defined by (5) corresponding to the number of bonds issued in period t. If $\tau_t > 0$, the revenues generated from taxation are used to pay part of the return on outstanding debt. If $\tau_t < 0$, young consumers receive a subsidy on their wage income which is financed by issuing additional debt. The state space $\mathbb{V}$ consists of all pairs $(w_t, b_t) \in \mathbb{R}_+^2$ such that $w_t > b_t$. Define a feasible policy as a pair $\pi := (\vartheta; \varepsilon_{\text{ref}})$ consisting of interest policy $\vartheta : \mathcal{E} \longrightarrow \mathbb{R}_{++}$ and some reference shock $\varepsilon_{\text{ref}} \in \mathcal{E}_s^\vartheta$ which identifies the set $\mathcal{M}^\pi := \mathcal{M}_{\varepsilon_{\text{ref}}}^\vartheta$ to be stabilized. By Theorem A.1, there exists a map $\psi^\pi := \psi_{\varepsilon_{\text{ref}}}^\vartheta : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ such that $\mathcal{M}^\pi = \text{graph}(\psi^\pi)$. Using this representation, we construct a stabilization policy such that $(w_t, b_t) \in \mathcal{M}^\pi$ for all t with probability one. Consider an arbitrary period $t \geq 0$. Let the previous net wage w_{t-1} and previous debt b_{t-1} together with the current realization of the shock $\varepsilon_t \in \mathcal{E}$ be given. These values define the gross wage and debt before taxation $(w_t^g, b_t^g) := \Phi^\vartheta(w_{t-1}, b_{t-1}; \varepsilon_t)$ with Φ^ϑ defined as in (18a,b). Assuming that $(w_t^g, b_t^g) \in \mathbb{V}$ we look for a value $\tau_t < w_t^g$ such that $(w_t^g - \tau_t, b_t^g - \tau_t) \in \mathcal{M}^\pi$ or, equivalently, $b_t^g - \tau_t = \psi^\pi(w_t^g - \tau_t)$.

Lemma 4.1
Given π, let the map $\psi^\pi : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ that represents $\mathcal{M}^\pi$ satisfy $\lim_{w\to\infty} \psi^{\pi\prime}(w) \neq 1$. Then, for each $(w, b) \in \mathbb{V}$ there exists a unique $\tau < w$ such that $b - \tau = \psi(w - \tau)$.

By Theorem A.1, ψ^π is strictly increasing with derivative $\psi^{\pi\prime}(w) \leq \frac{\partial_w \mathcal{K}^\vartheta(w, \psi^\pi(w))}{-\partial_b \mathcal{K}^\vartheta(w, \psi^\pi(w))} < 1$ for all $w > 0$. Hence, the additional requirement in Lemma 4.1 should generically be satisfied. Then, given π there exists a map $\mathcal{T}^\pi : \mathbb{V} \longrightarrow \mathbb{R}$ that defines for each point $(w, b) \in \mathbb{V}$ the corresponding tax adjustment $\tau = \mathcal{T}^\pi(w, b)$ such that $(w - \tau, b - \tau) \in \mathcal{M}^\pi$. Letting $\mathbb{V}^\pi := \{(w, b) \in \mathbb{V} \,|\, \Phi^\vartheta(w, b; \varepsilon) \in \mathbb{V} \ \forall \varepsilon \in \mathcal{E}\}$, the stabilized dynamics are determined by the mapping $\Psi^\pi = (\Psi_w^\pi, \Psi_b^\pi) : \mathbb{V}^\pi \times \mathcal{E} \longrightarrow \mathcal{M}^\pi$

$$w_{t+1} = \Psi_w^\pi(w_t, b_t; \varepsilon_{t+1}) := \Phi_w^\vartheta(w_t, b_t; \varepsilon_{t+1}) - \mathcal{T}^\pi(\Phi^\vartheta(w_t, b_t; \varepsilon_{t+1})) \tag{21a}$$

$$b_{t+1} = \Psi_b^\pi(w_t, b_t; \varepsilon_{t+1}) := \Phi_b^\vartheta(w_t, b_t; \varepsilon_{t+1}) - \mathcal{T}^\pi(\Phi^\vartheta(w_t, b_t; \varepsilon_{t+1})). \tag{21b}$$

The following figure illustrates how the proposed tax-policy stabilizes the set $\mathcal{M}^\pi$.

Given a feasible policy π, any initial value can be tax-adjusted to lie on the set $\mathcal{M}^\pi$ and the state (w_t, b_t) remains in $\mathcal{M}^\pi$ for all $t \geq 0$. Hence, for $(w_0, b_0) \in \mathcal{M}^\pi$ the dynamics (21a,b) are essentially one-dimensional and governed by the map $\phi^\pi : \mathbb{R}_{++} \times \mathcal{E} \longrightarrow \mathbb{R}_{++}$

$$w_{t+1} = \phi^\pi(w_t; \varepsilon_{t+1}) := \Psi_w^\pi(w_t, \psi^\pi(w_t); \varepsilon_{t+1}) \tag{22}$$

while debt is given by $b_{t+1} = \psi^\pi(w_{t+1})$. The next result establishes properties of ϕ^π.

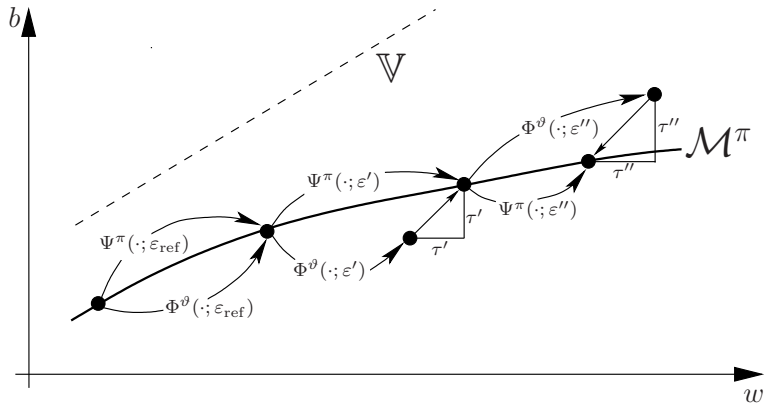


Figure 2: Tax-stabilization of the set $\mathcal{M}^\pi$.

Lemma 4.2
Suppose that policy π is feasible and (a) $\psi^{\pi\prime}(w) < \frac{\varepsilon}{\varepsilon_{\text{ref}}}\frac{\vartheta(\varepsilon_{\text{ref}})}{\vartheta(\varepsilon)}$ and (b) ϑ is differentiable with derivative $\vartheta'(\varepsilon) \leq \frac{\vartheta(\varepsilon_{\text{ref}})}{\varepsilon_{\text{ref}}}$ for $w > 0$ and $\varepsilon \in \mathcal{E}$. Then, the following holds:

(i) The map $w \longmapsto \phi^\pi(w;\varepsilon)$ is strictly increasing at all (w,ε) which satisfy (a).

(ii) The map $\varepsilon \longmapsto \phi^\pi(w;\varepsilon)$ is strictly increasing at all (w,ε) which satisfy (b).

Condition (a) is automatically satisfied in the CE case where $\vartheta = \text{id}_{\mathcal{E}}$ while (b) holds, e.g., if $\vartheta(\varepsilon) = \lambda\varepsilon + (1-\lambda)\bar{\varepsilon}\ \forall \varepsilon \in \mathcal{E}$, $\lambda \in [0,1]$, $\bar{\varepsilon} := \mathbb{E}_\nu[\varepsilon_t]$. For this class, (a) is satisfied if the range of noise is not too large such that $\psi^{\pi\prime}(w) < \frac{\varepsilon_{\min}}{\varepsilon_{\max}}$ for $w > 0$. Assuming that both (a) and (b) are satisfied, we seek to study the long-run properties of the dynamics defined by (22) and the existence of invariant distributions corresponding to stable sets of ϕ^π. For a formal definition of these concepts, the reader is referred to Brock & Mirman (1972) and Wang (1993). The following final result draws heavily on their findings.

Theorem 4.1
Let π be a feasible policy under which (a) and (b) in Lemma 4.2 hold. In addition, suppose $\lim_{w\to\infty} \phi^\pi(w;\varepsilon_{max})/w < 1 < \lim_{w\to 0} \phi^\pi(w;\varepsilon_{min})/w$. Then, the following holds:

(i) There exists a unique stable set $\bar{W}^\pi \subset \mathbb{R}_{++}$ for the family $\phi^\pi = (\phi^\pi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$.

(ii) There exists a unique invariant distribution μ^π of the dynamical system (22) which is supported on $\bar{W}^\pi$ and which is stable in the weak convergence sense.

5 Optimal Debt Policies

Based on the previous results this section develops a welfare criterion that allows to select an optimal policy. Let $\pi = (\vartheta, \varepsilon_{\text{ref}})$ satisfy the hypotheses of Theorem 4.1. For $(w,b) \in \mathbb{V}$ and $\varepsilon_+ \in \mathcal{E}$, denote first and planned second period consumption as $c^y(w,b;\vartheta) := w - b - \mathcal{K}^\vartheta(w,b)$ and $c^o(w,b,\varepsilon_+;\vartheta) := b\,\mathcal{Z}^\vartheta(w,b)\vartheta(\varepsilon_+) + \mathcal{K}^\vartheta(w,b)\mathcal{R}(\mathcal{K}^\vartheta(w,b),\varepsilon_+)$ defining

ex-ante lifetime utility $V(w,b;\vartheta) := u(c^y(w,b;\vartheta)) + \mathbb{E}_\nu[v(c^o(w,b,\cdot;\vartheta))]$. By Theorem 4.1, there exists a unique probability distribution μ^π supported on $\bar{\mathbb{W}}^\pi \subset \mathbb{R}_{++}$ which is invariant under (22) and determines long-run expected utility

$$U(\vartheta, \varepsilon_{\text{ref}}) := \int_{\mathbb{R}_{++}} V(w, \psi^\pi(w); \vartheta)\mu^\pi(dw). \tag{23}$$

Note that U is well-defined by continuity of the integrand and compactness of $\bar{W}^\pi$. The value $U(\vartheta, \varepsilon_{\text{ref}})$ can be interpreted as the average utility that consumers attain under policy π. With reference to the introduction, the interest policy ϑ determines the *risk-sharing* of debt while the *level of debt* relative to the net wage obtains as $b_t = \psi^\pi(w_t)$ for all t. Moreover, the domain of U can be extended to arbitrary policies by setting $\psi^\pi \equiv 0$ if $\pi = (\vartheta, \varepsilon_{\text{ref}})$ is non-feasible, i.e., $\varepsilon_{\text{ref}} \in \mathcal{E}_x^\vartheta$. In this case, (23) yields utility at the trivial equilibrium which is independent of π. With this extension, the criterion (23) is suitable to investigate the long-run welfare effects of arbitrary debt policies $\pi = (\vartheta, \varepsilon_{\text{ref}})$.
The remainder of the paper reports simulation results for a simple parametrization of the model.[6] We assume the scenario in Section 3 with power utilities $u(c) = c^\theta$, $v(c) = \gamma u(c)$, CES technology $g(k) = [1 - A + Ak^\alpha]^{\frac{1}{\alpha}}$, and three possible shocks $\mathcal{E} = \{\varepsilon^{\min}, \varepsilon^{\text{med}}, \varepsilon^{\max}\}$ drawn with probabilities $p^{\min}$, p^{med}, and $p^{\max}$. Table 1 lists the parameter values.

Parameter	Value	Parameter	Value	Parameter	Value
$\varepsilon^{\min}$	0.9	$\varepsilon^{\max}$	1.1	γ	1
ε^{med}	1	$p^{\min}$, p^{med}	1/3	A, α, θ	0.5

Table 1: Parameter set used in the simulations.

We confine attention to the class of interest policies $\vartheta_\lambda(\varepsilon) := \lambda\varepsilon + (1-\lambda)\bar{\varepsilon}$, $\varepsilon \in \mathcal{E}$ parameterized in $\lambda \in [0,1]$. In particular, debt is capital equivalent if $\lambda = 1$ and riskless if $\lambda = 0$. Under the previous parametrization, Assumption 3.1 holds and all policies are feasible, i.e., $\mathcal{E}_s^{\vartheta_\lambda} \equiv \mathcal{E}$. Hence, an optimal policy exists if U is continuous.
To quantify the welfare effects of different policies, Table 2 reports the utilities defined in (23) expressed as percentage deviations from utility at the trivial equilibrium. The latter exists and is unique by the results of Wang (1993) and Assumption 3.1.[7]

	$\lambda =$	0 (safe debt)	0.25	0.5	0.75	1 (CE debt)
	$\varepsilon^{\min}$	0.7508%	0.8071%	0.8196%	0.7838%	0.6944%
$\varepsilon^{\text{ref}} =$	ε^{med}	0.8175%	0.8197%	0.8210%	0.8212%	0.8204%
	$\varepsilon^{\max}$	0.7326%	0.8058%	0.8185%	0.7753%	0.6799%

Table 2: Long-run utilities under different debt policies.

[6] To replicate and verify the subsequent results, the reader is invited to download the $C++$ simulation files from my homepage `http://www.marten-hillebrand.de/research/TC/TC.htm`.

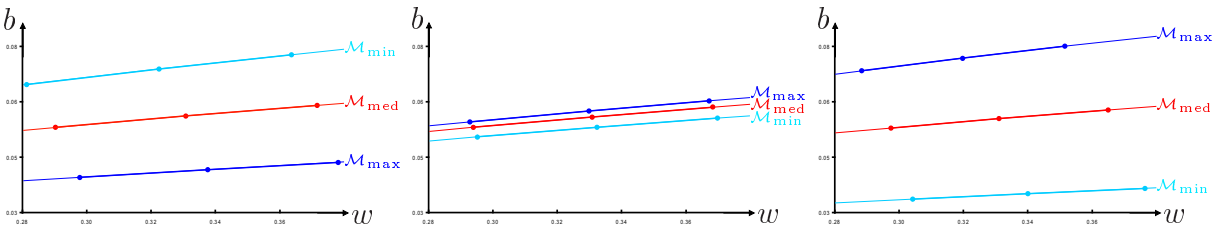
[7] The utilities defined in (23) were calculated as time averages of ex-ante utilities $V_t := V(w_t, b_t; \vartheta)$. Since the state process follows an ergodic Markov process, these averages converge to the corresponding expected utility by means of the ergodic theorem.

The values in Table 2 identify a unique optimal policy $\pi^\star$ given by $\varepsilon^\star_{\text{ref}} = \varepsilon_{\text{med}}$ and $\lambda^\star = .75$. Additional simulations permit to refine the latter value to $\lambda^\star \approx .7$. Note that for any fixed reference shock there exists a unique optimal interest policy determined by an *interior* value of λ. As a consequence, a riskless debt return is never optimal.
Since the stabilization policy was designed to keep taxation as minimal as possible, it seems worthwhile to confirm this property for the present case. In this regard, Table 3 displays the absolute values of taxes expressed as a percentage of debt.

$\lambda =$	0 (safe debt)	0.25	0.5	0.75	1 (CE debt)
ε^{min}	4.23%	1.88%	0.65%	3.41%	6.44%
$\varepsilon^{\text{ref}} = \varepsilon^{\text{med}}$	3.12%	1.31%	0.49%	2.29%	4.09%
ε^{max}	5.21%	2.05%	0.83%	3.46%	5.89%

Table 3: Average absolute stabilization taxes expressed as a percentage of debt.

For any reference value, taxes are least for $\lambda = .5 < \lambda^\star$. Again, this value may further be refined to $\lambda \approx .4$. Moreover, for all scenarios considered in Table 3 taxes remain small ($< 13\%$) throughout the entire sample. Nevertheless, note that the interest policy has a crucial impact on the size of stabilization taxes for any reference shock.
To provide some intuition for the last result, Figure 3 portrays the stable manifolds (20) under different interest policies. The bold sections represent the support of the invariant distribution which is bordered by the (smallest and largest) fixed points of $\Psi^\pi(\cdot;\varepsilon)$ respectively $\phi^\pi(\cdot;\varepsilon)$ which are also depicted. Intuitively, if the shock $\varepsilon_t = \varepsilon$ occurs at



(a) $\lambda = 0$ (riskless) (b) $\lambda = 0.5$ (tax-minimal) (c) $\lambda = 1$ (CE)

Figure 3: Stable sets $\mathcal{M}_{\text{min}} := \mathcal{M}^{\vartheta_\lambda}_{\varepsilon_{\text{min}}}$, $\mathcal{M}_{\text{med}} := \mathcal{M}^{\vartheta_\lambda}_{\varepsilon_{\text{med}}}$, $\mathcal{M}_{\text{max}} := \mathcal{M}^{\vartheta_\lambda}_{\varepsilon_{\text{max}}}$.

time t, taxes τ_t are large (in absolute value) if the previous state (w_{t-1}, b_{t-1}) is far away from the set $\mathcal{M}^\vartheta_\varepsilon$ and small for (w_{t-1}, b_{t-1}) close to $\mathcal{M}^\vartheta_\varepsilon$ (and zero if $(w_{t-1}, b_{t-1}) \in \mathcal{M}^\vartheta_\varepsilon$). As a consequence, taxes are least in Figure 3(b) where the stable manifolds are close together. Ideally, they would coincide as in the example of Section 2 in which case there would be no need for stabilization. Albeit this can not be achieved in the present case, the interest policy can be chosen as in Figure 3(b) such that taxes become negligible ($< .9\%$ of debt throughout and even smaller for $\lambda = .4$).
Another observation from Figures 3(a,c) is that the order of the stable sets is reversed which affects the sign of taxes depending on the shock. If the set $\mathcal{M}_{\text{med}}$ is stabilized ($\varepsilon_{\text{ref}} = \varepsilon_{\text{med}}$) and debt is capital-equivalent ($\lambda = 1$), young consumers are taxed ($\tau_t > 0$)

in 'good times' ($\varepsilon_t = \varepsilon_{\max}$) and receive a subsidy ($\tau_t < 0$) in 'bad times' ($\varepsilon_t = \varepsilon_{\min}$) while the opposite holds if debt is riskless ($\lambda = 0$). Moreover, if $\mathcal{M}_{\max}$ is stabilized, taxes are uniformly positive if $\lambda = 0$ and negative if $\lambda = 1$ while the opposite holds for $\mathcal{M}_{\min}$.

6 Conclusions

The results obtained in this paper suggest that any sustainable debt policy must be accompanied by a stabilizing tax-policy that prevents debt from exploding or converging to zero due to unfavorable random shocks. Based on this insight, a stabilization policy was developed designed to keep taxation as minimal as possible by exploiting the inherent stabilizing forces of the underlying dynamical system. A golden-rule type welfare criterion was suggested which measures consumer welfare at the stationary solution of the stabilized equilibrium and permits to simultaneously determine the optimal level of debt and an optimal interest policy. First simulation results indicate that (i) a unique optimal policy can be determined, (ii) offering a riskless return on debt is never optimal, (iii) taxes remain small under the proposed stabilization concept and depend crucially on the interest policy (iv) the latter can be chosen such that taxes become negligible. The finding that a riskless return on debt is not optimal confirms existing results in the literature, cf. Bohn (1991). Nevertheless, a better theoretical understanding of the previous numerical results seems necessary and is the primary objective of future research. In addition, recent research in the literature (e.g., Kunieda (2008)) has attempted to explain the emergence of bubbles in dynamically efficient economies by hypothesizing certain imperfections in the credit market. Another objective of future research is to analyze whether such imperfections could also explain the existence of non-trivial equilibria with governmental debt even if the trivial equilibrium is efficient.

A Mathematical Appendix

A.1 Proof of Lemma 1.1

For $t \geq 0$, define taxes $\tau_t^y := w_t^g - c_t^y - k_{t+1} - b_t < w_t$ and $\tau_t^o := b_{t-1} r_t^\star + k_t r_t - c_t^o$ which are feasible in the sense of Definition 1.1(ii). Using the corresponding expressions for c_t^y and c_t^o together with (3) and (4) in the aggregate feasibility condition shows that debt evolves according to equation (5). Since Definition 1.1(i) is satisfied by assumption, it remains to show that (b_t, s_t) solves (10). Since $s_t > 0$ and there are no short-sale constraints, it suffices to show that the first-order conditions are satisfied. This follows from the intertemporal efficiency condition and (11) by direct substitution. ■

A.2 Proof of Lemma 2.1

Given $w > b \geq 0$, define $\bar{k} := w - b > 0$ and, for brevity, $c^o(k, b, \varepsilon) := \mathcal{R}(k; \varepsilon)(b + k)$. We show that $H(\cdot; w, b)$ is strictly increasing and can, therefore, have at most one zero

in the interval $]0, \bar{k}[$. By (P1), the partial derivative[8] of (12) takes the form

$$\begin{aligned}\partial_k H(k;w,b) &= -u''(w-b-k) - \mathbb{E}_\nu\left[f_k(k;\varepsilon)^2 v''\big(c^o(k,b,\cdot)\big)\right] \\ &\quad -\mathbb{E}_\nu\Big[f_{kk}(k;\varepsilon)\big(v'\big(c^o(k,b,\cdot)\big) + c^o(k,b,\cdot)v''\big(c^o(k,b,\cdot)\big)\big)\Big] > 0. \quad \text{(A.1)}\end{aligned}$$

We show that $H(\cdot;w,b)$ has a zero in the interval $]0,\bar{k}[$. As existence of a zero for the case $b=0$ is established in Wang (1993), suppose $b>0$. Since there exists a unique zero for $b=0$ and $\partial_k H(k;w,0)>0$, there exists a zero also for $b>0$ sufficiently small by virtue of the implicit function theorem. Define the critical value $0 < b^{max}(w) \leq w$ as

$$b^{max}(w) := \sup\Big\{b \in]0,\ w[\ |\ H(k;w,b) = 0 \ \text{ for some } k \in]0, w-b[\Big\}.$$

Since the supremum is taken over a non-empty set bounded by w, $b^{max}(w)$ is well-defined. We claim that H has a zero for each $b \in]0, b^{max}(w)[$ which, by the previous results, is unique. By way of contradiction, suppose that for, say, $0 < b' < b^{max}(w)$ there is no zero of $H(\cdot;w,b')$. The Inada-assumption in (9) implies $\lim_{k\to\bar{k}} H(k;w,b') = \infty$ such that $H(k;w,b')>0$ for all $0<k<w-b'$. The derivative with respect to b satisfies

$$\partial_b H(k;w,b) = -u''(w-b-k) - \mathbb{E}_\nu\left[\mathcal{R}(k;\varepsilon)^2 v''\big(c^o(k,b,\varepsilon)\big)\right] > 0.$$

Let $b'' > b'$. Then, $H(k;w,b'') > H(k;w,b') > 0$ for all $0 < k < w-b'' < w-b'$. Hence, $H(\cdot;w,b'')$ has no zero for any $b'' > b'$. But then $b' \geq b^{max}(w)$, which is a contradiction. Finally, $\lim_{k\to 0} c^o(k,b,\varepsilon) = \lim_{k\to 0}(bf_k(k;\varepsilon) + kf_k(k;\varepsilon)) \geq \lim_{k\to 0} bf_k(k;\varepsilon) = \infty$ for each fixed $\varepsilon \in \mathcal{E}$. Hence, if (P3) holds, the left limit computes

$$\lim_{k\to 0} H(k;w,b) = u'(w-b) - \lim_{k\to 0}\left(\frac{1}{b+k}\mathbb{E}_\nu\left[c^o(k,b,\cdot)v'\big(c^o(k,b,\cdot)\big)\right]\right) = -\infty. \quad \text{(A.2)}$$

In this case, there exists a zero for all $0<b<w$ implying that $b^{max}(w) = w$. ■

A.3 Proof of Lemma 2.2

Define $c^o(k,b,\varepsilon)$ as in the previous proof. The claim follows by applying the implicit function theorem. The partial derivatives of the map H defined in (12) compute

$$\begin{aligned}\partial_w H(k;w,b) &= u''(w-b-k) < 0 \quad \text{(A.3)} \\ \partial_b H(k;w,b) &= -\partial_w H(k;w,b) - \mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)^2 v''\big(c^o(k,b,\cdot)\big)\right] > -\partial_w H(k;w,b). \quad \text{(A.4)}\end{aligned}$$

Moreover, by (A.1) the derivative with respect to k may be written as

$$\partial_k H(k;w,b) = \partial_b H(k;w,b) - \mathbb{E}_\nu\left[f_{kk}(k;\cdot)(v'(c^o(k,b,\cdot)) + (c^o(k,b,\cdot)\, v''(c^o(k,b,\cdot)))\right] \quad \text{(A.5)}$$

showing that $\partial_k H(k;w,b) \geq \partial_b H(k;w,b)$ by (P1). Combining (A.3) – (A.5) yields

$$0 < \partial_w \mathcal{K}(w,b) = \frac{-\partial_w H(k;w,b)}{\partial_k H(k;w,b)} < -\partial_b \mathcal{K}(w,b) = \frac{\partial_b H(k;w,b)}{\partial_k H(k;w,b)} \leq 1. \quad \blacksquare$$

[8] Note that interchanging differentiation with the expectations operator $\mathbb{E}_\nu[\cdot]$ is legitimate since the integrand is continuously differentiable and integration is over a compact set.

A.4 Proof of Lemma 2.3

Let $\varepsilon \in \mathcal{E}^s$ be fixed. For brevity, we omit the subscript ε such that $\bar{w}^0 > 0$ denotes the trivial steady state. In addition, define $\underline{w} := \mathcal{W}(0;\varepsilon) \geq 0$ and $\bar{k}^0 := \mathcal{K}(\bar{w}^0, 0)$.
(i) We determine unique values $\bar{k} > 0$ and $0 < \bar{b} < \bar{w}$ solving $k = \mathcal{K}(w,b)$, $w = \mathcal{W}(k,\varepsilon)$, and $\mathcal{R}(k,\varepsilon) = 1$. Since $\lim_{k\to 0} \mathcal{R}(k,\varepsilon) = \infty$ and $\mathcal{R}(\bar{k}^0,\varepsilon) < 1$, the last condition has a solution $\bar{k} \in]0, \bar{k}^0[$ which is unique by strict concavity of $f(\cdot;\varepsilon)$ and determines $\bar{w} := \mathcal{W}(\bar{k},\varepsilon) < \bar{w}^0$. We determine $\bar{b}$ as a solution to $\bar{w} = \mathcal{W}(\mathcal{K}(\bar{w},b),\varepsilon)$. By Lemma 2.2, there can be at most one such solution. By uniqueness and stability of $\bar{w}^0$, $\phi_0(w;\varepsilon) > w$ for all $w \in]\underline{w}, \bar{w}^0[$. Hence, $\bar{w} < \bar{w}^0$ implies $\lim_{b\to 0} \mathcal{W}(\mathcal{K}(\bar{w},b),\varepsilon) = \mathcal{W}(\mathcal{K}(\bar{w},0),\varepsilon) > \bar{w}$. Since $\mathcal{K}(\bar{w},b) < \bar{w} - b$, $\lim_{b\to\bar{w}} \mathcal{W}(\mathcal{K}(\bar{w},b),\varepsilon) = \underline{w} < \bar{w}$ proving that a unique non-trivial steady state exists. The Jacobian at the steady state computes

$$\bar{J} := D\Phi(\bar{w},\bar{b};\varepsilon) = \begin{bmatrix} -\bar{k} f_{kk}(\bar{k};\varepsilon)\partial_w \mathcal{K}(\bar{w},\bar{b}) & -\bar{k} f_{kk}(\bar{k};\varepsilon)\partial_b \mathcal{K}(\bar{w},\bar{b}) \\ b f_{kk}(\bar{k};\varepsilon)\partial_w \mathcal{K}(\bar{w},\bar{b}) & 1 + b f_{kk}(\bar{k};\varepsilon)\partial_b \mathcal{K}(\bar{w},\bar{b}) \end{bmatrix}.$$

By Lemma 2.2, the determinant and trace satisfy $\det \bar{J} = -\bar{k} f_{kk}(\bar{k};\varepsilon)\partial_w \mathcal{K}(\bar{w},\bar{b}) > 0$ and $\mathrm{tr}\bar{J} = 1 + \det \bar{J} + b f_{kk}(\bar{k};\varepsilon)\partial_b \mathcal{K}(\bar{w},\bar{b}) > 1 + \det \bar{J}$. The latter inequality implies $0 \leq (1 - \mathrm{tr}\bar{J}/2)^2 = 1 - \mathrm{tr}\bar{J} + (\mathrm{tr}\bar{J})^2/4 < -\det \bar{J} + (\mathrm{tr}\bar{J})^2/4$ ensuring real and distinct Eigenvalues of $\bar{J}$. By Galor (2007, p.88), these properties imply saddle-path stability.
(ii) Let $\varepsilon \in \mathcal{E}_x$ be fixed. By contradiction, suppose there exists $(w,b) \in \mathbb{V}_+$ such that $(\tilde{w}_t, \tilde{b}_t) := \Phi^t(w,b;\varepsilon) \in \mathbb{V}$ for all $t \geq 0$. Defining $\hat{w}_t := \phi_0^t(w;\varepsilon)$, Lemma 2.2 implies $\tilde{w}_t < \hat{w}_t$ and $\tilde{b}_t > 0$ for all t. By stability, $\lim_{t\to\infty} \hat{w}_t = \bar{w}^0$. This and continuity of $\mathcal{R}(\cdot;\varepsilon)$ imply existence of $T > 0$ such that $\mathcal{R}(\mathcal{K}(\hat{w}_t,0);\varepsilon) > 1$ for all $t \geq T$ implying that $\tilde{b}_{t+1}/\tilde{b}_t = \mathcal{R}(\mathcal{K}(\tilde{w}_t,\tilde{b}_t);\varepsilon) > \mathcal{R}(\mathcal{K}(\hat{w}_t,0);\varepsilon) > 1$. Hence, $(\tilde{b}_t)_{t\geq 0}$ is eventually strictly increasing and, therefore, either diverges or converges. But convergence to some finite value, say $b_\infty > 0$ would imply $\lim_{t\to\infty} \mathcal{R}(\mathcal{K}(\tilde{w}_t,\tilde{b}_t);\varepsilon) = 1$ requiring by the strict monotonicity of $w \mapsto \mathcal{R}(\mathcal{K}(\cdot,b);\varepsilon)$, $b > 0$ that $\lim_{t\to\infty} \tilde{w}_t = w_\infty \leq \bar{w}^0$ and $\mathcal{R}(\mathcal{K}(w_\infty,b_\infty);\varepsilon) = 1$, which is impossible. Hence, $\lim_{t\to\infty} \tilde{b}_t = \infty$ which contradicts $\tilde{b}_t < \tilde{w}_t < \hat{w}_t$ for all t. ■

A.5 Properties of the Stable Manifold

This section establishes properties of the stable manifold $\mathcal{M}_\varepsilon^\vartheta$ defined in (20). Many of the employed concepts and a somewhat related analysis may be found in Galor (1992). While the formal arguments adopt the setup and notation of Section 3, neither the multiplicative structure of f nor the additional assumptions (P2)–(P6) are used. Therefore, Theorem A.1 also applies for the scenario of Section 2 under the (weaker) hypotheses of Lemma 2.3 with the stable manifold $\mathcal{M}_\varepsilon$ defined as in (15). The main result is

Theorem A.1
Given some interest policy ϑ, let the hypotheses of Lemma 3.3 be satisfied. In addition, suppose $\lim_{c\to\infty} u'(c) = 0$. Then, for each $\varepsilon \in \mathcal{E}_s^\vartheta$ the following holds:

(i) The set $\mathcal{M}_\varepsilon^\vartheta$ defined in (20) is the graph of a C^1-map $\psi_\varepsilon^\vartheta : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$.

(ii) The map $\psi_\varepsilon^\vartheta$ is strictly increasing and satisfies $\lim_{w\to 0} \psi_\varepsilon^\vartheta(w) = 0$.

(iii) For all $\hat{\varepsilon} \in \mathcal{E}$ the map $w \longmapsto \mathcal{W}(\mathcal{K}^{\vartheta}(w, \psi_{\varepsilon}^{\vartheta}(w)); \hat{\varepsilon})$ is strictly increasing as well.

In what follows fix ϑ and $\varepsilon \in \mathcal{E}_s^{\vartheta}$ and suppress these parameters writing $\Phi = \Phi^{\vartheta}(\cdot; \varepsilon)$, etc. For ease of notation, the restriction of $F : \mathbb{X} \longrightarrow \mathbb{R}$ to $\mathbb{A} \subsetneq \mathbb{X}$ is denoted by the same letter F instead $F_{|\mathbb{A}}$. Furthermore, let $\underline{w}^0 := \mathcal{W}(0; \varepsilon) \geq 0$, $\bar{w}^{\infty} := \lim_{k \to \infty} \mathcal{W}(k; \varepsilon) \leq \infty$, $\mathbb{W}^+ :=]\underline{w}^0, \bar{w}^{\infty}[$, $\mathbb{U} := \mathbb{W}^+ \times \mathbb{R}_{++}$, and $\bar{\mathbb{V}} := \{(w, b) \in \mathbb{V} \,|\, \Phi^n(w, b) \in \mathbb{V} \ \forall n \geq 1\} \subset \mathbb{V}$. Obviously, $\Phi : \bar{\mathbb{V}} \longrightarrow \bar{\mathbb{V}}$ and the stable set $\mathcal{M} := \{(w, b) \in \bar{\mathbb{V}} \,|\, \lim_{n \to \infty} \Phi^n(w, b) = (\bar{w}, \bar{b})\}$ is a subset of $\bar{\mathbb{V}}$ containing $(\bar{w}, \bar{b})$. The proof is prefaced by the following results.

Lemma A.1
Let $(\hat{w}, \hat{b}) \neq (\tilde{w}, \tilde{b}) \in \bar{\mathbb{V}}$ be such that $\hat{w} \geq \tilde{w}$ and $\hat{b} \leq \tilde{b}$ with at leat one strict inequality. Then the sequences $\{\Phi^n(\hat{w}, \hat{b})\}_{n \geq 0}$ and $\{\Phi^n(\tilde{w}, \tilde{b})\}_{n \geq 0}$ can not converge to the same limit.

Proof of Lemma A.1. Let $(\hat{w}_n, \hat{b}_n) := \Phi^n(\hat{w}, \hat{b})$ and $(\tilde{w}_n, \tilde{b}_n) := \Phi^n(\tilde{w}, \tilde{b})$, $n > 0$. By induction and Lemma 3.2, $\hat{w}_n > \tilde{w}_n$, $\hat{b}_n < \tilde{b}_n$ and $\beta_n := \hat{b}_n / \tilde{b}_n$ satisfy $\beta_{n+1} = \frac{\mathcal{Z}(\hat{w}_n, \hat{b}_n)}{\mathcal{Z}(\tilde{w}_n, \tilde{b}_n)} \beta_n < \beta_n$ for $n \geq 0$. Since $\{\beta_n\}_{n \geq 0}$ is strictly decreasing and bounded by zero, the limit $\beta_{\infty} := \lim_{n \to \infty} \beta_n$ exists and $0 \leq \beta_{\infty} < \beta_1 < 1$. If $(\hat{w}_n, \hat{b}_n)_{n \geq 1}$ and $(\tilde{w}_n, \tilde{b}_n)_{n \geq 1}$ both converge, then $\lim_{n \to \infty} \frac{\hat{b}_n}{\tilde{b}_n} = \beta_{\infty} < 1$ which implies $\lim_{n \to \infty} \hat{b}_n \neq \lim_{n \to \infty} \tilde{b}_n$. □

Lemma A.2
Suppose $\lim_{c \to \infty} u'(c) = 0$. Then the map $\Phi : \mathbb{V}_+ \longrightarrow \mathbb{U}$ is a C^1-diffeomorphism.

Proof of Lemma A.2. Given some $(w', b') \in \mathbb{U}$ we determine a unique $(w, b) \in \mathbb{V}_+$ such that $\Phi(w, b) = (w', b')$. The condition $w' = \Phi_w(w, b)$ determines a unique $k' = \mathcal{K}(w, b)$ such that $w' = \mathcal{W}(k'; \varepsilon)$. The value $z' = \mathcal{Z}(w, b)$ then follows from the first order conditions $\mathbb{E}_{\nu}[z' \vartheta(\cdot) v'(b' + k' \mathcal{R}(k'; \cdot))] = \mathbb{E}_{\nu}[\mathcal{R}(k'; \cdot) v'(b' + k' \mathcal{R}(k'; \cdot))]$ from which $b = \frac{b'}{z' \vartheta(\varepsilon)}$ can be inferred. Finally, w is the unique solution to $u'(w - b - k') = \mathbb{E}_{\nu}[z' \vartheta(\cdot) v'(b' + k' \mathcal{R}(k'; \cdot))]$. Hence, Φ^{-1} is a well-defined function. Φ is clearly C^1 by the IFT. To see that Φ^{-1} is C^1, it is straightforward to show that the Jacobian $D\Phi(w, b)$ has determinant $\det D\Phi(w, b) > 0$ for each $(w, b) \in \mathbb{V}_+$. This yields the derivative of the inverse $D\Phi^{-1}(w', b') = [D\Phi(w, b)]^{-1}$ which is a continuous function. □

Proof of Theorem A.1.
Step 1: We show that $\mathcal{M}$ is a C^1 manifold. Since $(\bar{w}, \bar{b})$ is saddle-path stable under Φ, the so-called *Stable Manifold Theorem* (cf. Nitecki (1971)) implies existence of the locally stable manifold $\mathcal{M}^{\text{loc}} := \{(w, b) \in \mathbb{V}_+ \,|\, \Phi^n(w, b) \in U \ \forall n \geq 1 \wedge \lim_{n \to \infty} \Phi^n(w, b) = (\bar{w}, \bar{b})\}$ for some open neighborhood $U \subset \mathbb{V}_+$ of $(\bar{w}, \bar{b})$ where $\mathcal{M}^{\text{loc}}$ is as smooth as Φ. It is well-known (cf. Nitecki (1971, p.89) or Galor (1992)) that the globally stable manifold obtains as $\mathcal{M} = \cup_{n \geq 0} \Phi^{-n}(\mathcal{M}^{\text{loc}})$. The result from Lemma A.2 implies that $\mathcal{M}$ inherits the smoothness of $\mathcal{M}^{\text{loc}}$ and is hence a C^1-manifold in $\mathbb{V}_+$. Therefore, the projection $\mathbb{W} := \{w > 0 \,|\, \exists b > 0 : (w, b) \in \mathcal{M}\}$ is an interval containing $\bar{w}$ as an interior point.

Step 2: We show that $\mathcal{M} = \text{graph}(\psi)$ for some C^1-function $\psi : \mathbb{W} \longrightarrow \mathbb{R}_{++}$. It suffices to show that for each $w_0 \in \mathbb{W}$ there exists a unique $0 < b_0 < w_0$ such that $(w_0, b_0) \in \mathcal{M}$. Suppose there are two such values, say $\hat{b}_0 < \tilde{b}_0 < w_0$. Then, by Lemma A.1, the sequences generated from $(w_0, \hat{b}_0)$ and $(w_0, \tilde{b}_0)$ under Φ can not converge to the same limit which is a contradiction to (20). The smoothness of $\mathcal{M}$ then implies that ψ is C^1.

Step 3: We show that ψ is strictly increasing. Let $\hat{w} < \tilde{w}$ be two points in $\mathbb{W}$ and suppose that $\hat{b} := \psi(\hat{w}) \geq \psi(\tilde{w}) =: \tilde{b}$. By Lemma A.1, the sequences generated from both points under Φ can not converge to the same limit which contradicts $(\hat{w}, \hat{b}), (\tilde{w}, \tilde{b}) \in \mathcal{M}$.

Step 4: We show that $w \longmapsto \hat{\mathcal{K}}(w) := \mathcal{K}(w, \psi(w))$ is strictly increasing. We first claim that $\psi'(w) \leq \frac{\partial_w \mathcal{K}(w,\psi(w))}{\partial_b \mathcal{K}(w,\psi(w))}$ for all interior $w \in \mathbb{W}$. Suppose this fails to hold at some interior $\hat{w} \in \mathbb{W}$. Then, by (A.16), $\psi'(\hat{w}) > \frac{\partial_w \mathcal{K}(\hat{w},\psi(\hat{w}))}{-\partial_b \mathcal{K}(\hat{w},\psi(\hat{w}))} \geq \frac{-\partial_w \mathcal{Z}(\hat{w},\psi(\hat{w}))}{\partial_b \mathcal{Z}(\hat{w},\psi(\hat{w}))}$. By continuity, $\hat{\Phi}_w(w) := \mathcal{W}(\mathcal{K}(w, \psi(w)); \varepsilon)$ is locally decreasing while $w \mapsto \mathcal{Z}(w, \psi(w))$ and, by Step 3, $\hat{\Phi}_b(w) := \psi(w)\mathcal{Z}(w, \psi(w))\vartheta(\varepsilon)$ are non-decreasing and increasing, respectively around $\hat{w}$. Let $\hat{w} < \tilde{w} \in \mathbb{W}$ be an interior point close to $\hat{w}$. Set $\tilde{b} := \psi(\tilde{w}) > \hat{b} := \psi(\hat{w})$. Then, $(\hat{w}, \hat{b}), (\tilde{w}, \tilde{b}) \in \mathcal{M}$ and $\hat{w}_1 := \Phi_w(\hat{w}, \hat{b}) = \hat{\Phi}_w(\hat{w}) < \hat{\Phi}_w(\tilde{w}) = \Phi_w(\tilde{w}, \tilde{b}) =: \tilde{w}_1$ and $\hat{b}_1 := \Phi_b(\hat{w}, \hat{b}) = \hat{\Phi}_b(\hat{w}) < \hat{\Phi}_b(\tilde{w}) = \Phi_b(\tilde{w}, \tilde{b}) =: \tilde{b}_1$. But $\mathcal{M}$ being self-supporting under Φ, $(\hat{w}_1, \hat{b}_1) = \Phi(\hat{w}, \hat{b}) \in \mathcal{M}$ and $(\tilde{w}_1, \tilde{b}_1) = \Phi(\tilde{w}, \tilde{b}) \in \mathcal{M}$ and, therefore, $\hat{b}_1 = \psi(\hat{w}_1)$ and $\tilde{b}_1 = \psi(\tilde{w}_1)$ which contradicts that ψ is strictly increasing proving the claim. To see that $\hat{\mathcal{K}}$ resp. $\hat{\Phi}_w$ are even strictly increasing, suppose there are $\hat{w} > \tilde{w}$ such that $\hat{\Phi}_w(\hat{w}) = \hat{\Phi}_w(\tilde{w})$. Then, $\hat{\Phi}_w$ must be constant on the interval $[\tilde{w}, \hat{w}]$ implying by (A.16) $\psi'(w) = \frac{\partial_w \mathcal{K}(w,\psi(w))}{\partial_b \mathcal{K}(w,\psi(w))} \geq \frac{-\partial_w \mathcal{Z}(\hat{w},\psi(\hat{w}))}{\partial_b \mathcal{Z}(\hat{w},\psi(\hat{w}))}$. By the same argument as before, $w \mapsto \hat{\Phi}_b(w)$ is strictly increasing on $[\tilde{w}, \hat{w}]$. Defining $\hat{w}_1 := \hat{\Phi}_w(\hat{w}) = \tilde{w}_1$, $\hat{b}_1 := \hat{\Phi}_b(\hat{w}) > \hat{\Phi}_b(\tilde{w}) =: \tilde{b}_1$, both $(\hat{w}_1, \hat{b}_1)$ and $(\tilde{w}_1, \tilde{b}_1)$ must lie on $\mathcal{M}$ which contradicts that $\mathcal{M} = \text{graph}(\psi)$. This proves that $\hat{\mathcal{K}}$ and, therefore, $w \longmapsto \mathcal{W}(\hat{\mathcal{K}}(w); \hat{\varepsilon})$ are strictly increasing for all $\hat{\varepsilon} \in \mathcal{E}$.

Step 5: We show that $\mathbb{W} = \mathbb{R}_{++}$. Note from (20) that $\Phi : \mathcal{M} \longrightarrow \mathcal{M}$ and $\Phi^{-1}(\mathcal{M}) \subset \mathcal{M}$. Lemma A.2 implies that $\Phi : \mathcal{M} \longrightarrow \mathcal{M} \cap \mathbb{U}$ is a homeomorphism, i.e., a continuous bijection with continuous inverse $\Phi^{-1} : \mathcal{M} \cap \mathbb{U} \longrightarrow \mathcal{M}$. Define $\hat{\Phi}_w$ as above and $\mathbb{W}^\star := \mathbb{W} \cap \mathbb{W}^+$. The next results follow from the previous observations, monotonicity of $\hat{\Phi}_w$, and uniqueness and stability (on $\mathcal{M}$) of $(\bar{w}, \bar{b})$.

Lemma A.3
The map $\hat{\Phi}_w : \mathbb{W} \longrightarrow \mathbb{W}^\star$ is a homeomorphism with inverse $\hat{\Phi}_w^{-1} : \mathbb{W}^\star \longrightarrow \mathbb{W}$ satisfying:

(i) $\Phi(w, b) = (\hat{\Phi}_w(w), \psi(\hat{\Phi}_w(w)))$ *for all* $(w, b) \in \mathcal{M}$.

(ii) $\Phi^{-1}(w, b) = (\hat{\Phi}_w^{-1}(w), \psi(\hat{\Phi}_w^{-1}(w)))$ *for all* $(w, b) \in \mathcal{M} \cap \mathbb{U}$.

(iii) $w \gtreqless \bar{w} \Rightarrow \hat{\Phi}_w(w) \lesseqgtr w$ *for all* $w \in \mathbb{W}$.

(iv) $w \gtreqless \bar{w} \Rightarrow \hat{\Phi}_w^{-1}(w) \gtreqless w$ *for all* $w \in \mathbb{W}^\star$.

Let $w_{min} := \inf(\mathbb{W}) < \bar{w} < w_{max} := \sup(\mathbb{W}) \leq \infty$ and $w^\star_{min} := \inf(\mathbb{W}^\star) < \bar{w} < w^\star_{max} := \sup(\mathbb{W}^\star) \leq \infty$. Note that $w^\star_{min} = \max\{w_{min}, \underline{w}^0\}$ and $w^\star_{max} = \min\{w_{max}, \bar{w}^\infty\}$.
We show that $\mathbb{W}$ is open, i.e., $\mathbb{W} =]w_{min}, w_{max}[$. By contradiction, suppose $w_{min} \in \mathbb{W}$. Let $(w_n)_{n\geq 0}$ be a strictly decreasing sequence in $[w_{min}, \bar{w}[\subset \mathbb{W}$ converging to w_{min}. Since $\hat{\Phi}_w$ is a strictly increasing bijection, the sequence $w^\star_n := \hat{\Phi}_w(w_n)$ in $\mathbb{W}^\star$ is strictly decreasing and converges to $w^\star_{min}$. Suppose $w^\star_{min} \in \mathbb{W}^\star$. Since $\mathbb{W}^+$ is open, this requires $w_{min} > \underline{w}^0$ in which case $w^\star_{min} = w_{min}$. Hence, $\lim_{n\to\infty} w_n = w_{min} \in \mathbb{W}$ and $\lim_{n\to\infty} \hat{\Phi}_w(w_n) = w_{min}$ which implies $\hat{\Phi}_w(w_{min}) = w_{min} < \bar{w}$ and contradicts Lemma

A.3(iii). Conversely, suppose $w^\star_{min} \notin \mathbb{W}^\star$, i.e., $\mathbb{W}^\star$ is left open. Since $\hat\Phi_w$ is a bijection between $\mathbb{W}$ and $\mathbb{W}^\star$, let $\tilde w := \hat\Phi_w(w_{min})$. Since $\tilde w$ is an interior point, $\tilde w - \delta \in \mathbb{W}^\star$ for $\delta > 0$ small. But, since $\hat\Phi_w$ is strictly increasing, $\hat\Phi_w(w) \geq \tilde w$ for all $w \in \mathbb{W}$ contradicting $\hat\Phi_w$ being a bijection. Hence, $w_{min} \notin \mathbb{W}$. An analogous argument shows that $w_{max} \notin \mathbb{W}$. Hence, $\mathbb{W} =]w_{min}, w_{max}[$ and also $\mathbb{W}^\star = \mathbb{W} \cap \mathbb{W}^+ =]w^\star_{min}, w^\star_{max}[$ are open intervals.
We show that $w_{min} \leq \underline{w}^0$ and $w_{max} \geq \bar w^\infty$. By contradiction, suppose first that $w_{min} > \underline{w}^0 \geq 0$ such that $\hat\Phi_w :]w_{min}, w_{max}[\longrightarrow]w_{min}, w^\star_{max}[$. Choose $\tilde w_0 \in]w_{min}, \bar w[\subset \mathbb{W}$ and define $\tilde w_{n+1} := \hat\Phi_w^{-1}(\tilde w_n)$ and $\tilde b_n := \psi(w_n)$ for $n \geq 0$. By Lemma A.3(iv), $\tilde w_n \in]w_{min}, \bar w[\subset \mathbb{W}^\star \cap \mathbb{W}$ for all n. Hence, both sequences are well-defined and strictly decreasing such that $\tilde w_\infty := \lim_{n\to\infty} \tilde w_n \geq w_{min} > 0$ and $\tilde b_\infty := \lim_{n\to\infty} \tilde b_n \leq \tilde w_\infty$ both exist. By construction and Lemma A.3(ii), $(\tilde w_n, \tilde b_n) = \Phi^{-1}(\tilde w_{n-1}, \tilde b_{n-1})$ or, equivalently, $\Phi(\tilde w_n, \tilde b_n) = (\tilde w_{n-1}, \tilde b_{n-1})$ for all n. Therefore, $\lim_{n\to\infty}(\tilde w_n, \tilde b_n) = \lim_{n\to\infty} \Phi(\tilde w_n, \tilde b_n) = (\tilde w_\infty, \tilde b_\infty)$. Suppose $\tilde b_\infty = \tilde w_\infty$, i.e., $\tilde w_n \searrow \tilde b_\infty > 0$. Then, by Lemma 3.2 and A.5(iii) $\lim_{n\to\infty} \Phi_b(\tilde w_n, \tilde b_n) \geq \lim_{n\to\infty} \Phi_b(\tilde w_n, \tilde b_\infty) = \infty$ contradicting $\lim_{n\to\infty} \Phi_b(\tilde w_n, \tilde b_n) = \tilde b_\infty$. Conclude from this that $(\tilde w_\infty, \tilde b_\infty) \in \mathbb{V}$ which implies $(\tilde w_\infty, \tilde b_\infty) = \Phi(\tilde w_\infty, \tilde b_\infty)$ by the previous properties and continuity of Φ, i.e., $(\tilde w_\infty, \tilde b_\infty)$ must be a fixed point of Φ. But this is impossible since $0 < \tilde w_\infty < \bar w < \bar w^0_\varepsilon$. This contradiction proves $w_{min} \leq \underline{w}^0$. An analogous argument shows $w_{max} \geq \bar w^\infty$. Hence, $\mathbb{W}^\star =]\underline{w}^0, \bar w^\infty[$. This proves the claim if $\underline{w}^0 = 0$ and $\bar w^\infty = \infty$. To see that $w_{min} = 0$ and $w_{max} = \infty$ also if $\underline{w}^0 > 0$ and /or $\bar w^\infty < \infty$, suppose by contradiction that $w_{min} > 0$. Let $(w_n)_{n\geq 0}$ be a strictly decreasing sequence in $]w_{min}, \bar w[\subset \mathbb{W}$ converging to $w_{min} > 0$. Since $\hat\Phi_w :]w_{min}, w_{max}[\longrightarrow]\underline{w}^0, \bar w^\infty[$ is a strictly increasing bijection, this implies that $\lim_{n\to\infty} \hat\Phi_w(w_n) = \underline{w}^0 = \mathcal{W}(0;\varepsilon)$ requiring $\lim_{n\to\infty} \mathcal{K}(w_n, \psi(w_n)) = 0$. At the same time, $(w_n, b_n) \in \mathcal{M}$ for all n which, using the same argument as in the previous paragraph, implies $b_{min} := \lim_{n\to\infty} \psi(w_n) < w_{min}$ such that $(w_{min}, b_{min}) \in \mathbb{V}$. But then, by continuity $\lim_{n\to\infty} \mathcal{K}(w_n, b_n) = \mathcal{K}(w_{min}, b_{min}) > 0$, a contradiction. Conclude that $w_{min} = 0$. A similar argument shows that $w_{max} = \infty$ completing the proof. ■

A.6 Proof of Lemma 2.4

Again we show the claim for the general scenario of Section 3 under the hypotheses of Lemma 3.3. The claim of Lemma 2.4 follows from the remarks made in Section A.5. Let ϑ be given and $\varepsilon \in \mathcal{E}^\vartheta_s$ be fixed. Dependence on these parameters will be suppressed.
(i) Given $w_0 > 0$, let $\hat b_0 < \psi(w_0) =: b_0$. Lemma 3.2 resp. 2.2 and an induction argument yield that $(\hat w_t, \hat b_t) := \Phi^t(w_0, \hat b_0)$ and $(w_t, b_t) := \Phi^t(w_0, b_0)$ satisfy $\hat w_t > w_t > b_t > \hat b_t \geq 0$ for all $t > 0$. Defining $\beta_t := \hat b_t / b_t$ gives $\beta_0 < 1$ and $\beta_{t+1} = \beta_t \mathcal{Z}(\hat w_t, \hat b_t)/\mathcal{Z}(w_t, b_t) < \beta_t$ for all $t \geq 0$. Hence, $\lim_{t\to\infty} \beta_t = \bar\beta < 1$ and $\lim_{t\to\infty} b_t = \bar b_\varepsilon > 0$ imply $\lim_{t\to\infty} \hat b_t = \lim_{t\to\infty} \Phi_b(\hat w_t, \hat b_t) = \hat b_\infty < \bar b_\varepsilon$. The latter implies either $\lim_{t\to\infty} \mathcal{Z}(\hat w_t, \hat b_t) = 1/\vartheta(\varepsilon)$ or $\lim_{t\to\infty} \hat b_t = 0$. But the first limit, supposing it exists, satisfies $\lim_{t\to\infty} \mathcal{Z}(\hat w_t, \hat b_t) < \lim_{t\to\infty} \mathcal{Z}(w_t, b_t) = 1/\vartheta(\varepsilon)$. Conclude that $\hat b_\infty = 0$ which, by continuity of Φ implies that the evolution of the sequence $(\hat w_t)_{t\geq 0}$ is asymptotically governed by (14) which converges to the trivial steady state for any initial value $w_0 > 0$. Conclude that $\lim_{t\to\infty} \hat w_t = \bar w^0_\varepsilon$.
(ii) Given $w_0 > 0$, let $\tilde b_0 > \psi(w_0) =: b_0$. By contradiction, let $(\tilde w_t, \tilde b_t) := \Phi^t(w_0, \tilde b_0) \in \mathbb{V}$

for all $t \geq 0$. Define (w_t, b_t) as in (i) and $\beta'_t := \tilde{b}_t/b_t > 1$, $t \geq 0$. By analogous reasonings, $b_t < \tilde{b}_t < \tilde{w}_t < w_t$ and $\lim_{t\to\infty} \beta'_t = \bar{\beta}' > 1$. Hence, $\lim_{t\to\infty} \tilde{b}_t = \bar{\beta}'\bar{b}_\varepsilon =: \tilde{b}_\infty > \bar{b}_\varepsilon$. Since $\tilde{b}_t < w_t$ for all t, $(\tilde{w}_t, \tilde{b}_t) \in \mathbb{V}$ requires $\tilde{b}_\infty < \infty$ and $\lim_{t\to\infty} \mathcal{Z}(\tilde{w}_t, \tilde{b}_t) = 1/\vartheta(\varepsilon)$. Monotonicity of $w \longmapsto \mathcal{Z}(\cdot, b)$ then requires $\lim_{t\to\infty} \tilde{w}_t = \tilde{w}_\infty$. Since $\tilde{w}_t > \tilde{b}_t$ implies $\tilde{w}_\infty \geq \tilde{b}_\infty$ and $\tilde{w}_\infty = \tilde{b}_\infty$ would imply $\lim_{t\to\infty} \mathcal{Z}(\tilde{w}_t, \tilde{b}_t) = \infty$ by Lemma A.5(iii), $(\tilde{w}_\infty, \tilde{b}_\infty) \in \mathbb{V}$ and must be a steady state of Φ. But no steady state satisfying $\tilde{b}_\infty > \bar{b}_\varepsilon$ exists and the claim follows. ■

A.7 Proof of Lemma 2.5

For $t \geq 0$, let $\zeta_t := \frac{b_t}{w_t}$. Using Φ_w, Φ_b gives $\zeta_{t+1} = \phi(\zeta_t) := \frac{\alpha}{1-\alpha}[\frac{\gamma}{1+\gamma} - \zeta_t]^{-1}\zeta_t$, $t \geq 0$. The map ϕ has a unique non-trivial fixed point $\bar{\zeta} := \frac{\gamma}{1+\gamma} - \frac{\alpha}{1-\alpha}$ which is unstable. Moreover, $\zeta_0 < \bar{\zeta}$ implies $\lim_{t\to\infty} \zeta_t = 0$ and $\zeta_0 > \bar{\zeta}$ implies that $\phi^{t_0}(\zeta_0) > \frac{\gamma}{1+\gamma}$ for finite t_0. Hence, $b_0 = \bar{\zeta} w_0$ is necessary for $\lim_{t\to\infty} \Phi^t(w_0, b_0; \varepsilon) = (\bar{w}_\varepsilon, \bar{b}_\varepsilon)$. Sufficiency follows from Theorem A.1 which implies existence of a corresponding $b_0 = \psi(w_0)$ for $w_0 > 0$. ■

A.8 Proof of Lemma 3.1

Given $(w, b) \in \mathbb{V}$, let $\bar{k} := w - b > 0$. The argument $c^o(z, k, b, \varepsilon) := b\, z\, \vartheta(\varepsilon) + k\mathcal{R}(k; \varepsilon)$ will be suppressed when convenient. Suppose $b = 0$. Then, H_1^ϑ is independent of z and ϑ and $H_1^\vartheta(z, k; w, 0) = H(k; w, 0)$ for all $k \in]0, \bar{k}[$ with H defined as in (12). Hence, existence of $k_+ \in]0, \bar{k}[$ to satisfy $H_1^\vartheta(z, k_+; w, 0) = 0$ is due to Lemma 2.1. Using k_+ condition $H_2^\vartheta(z, k_+; w, 0) = 0$ can be solved explicitly for $z > 0$ proving the case $b = 0$. Suppose $b > 0$. The strategy is to use (17b) to eliminate z reducing (17a) to a one-dimensional problem. Fixing $\hat{k} \in]0, \bar{k}[$ we determine $\hat{z} > 0$ to satisfy $H_2^\vartheta(\hat{z}, \hat{k}; w, b) = 0$. Noting that $\lim_{z\to\infty} c^o(z, k, b, \varepsilon) = \infty$ for each fixed $\varepsilon \in \mathcal{E}$. Therefore, (P2) implies

$$\lim_{z\to\infty} z\, \vartheta(\varepsilon)\, v'(-) = b^{-1} \lim_{z\to\infty} c^o(z, \hat{k}, b, \varepsilon) v'(-) - b^{-1}\hat{k}\mathcal{R}(\hat{k}, \varepsilon) \lim_{z\to\infty} v'(-) = \infty.$$

This being true for all $\varepsilon \in \mathcal{E}$ implies $H_2^\vartheta(z, \hat{k}; w, b) < 0$ for z sufficiently large. Since $H_2^\vartheta(0, \hat{k}; w, b) = u'(w - b - \hat{k}) > 0$ this proves existence of $\hat{z}$. To show uniqueness, we prove that $z \longmapsto H_2^\vartheta(z, k; w, b)$ is strictly decreasing. Using (P1), the derivative satisfies

$$\begin{aligned} \partial_z H_2^\vartheta(z, k; w, b) &= -\mathbb{E}_\nu\Big[\vartheta(\cdot)\, v'\big(c^o(z, k, b, \cdot)\big) + b\, z\, \vartheta(\cdot)^2\, v''\big(c^o(z, k, b, \cdot)\big)\Big] \qquad \text{(A.6)} \\ &< -\mathbb{E}_\nu\Big[\vartheta(\cdot)\big(v'\big(c^o(z, k, b, \cdot)\big) + c^o(z, k, b, \cdot) v''\big(c^o(z, k, b, \cdot)\big)\big)\Big] \leq 0. \end{aligned}$$

These results imply the existence of a map $\hat{Z}(\cdot; w, b) :]0, \bar{k}[\longrightarrow \mathbb{R}_{++}$ which determines a value $\hat{z}$ for each $\hat{k} \in]0, \bar{k}[$ such that $H_2^\vartheta(\hat{z}, \hat{k}; w, b) = 0$. Using (4) yields the derivative

$$\partial_k H_2^\vartheta(z, k; w, b) = u''(w - b - k) - \mathbb{E}_\nu\Big[\mathcal{R}(k; \cdot)\, z\, \vartheta(\cdot) v''(-)\big(1 + E_{g'}(k)\big)\Big] > 0 \qquad \text{(A.7)}$$

where the second term is positive by (P3). By the IFT, $\hat{Z}(\cdot; w, b)$ is C^1 and strictly increasing on $]0, \bar{k}[$ since $\partial_k \hat{Z}(k; w, b) = -\partial_k H_2^\vartheta(z, k; w, b)/\partial_z H_2^\vartheta(z, k; w, b) > 0$.

Using these results, let $\hat{H}_1(\cdot;w,b):]0,\bar{k}[\longrightarrow\mathbb{R}$, $\hat{H}_1(k;w,b):=H_1^\vartheta(\hat{Z}(k;w,b),k;w,b)$. We determine a unique $k_+\in]0,\bar{k}[$ that solves $\hat{H}_1(k_+;w,b)=0$. Since v' is strictly decreasing, $\mathcal{R}(k;\varepsilon)v'\big(b\,\hat{Z}(k;w,b)\,\vartheta(\varepsilon)+k\mathcal{R}(k;\varepsilon)\big)<\mathcal{R}(k;\varepsilon)v'\big(k\mathcal{R}(k;\varepsilon)\big)$ for all $\varepsilon\in\mathcal{E}$ which implies $\hat{H}_1(k;w,b)>u'(w-b-k)-\mathbb{E}_\nu\big[\mathcal{R}(k;\cdot)v'\big(k\mathcal{R}(k;\cdot)\big)\big]$ for all $k\in]0,\bar{k}[$. Therefore, by (9)

$$\lim_{k\to\bar{k}}\hat{H}_1(k;w,b)\geq\lim_{k\to\bar{k}}\big(u'(w-b-k)-\mathbb{E}_\nu\big[\mathcal{R}(k;\cdot)v'\big(k\mathcal{R}(k;\cdot)\big)\big]\big)=\infty. \tag{A.8}$$

Let $(k_n)_{n\geq1}$ be a sequence in $]0,w-b[$ with $\lim_{n\to\infty}k_n=0$. Since $k\mapsto\hat{Z}(k;w,b)$ and, by (P3), $k\mapsto k\mathcal{R}(k;\varepsilon)$ are increasing, $c_n(\varepsilon):=b\,\hat{Z}(k_n;w,b)\,\vartheta(\varepsilon)+k_n\mathcal{R}(k_n,\varepsilon)$ is bounded from above for all $\varepsilon\in\mathcal{E}$ which implies $\lim_{n\to\infty}\mathcal{R}(k_n,\varepsilon)\,v'\big(c_n(\varepsilon)\big)=\infty$. This being true for all $\varepsilon\in\mathcal{E}$ gives $\lim_{n\to\infty}\mathbb{E}_\nu\big[\mathcal{R}(k_n,\cdot)v'\big(c_n(\cdot)\big)\big]=\infty$ and $\lim_{n\to\infty}\hat{H}_1(k_n;w,b)=-\infty$. Since $(k_n)_{n\geq1}$ was arbitrary, $\lim_{k\to0}\hat{H}_1(k;w,b)=-\infty$. This and (A.8) yields existence of a zero of $\hat{H}_1(\cdot;w,b)$. Finally, using (P2) the partial derivatives of $H_1^\vartheta(\cdot;w,b)$ compute

$$\partial_kH_1^\vartheta(z,k;w,b)\;=\;-u''(-)-\mathbb{E}_\nu\big[f_{kk}(k;\cdot)\,v'(-)+(1+E_{g'}(k))\mathcal{R}(k;\cdot)^2\,v''(-)\big]>0 \tag{A.9}$$

$$\partial_zH_1^\vartheta(z,k;w,b)\;=\;-\mathbb{E}_\nu\Big[\mathcal{R}(k,\cdot)\,b\,\vartheta(\cdot)\,v''(-)>0. \tag{A.10}$$

Combining (A.9) and (A.10) with the monotonicity of $\hat{Z}(\cdot;w,b)$ yields $\partial_k\hat{H}_1(k;w,b)=\partial_kH_1^\vartheta(z,k;w,b)+\partial_zH_1^\vartheta(z,k;w,b)\partial_k\hat{Z}(k;w,b)>0$ where $z=\hat{Z}(k;w,b)$. Hence, k_+ is the unique zero of $\hat{H}_1(\cdot;w,b)$ on $]0,\bar{k}[$. Setting $z=\hat{Z}(k_+;w,b)$ completes the proof. ■

A.9 Proof of Lemma 3.2

As in the previous proof, the argument $c^o(z,k,b,\varepsilon)$ defined as before is omitted when convenient. We preface the proof by the following technical result.

Lemma A.4
For the scenario of Section 3, let (P1)–(P4) hold and $\vartheta:\mathcal{E}\longrightarrow\mathbb{R}_{++}$ be continuous. Then, for all $(w,b)\in\mathbb{V}$, $z:=\mathcal{Z}^\vartheta(w,b)$ and $k:=\mathcal{K}^\vartheta(w,b)$ the following holds:

(i) $k\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot)-z\vartheta(\cdot))\mathcal{R}(k;\cdot)|v''(-)|\right]=-b\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot)-z\vartheta(\cdot))z\vartheta(\cdot)|v''(-)|\right]$.

(ii) $\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot)-z\vartheta(\cdot))\mathcal{R}(k;\cdot)|v''(-)|\right]\geq0\geq\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot)-z\vartheta(\cdot))z\vartheta(\cdot)|v''(-)|\right]$.

Proof of Lemma A.4.
(i) Equations (17a) and (17b) give $H_1^\vartheta(z,k;w,b)=H_2^\vartheta(z,k;w,b)=0$ and, therefore, $0=H_1^\vartheta(z,k;w,b)-H_2^\vartheta(z,k;w,b)=\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot)-z\vartheta(\cdot))v'(-)\right]$. By (P4), $v'(c)=\theta^{-1}c|v''(c)|$ for all $c=bz\vartheta(\varepsilon)+k\mathcal{R}(k,\varepsilon)>0$ which implies (i).
(ii) We have $0\leq\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot)-z\vartheta(\cdot))^2|v''(-)|\right]$ which can equivalently be written as $\mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)(\mathcal{R}(k;\cdot)-z\vartheta(\cdot))|v''(-)|\right]\geq\mathbb{E}_\nu\left[z\vartheta(\cdot)(\mathcal{R}(k;\cdot)-z\vartheta(\cdot))|v''(-)|\right]$. Since, by (i), one of the terms must be non-positive, the claim follows immediately. □

Let $(w,b)\in\mathbb{V}$ be arbitrary and set $z:=\mathcal{Z}^\vartheta(w,b)$ and $k:=\mathcal{K}^\vartheta(w,b)$ noting that $z>0$ and $0<k<w-b$. Write $H^\vartheta=(H_1^\vartheta,H_2^\vartheta)$ and $\xi=(z,k)$. The signs of the derivatives in (A.6), (A.7), (A.9), and (A.10) imply that the Jacobian matrix

$$D_\xi H^\vartheta(z,k;w,b)=\begin{bmatrix}\partial_zH_1^\vartheta(z,k;w,b) & \partial_kH_1^\vartheta(z,k;w,b)\\ \partial_zH_2^\vartheta(z,k;w,b) & \partial_kH_2^\vartheta(z,k;w,b)\end{bmatrix}.$$

has determinant $\det D_\xi H^\vartheta(z,k;w,b) > 0$ and is hence invertible. The inverse computes

$$[D_\xi H^\vartheta(z,k;w,b)]^{-1} = \frac{1}{\det D_\xi H^\vartheta(z,k;w,b)} \begin{bmatrix} \partial_k H_2^\vartheta(z,k;w,b) & -\partial_k H_1^\vartheta(z,k;w,b) \\ -\partial_z H_2^\vartheta(z,k;w,b) & \partial_z H_1^\vartheta(z,k;w,b) \end{bmatrix}.$$

The partial derivatives with respect to w and b take the form

$$\begin{aligned} \partial_w H_1^\vartheta(z,k;w,b) &= \partial_w H_2^\vartheta(z,k;w,b) = u''(w-b-k) < 0 && \text{(A.11)} \\ \partial_b H_1^\vartheta(z,k;w,b) &= -u''(w-b-k) - \mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)\, z\,\vartheta(\cdot) v''(-)\right] > 0 && \text{(A.12)} \\ \partial_b H_2^\vartheta(z,k;w,b) &= -u''(w-b-k) - \mathbb{E}_\nu\left[(z\,\vartheta(\cdot))^2 v''(-)\right] > 0. && \text{(A.13)} \end{aligned}$$

By the implicit function theorem, omitting the arguments for notational convenience

$$\begin{aligned} \partial_w \mathcal{Z}^\vartheta(w,b) &= \frac{-\partial_w H_1^\vartheta[\partial_k H_2^\vartheta - \partial_k H_1^\vartheta]}{\det D_\xi H^\vartheta}, \quad \partial_b \mathcal{Z}^\vartheta(w,b) = \frac{\partial_k H_1^\vartheta \partial_b H_2^\vartheta - \partial_k H_2^\vartheta \partial_b H_1^\vartheta}{\det D_\xi H^\vartheta} \\ \partial_w \mathcal{K}^\vartheta(w,b) &= \frac{-\partial_w H_1^\vartheta[\partial_z H_1^\vartheta - \partial_z H_2^\vartheta]}{\det D_\xi H^\vartheta}, \quad \partial_b \mathcal{K}^\vartheta(w,b) = \frac{\partial_z H_2^\vartheta \partial_b H_1^\vartheta - \partial_z H_1^\vartheta \partial_b H_2^\vartheta}{\det D_\xi H^\vartheta}. \end{aligned}$$

(i) As $\det D_\xi H^\vartheta = \partial_z H_1^\vartheta\, \partial_k H_2^\vartheta - \partial_k H_1^\vartheta\, \partial_z H_2^\vartheta > 0$, $\partial_z H_2^\vartheta < 0 < \partial_z H_1^\vartheta$ by (A.6) and (A.10), and $-\partial_w H_1^\vartheta < \partial_b H_i^\vartheta$, $i = 1, 2$, it follows that

$$0 < \partial_w \mathcal{K}^\vartheta(w,b) = \frac{-\partial_w H_1^\vartheta[\partial_z H_1^\vartheta - \partial_z H_2^\vartheta]}{\det D_\xi H^\vartheta} < \frac{\partial_z H_1^\vartheta \partial_b H_2^\vartheta - \partial_z H_2^\vartheta \partial_b H_1^\vartheta}{\det D_\xi H^\vartheta} = -\partial_b \mathcal{K}^\vartheta(w,b).$$

(ii) If, in addition, (P4) holds, straightforward calculations and Lemma A.4 imply

$$\begin{aligned} \partial_k H_1^\vartheta - \partial_k H_2^\vartheta &= \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))\mathcal{R}(k;\cdot)|v''(-)|(1+E_{g'}(k)) - f_{kk}(k;\cdot)v'(-)\right] > 0 && \text{(A.14)} \\ \partial_b H_1^\vartheta - \partial_b H_2^\vartheta &= \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))z\vartheta(\cdot)|v''(-)|\right] \le 0. && \text{(A.15)} \end{aligned}$$

By (A.11) and (A.14), $\partial_w \mathcal{Z}^\vartheta(w,b) < 0$ and, by (A.12) – (A.15), $\partial_b \mathcal{Z}^\vartheta(w,b) > 0$. Finally,

$$\partial_w \mathcal{K}^\vartheta(w,b)\partial_b \mathcal{Z}^\vartheta(w,b) - \partial_b \mathcal{K}^\vartheta(w,b)\partial_w \mathcal{Z}^\vartheta(w,b) = \frac{-\partial_w H_1^\vartheta}{\det D_\xi H^\vartheta}(\partial_b H_2^\vartheta - \partial_b H_1^\vartheta) \ge 0 \quad \text{(A.16)}$$

which follows from direct calculations and shows that $\frac{-\partial_w \mathcal{Z}^\vartheta(w,b)}{\partial_b \mathcal{Z}^\vartheta(w,b)} \le \frac{\partial_w \mathcal{K}^\vartheta(w,b)}{-\partial_b \mathcal{K}^\vartheta(w,b)} < 1$. ■

Remark A.1
Since $\mathcal{Z}^\vartheta$ and $\mathcal{K}^\vartheta$ are well-defined and the matrix $D_\xi H^\vartheta(z,k;w,b)$ is non-singular also at any boundary point $(w,0)$ of $\mathbb{V}$, the implicit function theorem implies that the mappings $\mathcal{Z}^\vartheta$ and $\mathcal{K}^\vartheta$ can locally be extended to an open neighborhood around $(w,0)$. Hence, their derivatives are well-defined and continuous also on the boundary of $\mathbb{V}$ where $b = 0$. Hence, Lemma 3.2 and also Lemma 2.2 indeed hold on the entire set $\mathbb{V}$.

A.10 Proof of Lemma 3.3

Given ϑ, let $\varepsilon \in \mathcal{E}$ be fixed. To alleviate the notation, these parameters will subsequently be suppressed. With this convention, denote the trivial steady state as $\bar{w}^0 > 0$ and let $\underline{w}_k := \mathcal{W}(0;\varepsilon) \ge 0$. By the monotonicity of $\mathcal{K}^\vartheta$ (cf. Lemma 3.2) and $\mathcal{W}(\cdot;\varepsilon)$, any steady state $(\bar{w},\bar{b}) \in \mathbb{V}_+$ satisfies $\underline{w}_k < \bar{w} < \bar{w}^0$. Further results are collected in the next lemma.

Lemma A.5
Under Assumption 3.1 and the hypotheses of Lemma 3.3, the following holds true:

(i) $w > \mathcal{W}(\mathcal{K}^\vartheta(w,0);\varepsilon)$ *for all* $w \in]\underline{w}_k, \bar{w}^0[$.

(ii) $\mathcal{W}(k;\varepsilon) \geq k\mathcal{R}(k;\varepsilon)$ *for all* $0 < k \leq \bar{k}^0 := \mathcal{K}^\vartheta(\bar{w}^0, 0)$.

(iii) For any $\hat{b} \geq 0$: $\lim_{w \searrow \hat{b}} \mathcal{Z}^\vartheta(w, \hat{b}) = \infty$.

Proof of Lemma A.5
(i) Uniqueness of $\bar{w}^0$ gives $w \neq \mathcal{W}(\mathcal{K}^\vartheta(w,0);\varepsilon) \forall w \in]\underline{w}_k, \bar{w}^0[$. Stability implies the claim. (ii) By (3) and (4), the claim is equivalent to $E_g(k) \leq \frac{1}{2}$ for all $k \in]0, \bar{k}^0]$. By Assumption 3.1, $E_g(\bar{k}^0) \leq \frac{1}{2}$. The derivative $E_g'(k) = (E_{g'}(k))^2/(kE_g(k))(1 - E_g(k) - |E_{g'}(k)|)$ is non-negative by (P5) implying that E_g is non-decreasing from which the claim follows. (iii) By (17a), (17b), for any $(w,b) \in \mathbb{V}$, defining $z := \mathcal{Z}^\vartheta(w,b)$ and $k := \mathcal{K}^\vartheta(w,b)$ and $c^o(z,k,b;\varepsilon)$ as above one has $\mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'(c^o(z,k,b;\cdot))\right] = \mathbb{E}_\nu\left[z\vartheta(\cdot)\, v'(c^o(z,k,b;\cdot))\right]$. This implies that there exists some $\tilde{\varepsilon} \in \mathcal{E}$ such that $z\vartheta(\tilde{\varepsilon}) \geq \tilde{\varepsilon} g'(k)$. Setting $\zeta := \min_\varepsilon\{\varepsilon/\vartheta(\varepsilon) \,|\, \varepsilon \in \mathcal{E}\} > 0$ (which is well-defined by continuity of ϑ and compactness of $\mathcal{E}$) gives the inequality $\mathcal{Z}^\vartheta(w,b) \geq \zeta g'(\mathcal{K}^\vartheta(w,b))$ for all $(w,b) \in \mathbb{V}$. Hence, for any $\hat{b} \geq 0$, $\lim_{w \searrow \hat{b}} \mathcal{K}(w,\hat{b}) = 0$ implies $\lim_{w \searrow \hat{b}} \mathcal{Z}^\vartheta(w,\hat{b})) \geq \zeta \lim_{w \searrow \hat{b}} g'(\mathcal{K}(w,\hat{b})) = \infty$. □

(i) *Existence.* Define $H_w : \mathbb{V} \longrightarrow \mathbb{R}$, $H_w(w,b) := w - \mathcal{W}(\mathcal{K}^\vartheta(w,b),\varepsilon)$ and the w-isocline $\mathbb{H}_w := \{(w,b) \in \mathbb{V} \,|\, H_w(w,b) = 0, w \in]\underline{w}_k, \bar{w}^0[\}$. Any interior steady state satisfies $(\bar{w}, \bar{b}) \in \mathbb{H}_w$. For each $\hat{w} \in]\underline{w}_k, \bar{w}^0[$ we claim there exists a unique $\hat{b} \in]0, \hat{w}[$ such that $H_w(\hat{w}, \hat{b}) = 0$. By Lemma A.5(i), $\lim_{b \to 0} H_w(\hat{w}, b) = \hat{w} - \mathcal{W}(\mathcal{K}^\vartheta(\hat{w}, 0), \varepsilon) < 0$ and $\lim_{b \to \hat{w}} \mathcal{K}^\vartheta(\hat{w}, b) = 0$ gives $\lim_{b \to \hat{w}} H_w(\hat{w}, b) = \hat{w} - \underline{w}_k > 0$ implying existence of $\hat{b}$. Uniqueness follows from Lemma 3.2 which implies that $H_w(w;\cdot)$ is strictly increasing. This result permits to define the solution $\hat{b}$ as a map $h_w :]\underline{w}_k, \bar{w}^0[\longrightarrow \mathbb{R}_{++}$ implying $\mathbb{H}_w = \text{graph}(h_w)$. By the implicit function theorem, h_w is C^1 with derivative

$$h_w'(w) = -\frac{\partial_w(H_w(w,b))}{\partial_b(H_w(w,b))} = -\frac{1 + kf_{kk}(k;\varepsilon)\partial_w\mathcal{K}(w,b)}{kf_{kk}(k;\varepsilon)\partial_b\mathcal{K}(w,b)},\ b = h_w(w), k = \mathcal{K}(w,b). \tag{A.17}$$

As $H_w(\bar{w}^0, 0) = 0$ and $\lim_{w \searrow \underline{w}_k} \mathcal{K}^\vartheta(w, \underline{w}_k) = 0$ yields $H_w(\underline{w}_k, \underline{w}_k) = 0$, continuity of H_w implies the boundary behavior $\lim_{w \to \bar{w}^0} h_w(w) = 0$ and $\lim_{w \to \underline{w}_k} h_w(w) = \underline{w}_k \geq 0$.
Analogously, let $H_b : \mathbb{V} \longrightarrow \mathbb{R}$, $H_b(w,b) := \mathcal{Z}^\vartheta(w,b) - \frac{1}{\vartheta(\varepsilon)}$. Note first that $\varepsilon \in \mathcal{E}_s^\vartheta$ implies $\lim_{w \to \bar{w}^0} H_b(\hat{w}, 0) = \mathcal{Z}^\vartheta(\bar{w}^0, 0) - \frac{1}{\vartheta(\varepsilon)} < 0$. By Lemma A.5(iii), $\lim_{w \to 0} H_b(\hat{w}, 0) = \infty$. As $w \mapsto \mathcal{Z}^\vartheta(w, 0)$ is strictly decreasing by Lemma 3.2(ii), a unique $0 < \underline{w}_z < \bar{w}^0$ satisfying $H_b(\underline{w}_z, 0) = 0$ exists. Define the b-isocline $\mathbb{H}_b := \{(w,b) \in \mathbb{V} \,|\, H_b(w,b) = 0, w \in]\underline{w}_z, \bar{w}^0[\}$. Any interior steady state satisfies $(\bar{w}, \bar{b}) \in \mathbb{H}_b$. Given $\hat{w} \in]\underline{w}_z, \bar{w}^0[$ we determine $\hat{b} \in]0, \hat{w}[$ such that $H_b(\hat{w}, \hat{b}) = 0$. Lemma 3.2(ii) implies $\lim_{b \to 0} H_b(\hat{w}, b) = \mathcal{Z}^\vartheta(\hat{w}, 0) - 1/\vartheta(\varepsilon) < \mathcal{Z}^\vartheta(\underline{w}_z, 0) - 1/\vartheta(\varepsilon) = 0$. Lemma A.5(iii) yields $\lim_{b \to \hat{w}} H_b(\hat{w}, b) = \infty$ implying existence of $\hat{b}$. Uniqueness follows from Lemma 3.2(ii) and monotonicity of $b \mapsto \mathcal{Z}^\vartheta(\hat{w}, b)$. This permits the solution $\hat{b}$ to be defined as a map $h_b :]\underline{w}_z, \bar{w}^0[\longrightarrow \mathbb{R}_{++}$ and $\mathbb{H}_b = \text{graph}(h_b)$. By the IFT, h_b is C^1 with derivative

$$h_b'(w) = -\frac{\partial_w H_b(w,b)}{\partial_b H_b(w,b)} = -\frac{\partial_w \mathcal{Z}(w,b)}{\partial_b \mathcal{Z}(w,b)} > 0, \quad b = h_b(w). \tag{A.18}$$

Recall that $H_b(\underline{w}_z, 0) = 0$ and $H_b(\bar{w}^0, 0) < 0$. By Lemma A.5(iii), the latter implies existence of a unique value $\bar{b}^0$ satisfying $H_b(\bar{w}^0, \bar{b}^0) = 0$. By continuity of H_b, this implies the boundary behavior $\lim_{w\to\bar{w}^0} h_b(w) = \bar{b}^0$ and $\lim_{w\to\underline{w}_z} h_b(w) = 0$.
Let $\underline{w} := \max\{\underline{w}_k, \underline{w}_z\} > 0$ and the map $\Delta :]\underline{w}, \bar{w}^0[\longrightarrow \mathbb{R}$, $\Delta(w) := h_w(w) - h_b(w)$. Since $(\bar{w}, \bar{b}) \in \mathbb{V}$ is an interior steady state iff $(\bar{w}, \bar{b}) \in \mathbb{H}_w \cap \mathbb{H}_b$, steady state values $\bar{w}$ are zeros of Δ while $\bar{b} = h_w(\bar{w}) = h_b(\bar{w})$. By the boundary behavior of h_w and h_b, $\lim_{w\to\bar{w}^0} \Delta(w) = -\lim_{w\to\bar{w}^0} h_b(w) = -\bar{b}^0 < 0$. As for the left limit, suppose $\underline{w} = \underline{w}_k > \underline{w}_z$. Then, $\lim_{w\to\underline{w}} \Delta(w) = \lim_{w\to\underline{w}_k} h_w(w) - h_b(\underline{w}_k) = \underline{w}_k - h_b(\underline{w}_k) > 0$. This also holds if $\underline{w}_k = \underline{w}_z$. Conversely, if $\underline{w} = \underline{w}_z > \underline{w}_k$, then $\lim_{w\to\underline{w}} \Delta(w) = h_w(\underline{w}_z) - \lim_{w\to\underline{w}_z} h_b(w) = h_w(\underline{w}_z) > 0$. Conclude that $\lim_{w\to\underline{w}} \Delta(w) > 0$ and a zero exists.

Uniqueness. Let $(\bar{w}, \bar{b}) \gg 0$ be an interior steady state. We show that $\Delta'(\bar{w}) < 0$ implying uniqueness by continuity of Δ'. Let $\bar{k} := \mathcal{K}^\vartheta(\bar{w}, \bar{b}) < \bar{k}^0$ and $\bar{z} := \mathcal{Z}^\vartheta(\bar{w}, \bar{b})$. By (A.17) and (A.18),

$$\Delta'(\bar{w}) = -\frac{\partial_b \mathcal{Z}^\vartheta(\bar{w}, \bar{b}) + \varepsilon \bar{k} g''(\bar{k}) \left[\partial_w \mathcal{K}^\vartheta(\bar{w}, \bar{b}) \partial_b \mathcal{Z}^\vartheta(\bar{w}, \bar{b}) - \partial_b \mathcal{K}^\vartheta(\bar{w}, \bar{b}) \partial_w \mathcal{Z}^\vartheta(\bar{w}, \bar{b})\right]}{\varepsilon \bar{k} g''(\bar{k}) \partial_b \mathcal{K}^\vartheta(\bar{w}, \bar{b}) \partial_b \mathcal{Z}^\vartheta(\bar{w}, \bar{b})}. \tag{A.19}$$

Since the denominator is strictly positive by Lemma 3.2, it suffices to show that the numerator is strictly positive as well. Using (A.16) and the definition of $\partial_b \mathcal{Z}^\vartheta(\bar{w}, \bar{b})$ from Lemma 3.2 and recalling that $\det D_\xi H^\vartheta > 0$, this is equivalent to showing that

$$M := \partial_k H_1 \partial_b H_2 - \partial_k H_2 \partial_b H_1 - \varepsilon \bar{k} g''(\bar{k}) \partial_w H_1 (\partial_b H_2 - \partial_b H_1) > 0 \tag{A.20}$$

where the arguments of the function have been omitted for convenience. In what follows, let $M_1 := \mathbb{E}_\nu[\bar{z}\vartheta(\cdot)\,|v'(-)|] = \mathbb{E}_\nu[\mathcal{R}(\bar{k};\cdot)\,|v'(-)|] > 0$, $M_2 := \mathbb{E}_\nu[\mathcal{R}(\bar{k};\cdot)^2\,|v''(-)|] > 0$, $M_3 := \mathbb{E}_\nu[(\bar{z}\vartheta(\cdot))^2\,|v''(-)|] > 0$ and $M_4 := \mathbb{E}_\nu[\mathcal{R}(\bar{k};\cdot)\,\bar{z}\vartheta(\cdot)\,|v''(-)|] > 0$. Using the functional forms of the derivatives from (A.7), (A.9), and (A.11) – (A.13), tedious but straightforward calculations imply that M can be written as $M = A + B + C$ where

$$\begin{aligned} A :=&\ \ |u''(-)|\Big[-\tfrac{g''(\bar{k})}{g'(\bar{k})} M_1 + m(M_3 - M_4) + (1 + E_{g'}(\bar{k}))(M_2 - M_4)\Big] \\ m :=&\ \ 1 + \varepsilon \bar{k} g''(\bar{k}),\ B := -\tfrac{g''(\bar{k})}{g'(\bar{k})} M_1 M_3,\ C := (1 + E_{g'}(\bar{k}))\Big[M_2 M_3 - \big(M_4\big)^2\Big]. \end{aligned}$$

By Lemma A.4(ii), $M_2 \geq M_4$ and $M_3 \geq M_4$ which implies $C \geq 0$ by (P3). Obviously, $B > 0$. Suppose $m \geq 0$. Then, $A > 0$ which implies $M > 0$. Conversely, suppose $m < 0$ such that $-mM_4 > 0$. By (P3), $(1 + E_{g'}(\bar{k}))(M_2 - M_4) \geq 0$. By (P4), $M_1 = \theta^{-1}(\bar{k}M_2 + \bar{b}M_3) > \bar{b}M_3$. By (P5) and the first order conditions, $M_1 = u'(\bar{w} - \bar{b} - \bar{k}) \geq |u''(\bar{w} - \bar{b} - \bar{k})|(\bar{w} - \bar{b} - \bar{k})$ which implies $B \geq |u''(-)|\frac{g''(\bar{k})}{g'(\bar{k})} M_3(\bar{w} - \bar{b} - \bar{k})$. Hence,

$$A + B > |u''(-)| M_3 \Big[(1 + E_{g'}(\bar{k})) - \frac{g''(\bar{k})}{g'(\bar{k})}(\bar{w} - \varepsilon \bar{k} g'(\bar{k}))\Big].$$

Both terms in brackets are non-negative due to (P3) and Lemma A.5(ii), respectively. Hence, $M > 0$ also in this case, proving uniqueness of the steady state.
Stability. The argument is similar to the one in Lemma 2.3. Computing the determinant and trace of the Jacobian $\bar{J}$ at the steady state gives, using Lemma 3.2 and (A.16)

$$\det \bar{J} = -\varepsilon \bar{k} g''(\bar{k}) \Big[\partial_w \mathcal{K}^\vartheta(\bar{w}, \bar{b}) + \frac{\bar{b}}{\bar{z}} \Big(\partial_w \mathcal{K}^\vartheta(\bar{w}, \bar{b}) \partial_b \mathcal{Z}^\vartheta(\bar{w}, \bar{b}) - \partial_b \mathcal{K}^\vartheta(\bar{w}, \bar{b}) \partial_w \mathcal{Z}^\vartheta(\bar{w}, \bar{b})\Big)\Big] > 0$$

$$\mathrm{tr}\bar{J} = 1+\det\bar{J}+\frac{\bar{b}}{\bar{z}}\Big(\partial_b\mathcal{Z}^\vartheta(\bar{w},\bar{b})+\varepsilon\bar{k}g''(\bar{k})\left[\partial_w\mathcal{K}^\vartheta(\bar{w},\bar{b})\partial_b\mathcal{Z}^\vartheta(\bar{w},\bar{b})-\partial_b\mathcal{K}^\vartheta(\bar{w},\bar{b})\partial_w\mathcal{Z}^\vartheta(\bar{w},\bar{b})\right]\Big).$$

As shown before, the numerator in (A.19) is positive which implies $\mathrm{tr}\bar{J} > 1+\det\bar{J}$. The same reasoning as in the proof of Lemma 2.3 gives the claim.
(ii) Replacing $\mathcal{R}(\mathcal{K}(w,b);\varepsilon)$ by $\vartheta(\varepsilon)\mathcal{Z}^\vartheta(w,b)$ and using Lemma 3.2 the proof is identical to the one of Lemma 2.3(ii). ■

A.11 Proof of Lemma 4.1

We look for a zero of the function $H(\tau;w,b) := b-\tau-\psi(w-\tau)$, $\tau < w$. Since H is C^1 and Theorem A.1 implies $0 < \psi'(w) < 1$, $\partial_\tau H(\tau;w,b) = -(1-\psi'(w-\tau)) < 0$ such that there is at most one such zero. Using Theorem A.1(ii) gives $\lim_{\tau\to w} H(\tau;w,b) = b-w < 0$. Moreover, $\lim_{\tau\to-\infty} H(\tau;w,b) = b+\lim_{\tau\to\infty}\tau(1-\frac{\psi(w+\tau)}{\tau})$. If $\lim_{\tau\to\infty}\psi(w+\tau) < \infty$, then $\lim_{\tau\to-\infty} H(\tau;w,b) = \infty$. If $\lim_{\tau\to\infty}\psi(w+\tau) = \infty$, then $\lim_{\tau\to\infty}(1-\frac{\psi(w+\tau)}{\tau}) = 1-\lim_{\tau\to\infty}\psi'(w+\tau) > 0$ by l'Hopital's rule. Hence, $\lim_{\tau\to-\infty} H(\tau;w,b) = \infty$ again. ■

A.12 Proof of Lemma 4.2

Let $\pi = (\vartheta, \varepsilon_{\mathrm{ref}})$ be given. For $w > 0$ and $\varepsilon \in \mathcal{E}$, define $\hat{\Phi}^\pi_w(w;\varepsilon) := \mathcal{W}(\mathcal{K}^\vartheta(w,\psi^\pi(w));\varepsilon)$ which is strictly increasing by Theorem A.1 and linear in ε. Let $w > 0$ be arbitrary but fixed and define $(w_{\mathrm{ref}}, b_{\mathrm{ref}}) := \Phi^\vartheta(w,\psi^\pi(w);\varepsilon_{\mathrm{ref}})$. Since $\mathcal{M}^\pi$ is self-supporting under $\Phi^\vartheta(\cdot;\varepsilon_{\mathrm{ref}})$, $b_{\mathrm{ref}} = \psi^\pi(w_{\mathrm{ref}})$. Moreover, the multiplicative structure implies that $\Phi^\vartheta(w,\psi^\pi(w);\varepsilon) = \big(\frac{\varepsilon}{\varepsilon_{\mathrm{ref}}}w_{\mathrm{ref}}, \frac{\vartheta(\varepsilon)}{\vartheta(\varepsilon_{\mathrm{ref}})}\psi^\pi\big(w_{\mathrm{ref}}\big)\big)$ for each $\varepsilon \in \mathcal{E}$. Using these relations, the stabilizing tax can be written as a function $\hat{\mathcal{T}}^\pi(w_{\mathrm{ref}},\varepsilon)$ defined implicitly by the condition $G(\tau;w_{\mathrm{ref}},\varepsilon) := \psi^\pi\big(w_{\mathrm{ref}}\big)\vartheta(\varepsilon)/\vartheta(\varepsilon_{\mathrm{ref}}) - \tau - \psi^\pi\big(w_{\mathrm{ref}}\,\varepsilon/\varepsilon_{\mathrm{ref}} - \tau\big) = 0$. By the IFT, the derivatives of $\phi^\pi(w;\varepsilon) = \hat{\Phi}^\pi_w(w;\varepsilon) - \hat{\mathcal{T}}^\pi(\hat{\Phi}^\pi_w(w;\varepsilon_{\mathrm{ref}}),\varepsilon)$ compute

$$\partial_w\phi^\pi(w;\varepsilon) = \partial_w\hat{\Phi}^\pi_w(w;\varepsilon)\frac{1-\frac{\varepsilon_{\mathrm{ref}}}{\varepsilon}\frac{\vartheta(\varepsilon)}{\vartheta(\varepsilon_{\mathrm{ref}})}\psi^{\pi\prime}(\phi^\pi(w;\varepsilon_{\mathrm{ref}}))}{1-\psi^{\pi\prime}(\phi^\pi(w;\varepsilon))} \tag{A.21a}$$

$$\partial_\varepsilon\phi^\pi(w;\varepsilon) = \frac{\hat{\Phi}^\pi_w(w;1) - \frac{\vartheta'(\varepsilon)}{\vartheta(\varepsilon_{\mathrm{ref}})}\psi^\pi(\hat{\Phi}^\pi_w(w;\varepsilon_{\mathrm{ref}}))}{1-\psi^{\pi\prime}(\phi^\pi(w;\varepsilon))}. \tag{A.21b}$$

Under (a), the fraction in (A.21a) is strictly positive such that $\phi^\pi(\cdot;\varepsilon)$ inherits the monotonicity properties of $\hat{\Phi}^\pi_w(\cdot;\varepsilon)$. Moreover, since $w > \psi^\pi(w)$ and $\hat{\Phi}^\pi_w(w;\cdot)$ is linear, the numerator in (A.21b) is strictly positive under (b). This proves the claim. ■

A.13 Proof of Theorem 4.1

First note that ϕ^π is strictly increasing such that both limits are well-defined. By the results from Theorem A.1, the map $\phi^\pi(w;\varepsilon_{\mathrm{ref}}) = \mathcal{W}(\mathcal{K}^\vartheta(w,\psi^\pi(w));\varepsilon_{\mathrm{ref}})$ has a unique fixed point $\bar{w}^\pi_{\mathrm{ref}} > 0$ and $\phi^\pi(w;\varepsilon_{\mathrm{ref}}) \gtreqless w$ iff $w \lesseqgtr \bar{w}^\pi_{\mathrm{ref}}$. Since $\varepsilon \longmapsto \phi^\pi(w;\varepsilon)$ is strictly increasing, this implies $\phi^\pi(w;\varepsilon_{\min}) < w$ for all $w \geq \bar{w}^\pi_{\mathrm{ref}}$ and $\phi^\pi(w;\varepsilon_{\max}) > w$ for all $w \leq \bar{w}^\pi_{\mathrm{ref}}$. Hence, non-trivial fixed points of $\phi^\pi(\cdot;\varepsilon_{\min})$ can only exist in $]0,\bar{w}^\pi_{\mathrm{ref}}[$ and do exist if

$\lim_{w\to 0}\phi^{\pi}(w;\varepsilon_{min})/w > 1$. Likewise, non-trivial fixed points of $\phi^{\pi}(\cdot;\varepsilon_{\max})$ can only exist in $]\bar{w}^{\pi}_{\text{ref}},\infty[$ and do exist if $\lim_{w\to 0}\phi^{\pi}(w;\varepsilon_{max})/w < 1$. Using the terminology of Brock & Mirman (1972), the map ϕ^{π} possesses a stable fixed-point configuration. Defining $\underline{w}^{\pi} := \max\{w > 0 |\, \phi^{\pi}(w;\varepsilon_{\min}) = w\} < \bar{w}^{\pi}_{\text{ref}} < \bar{w}^{\pi} := \min\{w > 0 |\, \phi^{\pi}(w;\varepsilon_{\max}) = w\}$ the set $\bar{\mathbb{W}}^{\pi} := [\underline{w}^{\pi}, \bar{w}^{\pi}]$ is the unique stable set of ϕ^{π} (for a definition, see Wang (1993, p.428)) which corresponds to a unique invariant distribution μ^{π} supported on $\bar{\mathbb{W}}^{\pi}$. ■

References

Bertocchi, G. (1994): "Safe Debt, Risky Capital", *Economica*, 61(244), 493–508.

Bertocchi, G. & Y. Wang (1994): "The Real Value of Money under Endogenous Beliefs", *Journal of Economic Theory*, 67, 205–222.

Bohn, H. (1991): "The Sustainability of Budget Deficits with Lump-Sum and with Income-Based Taxation", *Journal of Money, Credit and Banking*, 23(3), 580–604.

Brock, W. A. & L. J. Mirman (1972): "Optimal Growth and Uncertainty: The Discounted Case", *Journal of Economic Theory*, 4, 479–513.

Bullard, J. & S. Russell (1999): "An empirically plausible model of low interest rates and unbacked governmenat debt", *Journal of Monetary Economics*, 44, 477–508.

Chari, V. & P. Kehoe (1999): "Optimal Fiscal and Monetary Policy", in *Handbook of Macroeconomics*, ed. by J. Taylor & M. Woodford, chap. 26. Elsevier Science B.V., Amsterdam a.o.

de la Croix, D. & P. Michel (2002): *A Theory of Economic Growth - Dynamics and Policy in Overlapping Generations*. Cambridge University Press.

Diamond, P. (1965): "National Debt in a Neoclassical Growth Model", *American Economic Review*, 55(5), 1126–1150.

Elmendorf, D. & G. Mankiw (1999): "Government Debt", in *Handbook of Macroeconomics*, ed. by J. Taylor & M. Woodford, chap. 25. Elsevier Science B.V., Amsterdam a.o.

Farmer, R. E. A. (1986): "Deficits and Cycles", *Journal of Economic Theory*, 40(1), 77–88.

Galor, O. (1992): "A two-sector overlapping-generations model: A global characterization of the dynamical system", *Econometrica*, 60(6), 1351–1386.

——— (2007): *Discrete Dynamical Systems*. Springer, Berlin, a.o.

Kunieda, T. (2008): "Asset bubbles and borrowing constraints", *Journal of Mathematical Economics*, 44, 112–131.

Michel, P. & B. Wigniolle (2003): "Temporary Bubbles", *Journal of Economic Theory*, 112, 173–183.

Nitecki, Z. (1971): *Differentiable Dynamics*. MIT Press, Cambridge, MA.

Rankin, N. & B. Roffia (1999): "Maximum Sustainable Government Debt in the Overlapping Generations Model", CEPR Discussion Paper 2076, Centre for Economic Policy Research.

Tirole, J. (1985): "Asset Bubbles and Overlapping Generations", *Econometrica*, 53(6), 1499–1528.

Wang, Y. (1993): "Stationary Equilibria in an Overlapping Generations Economy with Stochastic Production", *Journal of Economic Theory*, 61(2), 423–435.