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Rarely Enjoyed? A Count Data Analysis of Transit Ridership in Germany's Public Transport

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Abstract. Focusing on adult members of German households, this paper investigates the determinants of public transit ridership with the aim of quantifying the effects of fuel prices, fares, person-level attributes, and characteristics of the transit system on transport counts over a five-day week. The reliance on individual data raises several conceptual and empirical issues, the most fundamental of which is the large proportion of null values in transit counts. To accommodate this feature of the data, we employ modeling procedures referred to as zero-inflated models (ZIMs), which order observations into two latent regimes defined by whether the individual never uses public transport. The model estimates reveal fuel prices to have a positive and substantial influence on transit ridership, though there is no evidence for a statistically significant impact of the fare. Methodologically, ZIMs are seen to have superior predictive accuracy over the classical count data models, and thus may serve as the method of choice when the aim is to predict trip frequency for modes that a large fraction of the population never uses.

JEL classification: D13, Q41.

Key words: Poisson Model, Negative Binomial Model, Zero-Inflated Models, Household Data.

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1 Introduction

Quality control management of manufacturing processes is a typical field where count data play a major role. When large product samples are drawn for quality control purposes, the number of defect products is usually low, as is clearly desirable from the perspective of manufacturers. The preponderance of zero counts, which indicates that the manufactured product fulfills the required quality standards, is therefore a distinguishing characteristic of quality control experiments. The classical approach to modeling rarely occurring events, such as a defect manufacturing output, is the Poisson regression model (PRM), which is based on the well-known Poisson distribution.\footnote{This distribution was developed in 1837 by the French mathematician Siméon Denis Poisson (1781-1840) and results from the binomial distribution as a limiting case to describe the probability of rare events. It was the Russian and in Germany living economist and statistician Ladislaus von Bortke- witsch (1868-1931), however, who first applied the Poisson distribution for the description of rare events such as deaths due to horse riding accidents in the Prussian army.}

Departing from this classical approach, many other models have been designed to account for the specific feature of count data for which the concrete values of the variable of interest are discrete, nonnegative numbers. Apart from the PRM, among the most popular are the negative binomial regression model (NBRM), as well as the so-called zero-inflated models (ZIMs), which are particularly appropriate for data sets with a substantial share of zero counts. There are two common ZIMs, referred to as the zero-inflated Poisson (ZIP) and the zero-inflated negative binomial (ZINB) models, both of which are generalizations of the PRM and NBRM.

Using household survey data from Germany, this paper applies zero-inflated modeling approaches to the issue of public transport patronage, another area in which the outcome of count processes is typically characterized by an overwhelmingly large number of zero counts. A basic question emerging from this data pattern is whether a zero count indicates an individual who never uses public transport, or alternatively the chance event that the individual does not use public transport during the sampling period. Zero-inflated modeling procedures take particular account of this distinction.
by ordering observations into two latent regimes defined by whether an individual never uses public transport and are therefore perfectly appropriate in this instance. By contrast, both the PRM and NBRM rest on the assumption that the probability for a positive value of the dependent variable is non-zero for every individual. With the exception of a handful of studies that mostly focus on accident rates (Shanker et al., 1997, Chin and Quddus, 2003), this feature of unobservable membership in either of two groups – the group of never-users and its complement – has rarely been addressed in the transportation literature.

We focus here on the determinants of adult transit ridership, with the specific aim of quantifying the effects of fuel prices and fares on public transport counts over a five-day week, while controlling for the effects of person-level attributes and characteristics of the transit system. A large empirical literature has emerged to address this issue, but, as with the literature on fuel price elasticities for automobile travel (Graham and Glaister, 2002), elasticity estimates for transit vary widely. Based on a comprehensive survey of the literature, Litman (2004) finds short-run elasticity estimates with respect to the fare varying between -0.2 and -0.5, with a subsequent meta-analysis by Holmgren (2007) finding the short-run elasticity to reach as high as -0.75 for Europe. The elasticity estimates of fuel prices tend to be lower, but also highly variable, ranging from 0.05 (Litman, 2004) to 0.4 (Holmgren, 2007).

As Nijkamp and Pepping (1998) note, the most important factor accounting for the differences in transit estimates is whether aggregate or disaggregate data is used. As aggregate data makes no allowance for the large variation of individual choices made in specific circumstances, it typically yields less precise estimates that are, moreover, more subject to bias. To date, however, the majority of empirical attempts to estimate price effects have drawn on country-level data or data aggregated at sub-national administrative districts, typically from the U.S., with a smaller pool of studies relying on household-level data. Departing from this reliance, our analysis is predicated on the notion that transit use is an individual decision, albeit one that is dependent on intra-household allocation processes. This tack is in line with a growing body of literature that has identified the importance of socioeconomic factors such as employment sta-

Among the key findings of our analysis is that ZIMs have superior predictive accuracy over the PRM and NBRM, and thus may serve as the method of choice when the aim is to predict trip frequency for modes that a large fraction of the population never uses. The model estimates reveal fuel prices to have a positive and substantial influence on transit ridership, though we find no evidence for a statistically significant impact of the fare. In this regard, our findings highlight the importance of referencing both the coefficients and associated marginal effects when interpreting the results. Due to the non-linearity of the model, the magnitude and significance level of these estimates can vary markedly from one another, requiring that inferences be cast in more specific terms than is conventionally the case.

The following section describes the data base used for estimating individual mobility behavior of adults. Section 3 explicates the econometric methods and model specifications, followed by the presentation and interpretation of the results in Section 4. The last section summarizes and concludes.

2 Data

The main data source used in this research is drawn from the German Mobility Panel (MOP 2010), an ongoing travel survey financed by the Federal Ministry of Transport, Building, and Urban Affairs. The panel is organized in waves, each comprising a group of households whose members are surveyed for a period of one week over each of three years. The data set includes twelve waves of the panel, spanning 1996 through 2007, and is limited to adult individuals who are at least 18 years old.

In total, the data set contains 8,577 individuals, 5,131 of whom participated in one year of the survey with the remaining 3,446 participating in two or three years. For this
latter group, we randomly selected a single year for inclusion in the data set to avoid repeat observations on the same individual. In this regard, it bears noting that public transit use and the variables that determine it vary little or not at all over the three years of the survey, dictating the use of pooled data in model estimation.

Individuals that participate in the survey are requested to fill out a questionnaire eliciting general household information and person-related characteristics, including zip code of residence, gender, age, employment status and relevant aspects of everyday travel behavior. In addition to this general survey, the MOP includes a separate survey focusing specifically on vehicle travel among a 50% sub-sample of randomly selected car-owning households. These households are drawn from the larger MOP-data set used in the present analysis. This so-called “tank survey” takes place over a roughly six-week period, during which time respondents record sundry automobile-related information, including the price paid for fuel.

As this variable is a potentially important determinant of transit pass ownership, it was linked with the larger sample of households in the MOP by using a Geographic Information System to create a coverage of spatially interpolated fuel prices (in real terms) for all of Germany. The coverage was then overlaid onto a map of household locations in the MOP data, thereby allowing for each household to be ascribed the locally prevailing fuel price. This process was repeated for each year of the data, yielding a data set of fuel prices that varies over space and time. An accuracy assessment of the data was undertaken by calculating the yearly average fuel prices and comparing these with those published for the German market by the oil company Aral (2009). The correspondence between the two sources is tight, deviating by an average of less than 1% over the 1996-2007 time-interval (see Frondel and Vance, 2010).

In addition to fuel prices, another important cost-determinant of transit use is the fare. Data on this variable was obtained by an internet-based survey that retrieved the price for a single-trip and monthly ticket for each of the 90 regional transit authorities in Germany. Each household was then assigned the fare of the transit authority to which it belongs. Fares, as well as fuel prices, were converted into real terms using...
a consumer price index published by the German Statistical Office (DESTATIS, 2010). From the same source, we also obtained a variable measuring the density of transit service that was merged with the MOP data. This variable is constructed by dividing the milage of transit travel for all modes by the area of the transit zone.

The remaining suite of variables selected for inclusion in the model measures the individual and household-level attributes that are hypothesized to influence the allocation of travel expenditures in maximizing utility. Variable definitions and descriptive statistics are presented in Table 1. As many of these variables could either positively or negatively affect the use of public transit, it is not always possible to state a priori which effects are expected to prevail. Negative signs are expected for the variables that either increase the opportunity- and/or transaction costs of transit use or decrease these costs for automobile use, including household income, the distance to the nearest transit stop, the fare ticket price, and dummies indicating driver-license holders and households in which the number of cars equals or exceeds the number of licensed drivers. Positive signs are expected for variables that are indicative of the availability or quality of public transit, including public transit density and the dummies for residence in a large city and for rail transit service at the nearest transit stop. Higher fuel prices are also expected to have a positive effect as they encourage the substitution of public transit for private car travel. The inclusion of time dummies in the model was also explored, but as these were individually and jointly insignificant, they were excluded from the final specification.

While the included explanatory variables afford reasonably broad coverage of the determinants of transit use, we cannot completely rule out the possibility that they are correlated with additional unobserved factors that impact travel. Such correlation would give rise to endogeneity bias and preclude us from ascribing a causative interpretation to the estimated coefficients. It is plausible, for example, that decisions pertaining to transit use and residential choice are jointly determined, implying that the coefficients of the urban form variables are partially picking up the effects of neighborhood

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2In the analysis that follows we use the monthly fare price, noting that our qualitative findings do not change when using the trip-based fare.
Table 1: Variable Definitions and Descriptive Statistics

<table>
<thead>
<tr>
<th>Variable Definition</th>
<th>Variable Name</th>
<th>Mean</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td># public transits during 5-day week</td>
<td>$y$</td>
<td>1.47</td>
<td>3.30</td>
</tr>
<tr>
<td>Real fuel price in € per liter</td>
<td>$real\ fuel\ price$</td>
<td>1.01</td>
<td>0.12</td>
</tr>
<tr>
<td>Real fare for a monthly ticket in €</td>
<td>$fare$</td>
<td>32.40</td>
<td>5.88</td>
</tr>
<tr>
<td>Density of the public transit service</td>
<td>$public\ transit\ density$</td>
<td>35.44</td>
<td>51.05</td>
</tr>
<tr>
<td>Dummy: 1 if person has a high school diploma</td>
<td>$high\ school\ diploma$</td>
<td>0.35</td>
<td>0.48</td>
</tr>
<tr>
<td>Dummy: 1 if person has a driver license</td>
<td>$license$</td>
<td>0.87</td>
<td>0.34</td>
</tr>
<tr>
<td>Age of adult</td>
<td>$age$</td>
<td>48.38</td>
<td>16.00</td>
</tr>
<tr>
<td>Dummy: 1 if person is employed in a full-time or part-time job</td>
<td>$employed$</td>
<td>0.54</td>
<td>0.50</td>
</tr>
<tr>
<td>Dummy: 1 if person is female</td>
<td>$female$</td>
<td>0.52</td>
<td>0.50</td>
</tr>
<tr>
<td>Net real household monthly income in 1,000 €</td>
<td>$income$</td>
<td>2.363</td>
<td>0.822</td>
</tr>
<tr>
<td>Dummy: 1 if number of cars in a household is at least equal to the number of licensed drivers</td>
<td>$enoughcars$</td>
<td>0.54</td>
<td>0.50</td>
</tr>
<tr>
<td>Number of children younger than 10</td>
<td>$#\ children&lt;10$</td>
<td>0.27</td>
<td>0.63</td>
</tr>
<tr>
<td>Dummy: 1 if household resides in a large city</td>
<td>$big\ city$</td>
<td>0.42</td>
<td>0.49</td>
</tr>
<tr>
<td>Walking time to the nearest public transportation stop</td>
<td>$minutes$</td>
<td>5.75</td>
<td>4.95</td>
</tr>
<tr>
<td>Dummy: 1 if household has a private parking space or garage</td>
<td>$parking\ space\ at\ home$</td>
<td>0.76</td>
<td>0.43</td>
</tr>
<tr>
<td>Dummy: 1 if household has a parking space at work</td>
<td>$parking\ space\ at\ work$</td>
<td>0.37</td>
<td>0.48</td>
</tr>
<tr>
<td>Dummy: 1 if there is a direct transit connection to work</td>
<td>$direct\ public\ transit\ to\ work$</td>
<td>0.16</td>
<td>0.36</td>
</tr>
<tr>
<td>Dummy: 1 if the nearest public transportation stop is serviced by rail transit</td>
<td>$rail\ transit$</td>
<td>0.13</td>
<td>0.33</td>
</tr>
</tbody>
</table>
hood preferences. We consequently abstain from making claims about causality, and instead apply a descriptive interpretation to the estimates.

3 Methodology

The reliance on individual data over a tightly circumscribed time interval raises several conceptual and empirical issues, the most fundamental of which is the presence of null values in the data. Slightly less than 75% of the adult individuals in the estimation sample do not use public transport systems during a given week and for whom the observation on transit counts is consequently recorded as zero. To accommodate this feature of the data, we employ modeling procedures referred to as zero-inflated models (ZIMs) that order observations into two latent regimes defined by whether the individual never uses public transport systems. As this kind of model builds on the classical count data models, we start with a brief description of the Poisson (PRM) and the negative binomial regression (NBRM) models and highlight the differences between these classical and the zero-inflated models, thereby following the general outline of CHIN and QUDDUS (2003).

Zero-inflated models assume that there is a substantial share of individuals with a zero probability that the dependent variable takes on positive values. For example, many citizens of rural areas suffer from the unavailability, or impracticability, of public transport systems, so that they are forced to the exclusive usage of cars for commuting. When employing the zero-inflated Poisson model (ZIPM) and the zero-inflated negative binomial model (ZINBM), one accounts for this feature, whereas the PRM and NBRM rest on the assumption that the probability for a positive value of the dependent variable is non-zero for every observation.
3.1 Classical Count Data Models

The univariate Poisson distribution is fundamental to understanding count data models. On this basis, the relationship between the expected count $\lambda$ and the probability of observing any count $y$ is given by

$$Pr(y|\lambda) = \frac{\lambda^y \exp\{-\lambda\}}{y!} \quad \text{for} \quad y = 0, 1, 2, \ldots, \quad (1)$$

where the mean $E(y) = \lambda > 0$ is the parameter that is uniquely defining this distribution. A limiting characteristic of the Poisson distribution, known as equidispersion, is that the variance $Var(y)$ is identical to the expected value: $Var(y) = E(y) = \lambda$. In practice, though, the variance of many count variables is greater than their mean, a fact that is called overdispersion.

The Poisson regression model (PRM) extends the Poisson distribution by allowing for each observation $i$ to have a different mean $\lambda_i$. The most common parameterization of the ideosyncratic means is the loglinear model (GREENE 2003: 740):

$$\lambda_i = E(y_i|x_i) = \exp\{x_i^T \beta\}, \quad (2)$$

where $\beta$ is a parameter vector to be estimated and observed heterogeneity is incorporated by the vector $x_i$, which includes the observable characteristics that determine the individual number of counts $y_i$. Note that taking the exponential of $x_i^T \beta$ in equation (2) ensures that the expected value $\lambda_i$ is positive, which is a natural property of count data.

While being a useful starting point, the PRM suffers from at least four shortcomings. First, it underestimates the number of zero counts, as can be seen from our empirical example presented in the subsequent section (see Table 2). Second, the PRM does not fit to real data in the case of overdispersion, i.e. if $Var(y) > E(y)$. Third, the standard errors pertaining to the PRM estimates are biased downward, resulting in spuriously large $z$-values and small $p$-values (CAMERON, TRIVEDI, 1986:31). Fourth, the PRM does not account for unobserved heterogeneity. These failures are circumvented by the NBRM, which addresses the last point by adding an error term $\varepsilon_i$ that is
assumed to be uncorrelated with the factors included in $x_i$:

$$\tilde{\lambda}_i = E(y_i|x_i) = E(\exp\{x_i^T\beta + \varepsilon_i\}) = \exp\{x_i^T\beta\}E(\delta_i), \quad (3)$$

where $\delta_i := \exp\{\varepsilon_i\}$. By assuming that $E(\delta_i) = 1$, which corresponds to the assumption $E(\varepsilon_i) = 0$ of the classical linear regression approach, the model is identified. From this assumption, it follows that the PRM and NBRM have the same mean structure:

$$\tilde{\lambda}_i = \exp\{x_i^T\beta\}E(\delta_i) = \lambda. \quad (4)$$

In the NBRM, the probability distribution of the counts $y_i$ is still Poisson, that is, $y_i$ obeys equation (1) with $\lambda_i$ being replaced by $\tilde{\lambda}_i$.\footnote{$\delta_i$ is commonly assumed to be drawn from a Gamma(1, $\alpha$) distribution, where $\alpha$ is the variance parameter (see CAMERON, TRIVEDI, 2009:555, for details). Then, the probability of the counts $Y$ follows the negative binomial distribution:

$$Pr(Y = y_i|x, \alpha) = \frac{\Gamma(y_i + \alpha^{-1})}{y_i\Gamma(\alpha^{-1})} \left(\frac{\alpha^{-1}}{\alpha^{-1} + \lambda}\right)^{\alpha^{-1}} \left(\frac{\lambda}{\alpha^{-1} + \lambda}\right)^y,$$

where $\Gamma(\cdot)$ denotes the Gamma function.}

### 3.2 Testing Overdispersion

A basic assumption of the PRM is equidispersion, i.e. the conditional mean equals the conditional variance:

$$Var(y_i|x_i) = E(y_i|x_i) = \lambda_i. \quad (5)$$

This rarely fulfilled assumption is relaxed in the NBRM, for which CAMERON and TRIVEDI (1986) show that there is a variety of alternatives to the constant-variance function given by (5). The most commonly used generalization is

$$Var(y_i|x_i) = \lambda_i + \alpha\lambda_i^2. \quad (6)$$

This term suggests examining the null hypothesis $H_0 : \alpha = 0$ in order to test for overdispersion. If the null holds true, equidispersion according to (5) prevails and the NBRM collapses to the PRM.
It bears noting that testing the null requires procedures other than the typical symmetric t-tests, as $\alpha$ must be non-negative. Instead, a Likelihood-Ratio (LR) test can be employed, where the test statistic follows a $\chi^2$-distribution and is computed in the usual manner:

$$LR = 2 \cdot (\ln L_{NBRM} - \ln L_{PRM}).$$

(7)

$\ln L_{NBRM}$ and $\ln L_{PRM}$ denote the Loglikelihood functions of the NBRM and PRM, respectively. The significance level of the test has to be adjusted to account for the truncated sampling distribution of $\hat{\alpha}$.

### 3.3 Zero-inflated Models

The NBRM improves upon the underprediction of zero counts in the PRM by increasing the conditional variance without altering the conditional mean. In contrast, zero-inflated models (ZIM) such as the ZIPM, which was introduced by Lambert (1992), change the mean structure, thereby increasing the probability of zero counts. Zero-inflated models assume that there are two latent groups, for which membership is unobservable: the *Always-Zero Group* $A$, for which

$$Pr(y_i = 0|A_i = 1, x_i) = 1,$$

(8)

where $A_i = 1$ designates membership of individual $i$ in Group $A$, and $A_i = 0$ indicates membership in the complementary group, for which $Pr(y_i = 0|A_i = 0, x_i)$ is the outcome of the PRM or NBRM in case of the ZIP or ZINBM, respectively.

Group membership is a binary outcome that can be modeled using standard logit or probit estimation procedures:

$$\psi_i := Pr(A_i = 1|z_i) = F(z_i^T \gamma),$$

(9)

where $\psi_i$ is the probability of being in Group $A$, $F(.)$ stands for the cumulative distribution function $\Phi(.)$ or $\Lambda(.)$ of the normal or logistic distribution, respectively. $\gamma$ is a parameter vector to be estimated, and vector $z_i$ includes variables that inflate the number of zero counts. Hence, they are referred to as inflation variables and (9) is called the
inflation equation. The vector of inflation variables $z_i$ may or may not differ from the determinants $x_i$ of the number of counts $y_i$.

If we knew probability $\psi_i$, the probability of a zero count could be calculated as follows:

$$Pr(y_i = 0|x_i, z_i) = Pr(A_i = 1|z_i) \cdot Pr(y_i = 0|A_i = 1, x_i)$$
$$+ Pr(A_i = 0|z_i) \cdot Pr(y_i = 0|A_i = 0, x_i)$$
$$= \psi_i \cdot 1 + (1 - \psi_i) \cdot Pr(y_i = 0|A_i = 0, x_i),$$

(10)

as $Pr(A_i = 0|z_i) = 1 - \psi_i$ and the probability of zero counts is 100% in the Always-Zero Group A: $Pr(y_i = 0|A_i = 1, x_i) = 1$. For outcomes $y_i = k > 0$,

$$Pr(y_i = k|x_i, z_i) = \psi_i \cdot 0 + (1 - \psi_i) \cdot Pr(y_i = k|A_i = 0, x_i)$$
$$= (1 - \psi_i) \cdot Pr(y_i = k|A_i = 0, x_i),$$

(11)

where we have used the assumption that the probability of non-zero counts is 0% in Group A: $Pr(y_i = k|A_i = 1, x_i) = 0$.

On the basis of these probability expressions, the unknown parameters $\beta$ and $\gamma$ can be estimated using maximum-likelihood methods. For instance, the Loglikelihood function of the ZIP reads

$$\ln L_{ZIP} = \sum_{y_i = 0} \log[\psi_i \cdot 1 + (1 - \psi_i) \cdot \exp\{-\lambda_i\}]$$
$$+ \sum_{y_i > 0} [y_i \log(\lambda_i) - \lambda_i - \log(y_i!)] \cdot \log(1 - \psi_i),$$

(12)

where $\lambda_i := \exp\{x_i^T \beta\}$ and $\psi_i := F(z_i^T \gamma)$, with $F(.)$ depending on whether a probit or logit model is specified. Note that one cannot separately estimate the parameters $\gamma$ in a first step, as we do not know those zero counts that originate from members of Group A. Instead, both parameter vectors, $\beta$ and $\gamma$, have to be estimated simultaneously.

Expected counts are computed in a way similar to that of the probabilities:

$$E(y_i|x_i, z_i) = \psi_i \cdot E(y_i|A_i = 1, x_i) + (1 - \psi_i) \cdot E(y_i|A_i = 0, x_i)$$
$$= \psi_i \cdot 0 + (1 - \psi_i) \cdot \lambda_i = (1 - \psi_i) \cdot \lambda_i,$$

(13)
where for the Always-Zero Group $A$, it is $E(y_i|A_i = 1, x_i) = 0$ and $E(y_i|A_i = 0, x_i) = \lambda_i$ for the complementary group, since the PRM and NBRM have the same mean structure (4). Because $0 \leq \psi_i \leq 1$, the expected value given by (13) will be smaller than $\lambda_i$, so that the mean structure of zero-inflated models differs from that of the PRM or NBRM. In fact, the expected count resulting from ZIMs is lower than that of the PRM and NBRM, thereby better fitting to the large number of zero counts in the empirical evidence on the dependent variable.

Given $E(y_i|x_i, z_i) = (1 - \psi_i) \cdot \lambda_i$, marginal effects can be readily calculated. For the case that the inflation regression is based on a logit model, i.e. $\psi_i = \Lambda(z_i^T \gamma)$, where $\Lambda(u) := 1/(1 + \exp(-u))$ is the logistic function whose derivative is given by $\Lambda'(u) = \Lambda(u)(1 - \Lambda(u))$, a marginal change in variable $x_k$ included in both $x$ and $z$ yields the following variation of the expected counts:

$$
\frac{\partial E}{\partial x_k} = (1 - \psi_i) \cdot \lambda_i \cdot \beta_k - \psi_i \cdot (1 - \psi_i) \cdot \lambda_i \cdot \gamma_k
= (1 - \psi_i) \cdot \lambda_i \cdot (\beta_k - \psi_i \cdot \gamma_k) = E(y_i|x_i, z_i) \cdot (\beta_k - \psi_i \cdot \gamma_k),
$$

(14)

as $\frac{\partial \lambda_i}{\partial x_k} = \frac{\partial}{\partial x_k}(\exp\{x_i^T \beta\}) = \lambda_i \cdot \beta_k$. This marginal effect collapses to

$$
\frac{\partial E}{\partial x_k} = E(y_i|x_i, z_i) \cdot \beta_k
$$

(15)

if $x_k$ is not included in $z$, that is, if $\gamma_k = 0$.

For the case that the inflation regression is based on a probit model, i.e. $\psi_i = \Phi_i := \Phi(z_i^T \gamma)$, where $\Phi(u)$ denotes the cumulative standard normal distribution and $\Phi'(u) = \phi(u)$ designates the density function of the standard normal distribution, the marginal effect reads:

$$
\frac{\partial E}{\partial x_k} = (1 - \Phi_i) \cdot \lambda_i \cdot \beta_k - \phi_i \cdot \lambda_i \cdot \gamma_k = E(y_i|x_i, z_i) \cdot \beta_k - \phi_i \cdot \lambda_i \cdot \gamma_k,
$$

(16)

with $\phi_i := \phi(z_i^T \gamma)$. If $x_k$ is not included in $z$, i.e. $\gamma_k = 0$, the marginal effect given by (16) collapses to formula (15).

The marginal effects are generally calculated at the mean of the regressors and can be requested in the output of most statistical software packages, though some
care must be taken in their calculation when interaction terms are involved (FRONDEL, VANCE, 2010). To explore interaction effects of two variables $x_1$ and $x_2$ in non-linear models, it does not suffice to include interaction terms $x_1 x_2$ in the model specification. Rather, investigating the interaction effect of two variables $x_1$ and $x_2$ requires computing the cross derivative $\frac{\partial^2 E}{\partial x_1 \partial x_2}$, whose general formulae are not implemented in standard statistical software packages. These formulae are consequently derived for the ZIMs in Appendix B for three cases, if the interacted variables are (i) both continuous, (ii) both dummy variables, and (iii) if one variable of each type is included in the interaction term.

### 3.4 VUONG Test of Non-Nested Models

Neither the NBRM is nested in the ZINBM, nor is the PRM nested in the ZIPM, as is pointed out by GREENE (1995). While the ZINBM, for instance, would collapse to the NBRM if $\psi_i$ were identical to zero for all observations $i$, this equality cannot hold in general and is, specifically, not fulfilled for $\gamma = 0$, as $\psi_i = E(z_i^T 0) = 0.5$. To the test the superiority of the ZINBM over the NBRM, as well as of the ZIPM over the PRM, GREENE consequently suggests using a test specified by VUONG (1989:319) for non-nested models.

This test is based on the asymptotically normal distributed Vuong test statistic given by

$$ V := \frac{\bar{m}}{s_m / \sqrt{N}}, $$

(17)

where $\bar{m}$ and $s_m$ designates the mean and standard deviation of the logged relationship of the predicted probabilities, $m_i$, obtained from two Models 1 and 2, respectively:

$$ m_i := \ln \left\{ \frac{P_1(y_i|x_i)}{P_2(y_i|x_i)} \right\}. $$

(18)

The Vuong test examines the null hypothesis $H_0 : E(m_i) = 0$. Large positive values of $V$ that exceed the well-known critical value of 1.96 of the normal distribution favor Model 1, whereas negative values of $V$ below the critical value of -1.96 are supportive of Model 2.
4 Empirical Results

Along the lines of the previous section, we estimate both the classical as well as the zero-inflated models and select the most appropriate approach by comparing the predicted probabilities for the range of public transit counts reported in Table 2. Beginning the discussion with the PRM, our empirical example is another confirmation for the fact that this most basic model typically underestimates the number of zero counts: While 74% of the adult individuals in the estimation sample are observed not to use public transport systems during a given week, the PRM predicts a markedly lower probability of 40% for this outcome. Conversely, the PRM drastically overestimates the probability for a single use and also overshoots for two to five transit counts a week.

The accordance of the observed frequencies and the predictions gleaned from the NBRM is clearly superior to the PRM, particularly for the predicted non-use of public transit systems. The superiority of the NBRM over the PRM is additionally confirmed by the Likelihood Ratio test on overdispersion, for which the chi-squared test statistic amounts to 16,000. While the existence of always-zero observations is ignored by both the PRM and NBRM, the special treatment of this feature by the ZIMs leads us to expect a further improvement in the fit due to their employment. And indeed, the observed frequency for zero counts is perfectly reproduced by both the ZINBM and ZIP model. Relative to the NBRM, the ZINBM also provides for a substantially better fit for a single count, whereas the predictions of the probabilities of \( y_i = 3, 4, 5 \) counts are somewhat worse. Therefore, an ultimate decision on whether the ZINBM is superior to the NBR model requires a VUONG test, whose large positive value of 21.75 for the standard-normal distributed normal test statistic favors the ZINB model. Finally, the probability-by-probability comparison of the ZIPM and ZINBM is clearly in favor of the ZINBM. This conclusion is confirmed by the Likelihood-Ratio test on overdispersion, for which the test statistic amounts to about 1,679.

Turning to the coefficient estimates of the ZINBM reported in Table 3, the majority are statistically significant and have signs that are consistent with intuition.\(^4\) Two

\(^4\)The coefficients from the inflation equation of the model are given in the Appendix.
Table 2: Comparison of Observed Frequencies with Predicted Probabilities Resulting from Various Counts Data Models.

<table>
<thead>
<tr>
<th></th>
<th>Observed Frequencies</th>
<th>ZINBM Predictions</th>
<th>NBRM Predictions</th>
<th>ZIPM Predictions</th>
<th>PRM Predictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{P}(y_i = 0) ):</td>
<td>0.740</td>
<td>0.740</td>
<td>0.721</td>
<td>0.740</td>
<td>0.400</td>
</tr>
<tr>
<td>( \hat{P}(y_i = 1) ):</td>
<td>0.028</td>
<td>0.034</td>
<td>0.103</td>
<td>0.011</td>
<td>0.280</td>
</tr>
<tr>
<td>( \hat{P}(y_i = 2) ):</td>
<td>0.065</td>
<td>0.036</td>
<td>0.048</td>
<td>0.022</td>
<td>0.144</td>
</tr>
<tr>
<td>( \hat{P}(y_i = 3) ):</td>
<td>0.019</td>
<td>0.034</td>
<td>0.029</td>
<td>0.033</td>
<td>0.073</td>
</tr>
<tr>
<td>( \hat{P}(y_i = 4) ):</td>
<td>0.025</td>
<td>0.030</td>
<td>0.019</td>
<td>0.038</td>
<td>0.039</td>
</tr>
<tr>
<td>( \hat{P}(y_i = 5) ):</td>
<td>0.013</td>
<td>0.025</td>
<td>0.013</td>
<td>0.038</td>
<td>0.023</td>
</tr>
<tr>
<td>( \hat{P}(y_i = 6) ):</td>
<td>0.019</td>
<td>0.020</td>
<td>0.001</td>
<td>0.033</td>
<td>0.014</td>
</tr>
<tr>
<td>( \hat{P}(y_i = 7) ):</td>
<td>0.012</td>
<td>0.016</td>
<td>0.008</td>
<td>0.026</td>
<td>0.009</td>
</tr>
<tr>
<td>( \hat{P}(y_i = 8) ):</td>
<td>0.013</td>
<td>0.013</td>
<td>0.006</td>
<td>0.020</td>
<td>0.006</td>
</tr>
<tr>
<td>( \hat{P}(y_i = 9) ):</td>
<td>0.014</td>
<td>0.010</td>
<td>0.005</td>
<td>0.014</td>
<td>0.004</td>
</tr>
</tbody>
</table>

notable exceptions are the coefficients on fuel prices and fares: neither appear to be important determinants of the number of public transits over the 5-day week. The statistically insignificant impact of fares maintains when considering the marginal effect (right-hand panel of Table 3). With reference to real fuel prices, however, a discrepancy emerges: the marginal effect is highly significant in this case, and suggests that a 1€ increase in fuel costs increases transit counts by 0.7 trips over the course of a 5-day week. Fuel cost shocks may thus be highly supportive of transit ridership.

That the effect of the fare does not mirror that of fuel prices might be attributed to the fact that the majority of public transit users buy lump-sum tickets that allow for the unlimited use of the transit system during their validation period. To explore whether insignificant effects of the fare main under alternative specifications, we estimated models that included interaction terms and calculated interaction effects according to the formulae of Appendix B. These specifications accommodated the possibility of differential effects of fuel and fare prices by income level, residential location, and car availability (Litman, 2004). In all cases, the interaction effects were found to be statistically insignificant.
Stark differences between coefficient estimates and marginal effects are to be observed for female persons, as well as in case of the existence of private parking spaces and the prevalence of rail transits, with the coefficient estimates being statistically insignificant for all these factors. Being female, for instance, seems to be irrelevant when focusing on coefficients, but, in fact, increases the number of transit counts by about 0.19 based on the marginal effect. A similar discrepancy is seen with respect to the rail transit dummy, whose coefficient estimate is statistically negligible. In fact, the existence of rail transit service, which tends to afford greater speed and comfort, significantly increases transit counts, by about 0.32. Such differences highlight the relevance of additionally reporting marginal effects, rather than merely coefficient estimates.

Table 3: Estimation Results of the Zero-Inflated Negative Binomial Model

<table>
<thead>
<tr>
<th></th>
<th>Coeff.</th>
<th>Robust Std. Errors</th>
<th>Marginal Effects</th>
<th>Robust Std. Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>female</td>
<td>-0.001</td>
<td>0.034</td>
<td>** 0.173</td>
<td>0.050</td>
</tr>
<tr>
<td>age</td>
<td>** -0.031</td>
<td>0.007</td>
<td>** -0.086</td>
<td>0.010</td>
</tr>
<tr>
<td>age squared</td>
<td>** 0.0002</td>
<td>0.000</td>
<td>** 0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>employed</td>
<td>** 0.380</td>
<td>0.056</td>
<td>** 0.486</td>
<td>0.084</td>
</tr>
<tr>
<td>high school diploma</td>
<td>* 0.096</td>
<td>0.039</td>
<td>** 0.465</td>
<td>0.066</td>
</tr>
<tr>
<td>license</td>
<td>** -0.225</td>
<td>0.043</td>
<td>** -1.440</td>
<td>0.131</td>
</tr>
<tr>
<td>employed × (parking space at work)</td>
<td>** -0.316</td>
<td>0.050</td>
<td>** -0.784</td>
<td>0.068</td>
</tr>
<tr>
<td>parking space at home</td>
<td>-0.081</td>
<td>0.047</td>
<td>** -0.500</td>
<td>0.084</td>
</tr>
<tr>
<td>enough cars</td>
<td>** -0.191</td>
<td>0.040</td>
<td>** -0.829</td>
<td>0.062</td>
</tr>
<tr>
<td>minutes</td>
<td>* -0.010</td>
<td>0.005</td>
<td>** -0.036</td>
<td>0.007</td>
</tr>
<tr>
<td>direct public transit to work</td>
<td>** 0.167</td>
<td>0.049</td>
<td>** 0.535</td>
<td>0.106</td>
</tr>
<tr>
<td>big city</td>
<td>* 0.098</td>
<td>0.045</td>
<td>** 0.424</td>
<td>0.070</td>
</tr>
<tr>
<td>rail transit</td>
<td>0.083</td>
<td>0.049</td>
<td>** 0.341</td>
<td>0.099</td>
</tr>
<tr>
<td># children &lt; 10</td>
<td>* -0.089</td>
<td>0.037</td>
<td>** -0.305</td>
<td>0.052</td>
</tr>
<tr>
<td>income</td>
<td>* -0.063</td>
<td>0.027</td>
<td>** -0.105</td>
<td>0.039</td>
</tr>
<tr>
<td>real fuel price</td>
<td>0.262</td>
<td>0.149</td>
<td>** 0.663</td>
<td>0.230</td>
</tr>
<tr>
<td>fare</td>
<td>0.001</td>
<td>0.003</td>
<td>-0.003</td>
<td>0.005</td>
</tr>
<tr>
<td>public transit density</td>
<td>** 0.002</td>
<td>0.000</td>
<td>** 0.007</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Note: * denotes significance at the 5 %-level and ** at the 1 %-level, respectively.
Number of observations used in estimation: 8,577.

With respect to the remaining coefficients, age is seen to have a non-linear effect,
which is initially negative up to about an age of 72 after which it becomes positive. The dummies for employed persons and those with a high school diploma both have positive signs and are roughly the same magnitude (at least with respect to the marginal effects), suggesting that these individuals have transit counts that are about 49% higher than their counterparts. Likewise, those living in a big city and with a direct transit connection from home to work are also more frequent users of public transport, as are individuals who live in regions with a denser transit network. Consistent with expectations, factors that increase the costs of transit use or that decrease the cost of car use have negative effects. These include the dummies indicating license holders, those with a parking space at work, and those who live in households with at least as many cars as licensed drivers.

5 Conclusion

In Germany, as elsewhere in the industrialized world, the promotion of public transit use is a central policy tool in the mitigation of pollution, congestion, and other automobile-caused externalities. Despite Germany’s relative success in capping emissions from transport, which rose by 1% between 1990 and 2005 compared to a 26% increase in the European Union (EEA, 2008), public transit use has been on the decline. Between 1994 and 2003, the percentage of trips traveled by transit dropped by 1%, contrasted by a 16% increase in motor vehicle trips (Destatis, 2006). To counter this trend, the country’s transport ministry has placed a high priority on improving the competitive position of public transit relative to the automobile (BMVBS, 2009).

An important step in this endeavor is to identify the economic and structural factors that draw or repel potential transit customers, thereby enabling the design of measures to increase ridership among those segments of the population where the scope for mode switching is greatest. From a planning perspective, one particularly important factor is the responsiveness of transit riders to both gas prices and fares. This paper investigates this issue with an analysis of the determinants of weekly transit
usage by drawing on household survey data from Germany.

To our knowledge, this is the first study to parameterize the effects of fares and fuel prices, as well as socioeconomic and geographic determinants, on the basis of individual-level data. Although necessarily neglected in studies on the effects of fuel and fare prices using aggregated data, the discrete decision to occasionally or regularly use public transit system appears to be of particular relevance in the analysis of individual data, as fuel price peaks may trigger a reduction of car use and a switch to public transit.

We have addressed this issue by employing zero-inflated modeling approaches, which is particularly appropriate when the question at hand requires distinguishing between those who never use public transit from those who have some non-zero probability of a positive trip count. Our estimates suggest that a 1€ increase in fuel prices increases transit use by 0.7 trips over a week, an effect that is statistically significant at the 1% level. Somewhat unexpectedly, we find that that the effect of the fare, by contrast, is not significantly different from zero, even when allowing for differential effects according to residential location, car ownership, and the income of the household. Taken together, these findings suggest that fuel prices are a more effective lever than fares for influencing transit ridership.

As this is one of the few studies to be conducted on this topic using micro-level data in a European context, it would be of interest to see whether the qualitative findings presented here are corroborated by studies using other data sets from within Germany and other European countries. A particularly useful line of inquiry would focus on distinguishing short- and long-run price responsiveness using micro-level data, which is not subject to the aggregation problems that commonly afflict regional-level temporal studies of transit use. Data constraints precluded such an analysis in the present study, but it is one that would further facilitate the formulation of pricing strategies to encourage transit use.
**Appendix A: Tables**

**Table A1: Regression Results of the Inflation Equation**

<table>
<thead>
<tr>
<th></th>
<th>Robust Coeff.s</th>
<th>Robust Std. Errors</th>
<th>Marginal Effects</th>
<th>Marginal Std. Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>female</td>
<td><strong>-0.242</strong></td>
<td>0.059</td>
<td><strong>-0.038</strong></td>
<td>0.009</td>
</tr>
<tr>
<td>age</td>
<td><strong>0.078</strong></td>
<td>0.013</td>
<td><strong>0.013</strong></td>
<td>0.002</td>
</tr>
<tr>
<td>age squared</td>
<td><strong>-0.001</strong></td>
<td>0.000</td>
<td><strong>0.000</strong></td>
<td>0.000</td>
</tr>
<tr>
<td>employed</td>
<td>-0.171</td>
<td>0.098</td>
<td>* -0.039</td>
<td>0.015</td>
</tr>
<tr>
<td>high school diploma</td>
<td><strong>-0.478</strong></td>
<td>0.067</td>
<td><strong>-0.082</strong></td>
<td>0.011</td>
</tr>
<tr>
<td>license</td>
<td><strong>1.173</strong></td>
<td>0.090</td>
<td><strong>0.232</strong></td>
<td>0.019</td>
</tr>
<tr>
<td>employed × (parking space at work)</td>
<td><strong>0.752</strong></td>
<td>0.088</td>
<td><strong>0.124</strong></td>
<td>0.013</td>
</tr>
<tr>
<td>parking space at home</td>
<td><strong>0.516</strong></td>
<td>0.081</td>
<td><strong>0.091</strong></td>
<td>0.015</td>
</tr>
<tr>
<td>enoughcars</td>
<td><strong>0.846</strong></td>
<td>0.065</td>
<td><strong>0.145</strong></td>
<td>0.010</td>
</tr>
<tr>
<td>minutes</td>
<td><strong>0.036</strong></td>
<td>0.008</td>
<td><strong>0.006</strong></td>
<td>0.001</td>
</tr>
<tr>
<td>direct public transit to work</td>
<td><strong>-0.415</strong></td>
<td>0.092</td>
<td><strong>-0.078</strong></td>
<td>0.017</td>
</tr>
<tr>
<td>big city</td>
<td><strong>-0.435</strong></td>
<td>0.079</td>
<td><strong>-0.076</strong></td>
<td>0.013</td>
</tr>
<tr>
<td>rail transit</td>
<td><strong>-0.315</strong></td>
<td>0.097</td>
<td><strong>-0.056</strong></td>
<td>0.017</td>
</tr>
<tr>
<td># children &lt; 10</td>
<td><strong>0.301</strong></td>
<td>0.059</td>
<td><strong>0.053</strong></td>
<td>0.009</td>
</tr>
<tr>
<td>income</td>
<td>0.062</td>
<td>0.046</td>
<td>0.013</td>
<td>0.007</td>
</tr>
<tr>
<td>real fuel price</td>
<td>*-0.565</td>
<td>0.265</td>
<td>*-0.095</td>
<td>0.042</td>
</tr>
<tr>
<td>fare</td>
<td>0.005</td>
<td>0.006</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>public transit density</td>
<td><strong>-0.007</strong></td>
<td>0.001</td>
<td><strong>-0.001</strong></td>
<td>0.000</td>
</tr>
</tbody>
</table>

*Note:* * denotes significance at the 5 % -level and ** at the 1 % -level, respectively.

Number of observations used in estimation: 8,577.

Note that the dependent variable of the model is 1 if the individual never uses transit during the 5-day week and zero otherwise.
Appendix B: Interaction Effects

To explore whether the effect of an explanatory variable \( z_1 \) on the expected value \( E[y] \) of the dependent variable \( y \) depends on the size of another explanatory variable \( z_2 \), it is indispensable to estimate the interaction effect given by the second derivative \( \frac{\partial^2 E[y]}{\partial z_2 \partial z_1} \).

In this section, we follow Frondel and Vance (2010) and derive general formulae for the interaction effects resulting from ZIMs if (a) \( z_1 \) and \( z_2 \) are both continuous variables, (b) \( z_1 \) is continuous, while \( z_2 \) is a dummy variable, and (c) both are dummy variables.

To this end, we depart from the expected value (13),

\[
E := E[y|z_1, z_2, w] = [1 - F(u)] \cdot \exp\{u\} = [1 - F(u)] \cdot \lambda(u),
\]

where \( u := \gamma_1 z_1 + \gamma_2 z_2 + \gamma_{12} z_1 z_2 + w^T \gamma \), and vector \( w \) excludes \( z_1 \) and \( z_2 \). \( F(u) \) equals the cumulative normal distribution \( \Phi(u) \), when the inflation equation is specified as a probit model and \( F(u) = \Lambda(u) = 1/(1 + \exp\{-u\}) \) for the logit model. As in the methodology section, we use the abbreviation \( \lambda(u) = \exp\{u\} \).

(a) If \( F(u) \) is a twice differentiable function, with the first and second derivatives being denoted by \( F'(u) \) and \( F''(u) \), respectively, the marginal effect with respect to \( x_1 \) reads:

\[
\frac{\partial E}{\partial z_1} = \{[1 - F(u)] - F'(u)\} \cdot \lambda(u) \cdot \frac{\partial u}{\partial x_1} = \{[1 - F(u)] - F'(u)\} \cdot \lambda(u) \cdot (\gamma_1 + \gamma_{12} z_2). \tag{20}
\]

The interaction effect of two continuous variables \( z_1 \) and \( z_2 \) is given by the second derivative:

\[
\frac{\partial^2 E}{\partial z_2 \partial z_1} = \{[1 - F(u)] - F'(u)\} \cdot \lambda(u) \cdot [(\gamma_2 + \gamma_{12} z_1) \cdot (\gamma_1 + \gamma_{12} z_2) + \gamma_{12}]
\]

\[
- [F'(u) + F''(u)] \cdot \lambda(u) \cdot (\gamma_1 + \gamma_{12} z_2)(\gamma_2 + \gamma_{12} z_1)
\]

(b) If \( z_1 \) is a continuous variable and \( z_2 \) is a dummy variable, the mixed interaction effect \( \Delta \Delta_{z_2} \left( \frac{\partial E}{\partial z_1} \right) \) can be computed on the basis of the first derivative (20) as follows:

\[
\frac{\Delta}{\Delta_{z_2} \left( \frac{\partial E}{\partial z_1} \right)} := \frac{\partial E}{\partial z_2}\bigg|_{z_2=1} - \frac{\partial E}{\partial z_2}\bigg|_{z_2=0}
\]

\[
= \{[1 - F(u_1)] - F'(u_1)\} \cdot \lambda(u_1) \cdot (\gamma_1 + \gamma_{12})
\]

\[
- \{[1 - F(u_0)] - F'(u_0)\} \cdot \lambda(u_0) \cdot \gamma_1, \tag{21}
\]
where \( u_0 := \gamma_1 z_1 + w^T \gamma \) and \( u_1 := (\gamma_1 + \gamma_{12}) z_1 + \gamma_2 + w^T \gamma \).

(c) The interaction effect \( \frac{\Delta^2 E}{\Delta z_2 \Delta z_1} \) of two binary variables \( z_1 \) and \( z_2 \) is obtained as follows:

\[
\frac{\Delta^2 E}{\Delta z_2 \Delta z_1} = \{ [E[y|z_1 = 1, z_2 = 1, w] - E[y|z_1 = 0, z_2 = 1, w] \} \\
- \{ [E[y|z_1 = 1, z_2 = 0, w] - E[y|z_1 = 0, z_2 = 0, w] \} \\
= [1 - F(\gamma_1 + \gamma_2 + \gamma_{12} + w^T \gamma)] \cdot \lambda(\gamma_1 + \gamma_2 + \gamma_{12} + w^T \gamma) \\
- [1 - F(\gamma_2 + w^T \gamma)] \cdot \lambda(\gamma_2 + w^T \gamma) - [1 - F(\gamma_1 + w^T \gamma)] \cdot \lambda(\gamma_1 + w^T \gamma) \\
+ [1 - F(w^T \gamma)] \cdot \lambda(w^T \gamma). \tag{22}
\]

For the case that the inflation regression is based on a logit model, i.e. if \( F(u) = \Lambda(u) := 1/(1+\exp\{-u\}) \), \( F'(u) = \Lambda'(u) = \Lambda(u)(1-\Lambda(u)) \) and \( F''(u) = \Lambda''(u) = \Lambda(u)(1-\Lambda(u))(1-2\Lambda(u)) \). For the case that the inflation regression is based on a probit model, \( F(u) \) equals the cumulative standard normal distribution \( \Phi(u) \), so that \( F'(u) = \Phi'(u) = \phi(u) \) is the density function of the standard normal distribution and \( F''(u) = \phi'(u) = -u\phi(u) \).

If the expected value \( E := E[y|z_1, z_2, w] = [1 - F(v)] \cdot \exp\{u\} = [1 - F(v)] \cdot \lambda(u) \), \( \tag{23} \)

differs from (19), because \( v := \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + w^T \gamma \) does not include the variables \( z_1 \) and \( z_2 \) occurring in \( u := \gamma_1 z_1 + \gamma_2 z_2 + \gamma_{12} z_1 z_2 + w^T \gamma \), the formulae for the interaction effects simplify slightly.
References


