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Auction design with endogenously correlated buyer types

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Abstract

This paper studies optimal auction design when the seller can affect the buyers' valuations through an unobservable ex ante investment. The key insight is that the optimal mechanism may have the seller play a mixed investment strategy so as to create correlation between the otherwise (conditionally) independent valuations of buyers. The paper establishes conditions under which the seller can, in fact, extract the first best surplus almost fully.

Keywords: Auction, ex ante investment, full surplus extraction, correlation, mechanism design.

JEL Classification No.: C72, D42, D44, D82

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1 Introduction

In many situations a seller can affect buyers' valuations by an unobservable ex ante investment in the object at sale. For example, buyers' valuations for a house will depend on the effort spent by the construction company. The company's effort, such as work care or the quality of materials, is typically not directly observable. Similarly, in public procurement the contractor's cost may depend on, for example, infrastructure investments or the quality of services provided by the government who acts as the procurer itself. Or, on second hand markets, buyers' valuations are affected by the unobservable care with which initial owners have treated the item they sell. A final example is persuasive advertising.

What is the revenue maximizing selling mechanism in this setting? This is the question I address in this paper. I study a private value environment in which the seller's investment raises buyers' valuations stochastically. Conditional on the seller's investment, valuations are conditionally independent. I assume that the seller's investment is unobservable for buyers and that buyers' valuations is their private information. Consequently, if the seller adopts a pure investment strategy, there are independent private values, and buyers can typically secure information rents.

The purpose of this paper is to show that the seller can reduce information rents by designing a mechanism that induces him to adopt a mixed investment strategy. The key insight is that if the seller randomizes, then, because buyers cannot observe investment, their valuations become correlated in equilibrium. The seller can use this correlation to reduce buyers' information rents. To make this point most transparent, I consider environments in which the seller can exploit this correlation in the most extreme way and design a mechanism which extracts the full surplus that is generated by his investment. In fact, if his mixed investment strategy places almost full probability mass on the efficient investment level, then the seller extracts the first best surplus almost fully.

To see intuitively why correlation emerges when the seller randomizes, one may think of an urn model where each urn corresponds to a pure investment strategy by the seller. Buyers' valuations are drawn independently from one urn, but if the seller randomizes and buyers do not observe the realized investment, they do not know what the true urn is. Therefore, the

realization of a buyer's valuation contains information about the true urn and thus about the valuation of the rival buyer.

The existence of full surplus extracting selling mechanisms when buyers' valuations are correlated is well-established (see Myerson, 1981, Cremer and McLean, 1988, McAfee and Perry, 1992). The basic idea behind these constructions is to elicit buyers' beliefs by offering contingent lotteries whose payoffs depend on the choice of lotteries by rival buyers. Many lotteries exist which induce buyers to report their beliefs truthfully without leaving them information rents. I derive conditions such that at least one of those lotteries can be found which, in addition, makes the seller indifferent between his investment options so that randomizing is optimal.

I begin the analysis with the simplest case with two buyers, two investment opportunities, and a binary distribution of buyer valuations. In this case, a geometric argument shows that the seller can implement any mixed investment strategy and fully extract the resulting surplus. I then demonstrate how this result extends to the case in which there are (weakly) less investments than possible buyer valuations. A driving force behind these results is that the number of instruments available to make the seller indifferent between his investments is equal to the number of buyer types. Hence, with less investments than types the number of instruments exceeds the number of indifference conditions to be satisfied. This is no longer true when there are more investments than possible buyer types. Therefore, for this case I confine myself with considering the model with two buyer types only and demonstrate that there is a mechanism which yields the seller the first best profit almost fully.

In a related paper, Obara (2008) studies an auction model where buyers can take (hidden) actions that influence the joint distribution of their valuations. He demonstrates that this generically prevents the seller from extracting full surplus by a mechanism that implements a pure action profile by buyers. However, almost full surplus extraction can be attained by a mechanism which implements a mixed action profile by buyers and has them report not only their valuation but also the realization of their actions. Similar to my construction, Obara's mechanism thus exploits correlation that is created through mixed strategies. In contrast, in my setup it is the seller who randomizes, and almost full surplus extraction is achieved without having the seller report about the realization of his action.

Full surplus extraction results have come under criticism from a variety of angles. First, full surplus extraction critically relies on risk–neutrality or unlimited liability of buyers (Roberts, 1991, Demougin and Garvie, 1991) or on the absence of collusion by buyers (Laffont and Martimort, 2000). In principle, these concerns apply to a literal interpretation of my construction, too. However, even if the conditions for full surplus extraction are not met, often the correlation among buyer valuations can still be exploited to some extent.¹ The spirit of my argument is likely to carry over to such situations. In this paper, I consider an environment in which full surplus extraction is possible, because this allows me to focus on the question if there is a mechanism which induces the seller to randomize at all.

Second, a recent literature points out that full surplus extraction depends on strong common knowledge assumptions with respect to the distribution of buyers’ valuations and their higher order beliefs. Neeman (2004) has shown that full surplus extraction relies on the property that an agent’s beliefs about other agents uniquely determine his payoff.² Parreiras (2005) finds that full surplus extraction fails if the precision of agents’ information is their private information.³ While some work studies optimal (“robust”) design with weaker common knowledge assumptions (Chung and Ely, 2007, Bergemann and Schlag, 2008) it is an open issue how a seller can exploit correlation in such environments. Note, however, that in my setup the joint distribution of buyers’ valuations emerges endogenously in equilibrium as a result of the seller’s investment. Therefore, if there are no significant exogenous information sources that affect buyers’ valuations and/or their beliefs, then the common knowledge assumption is simply embodied in the equilibrium concept, it is not an ad hoc assumption on players’ exogenous beliefs.

A question related to mine has been raised in industrial economics by Spence (1975) who studies the incentives of a monopolist to invest in product quality. The difference is that

¹See e.g. Bose and Zhao (2007) who study optimal design when the agents’ beliefs violate the spanning condition by Cremer and McLean (1988), or Dequiedt and Martimort (2009) who consider the case when the designer cannot commit to a grand mechanism but only to bilateral contracts with each agent.

²Neeman and Heifetz (2006) and Barelli (2009) demonstrate that this property is generic. To the contrary, Gizatulina and Hellwig (2009) point out that the genericity of the “beliefs determine preferences” property depends on the assumption that beliefs and payoffs are exogenous features of an abstract “type” of the agent. They show that when an agent’s beliefs derive from available information, then generically beliefs do uniquely determine payoffs.

³For a related observation when agent’s can acquire information about each other see Bikhchandani (2009).

in Spence the monopolist cannot price discriminate between consumers. There seems to be surprisingly little work in the mechanism design literature that considers optimal design with an ex ante action by the designer. Instead, most work focusses on optimal design with ex ante actions by agents, such as investments in their valuation or information acquisition (e.g., Rogerson, 1992, Cremer et al., 1998, Bergemann and Välimäki, 2002, to name only a few).

The paper is organized as follows. The next section presents the model. Section 3 derives the first best benchmark. In section 4, the seller's problem is described, and section 5 contains the main argument. Section 6 concludes. All proofs are in the appendix.

2 Model

There are one risk-neutral seller and two risk-neutral buyers $i = 1, 2$. The seller has one good for sale. Buyer i 's valuation for the good is denoted by θ^i . For simplicity, buyers' valuations are assumed to be symmetric and can take on the values $0 < \theta_1 < \dots < \theta_K$. In what follows, $i, j \in \{1, 2\}$ indicates a buyer's identity, and $k, \ell \in \{1, \dots, K\}$ indicates a buyer's type. The distribution of buyers' valuations depends on a costly ex ante investment $z \in \{z_1, \dots, z_M\}$ by the seller. Investing z_m costs $c(z_m) = c_m$. Given z_m , the probability with which a buyer has valuation θ_k is p_{mk} . I assume $p_{mk} > 0$ for all m, k . This rules out deterministic investment technologies and captures buyer heterogeneity. Let

$$p_m = \begin{pmatrix} p_{m1} \\ \vdots \\ p_{mK} \end{pmatrix}$$

be the *type distribution* conditional on investment z_m . Buyers' valuations are assumed to be conditionally independent, conditional on z . Moreover, I assume that buyers cannot directly observe the seller's investment choice.

The seller may randomize between investments. A mixed investment profile is denoted by $\zeta = (\zeta_m)_m$, where ζ_m is the probability with which the seller chooses z_m . In the analysis, an important role will be played by the set of *totally mixed* investment profiles denoted by

$$\mathring{\Delta}_M = \{\zeta \mid \sum_m \zeta_m = 1, \quad \zeta_m > 0 \quad \forall m = 1, \dots, M\}.$$

If the seller adopts ζ , the unconditional probability that a buyer's valuation is θ_k is $\sum_n p_{nk}\zeta_n$. By Bayes' rule, conditional on observing valuation θ_k , a buyer's belief that investment is z_m is $q_{km} = p_{mk}\zeta_m / \sum_n p_{nk}\zeta_n$, and his belief that his rival has valuation θ_ℓ is $\mu_{k\ell}(\zeta) = \sum_m q_{km}p_{m\ell}$. Let $\mu_k(\zeta)$ be the corresponding belief (column) vector. Hence, we can write

$$\mu_k(\zeta) = \sum_m q_{km}p_m. \quad (1)$$

Since $\sum_m q_{km} = 1$, this means that the buyer's belief about his rival is a convex combination of the type distributions. Intuitively, this is because one's own valuation is a noisy signal of the true investment. This observation will be useful below.

The basic point of the paper rests on the insight that if the seller adopts a mixed investment strategy, then valuations are correlated from the point of view of buyers. The reason is that a buyer cannot observe the investment realization. One may think of an urn model where each pure investment corresponds to one urn. Buyers' valuations are drawn independently from from one urn, but a buyer does not know from which one. Therefore, the realization of a buyer's own valuation contains information about the urn and thus about the valuation of the rival buyer.

The objective of the paper is to explore whether the seller can exploit this correlation to extract full surplus. Cremer and McLean (1988, Theorem 2) have shown that full surplus extraction is closely related to a certain form of correlation which requires that beliefs be convexly independent. Formally, a set of vectors $(v_k)_k$ is *convexly independent* if no vector is the convex combination of the other vectors, that is, for no k there are weights $\beta_\ell \geq 0$ with $\sum_{\ell \neq k} \beta_\ell = 1$ so that

$$v_k = \sum_{\ell \neq k} \beta_\ell v_\ell.$$

3 First best

As a benchmark, consider the situation in which the buyers' valuation is public information. In that case, the seller optimally offers the good to the buyer with the maximal valuation at a price equal to that valuation. Therefore, for each realization of valuations, the seller can extract the full ex post surplus $\max\{\theta^1, \theta^2\}$, yielding a profit of

$$\pi^{FB}(z) = E[\max\{\theta^1, \theta^2\} \mid z] - c(z).$$

At the investment stage, the seller chooses z so as to maximize $\pi(z)$. Suppose there is a unique first best investment level $z_{\bar{m}}$ given by

$$z_{\bar{m}} = \arg \max_z \pi^{FB}(z).$$

4 Seller's problem

I now turn to the case in which the seller's investment is unobservable and the buyers' valuations are their private information. Therefore, the seller designs a mechanism which makes the assignment of the good and payments conditional on communication by the buyers. I consider the following timing.⁴

1. Seller proposes and commits to a mechanism.
2. Seller privately chooses an investment.⁵
3. Buyers privately observe their valuation.
4. Buyers reject or accept the contract.
 - If a buyer rejects, he gets his outside option of zero.
5. If buyers accept, the mechanism is implemented.

In general, a mechanism specifies for each buyer a message set, the probability with which a buyer gets the object, and payments from buyers to the seller contingent on messages submitted by the buyers in stage 5. A mechanism induces a Bayesian game between the players which starts at date 2. In equilibrium the seller's investment is a best reply against buyers' reporting strategies, and buyer's reporting strategies are best replies against the seller's investment and the rival buyer's reporting strategy. The objective of the seller is to design a revenue maximizing mechanism subject to the constraint that an equilibrium is played in the induced game.⁶

The revelation principle implies that an equilibrium outcome of any mechanism can also be attained as an equilibrium outcome of a direct and incentive compatible mechanism. A direct

⁴A similar timing is adopted in Cremer et al. (1998). If the stages 1 and 2 are swapped, signaling issues may contaminate the analysis.

⁵My results would hold a fortiori and under weaker assumptions if the seller could ex ante commit to an investment strategy.

⁶Implicit in this formulation of the seller's problem is the (standard) assumption that the seller can select his most preferred equilibrium.

mechanism asks each buyer to announce his type after stage 3 and before stage 4, and consists of an assignment rule

$$x_{k\ell} = (x_{k\ell}^1, x_{k\ell}^2), \quad 0 \leq x_{k\ell}^1, x_{k\ell}^2 \leq 1, \quad x_{k\ell}^1 + x_{k\ell}^2 \leq 1,$$

which specifies the probabilities $x_{k\ell}^1, x_{k\ell}^2$ with which buyer 1 and 2 obtain the good, conditional on the buyers' type announcements $(\theta^1, \theta^2) = (\theta_k, \theta_\ell)$. Moreover, it consists of a transfer rule

$$t_{k\ell} = (t_{k\ell}^1, t_{k\ell}^2),$$

which specifies the transfers $t_{k\ell}^1, t_{k\ell}^2$ which buyer 1 and 2 pay to the seller, conditional on the buyers' type announcements $(\theta^1, \theta^2) = (\theta_k, \theta_\ell)$. In vector notation:

$$x_k^i = \begin{pmatrix} x_{k1}^i \\ \vdots \\ x_{kK}^i \end{pmatrix}, \quad t_k^i = \begin{pmatrix} t_{k1}^i \\ \vdots \\ t_{kK}^i \end{pmatrix}.$$

A mechanism is incentive compatible if each buyer has an incentive to announce his type truthfully, given his beliefs about the rival buyer's type. Note that since a buyer's beliefs about the rival buyer's type depend upon his conjectures about the seller's investment strategy, incentive compatibility has to be defined for given conjectures.⁷ I denote by $\zeta^c = (\zeta_m^c)$ the conjecture of a buyer that the seller has chosen investment z_m with probability ζ_m^c . The expected probability of winning and the expected transfers of type k of buyer i , conditional on ζ^c , when he announces type ℓ are then respectively given as

$$\sum_{r=1}^K x_{\ell r}^i \mu_{kr}(\zeta^c) = \langle x_\ell^i, \mu_k(\zeta^c) \rangle, \quad \sum_{r=1}^K t_{\ell r}^i \mu_{kr}(\zeta^c) = \langle t_\ell^i, \mu_k(\zeta^c) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product. The mechanism is incentive compatible, conditional on ζ^c if for all i, k, ℓ :

$$\theta_k \langle x_k^i, \mu_k(\zeta^c) \rangle - \langle t_k^i, \mu_k(\zeta^c) \rangle \geq \theta_k \langle x_\ell^i, \mu_k(\zeta^c) \rangle - \langle t_\ell^i, \mu_k(\zeta^c) \rangle. \quad (\text{IC}^c)$$

Finally, the seller needs to make sure that buyers participate in the mechanism at stage 4. The mechanism is individually rational, conditional on ζ^c , if for all i, k :⁸

$$\theta_k \langle x_k^i, \mu_k(\zeta^c) \rangle - \langle t_k^i, \mu_k(\zeta^c) \rangle \geq 0. \quad (\text{IR}^c)$$

⁷I use the word "conjecture" for a buyer's beliefs about the seller's strategy so as to distinguish these beliefs from his beliefs about the rival buyer's type.

⁸If it is optimal for the seller to exclude one buyer i , he can do so by setting $x^i = t^i = 0$.

A mechanism that is incentive compatible and individually rational, conditional on ζ^c , is called feasible, conditional on ζ^c .

The seller's problem is to choose a mechanism (x, t) and an investment profile ζ which maximizes his profit subject to the constraint that the mechanism be feasible, conditional on the buyers' conjecture ζ^c and the equilibrium requirement that the buyers' conjecture be correct: $\zeta^c = \zeta$. Let

$$\pi_m = \sum_{k, \ell} [t_{k\ell}^1 + t_{\ell k}^2] p_{mk} p_{m\ell} - c_m \quad (2)$$

be the seller's profit from investment z_m . The seller's problem is thus

$$\max_{x, t, \zeta} \sum \pi_m \zeta_m \quad s.t. \quad (IC^c), (IR^c), \zeta^c = \zeta.$$

The constraint $\zeta^c = \zeta$ can be eliminated by inserting it directly in the feasibility constraints (IC^c) , (IR^c) . In what follows, I can therefore ignore the distinction between the actual and the conjectured investment strategy and consider the problem

$$\max_{x, t, \zeta} \sum \pi_m \zeta_m \quad s.t.$$

$$\theta_k \langle x_k^i, \mu_k(\zeta) \rangle - \langle t_k^i, \mu_k(\zeta) \rangle \geq \theta_k \langle x_\ell^i, \mu_k(\zeta) \rangle - \langle t_\ell^i, \mu_k(\zeta) \rangle, \quad (IC)$$

$$\theta_k \langle x_k^i, \mu_k(\zeta) \rangle - \langle t_k^i, \mu_k(\zeta) \rangle \geq 0. \quad (IR)$$

5 Mechanisms with endogenous correlation

When the seller is restricted to use a pure investment strategy z_m , a buyer's belief is independent of his type: $\mu_k = p_m$ for all k . In that case, a buyer can typically secure an information rent. The basic insight of this paper is that randomizing between investments may allow the seller to concede no information rent to buyers. For randomizing to occur in equilibrium, the mechanism has to leave the seller indifferent between all investments that he uses with positive probability. My approach to the seller's problem is to first examine if there are mechanisms that permit an equilibrium of the induced game in which the seller randomizes and extracts the full surplus. If that is the case, I say that the respective investment strategy is *FSE-implementable*.

I focus on extraction of the full *ex post efficient* surplus. From now on fix x to be the ex post efficient allocation rule which assigns the object to the buyer with the highest valuation.

I assume that ties are broken by tossing a fair coin. Thus,

$$x_{k\ell}^i = \begin{cases} 0 & \text{if } k < \ell \\ 1/2 & \text{if } k = \ell \\ 1 & \text{if } k > \ell. \end{cases}$$

I say that an investment strategy $\zeta = (\zeta_m)_m$ is *FSE-implementable* if there is a feasible mechanism (x, t) so that

$$(IR) \text{ is binding for all types } k, \tag{FSE}$$

$$\pi_m = \pi_n \text{ if } \zeta_m, \zeta_n > 0, \quad \text{and} \quad \pi_m \geq \pi_n \text{ if } \zeta_m > 0, \zeta_n = 0. \tag{IND}$$

Condition (FSE) means that, given ζ , agents get no rents. Hence, the seller extracts the full surplus. (IND) means that, given the mechanism, it is optimal for the seller to adopt the investment strategy ζ . To describe the set of FSE-implementable strategies, I proceed in two steps. I first construct mechanisms which satisfy (FSE) for given ζ and then look among those for one which satisfies (IND).

Ex post efficient surplus extracting mechanisms

The construction of surplus extracting mechanisms follows the existing literature. I consider payment rules where buyer i 's payment consists of a base payment b_k^i that depends on his own report θ_k only and a contingent payment $\tau_{k\ell}^i$ which depends on the rival's announcement θ_ℓ as well. Intuitively, contingent payments serve to elicit a buyer's belief: when valuations are correlated, there exist contingent payments that entail zero expected payments when telling the truth yet (arbitrarily) large expected payments when lying. Base payments then serve to extract the buyer's gross utility from truth-telling and may be interpreted as entry fees.

I focus on symmetric mechanisms which treat buyers symmetrically: $b_k^1 = b_k^2$ and $\tau_{k\ell}^1 = \tau_{k\ell}^2$. This allows me to consider only buyer 1 and omit the superindex i . All arguments carry over to buyer 2. The vector of contingent payments is

$$\tau_k = \begin{pmatrix} \tau_{k1} \\ \vdots \\ \tau_{kK} \end{pmatrix}.$$

The next result shows that for constructing full surplus extracting mechanisms, one needs only construct contingent payments which (a) are orthogonal to buyer type k 's belief μ_k and (b) whose projection on the other buyer types' beliefs is sufficiently large. Base payments are then automatically determined by the efficient allocation rule and the buyers' beliefs.

Lemma 1 *A feasible full surplus extracting mechanism (x, t) exists if and only if there are contingent payments τ with⁹*

$$\langle \tau_k, \mu_k \rangle = 0, \quad (\text{SE1})$$

$$\langle \tau_\ell, \mu_k \rangle \geq \theta_k \langle x_\ell, \mu_k \rangle - \theta_\ell \langle x_\ell, \mu_\ell \rangle, \quad \ell \neq k \quad (\text{SE2})$$

In this case, base payments are pinned down by $b_k = \theta_k \langle x_k, \mu_k \rangle$.

The condition (SE1) and $b_k = \theta_k \langle x_k, \mu_k \rangle$ make sure that the buyer's utility from reporting truthfully is zero, thus the seller extracts the full ex post surplus. The condition (SE2) guarantees truthtelling: the expected payments of a lie of type k , $\langle \tau_\ell, \mu_k \rangle + b_\ell$, are large enough so that lying would give negative utility.

Seller indifference

The second step is to ask when there are transfers that leave the seller indifferent between his investment opportunities. By re-arranging (2), the seller's profit from investment z_m can be written as

$$\pi_m = 2 \sum_k p_{mk} \langle \tau_k, p_m \rangle + 2 \langle b, p_m \rangle - c_m, \quad (3)$$

where b is the (column) vector whose k -th component is b_k . $\langle \tau_k, p_m \rangle$ can be interpreted as the expected (contingent) payment from buyer 1 conditional on buyer 1 being type k , where the expectation is taken from the seller's perspective, i.e. with respect to p_m .

5.1 Full surplus extraction and seller indifference

I now look for FSE-implementable strategies. By Lemma 1 this amounts to looking for contingent payments that satisfy (SE1), (SE2), and (IND) with π_m given by (3). To build intuition, I begin with the "binary-binary case" in which there are only two types and two

⁹If it does not create confusion, I shall drop the dependency of μ on ζ .

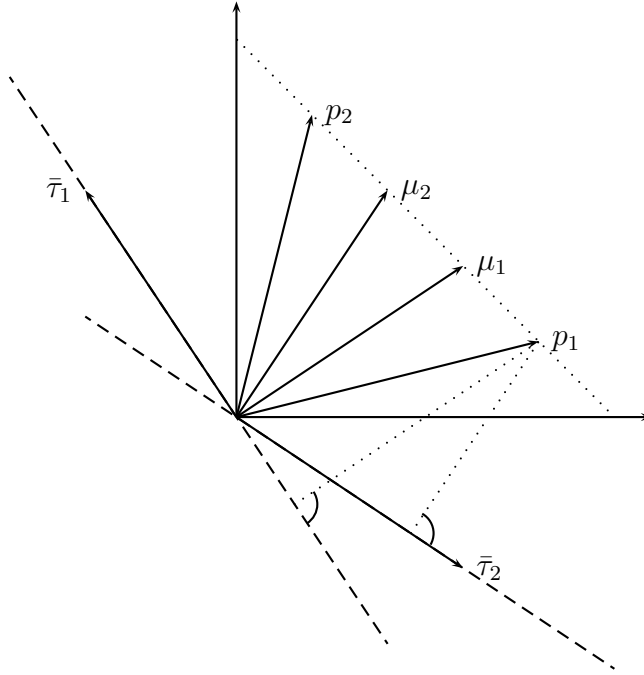


Figure 1: Full ex post surplus extracting mechanisms.

investments. A geometric argument shows that any mixed investment strategy by the seller is FSE-implementable in the binary–binary case.

5.1.1 The binary–binary case

Suppose there are two types $k = 1, 2$ and two investments z_1, z_2 . I assume that the “low” investment z_1 is more likely than the “high” investment z_2 to bring about the low valuation θ_1 : $p_{11} > p_{21}$. This implies that a low valuation buyer assigns a higher probability than a high valuation buyer to the event that he faces a low valuation rival buyer: $\mu_{11} > \mu_{21}$ for all $\zeta \in \mathring{\Delta}_2$.

Figure 1 illustrates the setup. The horizontal axis displays the first, and the vertical axis the second component of a vector. As probability vectors, p_1, p_2, μ_1, μ_2 are located on a line where the components sum to one. Since μ_k is a convex combination of p_1 and p_2 , μ_k is in between p_1 and p_2 . Moreover, since observing θ_1 (resp. θ_2) increases the likelihood that investment is low (resp. high), μ_1 is flatter than μ_2 .

I now illustrate the construction of surplus extracting mechanisms. By (SE1), the contingent payments τ_k are orthogonal to the beliefs μ_k . By (SE2), they have to be directed so that the projection on μ_ℓ , $\ell \neq k$ is large enough. In the figure, $\bar{\tau}_k$ is meant to indicate the shortest transfer vector so that the projection on μ_ℓ is just long enough to meet (SE2). Hence, all vectors τ_k on the dashed lines that are longer than $\bar{\tau}_k$ extract the full surplus.

Next, consider seller indifference. By (3):

$$\pi_m = 2[p_{m1}\langle\tau_1, p_m\rangle + p_{m2}\langle\tau_2, p_m\rangle] + 2\langle b, p_m\rangle - c_m.$$

The key observation is that the sign of the “projection” $\langle\tau_k, p_m\rangle$ is determined by the fact that the beliefs μ_1 and μ_2 are convex combinations of the type distributions p_1 and p_2 . Note that this implies that the line (hyperplane) through μ_k separates the points μ_k, p_k jointly from the points μ_ℓ, p_ℓ , $\ell \neq k$. So, because contingent payments are orthogonal to beliefs, the projections of the type distributions on contingent payments can be signed unambiguously:

$$\begin{aligned}\langle\tau_1, p_1\rangle &< 0, & \langle\tau_2, p_1\rangle &> 0, \\ \langle\tau_1, p_2\rangle &> 0, & \langle\tau_2, p_2\rangle &< 0.\end{aligned}$$

As a consequence, π_1 decreases in the length of τ_1 and increases in the length of τ_2 , while π_2 increases in the length of τ_1 and decreases in the length of τ_2 . Thus, an intermediate value argument implies that a solution to the indifference condition $\pi_1 = \pi_2$ can be found by either increasing τ_1 or τ_2 . Accordingly:

Proposition 1 *In the binary–binary case, any $\zeta \in \mathring{\Delta}_2$ is FSE-implementable.*

Let me emphasize the three main properties used. First, beliefs are convexly independent. Second, the type distributions and beliefs are ordered in a way that μ_k and p_k can jointly be separated from μ_ℓ and p_ℓ , $\ell \neq k$. Third, the seller can be made indifferent by an intermediate value argument. Next, I turn to the case when there are (weakly) fewer types than investments.

5.2 Less investments than types: $K \leq M$

I develop the argument according to the three properties used in the binary–binary case.

Convex independence of beliefs

Cremer and McLean (1988, Theorem 2) have shown that, if beliefs are not convexly independent, full surplus extraction mechanism may not exist. In my setup, convex independence of beliefs might, in principle, depend on the endogenous investment strategy ζ . However, the next lemma shows that for totally mixed investment strategies this is not the case. Rather, convex independence is a property of the primitives $(p_m)_m$ only. To state the lemma, define by

$$\bar{q}_{km} = \frac{p_{mk}}{\sum_n p_{nk}}$$

the probability with which investment z_m has realized conditional on observing θ_k when the seller adopts the uniform investment strategy which places weight $1/M$ on each investment. Denote by \bar{q}_k the corresponding probability (column) vector.

Lemma 2 *Let $(p_m)_m$ be linearly independent. Then $(\mu_k(\zeta))_k$ is convexly independent for all $\zeta \in \mathring{\Delta}_M$ if and only if $(\bar{q}_k)_k$ is convexly independent.*

The proof uses Farkas' lemma and the fact that beliefs are convex combinations of the type distributions. In light of Theorem 2 by Cremer and McLean (1988), Lemma 2 makes clear that if $(\bar{q}_k)_k$ is not convexly independent, then the seller will, in general, not be able to create correlation that allows him to extract full surplus. While the seller might still benefit from creating correlation also in this case, the construction of optimal mechanisms is demanding already in the case with exogenous beliefs (see Bose and Zhao, 2007). To focus on the question when randomizing by the seller can occur at all, I restrict attention to the case in which beliefs are convexly independent, and impose:

Assumption 1 *$(p_m)_m$ is linearly independent, and $(\bar{q}_k)_k$ is convexly independent.*

Notice that if there are only two investments and more than two types, then Assumption 1 cannot be satisfied. The reason is that beliefs are convex combinations of the type distributions. Thus, all beliefs are located on the line connecting p_1 and p_2 , and are thus not convexly independent.

Ordering of type distributions and beliefs

In the binary–binary case, the ordering of beliefs and type distributions allows a separation argument. More precisely, the type distribution associated to an investment and the belief of the type which provides the strongest evidence for this investment can jointly be separated

from all other type distributions and beliefs. I now show how this feature carries over to the general case. Recall that a buyer of type k assigns probability $q_{km} = p_{mk}\zeta_m / \sum_n p_{nk}\zeta_n$ to the event that investment z_m has realized. Consider a type which provides, among all types, the strongest evidence that z_m has occurred:

$$\kappa(\zeta, m) \in \arg \max_k q_{km}.$$

It turns out that, as in the binary-binary case, p_m and $\mu_{\kappa(\zeta, m)}$ can be jointly separated from all other type distributions and beliefs in environments in which $\kappa(\zeta, m)$ is independent of ζ . In what follows, I will focus on such environments.¹⁰ The next lemma shows that $\kappa(\zeta, m)$ is independent of ζ if and only if a monotone likelihood ratio condition is satisfied.

Assumption 2 *For all m there is a $k = k(m)$ so that for all ℓ, n , $n \neq m$, $\ell \neq k$:*

$$\frac{p_{mk}}{p_{m\ell}} \geq \frac{p_{nk}}{p_{n\ell}}. \quad (4)$$

Lemma 3 *$\kappa(\zeta, m)$ is independent of ζ if and only if Assumption 2 holds. In this case, $\kappa(\zeta, m) = k(m)$.*

Assumption 2 says that for each investment z_m there is one observation k whose likelihood ratio relative to any other observation ℓ is the highest among *all* investments. Lemma 3 says that this is the case if and only if observation k provides the strongest evidence for investment z_m irrespective of the “prior” distribution ζ with which investments are drawn.

I can now show that the type distribution p_m and the belief $\mu_{\kappa(m)}$ can jointly be separated from all other vectors p_n , $n \neq m$, and μ_ℓ , $\ell \neq \kappa(m)$:

Lemma 4 *Suppose Assumptions 1 and 2 hold, and let $\zeta \in \mathring{\Delta}_M$. Then for all m and for all numbers $s_{mn} > 0$, $n \neq m$, there is a vector $\mu_{\kappa(m)}^\perp \in \mathbb{R}^K$ so that*

$$\langle \mu_{\kappa(m)}^\perp, \mu_{\kappa(m)} \rangle = 0, \quad (5)$$

$$\langle \mu_{\kappa(m)}^\perp, p_m \rangle < 0, \quad (6)$$

$$\langle \mu_{\kappa(m)}^\perp, p_n \rangle = s_{mn} \quad \text{for } n \neq m. \quad (7)$$

Moreover,

$$\langle \mu_{\kappa(m)}^\perp, \mu_\ell \rangle > 0 \quad \text{for } \ell \neq \kappa(m). \quad (8)$$

¹⁰This is somewhat stronger than needed, but makes the argument more transparent.

Recall that two vectors are separated by a hyperplane if the scalar products of these vectors with the hyperplane's orthogonal vector have opposite signs. In Lemma 4, $\mu_{\kappa(m)}^\perp$ is an orthogonal vector of a hyperplane, which by (5) passes through $\mu_{\kappa(m)}$. The properties (6) and (7) say that the hyperplane separates the type distribution p_m from all other p_n , $n \neq m$, and, moreover, that the projections of the orthogonal vector on the type distributions p_n , $n \neq m$ are of a given magnitude s_{mn} . These properties follow from the fact that the type distributions are linearly independent and beliefs are convex combinations of the type distributions. Below, the numbers s_{mn} will be chosen in the construction of transfers.

Property (8) says that the belief vectors μ_ℓ , $\ell \neq \kappa(m)$ are all located on the same side of the hyperplane. In fact, (7) implies that they are located on the same side as the type distributions p_n , $n \neq m$. Together with (5) and (6), this means that the hyperplane separates $\mu_{\kappa(m)}$ and p_m from all other vectors. Also property (8) is due to beliefs being convex combinations of the type distributions.

The left panel of Figure 2 illustrates Lemma 4 for the case $M = K = 3$. Each point in the simplex corresponds to a probability vector over $\{1, 2, 3\}$. Any line through a point corresponds to a hyperplane that passes through this point. The dashed line through $\mu_{\kappa(1)}$ indicates a hyperplane with normal vector $\mu_{\kappa(1)}^\perp$. The hyperplane separates the points p_1 and $\mu_{\kappa(1)}$ jointly from all the other points. Lemma 4 says that under Assumptions 1 and 2 such a separation is always possible.

Lemma 4 constitutes the first step in the construction of transfers. As in the binary–binary case, I will define the transfer τ_k as an appropriate orthogonal vector of type k 's belief. Consider first types k for which there is an m with $k = \kappa(m)$. Lemma 4 specifies orthogonal vectors for these types, yet not uniquely. To guarantee that for each type k there is a unique orthogonal vector, I assume that no type can provide the strongest evidence for two different investments.

Assumption 3 *If $m \neq m'$, then $\kappa(m) \neq \kappa(m')$.*

Together with Lemma 4, Assumption 3 implies that for each type distribution, there is a *different* belief so that the two can be separated from all other type distributions and beliefs. This is again illustrated by the left panel of Figure 2. Observe that for each m a distinct $\mu_{\kappa(m)}$ can be found so that the points p_m and $\mu_{\kappa(m)}$ can jointly be separated from all other points.

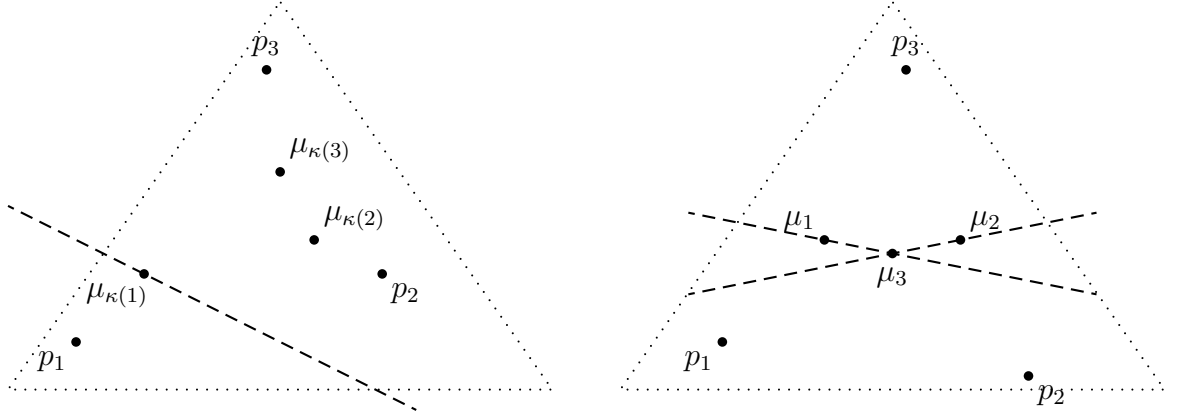


Figure 2: Orderings of beliefs and type distributions.

The right panel in Figure 2 depicts a constellation which is not covered by Assumption 3. There is no hyperplane that separates μ_3 and exactly one type distribution p_m jointly from all other points. Indeed, consider any hyperplane passing through μ_3 that leaves μ_1 and μ_2 on the same side. Such a hyperplane corresponds to a line located in between the two dashed lines. Therefore, if μ_1 and μ_2 are separated from μ_3 , the points p_1 and p_2 are always on the “other” side. It follows that there is no m with $\kappa(m) = 3$. With three investments and three types, this implies that Assumption 3 is violated.

Assumptions 2 and 3 hold, for example, under symmetry conditions. Suppose that there are as many investments as types, and investment z_m brings about type $k = m$ with some probability ρ and any other type with probability $(1 - \rho)/(K - 1) < \rho$. Then $\kappa(m) = m$, implying Assumptions 2 and 3.

After having specified orthogonal vectors for all μ_k with $k = \kappa(m)$ for some m , I next specify orthogonal vectors for types ℓ for which there is no m with $\ell = \kappa(m)$. For such a type, I define the corresponding orthogonal vector such that the projection on all other belief vectors is positive. By the separating hyperplane theorem, this is always possible since beliefs are convexly independent. Formally, choose μ_ℓ^\perp so that

$$\langle \mu_\ell, \mu_\ell^\perp \rangle = 0 \quad \text{and} \quad \langle \mu_k, \mu_\ell^\perp \rangle > 0 \quad \forall k \neq \ell. \quad (9)$$

Hence, orthogonal vectors μ_k^\perp are now defined for all k , and I set

$$\tau_k = \lambda_k \mu_k^\perp.$$

Then (5) and the left part of (9) imply (SE1). Moreover, (8) and the right part of (9) imply (SE2) whenever λ_k is large enough, that is,

$$\lambda_k \geq \frac{\theta_\ell \langle x_k, \mu_\ell \rangle - \theta_\ell \langle x_\ell, \mu_\ell \rangle}{\langle \mu_k^\perp, \mu_\ell \rangle} \equiv \bar{\lambda}_k.$$

Intermediate value argument

In the final step of the construction, I choose the numbers s_{mn} in Lemma 4 so that, similarly to the binary–binary case, an intermediate value argument can be used to make the seller indifferent. Consider profits (3). I split up the sum in types k for which there is an m with $k = \kappa(m)$ and all other types:

$$\begin{aligned} \pi_m &= 2p_{m\kappa(m)} \underbrace{\langle \mu_{\kappa(m)}^\perp, p_m \rangle}_{< 0} \lambda_{\kappa(m)} + 2 \sum_{n \neq m} p_{m\kappa(n)} \underbrace{\langle \mu_{\kappa(n)}^\perp, p_m \rangle}_{= s_{nm} > 0} \lambda_{\kappa(n)} + \\ &\quad + 2 \sum_{\ell: \nexists n \text{ with } \kappa(n)=\ell} p_{m\ell} \langle \mu_\ell^\perp, p_m \rangle \lambda_\ell + 2\langle b, p_m \rangle - c_m. \end{aligned} \tag{10}$$

Observe that by Lemma 4, $\langle \mu_{\kappa(m)}^\perp, p_m \rangle < 0$ and $\langle \mu_{\kappa(n)}^\perp, p_m \rangle = s_{nm} > 0$. Because $\kappa(m) \neq \kappa(n)$ by Assumption 3, this implies that π_m decreases in $\lambda_{\kappa(m)}$ and increases in $\lambda_{\kappa(n)}$, $n \neq m$. This property can be used to construct transfers that leave the seller indifferent. I have the following proposition.

Proposition 2 *Under Assumptions 1 to 3, any $\zeta \in \mathring{\Delta}_M$ is FSE-implementable.*

To see how the construction works, consider the case with $M = 3$ investments. The coefficients s_{nm} can be chosen in such a way that a change of $\lambda_{\kappa(m)}$ changes all profits π_n , $n \neq m$ at the *same* rate.¹¹ Now, consider some constellation with $\lambda_k \geq \bar{\lambda}_k$, and suppose that profits are ranked, e.g., as $\pi_3 > \pi_1 > \pi_2$. Now increase $\lambda_{\kappa(1)}$. Since this decreases π_1 and raises π_2 , there will be a point $\lambda'_{\kappa(1)}$ at which equality holds: $\pi_1 = \pi_2$. Moreover, at this point it is still true that $\pi_3 > \pi_1$, because also π_3 increases in $\lambda_{\kappa(1)}$. Next, increase $\lambda_{\kappa(3)}$. This decreases π_3 and raises π_1 and π_2 . In fact, because π_1 and π_2 increase at the same rate, the equality $\pi_1 = \pi_2$ is maintained when $\lambda_{\kappa(3)}$ is increased. Hence, there will be a point $\lambda'_{\kappa(3)}$ at which equality holds between all three profits, and the seller is indifferent.

¹¹This works for $s_{nm} = \prod_{n' \neq m} p_{n'\kappa(n)}$.

Proposition 2 provides conditions so that *any* totally mixed strategy is FSE-implementable. Therefore, it directly implies that the seller can attain a profit arbitrarily close to the (ex ante) first best surplus by implementing an investment strategy which places almost full mass on the first best investment:

Proposition 3 *Under the conditions of Proposition 2, the seller can approximately attain the ex ante first best surplus π^{FB} .*

For the seller to attain approximate first best, only investment strategies which are close to the first best investment need to be FSE-implementable. Therefore, Proposition 3 holds under weaker conditions than those of Proposition 2, which guarantees FSE-implementability for *all* totally mixed strategies. In fact, suppose that $\kappa(\zeta, m)$ is independent of ζ and that Assumption 3 holds in a neighborhood around the first best investment strategy (instead of globally for all ζ). Then the same argument that I presented is applicable to show that any totally mixed strategy in this neighborhood is FSE-implementable, and accordingly, the seller can approximately attain the first best.

5.3 More investments than types

The previous section shows that when there are fewer investments than types, any totally mixed strategy is FSE-implementable. When there are more investments than types, this will no longer be true in general. The reason is that the number of transfers available to make the seller indifferent is equal to the number of types. Thus, there are K instruments only to satisfy $M - 1 \geq K$ equations (plus non-negativity constraints on the instruments).

For this reason, I now specialize the analysis in two respects. First, I focus exclusively on the question whether the seller can approximately attain the first best profit. Second, for tractability reasons I confine myself with considering the case with two types $k = 1, 2$.

5.3.1 Two types

Assume that “low” investments are more likely to bring about the low valuation: $p_{m1} > p_{n1}$ if $m < n$. The next result shows that the seller can almost fully extract the ex ante first best surplus.

Proposition 4 *Suppose there are two types. Then the seller can approximately attain the first best profit $\pi^F B$*

The idea behind the proof is to let the seller randomize between the first best investment $z_{\bar{m}}$ and, say, the next smaller investment $z_{\bar{m}-1}$. As in the binary–binary case, transfers can be found which make the seller indifferent between these two investments. In fact, any multiple of these transfers leaves the seller indifferent, too. The remaining question is then whether within this set of transfers one can be found so that, in addition, the seller (weakly) prefers the investment $z_{\bar{m}}$ over all other investments z_m . This essentially boils down to the question how the difference in profits $\pi_{\bar{m}} - \pi_m$ changes as transfers are increased. It turns out that this change is always positive, once the probability $\zeta_{\bar{m}}$ with which the seller plays $z_{\bar{m}}$ is sufficiently large. Therefore, if transfers are increased, $\pi_{\bar{m}} - \pi_m$ becomes arbitrarily large, making the seller prefer $z_{\bar{m}}$ over z_m .

What facilitates the analysis in the two type case is that all type distributions and beliefs are ordered on a one–dimensional line. This imposes enough structure to determine the size of the projections of type distribution p_m on transfers τ_k , which, in turn, is needed to figure out the change in the profit difference $\pi_{\bar{m}} - \pi_m$. In higher dimensions there is no such restriction on the location of type distributions and beliefs.

6 Discussion and Conclusion

Let me discuss some assumptions underlying the analysis. First, while I have considered a private values auction model, all my results will go through for more general mechanism design problems with general allocation spaces and (gross) utility functions of agents, including interdependent values models. The reason is that by Lemma 1, it is essentially enough to construct appropriate contingent transfers which are orthogonal to an agent’s *beliefs*. The specific form of agents’ *willingness to pay* is irrelevant for the construction of contingent payments, it only pins down base payments. Similarly, the restriction to two symmetric buyers is not substantial and just keeps notation simple.

What is more substantial is the restriction to simple type spaces with the property that any belief goes along with a distinct valuation (“beliefs determine preferences”). A situation in

which this need not be true is when buyers possess (imperfect) private information *ex ante*, and the information they receive after the seller's investment is only *additional* information. In such a case, convex independence of beliefs and thus full surplus extraction may fail (see, e.g. Neeman 2004, or Parreiras 2005). As argued earlier, then the construction of optimal mechanisms is demanding already in the case in which the seller cannot affect beliefs. I therefore focus on simple type spaces.

Furthermore, I have confined the analysis to mechanisms which do not condition on a report by the seller. While under the assumptions of Propositions 2 and 4, mechanisms with buyer reports only cannot be improved upon, this may change once these assumptions are violated: because the seller, after having observed the realization of his investment, holds private information, too, allowing for mechanisms with seller reports could extend the set of implementable outcomes. In fact, Obara (2008) shows that when it is buyers who choose *ex ante* actions, having them report about the realizations of their actions, can be beneficial. However, in Obara's setup, agents' transfers in the extended mechanism can be constructed by standard orthogonality conditions. In my setup, this is not true, because here the seller's transfers are the payments by buyers. Therefore, in my setup, the transfers in the extended mechanism have to respect, in addition, a sort of budget balance condition. Dealing with this requires a rather different line of argument. I leave the full analysis for future research.

I conclude with noting that my analysis raises the more general question about strategies by which a mechanism designer can influence the joint distribution of the agents' valuations. A case in point is disclosure. Standard models of disclosure (e.g. Bergemann and Pesendorfer, 2007, Ganuza and Penalva, 2009) typically fix a selling format such as a first price auction and ask how much information the seller optimally wants to disclose to bidders. If the seller has some discretion over the selling format, then disclosing information in ways such that bidders' information is correlated may be beneficial.

Appendix

Proof of Lemma 1 For a mechanism with base payments b_k and contingent payments τ_k , the feasibility constraints (IC) and (IR) are respectively:

$$\theta_k \langle x_k, \mu_k \rangle - \langle \tau_k, \mu_k \rangle - b_k \geq \theta_k \langle x_\ell, \mu_k \rangle - \langle \tau_\ell, \mu_k \rangle - b_\ell \quad \forall k, \ell, \quad (11)$$

$$\theta_k \langle x_k, \mu_k \rangle - \langle \tau_k, \mu_k \rangle - b_k \geq 0 \quad \forall k, \quad (12)$$

where (12) is binding if the full surplus is extracted. With this, the if-part is obvious (simply define $t_{k\ell} = \tau_{k\ell} + b_k$). For the only-if-part, define $b_k = \theta_k \langle x_k, \mu_k \rangle$ and $\tau_{k\ell} = t_{k\ell} - b_k$. It is straightforward to verify (SE1) and (SE2). Q.E.D.

Proof of Lemma 2 Let $\zeta \in \mathring{\Delta}_M$. Define by $\alpha_k = \sum_n p_{nk} \zeta_n$ the probability of type k , and recall that $q_{km} = p_{mk} \zeta_m / \alpha_k$ is the posterior over investments conditional on k given ζ . Let q_k be the corresponding probability (column) vector. The lemma follows from the two following claims:

- (a) $(\bar{q}_k)_k$ is convexly independent if and only if $(q_k)_k$ is convexly independent.
- (b) If $(p_m)_m$ is linearly independent, $(\mu_k(\zeta))_k$ is convexly independent if and only if $(q_k)_k$ is convexly independent.

As for (a). By Farkas' lemma (see Ok, 2007, p. 482), $(\bar{q}_k)_k$ is convexly independent if and only if for all k there is a $x_k \in \mathbb{R}^M$ so that

$$\langle x_k, \bar{q}_\ell \rangle \geq 0 \quad \forall \ell \neq k \quad \text{and} \quad \langle x_k, \bar{q}_k \rangle < 0. \quad (13)$$

Observe that for all ℓ

$$\bar{q}_{\ell m} = \frac{\alpha_\ell}{\zeta_m \sum_n p_{n\ell}} \frac{p_{m\ell} \zeta_m}{\alpha_\ell} = \frac{\alpha_\ell}{\zeta_m \sum_n p_{n\ell}} q_{\ell m}. \quad (14)$$

Let $y_k \in \mathbb{R}^M$ be defined by the components $y_{km} = x_{km} / \zeta_m$. Then (13) is equivalent to

$$\frac{\alpha_\ell}{\sum_n p_{n\ell}} \langle y_k, q_\ell \rangle \geq 0 \quad \forall \ell \neq k \quad \text{and} \quad \frac{\alpha_k}{\sum_n p_{nk}} \langle y_k, q_k \rangle < 0 \quad (15)$$

$$\Leftrightarrow \langle y_k, q_\ell \rangle \geq 0 \quad \forall \ell \neq k \quad \text{and} \quad \langle y_k, q_k \rangle < 0. \quad (16)$$

Consequently, using again Farkas' lemma, $(q_\ell)_\ell$ is convexly independent, and this proves (a).

As for (b). Consider an arbitrary index k and probability weights β_ℓ , $\ell \neq k$. By (1):

$$\mu_k = \sum_{\ell \neq k} \beta_\ell \mu_\ell \Leftrightarrow \sum_m q_{km} p_m = \sum_{\ell \neq k} \beta_\ell \sum_m q_{\ell m} p_m \Leftrightarrow \sum_m (q_{km} - \sum_{\ell \neq k} \beta_\ell q_{\ell m}) p_m = 0. \quad (17)$$

Now, if $(q_k)_k$ is convexly independent, then there is an m' so that $q_{km'} - \sum_{\ell \neq k} \beta_\ell q_{\ell m'} \neq 0$. Since $(p_m)_m$ is linearly independent, the right equation is, therefore, violated. Hence, also the first equation is violated, but this means that $(\mu_k)_k$ is convexly independent. As for the reverse, suppose $(\mu_k)_k$ is convexly independent so that the first equation is violated. Then also the third equation is violated, and hence $q_{km} - \sum_{\ell \neq k} \beta_\ell q_{\ell m} \neq 0$ for some m . But this means that $(q_k)_k$ is convexly independent. Q.E.D.

Proof of Lemma 3 Note that $\kappa(\zeta, m) = k$ if and only if

$$\frac{p_{mk}\zeta_m}{\sum_n p_{nk}\zeta_n} \geq \frac{p_{m\ell}\zeta_m}{\sum_n p_{n\ell}\zeta_n} \quad \forall \ell \neq k \quad (18)$$

$$\Leftrightarrow \sum_n (p_{mk}p_{n\ell} - p_{m\ell}p_{nk})\zeta_n \geq 0 \quad \forall \ell \neq k. \quad (19)$$

Observe that the inequality is true for all ζ if and only if the term in brackets under the sum are positive for all $n \neq m$. But this is equivalent to Assumption 2. Q.E.D.

Proof of Lemma 4 As to (5) to (7). Fix m . First, I write the system of M equations given by (5) and (7) in matrix notation. Define the $K \times M$ -matrix $A = (\mu_{\kappa(m)}, p_1, \dots, p_{m-1}, p_{m+1}, \dots, p_M)$ and the row vector $v = (0, s_{m1}, \dots, s_{m,m-1}, s_{m,m+1}, \dots, s_{mM}) \in \mathbb{R}^M$. Then the equations in (5) and (7) can be stated as

$$(\mu_{\kappa(m)}^\perp)^T A = v, \quad (20)$$

where the superscript T indicates the transposed. I have to show that for all v there is a solution $\mu_{\kappa(m)}^\perp$ to (20). Indeed, observe that the M vectors $\mu_{\kappa(m)}, p_n, n \neq m$, are linearly independent. This is so since the $M - 1$ vectors $p_n, n \neq m$, are linearly independent by Assumption 1, and $\mu_{\kappa(m)}$ is a convex combination of *all* p_n 's with a positive weight on p_m . Hence, the matrix A has rank M , and a solution $\mu_{\kappa(m)}^\perp$ to (20) exists.

As for (8). I begin with the remark that under Assumption 1, for all k, ℓ there is a strict inequality in (4) for some m and n . To the contrary, suppose that there are $k, \ell \neq k$ so that $p_{mk}/p_{nk} = p_{m\ell}/p_{n\ell}$ for all m and n , then

$$(\bar{q}_{km})^{-1} = \frac{\sum_n p_{nk}}{p_{mk}} = \frac{\sum_n p_{n\ell}}{p_{m\ell}} = (\bar{q}_{\ell m})^{-1}. \quad (21)$$

It follows that $\bar{q}_k = \bar{q}_\ell$, a contradiction to the convex independence of $(\bar{q}_k)_k$ posited in Assumption 1.

I now show inequality (8). The fact that $\langle \mu_{\kappa(m)}^\perp, \mu_{\kappa(m)} \rangle = 0$ and (1) imply

$$\langle \mu_{\kappa(m)}^\perp, p_m \rangle = -\frac{\alpha_{\kappa(m)}}{p_{m\kappa(m)}\zeta_m} \sum_{n \neq m} \frac{p_{n\kappa(m)}\zeta_n}{\alpha_{\kappa(m)}} \langle \mu_{\kappa(m)}^\perp, p_n \rangle = \sum_{n \neq m} \frac{p_{n\kappa(m)}\zeta_n}{p_{m\kappa(m)}\zeta_m} \langle \mu_{\kappa(m)}^\perp, p_n \rangle. \quad (22)$$

Using this in (1) for μ_ℓ gives

$$\langle \mu_{\kappa(m)}^\perp, \mu_\ell \rangle = \frac{p_{m\ell}\zeta_m}{\alpha_\ell} \langle \mu_{\kappa(m)}^\perp, p_m \rangle + \sum_{n \neq m} \frac{p_{n\ell}\zeta_n}{\alpha_\ell} \langle \mu_{\kappa(m)}^\perp, p_n \rangle \quad (23)$$

$$= \sum_{n \neq m} \left[-\frac{p_{n\kappa(m)}\zeta_n}{p_{m\kappa(m)}\zeta_m} \cdot \frac{p_{m\ell}\zeta_m}{\alpha_\ell} + \frac{p_{n\ell}\zeta_n}{\alpha_\ell} \right] \langle \mu_{\kappa(m)}^\perp, p_n \rangle \quad (24)$$

$$= \sum_{n \neq m} \frac{\zeta_n}{\alpha_\ell} \left[-\frac{p_{n\kappa(m)}p_{m\ell}}{p_{m\kappa(m)}} + p_{n\ell} \right] \langle \mu_{\kappa(m)}^\perp, p_n \rangle. \quad (25)$$

By Assumption 2, the term in the square bracket is non-negative for all n, ℓ , and, by the remark at the beginning, is positive for some n, ℓ . Together with the fact that $\langle \mu_{\kappa(m)}^\perp, p_n \rangle > 0$ by (7), this yields the claim. Q.E.D.

Proof of Proposition 2 I construct $\lambda_k \geq \bar{\lambda}_k$ so that $\pi_m = \pi_n$ for all m, n . Let

$$s_{nm} = \prod_{n' \neq m} p_{n'\kappa(n)}, \quad s_n = \prod_{n'} p_{n'\kappa(n)}. \quad (26)$$

Then profits in (10) become

$$\begin{aligned} \pi_m &= 2p_{m\kappa(m)} \langle \mu_{\kappa(m)}^\perp, p_m \rangle \lambda_{\kappa(m)} + 2 \sum_{n \neq m} s_n \lambda_{\kappa(n)} + \\ &\quad + 2 \sum_{\ell: \nexists n \text{ with } \kappa(n)=\ell} p_{m\ell} \langle \mu_\ell^\perp, p_m \rangle \lambda_\ell + 2\langle b, p_m \rangle - c_m. \end{aligned} \quad (27)$$

Observe that if $\lambda_{\kappa(m)}$ is increased, then, since $\langle \mu_{\kappa(m)}^\perp, p_m \rangle < 0$ and because $\kappa(m) \neq \kappa(n)$, $n \neq m$, the profit π_m decreases, while all π_n , $n \neq m$, increase. Moreover, all π_n , $n \neq m$ increase at the *same* rate $2s_m$. Therefore, if $\lambda_{\kappa(m)}$ is increased, then the difference $\pi_n - \pi_{n'}$ is unaffected for all $n, n' \neq m$. I now exploit this property to construct the desired λ step by step.

Let $\lambda' \in \mathbb{R}^K$ with $\lambda'_k \geq \bar{\lambda}_k$ for all k . If $\pi_m(\lambda') = \pi_n(\lambda')$ for all m, n we are done. Otherwise, consider the investments that give the lowest payoff:

$$N_{\min}(\lambda') = \{n \mid \pi_n(\lambda') = \min_m \pi_m(\lambda')\}. \quad (28)$$

Moreover, denote by \hat{m} the next best investment, which is given by:

$$\pi_{\hat{m}}(\lambda') > \min_m \pi_m(\lambda'), \quad \pi_{\hat{m}}(\lambda') \leq \pi_m(\lambda') \quad \text{for all } m \notin N_{\min}(\lambda'). \quad (29)$$

Now consider an arbitrary $n_0 \in N_{\min}(\lambda')$. Because $\pi_{\hat{m}}$ decreases and π_{n_0} increases continuously in $\lambda_{\kappa(\hat{m})}$, there is a $\lambda''_{\kappa(\hat{m})} > \lambda'_{\kappa(\hat{m})}$ so that $\pi_{\hat{m}}(\lambda'') = \pi_{n_0}(\lambda'')$, where λ'' has the same components as λ' except of the $\kappa(\hat{m})$ -th component, which is $\lambda''_{\kappa(\hat{m})}$.

Moreover, because all π_n , $n \neq \hat{m}$ increase at the same rate as $\lambda'_{\kappa(m)}$ is increased to $\lambda''_{\kappa(m)}$, we also have that $\pi_{\hat{m}}(\lambda'') = \pi_n(\lambda'')$ for all $n \in N_{\min}(\lambda')$, so that at λ'' :

$$N_{\min}(\lambda'') = N_{\min}(\lambda') \cup \{\hat{m}\}. \quad (30)$$

Now proceed repeatedly in the same manner. After at most M steps, this yields a λ with $\lambda_k \geq \bar{\lambda}_k$ for all k , at which all π_m are the same. Q.E.D.

Proof of Proposition 4 Consider the investment strategy ζ which places probability $1 - \eta$ on the first best investment $z_{\bar{m}}$ and probability η on $z_{\bar{m}-1}$. (If $\bar{m} = 1$, then randomizing between $z_{\bar{m}}$ and $z_{\bar{m}+1}$ works.) I show that for all η smaller than some $\bar{\eta} > 0$, ζ is FSE-implementable. Consequently, the seller's ex ante profit gets arbitrarily close to the first best profit as η goes to zero.

By (3), the profit difference between investment \bar{m} and any other investment $n \neq \bar{m}$ is

$$\begin{aligned} \pi_{\bar{m}} - \pi_n &= 2[p_{\bar{m}1}\langle\mu_1^\perp, p_{\bar{m}}\rangle - p_{n1}\langle\mu_1^\perp, p_n\rangle]\lambda_1 \\ &\quad + 2[p_{\bar{m}2}\langle\mu_2^\perp, p_{\bar{m}}\rangle - p_{n2}\langle\mu_2^\perp, p_n\rangle]\lambda_2 \\ &\quad + T_1, \end{aligned} \quad (31)$$

where T_1 is a constant independent of λ_1, λ_2 . Hence, the seller is indifferent between \bar{m} and $\bar{m} - 1$ if and only if

$$\lambda_1 = -\frac{p_{\bar{m}2}\langle\mu_2^\perp, p_{\bar{m}}\rangle - p_{\bar{m}-1,2}\langle\mu_2^\perp, p_{\bar{m}-1}\rangle}{p_{\bar{m}1}\langle\mu_1^\perp, p_{\bar{m}}\rangle - p_{\bar{m}-1,1}\langle\mu_1^\perp, p_{\bar{m}}\rangle}\lambda_2 + T_2 \quad (32)$$

for some constant T_2 . With this, I can compare the profit of m and any other investment n :

$$\begin{aligned} \pi_{\bar{m}} - \pi_n &= 2 \left\{ \frac{p_{\bar{m}2}\langle\mu_2^\perp, p_{\bar{m}}\rangle - p_{\bar{m}-1,2}\langle\mu_2^\perp, p_{\bar{m}-1}\rangle}{p_{\bar{m}1}\langle\mu_1^\perp, p_{\bar{m}}\rangle - p_{\bar{m}-1,1}\langle\mu_1^\perp, p_{\bar{m}}\rangle} \cdot [p_{\bar{m}1}\langle\mu_1^\perp, p_{\bar{m}}\rangle - p_{n1}\langle\mu_1^\perp, p_n\rangle] \right. \\ &\quad \left. + [p_{\bar{m}2}\langle\mu_2^\perp, p_{\bar{m}}\rangle - p_{n2}\langle\mu_2^\perp, p_n\rangle] \right\} \cdot \lambda_2 + T_3, \end{aligned} \quad (33)$$

where T_3 is some constant. I now show that when $\eta = 0$, the coefficient in the curly brackets in front of λ_2 is strictly positive for all $n \neq \bar{m}, \bar{m} - 1$. By continuity, this coefficient will still be strictly positive for all small $\eta > 0$. This implies that by raising λ_2 , the difference $\pi_{\bar{m}} - \pi_n$ can

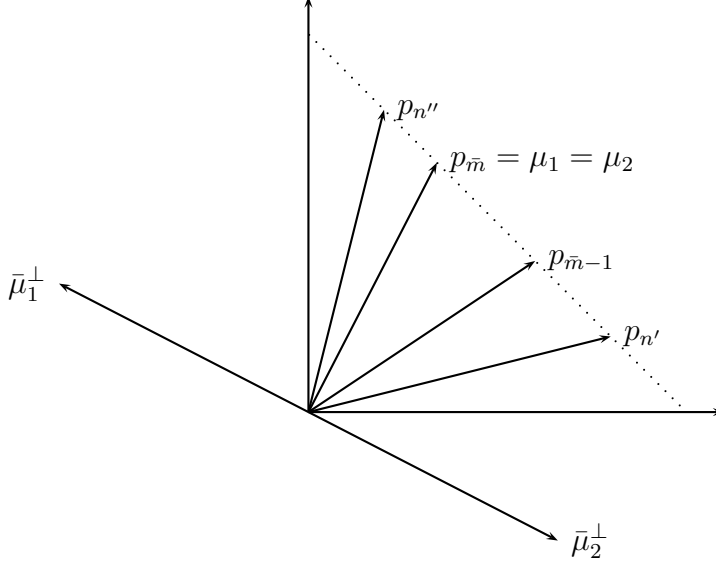


Figure 3: Ordering of beliefs and type distributions for $M > K = 2$ and $\eta = 0$.

be made arbitrarily large while keeping $\pi_{\bar{m}} - \pi_{\bar{m}-1}$ equal to zero. Therefore, the seller prefers the investments $z_{\bar{m}}$ and $z_{\bar{m}-1}$ over all other investments, and randomizing between $z_{\bar{m}}$ and $z_{\bar{m}-1}$ is optimal.

Indeed, for $\eta = 0$, it holds that $\mu_1 = \mu_2 = p_{\bar{m}}$ and, given the orientation of μ_1^\perp and μ_2^\perp :

$$\mu_1^\perp = \begin{pmatrix} -p_{\bar{m}2} \\ p_{\bar{m}1} \end{pmatrix}, \quad \mu_2^\perp = \begin{pmatrix} p_{\bar{m}2} \\ -p_{\bar{m}1} \end{pmatrix}. \quad (34)$$

Thus, we have that $\langle \mu_1^\perp, p_{\bar{m}} \rangle = \langle \mu_2^\perp, p_{\bar{m}} \rangle = 0$ and $\langle \mu_1^\perp, p_n \rangle = -\langle \mu_2^\perp, p_n \rangle$ for all n . Figure 3 illustrates.

With this, the coefficient in front of λ_2 gets

$$\left\{ -\frac{-p_{\bar{m}-1,2} \langle \mu_2^\perp, p_{\bar{m}-1} \rangle}{-p_{\bar{m}-1,1} \langle \mu_1^\perp, p_{\bar{m}-1} \rangle} \cdot [-p_{n1} \langle \mu_1^\perp, p_n \rangle] + [-p_{n2} \langle \mu_2^\perp, p_n \rangle] \right\} \quad (35)$$

$$= \langle \mu_1^\perp, p_n \rangle \left[-\frac{p_{\bar{m}-1,2}}{p_{\bar{m}-1,1}} \cdot p_{n1} + p_{n2} \right]. \quad (36)$$

I now argue that this expression is positive. Observe first that the term in the square brackets is positive if and only if

$$-(1 - p_{\bar{m}-1,1})p_{n1} + p_{\bar{m}-1,1}(1 - p_{n1}) > 0 \quad \Leftrightarrow \quad p_{\bar{m}-1,1} > p_{n1} \quad \Leftrightarrow \quad n > \bar{m} - 1. \quad (37)$$

Observe moreover (see Figure 3),

$$\langle \mu_1^\perp, p_n \rangle > 0 \quad \Leftrightarrow \quad n > \bar{m}. \quad (38)$$

These two observations imply that the term in front of λ_2 in (33) is strictly positive for all $n \neq \bar{m}, \bar{m} - 1$. This completes the proof. Q.E.D.

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