

Klose, Bettina; Kovenock, Dan

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## Extremism Drives Out Moderation

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# Extremism drives out Moderation

**Bettina Klose**

Department of Economics, Purdue University, West Lafayette, IN 47907, USA

**Dan Kovenock**

Department of Economics, University of Iowa, Iowa City, IA 52242, USA

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## **Abstract**

This article investigates the impact of societal structure on behavior in competitions that can be modeled as all-pay auctions with identity-dependent externalities. The consideration of identity-dependent externalities, which naturally arise in the most common applications of all-pay auctions, enables us to define players' individual characteristics in society (in particular radicalism and centrism) not only for arrangements on the line but more generally. We find that even with a high ratio of centrists in comparison to radicals extremism, characterised by higher expenditure by radicals in comparison to centrists, persists. Moreover, for environments with two radical players we show that there exists a symmetric equilibrium in which all moderates bid zero with certainty. This equilibrium is the unique symmetric equilibrium if there is only a single centrist player. Our results suggest that the active participation of centrists in equilibrium remains characteristic for tullock lottery success functions in the presence of identity-dependent externalities, and therefore that the chosen contest success function was crucial for predictions about extremism and moderation that can be found in the political economy literature.

# 1 Introduction

It is axiomatic that the nature of conflict depends on the institutions of conflict. In this paper we examine conflicts in which economic agents expend scarce resources in order to achieve their preferred outcome among a set of alternatives. If an agent secures his preferred alternative we say that the agent "wins." Otherwise, the agent "loses." In this respect the conflicts that we examine are contests as defined, say, in Konrad (2009). Our approach differs from much of the literature on contests in that agents are not indifferent to the identity of the winning agent in the event that they themselves lose. That is, we examine contests with identity-dependent externalities.

In much of the theoretical work on conflict to date the institutions of conflict have been black-boxed by the application of a contest success function - a function that maps the vector of agents' resource expenditures in the conflict into their respective probabilities of winning their preferred outcome. Two prominent types of contest success functions (henceforth, CSFs) employed in the literature are the "lottery" CSF, in which the probability that an agent wins his preferred outcome equals the ratio of the agent's expenditure to the sum of all agents' expenditures, and the "auction" CSF, in which the agent with the greatest expenditure wins his preferred outcome with certainty. The lottery CSF is a popular method of modeling conflicts in which the outcome is determined not just by the respective expenditures of resources, but also a substantial random component. An auction CSF may be viewed as approximating environments in which random exogenous factors play little role in influencing the outcome of the conflict. Because of the discontinuity in the auction CSF when agents are tied for the highest expenditure, small differences in (positive) expenditure may lead to large differences in the probability of winning. That is, in contests, the auction CSF represents cutthroat competition in sunk expenditure, much the way that classical Bertrand competition is cutthroat competition in price. With the lottery CSF competition is softened by randomness in the outcome,

conditional on the profile of expenditures.

Contests with identity-dependent externalities utilizing a lottery CSF have been examined by Linster (1993) and Esteban and Ray (1999). Linster (1993) demonstrates that with a constant unit cost of expenditure, pure strategy Nash equilibrium profiles of expenditures may be obtained as the solution to a nonlinear system of equations.<sup>1</sup> He analyzes two three-player environments in more detail, including a comparative statics analysis that links total conflict and social surplus to the extent of the externalities. Esteban and Ray (1999) extend Linster's (1993) model by considering groups of agents, with heterogeneous preferences across groups but homogeneous preferences within each group. Each agent has an identical strictly convex cost of expenditure function, and free-rider problems are assumed away by requiring that each group of agents acts as a single agent with the group's aggregate cost of expenditure function (and dividing the resulting expenditure equally). Hence, larger groups have lower costs. The current contribution reexamines several of the issues addressed in these papers applying the auction CSF. That is, we examine all-pay auctions with identity-dependent externalities.

To the best of our knowledge, we are the first to study equilibria of the all-pay auction with identity-dependent externalities.<sup>2</sup> In this sense we provide a bridge between models of conflict such as Linster (1993) and Esteban and Ray (1999) that utilize a lottery CSF and the growing literature on winner-pay auctions with identity-dependent externalities in which agents place bids, an auction CSF is employed, but generally all bids except for the winner's are refunded. Jehiel and Moldovanu(2006) review this literature and note that the endogeneity of valuations in winner-pay auctions is the main driving force behind many new, and interesting phenomena that arise even in complete

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<sup>1</sup>Linster (1993) argues that such a solution exists unless the contest is degenerate in the sense that players are indifferent to the outcome.

<sup>2</sup>Konrad(2006) examines the effect of silent shareholdings in an all-pay auction framework with complete information and finds that the social value may increase or decrease depending on the identity of the firm that holds a share in its competitor. However, Konrad does not further analyze settings in which three firms are active in equilibrium and allows only one player's valuation to be endogenous.

information settings. A comprehensive treatment of the (first-price) winner-pay auction with identity-dependent externalities and complete information appears in Funk (1996).

As noted by Esteban and Ray (1999) identity-dependent externalities can, under certain conditions, impart a natural "metric" measuring the distance between players. If, for every  $i = 1, 2, \dots, n$ ,  $v_i = (v_{i1}, v_{i2}, \dots, v_{in})$  is the vector of payoffs received by player  $i$  when players  $1, 2, \dots, n$ , respectively, win their preferred option, it is natural to extend the definition of "reach" due to Siegel (2009) to account for the identity-dependent externalities. More precisely, let  $r_{ij} = v_{ii} - v_{ij}$  be player  $i$ 's reach with respect to player  $j$ . That is,  $r_{ij}$  is the maximum amount that player  $i$  would be willing to expend in order to win with certainty rather than have player  $j$  win with certainty. Under the assumption of symmetry players' reaches may be interpreted as a distance between players with two players being close if they value the success of one another in terms similar to their own. Generally, this interpretation sheds light on the societal order amongst players. Intuitively, a player who favors a radical outcome will generally face stronger opposition from his rivals, and in turn be willing to expend high effort to support it. Based on this idea, we provide a definition of players' societal characteristics which is established upon players' willingness to outbid each other.

In the following section we describe a model which embeds societal order in all-pay auctions. We analyze simple, symmetric, three player scenarios similar to those in Esteban and Ray (1999), and find that centrist players in general either completely stay out of the contest or at least on average expend lower effort than radical players, thus, extremism drives out moderation<sup>3</sup> if an all-pay auction rather than a lottery CSF is employed.

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<sup>3</sup>Following the definitions of "extremism" and "moderation" in Esteban and Ray (1999)

## 2 The Model

We examine all-pay auctions with identity-dependent externalities under complete information. In an all-pay auction all players make their bids simultaneously, the player with the highest bid wins the prize, and all players pay their bid. In order to capture the idea that a player is not indifferent to who wins the prize if he does not, we represent a player's valuation of the prize as an  $n$ -dimensional vector rather than a scalar. Each player's valuation of the prize is a vector  $v_i = (v_{i1}, v_{i2}, \dots, v_{in})$ ,  $i \in I = \{1, \dots, n\}$ , where  $v_{ij}$  is the value to player  $i$  if player  $j$  wins the prize. Externalities are not restricted to being positive or negative only, but we assume that players strictly prefer to win the prize, i.e.  $\forall i \in I : v_{ii} > v_{ij} \forall j \in I, j \neq i$ .

We aim to analyze the effects that the presence of radicals has on strategic behavior in all-pay auctions and for this purpose focus on three-player environments ( $I = \{1, 2, 3\}$ ) and for those environments define radicalism and centrism based on players' valuations. More precisely, let  $r_{ij} := v_{ii} - v_{ij}$ ,  $i, j \in I$ , be player  $i$ 's *reach*<sup>4</sup> with respect to player  $j$ , meaning that  $r_{ij}$  is the maximum player  $i$  would be willing to bid in order to outbid player  $j$ , if players  $i$  and  $j$  were the only actively competing players.

**Definition 1.** A player  $i \in I$  is called *radical*, if

$$i \in \bigcap_{r_{st} = \max_i \max_j r_{ij}} \{s, t\}$$

**Definition 2.** A player  $i \in I$  is called *centrist*, if  $i$  is not radical.

If we assume that inter-agent antagonism is symmetric, i.e.  $r_{ij} = r_{ji} \forall i, j \in I$ , then  $d(i, j) := r_{ij}$  can be interpreted as some distance between players that reflects preferences over outcomes in the sense that player  $i$  (weakly) prefers the outcome where  $j$  wins over the outcome where  $k$  wins if and only if  $d(i, j) \leq d(i, k)$ ,  $i, j, k \in I$ .

Following Esteban and Ray(1999), we refer to *extremism* as a situation where radical

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<sup>4</sup>This definition is based on Siegel(2009) but accounts for the identity-dependent externalities.

players expend more effort, in the sense of first-order stochastic dominance, in order to reach their preferred outcome. Alternatively, a situation in which centrists compete more aggressively would be referred to as *moderate*. In the following paragraphs we separately consider the cases of (A)two radical players and one centrist, (B)one radical player and two centrists, and (C)the all-pay auction without radical players.

## 2.1 Two Radicals

Let players 1 and 3 be radical and player 2 be the centrist. That is, the players are arranged either on the line or in an obtuse triangle as illustrated in Figure 1. Without loss of generality we consider the case where  $d(2, 1) = d(2, 3)$ .

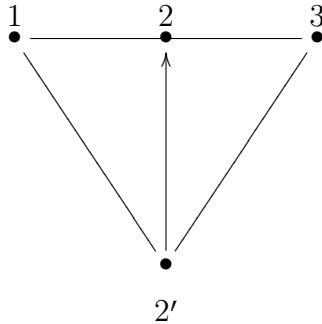


FIGURE 1:Diagram to illustrate the case of two radical players and one centrist.

We find that in any equilibrium both radicals will compete, while there exists a symmetric equilibrium, which is unique and has the property that the centrist player does not actively take part in the all-pay auction. This stands in contrast to a first-price winner-pay auction in this environment. Funk(1996) shows that there exists a pure-strategy equilibrium in the environment described above, in which player 2 wins the price with a bid of  $r_{2j}, j \in \{1, 3\}$ .

**Proposition 1** (Moderation does not drive out extremism). *In any equilibrium of the environment described above, both radicals will participate in the auction with strictly positive probability.*

*Proof.* By way of contradiction, assume that one of the radical players stays out, without loss of generality let that player be player 1, i.e.  $F_1(0) = 1$ . Given player 1's strategy

player 2 and 3 would randomize up to  $r_{23} = r_{32} < r_{31} = r_{13}$ . Player 1's payoff if he stays out of the competition will be in the interval  $(v_{13}, v_{12})$  and he could strictly improve by bidding  $r_{23}$  which would guarantee him a payoff of  $v_{11} - r_{23} = v_{11} - r_{12} = v_{12}$ .  $\square$

**Proposition 2** (Cut-throat Competition). *1. (Existence) There exists an equilibrium in which the centrist stays out, i.e. bids zero with probability one, and the radical players randomize up to their common reach,  $r_{13} = r_{31}$ .*

*2. (Uniqueness) This "cut-throat" competition is the only symmetric equilibrium (in the sense that identical players apply identical strategies).*

*Proof.* The outline of a proof can be found in the Appendix.  $\square$

**Lemma 3.** *The equilibrium described in Proposition 2.1. persists if more players, who are centrists in comparison with players 1 and 3, are added to the environment.*

*Proof.* Let player  $m$  be an additional player, who is centrist in comparison with players 1 and 3, i.e.  $r_{mj} \leq r_{jk}$  for all  $j, k \in \{1, 3\}, j \neq k$  and it exists  $j \in \{1, 3\}$  such that the inequality is strict. If player  $m$  bids zero and all other players follow the equilibrium strategies described in proposition 2, then  $m$ 's expected payoff is  $\frac{1}{2}(v_{m1} + v_{m3})$ . If player  $m$  made a strictly positive bid,  $b \leq r_{jk}$ , his expected payoff would be

$$\begin{aligned}
u_m(b) &= F(b)^2 v_{mm} + (1 - F(b)^2) \left[ \frac{v_{m1} + v_{m3}}{2} \right] - b \\
&= \frac{v_{m1} + v_{m3}}{2} + F(b)^2 \left[ \frac{r_{m1} + r_{m3}}{2} \right] - b \\
&= \frac{v_{m1} + v_{m3}}{2} + \left( \frac{b}{r_{jk}} \right)^2 \left[ \frac{r_{m1} + r_{m3}}{2} \right] - b \\
&= \frac{v_{m1} + v_{m3}}{2} + b \underbrace{\left[ \underbrace{\frac{b}{r_{jk}}}_{\leq 1} \underbrace{\left( \frac{\frac{1}{2}(r_{m1} + r_{m3})}{r_{jk}} \right)}_{< 1} - 1 \right]}_{< 0} \\
&< \frac{v_{m1} + v_{m3}}{2}.
\end{aligned}$$



If player  $m$  bids more than  $r_{jk}$ , then his payoff is

$$u_m(b) = v_{mm} - b < v_{mm} - r_{jk} < v_{mm} - \frac{1}{2}(r_{m1} + r_{m3}) = \frac{v_{m1} + v_{m3}}{2}.$$

□

Before proceeding to the next setting, we provide an example which illustrates the results above and allows us to compare the all-pay auction with a different form of all-pay contest, namely a Tullock-type model with a lottery contest success function.

For the purpose of comparison we consider an example given by Linster(1993) which applies to this setting.

**Example 1.** Consider three players and normalize the value of the prize to one, such that we can think of the contributions as fractions of the common "own value" for winning the prize. Players' valuations are  $v_1 = (1, \gamma, \gamma^2)$ ,  $v_2 = (\gamma, 1, \gamma)$ ,  $v_3 = (\gamma^2, \gamma, 1)$  where  $\gamma \in [0, 1)$ . The order of players' reaches is illustrated in the following diagram, which shows that player 2 is a centrist player and players 1 and 3 are radical.

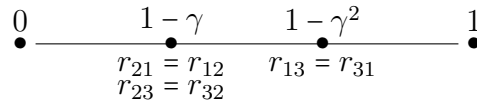


FIGURE 2:Diagram to illustrate players preferences in Example 1.

By Proposition 2 in the unique symmetric equilibrium player 2 stays out, i.e.  $F_2(x) = 1$  for all  $x \geq 0$ , and players 1 and 3 randomize symmetrically over  $[0, r_{13}]$  using cdf

$$F_1(x) = F_3(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{1-\gamma} & 0 \leq x \leq 1-\gamma \\ 1 & x > 1-\gamma \end{cases} .$$

Note that in the equilibrium of the all-pay auction described above, the socially optimal outcome will be achieved with probability zero as compared to  $(1-\gamma)/(3-\gamma)$  in the Tullock

game with lottery contest success function as considered by *Linster(1993)*. Moreover, the expected sum of bids, which *Esteban and Ray(1999)* use to measure societal conflict, is strictly greater in the all-pay auction  $\left(1 - \gamma > \frac{2}{3-\gamma} \cdot (1 - \gamma)\right)$ , although the moderate bids  $2\frac{(1-\gamma)^2}{(3-\gamma)^2}$  in the lottery contest, which is strictly greater than zero under our assumption that players prefer winning to losing. The socially optimal outcome will be realized with strictly positive probability only in the lottery contest.

## 2.2 One Radical

Now consider a three player setup with only one radical player. Given our assumptions, players must be arranged on an acute-angled triangle. Without loss of generality assume player 2 is the radical player and that the two centrist players, 1 and 3, are symmetric<sup>5</sup>. Figure 3 below illustrates players' preferences over outcomes.

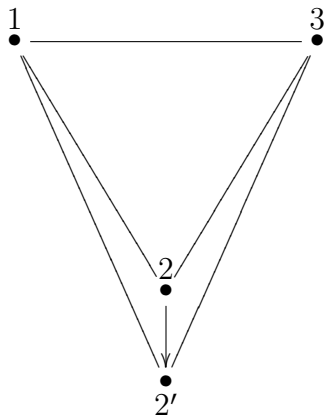


FIGURE 3: Diagram to illustrate the case of one radical player and two centrists.

We find that there exists an equilibrium in this environment where one of the centrist players does not actively participate in the contest, while the radical player never stays out in equilibrium. Moreover, even in a symmetric equilibrium (in which all players participate) extremism persists.

**Proposition 4** (Moderation does not drive out extremism). *In the environment described above, player 2 always participates in equilibrium.*

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<sup>5</sup>If players 1 and 3 were not symmetric, the identity of the player who stays out in the equilibrium described in Proposition 3 would be determined.

*Proof.* Assume that player 2 would stay out of the contest, then his expected payoff would be  $v_{2j}, j \in \{1, 3\}$  and players 1 and 3 would randomize uniformly over  $[0, r_{jk}]$ ,  $j, k \in \{1, 3\}, j \neq k$ . Thus, if player 2 would bid  $x = r_{jk}$ , he would win with certainty and receive expected payoff  $v_{22} - r_{jk} > v_{22} - r_{2j} = v_{2j}$ .  $\square$

**Proposition 5** (Extremism drives out moderation). *In the environment described above, there exists an equilibrium in which one of the centrist players stays out, i.e.  $\exists i \in \{1, 3\} : F_i(x) = 1$  for all  $x \geq 0$ .*

*Proof.* Without loss of generality assume that  $F_1(x) = 1$  for all  $x \geq 0$ . Then players 2 and 3 randomize uniformly over  $[0, r_{23}]$ . Both players must earn their equilibrium payoff at the upper bound of the support of their equilibrium strategies. Thus, player 3's expected payoff from a bid  $x \in [0, r_{23}]$  must be  $v_{32}$  and player 2's expected payoff from a bid  $x \in [0, r_{23}]$  must be  $v_{23}$ . Their equilibrium strategies are

$$F_2(x) = F_3(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{r_{23}} & 0 \leq x \leq r_{23} \\ 1 & x > r_{23} \end{cases} .$$

Given these strategies it is optimal for player 1 to bid zero and receive expected payoff  $\frac{1}{2}(v_{12} + v_{13})$ , because any bid  $x \in (0, r_{13})$  would yield an expected payoff of

$$\begin{aligned} u_1(x, F_{-1}) &= -x + v_{11}F_2(x)F_3(x) + v_{13}F_2(x)(1 - F_3(x)) + v_{12}F_3(x)(1 - F_2(x)) \\ &\quad + v_{13} \int_x^{r_{13}} (1 - F_3(s))f_2(s)ds + v_{12} \int_x^{r_{13}} (1 - F_2(s))f_3(s)ds \\ &= \frac{v_{12} + v_{13}}{2} - x \left[ 1 - \frac{x}{r_{31}^2} \left( v_{11} - \frac{v_{12} + v_{13}}{2} \right) \right] < \frac{v_{12} + v_{13}}{2}. \end{aligned}$$

$\square$

In the environment with one radical player, it is not necessary that a centrist player completely stays out of the contest. However, even in the symmetric equilibrium in which both centrists actively participate, extremism is present.

**Proposition 6** (Extremism). *In the environment described above, there exists a symmetric equilibrium (in the sense that identical players use identical strategies). This equilibrium exhibits extremism, i.e. a radical player expends more effort than a centrist player in the sense of first order stochastic dominance.*

*Proof.* See Appendix. □

Linster's (1993) second example takes on exactly this setup. Let us compare the all-pay auction once more to the Tullock-Lottery contest.

**Example 2.** *Consider three players and normalize the value of the prize to one, players' valuations are  $v_1 = (1, 0, \gamma), v_2 = (0, 1, 0), v_3 = (\gamma, 0, 1)$  where  $\gamma \in [0, 1)$ . The order of players' reaches is illustrated in the diagram below, showing that player 2 is a radical player and players 1 and 3 are centrists.*

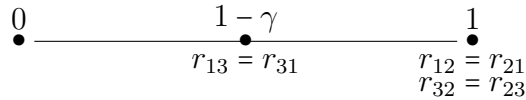


FIGURE 4: Diagram to illustrate players preferences in Example 2.

Linster(1993) computes for this example that the moderates bid  $\frac{2}{(3+\gamma)^2}$  each and the extremist bids  $(1+\gamma)\frac{2}{(3+\gamma)^2}$ , the expected sum of bids is  $2/(3+\gamma) \in (\frac{1}{2}, \frac{2}{3})$ , and player 2 wins with probability  $\frac{1+\gamma}{3+\gamma} \in (\frac{1}{2}, \frac{1}{3})$  which is increasing in  $\gamma$ .

On the other hand, in the all-pay auction in the asymmetric equilibrium (described in Proposition 5) players expend on average higher effort (the expected sum of bids is 1) and the least socially desirable outcome, i.e. player 2 wins, is more likely to occur (2 wins with probability  $\frac{1}{2}$ ). The symmetric equilibrium yields higher payoffs to the players who participate in the asymmetric equilibrium in which one moderate stays out. In this example all players have equal expected payoff,  $(\frac{\gamma}{2})^{\frac{2}{2-\gamma}}$ , in the symmetric equilibrium, while both active players in the asymmetric equilibrium have an expected payoff of zero. The moderate who stays out receives on average  $\frac{\gamma}{2} > (\frac{\gamma}{2})^{\frac{2}{2-\gamma}}$  in the asymmetric equilibrium. However, the sum of expected payoffs is strictly greater in the symmetric equilibrium.

## 2.3 No Radicals

Under the assumption of symmetric antagonism, there is only one three-player environment without any radical players. All reaches  $r_{ij}$ ,  $i, j \in I$ ,  $i \neq j$ , must coincide. This case is illustrated in Figure 5 below.

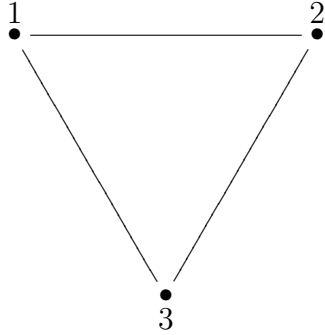


FIGURE 5: Diagram to illustrate the case without radical players and with symmetric antagonism.

Note that all players are equidistant from each other, thus  $r_{ij} = r_{kl}$ ,  $\forall i, j, k, l \in I$ ,  $i \neq j$ ,  $k \neq l$ . This case is equivalent to a three player all-pay auction without identity-dependent externalities in which players are symmetric and value the prize at  $r_{ij}$ ,  $i, j \in I$ ,  $i \neq j$ . Baye, Kovenock and DeVries (1996) show that there exists a unique symmetric equilibrium as well as a continuum of asymmetric equilibria. All equilibria however yield the same expected payoff ( $v_{ij}$ ,  $i \neq j$  after rescaling) for each player and the same expected revenue to the auctioneer.

## 3 Conclusion

In the previous section we demonstrated that societal order highly influences players' behavior in all-pay auctions. Specifically, extremism drives out moderation in the sense that whenever at least one player is in a radical position there exists an equilibrium in which at least one centrist player does not actively take part in the contest. This type of strategic behavior typically results in a less efficient outcome, i.e. total conflict is high and the socially optimal outcome will not be achieved. Two examples illustrated similarities (e.g. the existence of extremism), as well as major differences (e.g. participation

versus non-participation of moderates), that resulted from the mechanism we employed in comparison to the lottery success function which is predominantly found in the literature.

We were able to generalize the findings in Osborne et al.(2000) from an environment where players' choice sets consist only of two choices (zero and the fixed cost,  $c$ ) and the outcome is determined by a tie-breaking rule (compromise) to one in which expenditure is variable and influences the outcome.

We leave it to future research to analyze social conflict in the all-pay auction with differently sized groups and identity-dependent externalities for a direct comparison with Esteban and Ray(1999). However, our results that moderation does not drive out extremism even with only a single extremist, and on the other hand, that the presence of one extremist is sufficient for driving out moderation in one equilibrium, suggest that the specific CSF (lottery) assumed by Esteban and Ray(1999) and Linstler(1993) was crucial for the active participation of moderates in equilibrium (if all players face identical cost structures). In conclusion, we need to be aware that the differences between ratio and difference-form success functions, which were found in pure contests (Hirshleifer (1989)), persist in contests with identity-dependent externalities.

# Appendix

*Proof.* (Proposition 2)

1. (*Existence*) The strategy profile in which 2 stays out completely (puts mass 1 on zero) and players 1 and 3 randomize uniformly over  $[0, r_{jk}]$  ( $j, k \in \{1, 3\}, j \neq k$ ) is a Nash equilibrium. Assume that 2 uses the strategy  $F_2(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$ . Then it is optimal for players 1 and 3 to randomize over  $[0, r_{jk}]$  according to

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x}{r_{jk}} & \text{for } 0 \leq x \leq r_{jk} \\ 1 & \text{for } x > r_{jk} \end{cases} .$$

Given that 1 and 3 apply this strategy player 2's payoff if he would make a strictly positive bid  $x \in (0, r_{2j}]$  is:

$$\begin{aligned} u_2^*(x) &= [F(x)]^2 v_{22} + (1 - [F(x)]^2) v_{2j} - x \\ &= v_{2j} + F(x)^2 r_{2j} - x \\ &= v_{2j} - x \left( 1 - \frac{x}{r_{jk}} \frac{r_{2j}}{r_{jk}} \right) \\ &< v_{2j}. \end{aligned}$$

2. (*Uniqueness*) Before we show the uniqueness of the symmetric equilibrium described under (1.), we make the following observations regarding symmetric equilibria:

(a)  $\underline{s}_1 = \underline{s}_2 = \underline{s}_3 = \mathbf{0}$

Assume  $\exists i, j$  such that  $\underline{s}_i < \underline{s}_j$ . For every bid  $x_i \in [\underline{s}_i, \underline{s}_j)$   $i$  loses with certainty.  $\Rightarrow \underline{s}_i = 0$  and player  $i$  does not put mass anywhere over  $(\underline{s}_i, \underline{s}_j)$ .  $\Rightarrow$  player  $j$  would improve by moving mass down.

(b) **Not all three players can have a mass point at zero**

Otherwise a player could increase his profit by moving mass up slightly.

(c)  $\bar{s}_j > \bar{s}_2, \mathbf{j} \in \{1, 3\}$

Assume  $\bar{s}_1 = \bar{s}_2 = \bar{s}_3 = \bar{s}$ .  $u_j^*(0) = \alpha v_{j2} + (1 - \alpha)v_{jk}$  for some  $\alpha \in (0, 1)$  ( $j \in \{1, 3\}, k \in \{1, 3\} \setminus \{j\}$ ).  $\Rightarrow u_j^* < v_{j2}$ . On the other hand,  $u_j^*(\bar{s}) = v_{jj} - \bar{s} \Rightarrow \bar{s} > r_{j2}$ . Contradiction ( $\bar{s}_2 \leq r_{2j}$ )  $\Rightarrow \bar{s}_j \neq \bar{s}_2 \Rightarrow \bar{s}_j > \bar{s}_2$ .

(d)  $\bar{s}_2 < \mathbf{r}_{2j}$

By (b) player 2 loses with strictly positive probability at  $\bar{s}_2$ . Suppose  $\bar{s}_2 \geq r_{2j}$ , then player 2's equilibrium payoff at  $\bar{s}_2$  is

$$u_2^*(\bar{s}_2) = [F(\bar{s}_2)]^2 v_{22} + (1 - [F(\bar{s}_2)]^2) v_{2j} - \bar{s}_2 \leq v_{2j} - (1 - [F(\bar{s}_2)]^2) r_{2j} < v_{2j}.$$

(e) **F (player 1's and 3's c.d.f.) is strictly increasing over  $[0, \bar{s}_j]$**

1 and 3 use identical cdf (symmetric equilibrium). If F had a flat spot then so would  $F_2$ , but then no player would make a bid in this interval and a player could improve by moving mass down from the upper end of the interval.

(f) **There are no point masses on the open interval  $(0, \bar{s}_j)$**

Suppose player  $i \in I$  has a mass point at  $x \in (0, \bar{s}_2]$ , then a player  $l \neq i$  could improve by moving mass from an  $\epsilon$ -neighborhood below  $x$  to some  $\delta$  neighborhood above  $x$  (remember that  $\bar{s}_2 < r_{2j}$  by (d)). Thus, there would be an  $\epsilon$ -neighborhood below  $x$  in which no other player would put mass. But then it is not optimal for  $i$  to put mass at  $x$ . Suppose 1 and 3 have a mass point at  $y \in (\bar{s}_2, \bar{s}_j)$ , then either would improve by moving his mass point up slightly (note that  $\bar{s}_j \leq r_{jk}$ ).

Assume that player 2 randomizes continuously over a finite number of disjoint intervals. Player 2's support can be written as  $\bigcup_{k=1}^{\bar{k}} [t_{n-2k}, t_{n-2k+1}] \cup [t_n, \bar{s}_2]$  where  $\bar{k} = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ \frac{n-1}{2} & \text{otherwise} \end{cases}$ ,  $t_0 = \underline{s}_2 = 0$ . Let  $[t, \bar{t}]$  be one of these intervals over which



all three players randomize continuously. Player  $j$ 's ( $j \in \{1, 3\}$ ) expected payoff for a bid,  $x \in [\underline{t}, \bar{t}]$ , is

$$\begin{aligned} u_j^*(x) &= v_{jj}F(x)F_2(x) + v_{jk}F_2(x)(1 - F(x)) + v_{j2}F(x)(1 - F_2(x)) \\ &\quad + v_{jk} \int_x^{\bar{s}_2} (1 - F(s)) f_2(s) ds + v_{j2} \int_x^{\bar{s}_2} (F(s) - F(x)) f_2(s) ds - x \\ &= r_{jk}F(x)F_2(x) + v_{jk} + (v_{j2} - v_{jk}) \int_x^{\bar{s}_2} f_2(s)F(s) ds - x \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{du_j^*(x)}{dx} &= r_{jk} (F_2'(x)F(x) + F_2(x)F'(x)) - (v_{j2} - v_{jk})F_2'(x)F(x) - x \\ \Rightarrow F_2'(x)F(x)r_{j2} + F_2(x)F'(x)r_{jk} &= 1 \\ \Rightarrow F_2(x) &= \frac{2r_{j2}}{r_{j2} + r_{jk}}F(x) + [F(\underline{t})]^{\frac{r_{jk}}{r_{j2}}} \left( F_2(\underline{t}) - \frac{2r_{j2}}{r_{j2} + r_{jk}}F(\underline{t}) \right) [F(x)]^{-\frac{r_{jk}}{r_{j2}}} \end{aligned}$$

By assumption  $r_{jk} > r_{j2}$ , thus  $\exists \epsilon > 0$  such that  $r_{jk} = (1 + \epsilon)r_{j2}$  and we can rewrite

$$F_2(x) = \underbrace{\frac{2}{2 + \epsilon}}_{<1} F(x) + \underbrace{\left( \frac{F(\underline{t})}{F(x)} \right)^{1+\epsilon}}_{>0} \left( F_2(\underline{t}) - \frac{2}{2 + \epsilon}F(\underline{t}) \right) \quad (\star)$$

From  $(\star)$  we can derive

$$F_2'(t) = \frac{2}{2 + \epsilon}F'(t) - \underbrace{(1 + \epsilon)}_{>0} \underbrace{F'(t)}_{>0} \underbrace{\left( \frac{F(\underline{t})}{F(x)} \right)^{1+\epsilon}}_{>0} \left( F_2(\underline{t}) - \frac{2}{2 + \epsilon}F(\underline{t}) \right) \underbrace{\frac{1}{F(t)}}_{>0}. \quad (\star\star)$$

By (c)  $F_2(\bar{s}_2) = 1 > F(\bar{s}_2)$ , a necessary condition for this to hold is that

$$F_2(t_n) - \frac{2}{2 + \epsilon}F(t_n) > 0.$$

It follows that  $F_2'(t) < F'(t)$  for  $t \in [t_n, \bar{s}_2] \Rightarrow F_2(t_n) > F(t_n)$ . By assumption  $F_2$  is

constant over  $(t_{n-1}, t_n)$  and  $F$  is increasing over the same interval

$$\begin{aligned}
&\Rightarrow F_2(t_{n-1}) > F(t_{n-1}) \\
&\Rightarrow F_2(t_{n-2}) - \frac{2}{2+\epsilon} F(t_{n-2}) > 0 \\
&\Rightarrow (\text{By } (\star\star)) F_2'(t) < F'(t) \text{ for } t \in [t_{n-2}, t_{n-1}] \\
&\Rightarrow F_2(t_{n-1}) > F(t_{n-1})
\end{aligned}$$

After a finite number of iterations of the argument above follows that  $F_2(t_0 = 0) > F(t_0 = 0)$  which implies that 2 has a mass point at zero and (by b) players 1 and 2 do not have a mass point at zero. Altogether  $F_2(t) > F(t)$  for all  $t \in [0, \bar{s}_2]$ .

We distinguish two cases:

(a)  $n$  is even, i.e. player 2 distributes mass strictly greater than zero over  $[0, t_1]$ .

In this case for  $t \in [0, t_1]$   $(\star)$  simplifies to

$$F_2(t) = \frac{2}{2+\epsilon} F(x) < F(t)$$

(b)  $n$  is odd, i.e. player 2 has does not make bids in the interval  $(0, t_1)$ .

In this case only players 1 and 3 randomize over  $(0, t_1)$ . Player  $j$ 's ( $j \in \{1, 3\}$ ) expected payoff for a bid,  $x \in (0, t_1)$  is:

$$\begin{aligned}
u_j^*(x) &= v_{jj} F(x) F_2(x) + v_{jk} F_2(x) (1 - F(x)) + v_{j2} F(x) (1 - F_2(x)) \\
&\quad + v_{j2} \int_{t_0}^{\bar{s}_2} (F(s) - F(x)) f_2(s) ds + v_{jk} \int_{t_0}^{\bar{s}_2} (F_2(s) - F_2(x)) f(s) ds - x \\
&= v_{jj} F(x) F_2(0) + v_{jk} F_2(0) (1 - F(x)) + v_{j2} F(x) (1 - F_2(0)) - v_{j2} F(x) [1 - F_2(0)] \\
&\quad + v_{j2} \int_{t_0}^{\bar{s}_2} F(s) f_2(s) ds - v_{jk} F_2(0) [F(\bar{s}_2) - F(t_0)] + v_{jk} \int_{t_0}^{\bar{s}_2} F_2(s) f(s) ds - x \\
&= v_{jk} F(x) F_2(0) + v_{jk} F_2(0) [1 - F(\bar{s}_2) + F(t_0)] + v_{j2} \int_{t_0}^{\bar{s}_2} F(s) f_2(s) ds \\
&\quad + v_{jk} \int_{t_0}^{\bar{s}_2} F_2(s) f(s) ds - x
\end{aligned}$$

$\frac{du_j^*(x)}{dx} = r_{jk}F_2(0)F'(x) - 1$ , thus  $F'(x) = \frac{1}{F_2(0)r_{jk}}$  for  $x \in (0, t_0)$ . By  $F(0) = 0$  players 1 and 3 randomize uniformly over  $[0, t_1]$  according to

$$F(x) = \frac{x}{F_2(0)r_{jk}}, x \in [0, t_1].$$

All three players randomize continuously over  $[t_1, t_2]$ . Player 2's expected payoff if he bids  $x \in [t_1, t_2]$  is:

$$\begin{aligned} u_2^*(x) &= [F(x)]^2 v_{22} + [1 - (F(x))^2] v_{2j} - x \\ &= v_{2j} + [F(x)]^2 r_{2j} - x \\ &= v_{2j} \\ \Rightarrow F(x) &= \left( \frac{x}{r_{2j}} \right)^{\frac{1}{2}} \text{ for } x \in [t_1, t_2] \end{aligned}$$

$F$  is continuous at  $t_1$  (by (f)), therefore

$$\left( \frac{t_1}{r_{2j}} \right)^{\frac{1}{2}} = \frac{t_1}{F_2(0)r_{jk}} \Rightarrow t_1 = F_2(0)^2 \frac{(r_{jk})^2}{r_{2j}}$$

By (★) for  $t \in [t_1, t_2]$

$$\begin{aligned} F_2(t) &= \underbrace{\frac{2}{2+\epsilon} F(t)}_{<1} + \underbrace{\left( \frac{F(t_1)}{F(t)} \right)^{1+\epsilon}}_{>0} \underbrace{\left( F_2(0) \left( 1 - \underbrace{\frac{2(1+\epsilon)}{2+\epsilon}}_{>1} \right) \right)}_{<0} \\ &< F(t) \end{aligned}$$

Altogether, player 2 does not randomize over any finite number of intervals. Hence, the equilibrium described in (1.) is the unique symmetric equilibrium. □

*Proof.* (Proposition 5)

Under the assumption that all three players make positive bids with strictly positive probability and players 1 and 3 use identical strategies, i.e.  $F_1 = F_3 =: F$ , we know that  $\underline{s}_1 = \underline{s}_2 = \underline{s}_3 = 0$  and  $\bar{s}_2 = \bar{s}_1 = \bar{s}_3 =: \bar{s}$ . Moreover,  $\bar{s} \in (r_{jk}, r_{2j})$ ,  $j, k \in \{1, 3\}$ ,  $j \neq k$ , and player 2 cannot have a masspoint at zero. Assume that all players randomize continuously over  $[0, \bar{s}]$ . All players must earn their equilibrium payoff at  $\bar{s}$ , therefore player 2's expected payoff from a bid  $b \in (0, \bar{s}]$ ,  $u_2(b, F) = v_{22}[F(b)]^2 + v_{2i}(1 - [F(b)]^2)$ , must be  $v_{22} - \bar{s}$ . This yields

$$F(x) = \begin{cases} 0 & x < 0 \\ \left[ \left(1 - \frac{\bar{s}}{r_{2j}}\right) + \frac{x}{r_{2j}} \right]^{\frac{1}{2}} & 0 \leq x \leq \bar{s} \\ 1 & x > \bar{x} \end{cases} .$$

Player  $j$ 's payoff must be  $v_{jj} - \bar{s}$ . Moreover, player  $j$  chooses his equilibrium strategy such that his expected payoff,  $u_j(b, F_2, F) = -b + v_{j2} + [v_{jj} - v_{j2}]F(x)F_2(x) + [v_{jk} - v_{j2}] \int_b^{\bar{s}} F_2(s)F'(s)ds$ , is maximized. The first order condition yields the first order differential equation

$$0 = F(x)F_2'(x)r_{j2} + F'(x)F_2(x)r_{jk} - 1.$$

Using the boundary conditions  $F_2(0) = 0$  and  $F_2(\bar{s}) = 1$  this yields

$$F_2(x) = \kappa F(x) - (\kappa - 1)F_1(x)^{-\frac{r_{jk}}{r_{j2}}}$$

with  $\kappa = \frac{2r_{j2}}{r_{j2} + r_{jk}} > 1$  and  $\bar{s} = r_{j2} \left[1 - \left(1 - \frac{1}{\kappa}\right)^\kappa\right]$ . Note that  $\bar{s} \in (r_{jk}, r_{j2})$  and  $F_2$  is strictly increasing.

In order to show that this equilibrium exhibits extremism, we need to show that  $F_2(x) \leq F_1(x) \forall x$ . All players' cdfs coincide for  $x < 0$  and  $x \geq \bar{x}$ . The centrist players put strictly positive mass on zero, thus  $F_2(0) < F_1(0)$ . For  $x \in (0, \bar{x})$ ,

$$F_2(x) = \kappa F_1(x) - (\kappa - 1)F_1(x)^{-\frac{r_{13}}{r_{12}}} = F_1(x) \underbrace{\left[ \kappa - (\kappa - 1) \overbrace{F_1(x)^{-\left(1 + \frac{r_{13}}{r_{12}}\right)}}^{>1} \right]}_{< \kappa - (\kappa - 1) = 1} < F_1(x).$$

Altogether,  $F_2$  first order stochastically dominates  $F_1$ . □

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