

Katsenos, Georgios

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# Optimal Reserve Prices in Sequential Auctions with Imperfect Commitment

Georgios Katsenos\*

Department of Economics and Management,  
Leibniz Universität Hannover

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## Abstract

In a sequential auction of perfect substitutes, we analyze the consequences of the seller's incapacity to commit perfectly to a reserve-price schedule. When facing such a seller, the bidders have strong incentives not to reveal during the earlier rounds of the auction any information about their valuations. If the seller observes only the winning bid in each round, as in a sequence of Dutch auctions, there is a symmetric monotone equilibrium in which the seller may lower the reserve price over time. The possibility of lower future reserve prices makes several bidder types abstain from the earlier rounds, even though their valuations exceed the requested price. In addition, because of the restriction in competition, the participating bidders shade their bids sharply. Thus, imperfect commitment results in revenue loss. The loss becomes more severe if the seller attempts to suppress some of the information revealed in the auction, for example, by learning only whether an item is sold. Finally, if the seller observes all bids, as in a sequence of sealed-bid auctions, a monotone equilibrium fails to exist, however small the imperfection of the seller's commitment is. Our results rationalize the adoption of costly commitment or privacy preserving measures, such as the use of a well-established auction house.

**JEL classification:** D44, D82.

**Keywords:** Sequential auctions, reserve price, non-commitment, information, participation, bid shading.

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*Email address:* katsenos@mik.uni-hannover.de.



# 1 Introduction

Several multi-unit sales are conducted by means of sequential auctions, carried out either in rapid succession or over long periods of time. For example, wine, art, condominium units, used cars, agricultural products and fish are often auctioned sequentially. On internet auction sites, sellers often auction in a sequence many units of the same consumer product. Furthermore, procurement contracts are also auctioned sequentially, as the need for each project arises. Finally, the radio spectrum auction in the United States, as well as similar spectrum auctions in numerous other countries, was conducted by means of a dynamic procedure.

In the study of such auctions, a typical assumption is that of intertemporal commitment. In particular, the seller is able to commit in a fully credible manner to a specific auction mechanism, or to a sequence of auction mechanisms, via which all sales will be made. The bidders can therefore reveal, in the earlier rounds of the auction process, private information about their valuations, without fearing that the seller will change the rules, for example, alter the reserve price, at their expense.

In reality, however, perfect intertemporal commitment is often infeasible. In many cases, the seller lacks the credibility that this assumption requires.<sup>1</sup> In addition, he may find it too costly to attempt to guarantee the rules of a sale by using an institution like a well-established auction house. Furthermore, even when the seller does commit to a certain selling scheme by means of a contract, he may still try to break the sale rules at a cost, if the revealed information makes it profitable for him to do so.<sup>2</sup> Finally, in several cases in which the seller announces the auction of only few units, possibly because at that time he can guarantee the delivery of only those units, the bidders may try to conceal some of their private information, in anticipation of similar sales in the future.

In this paper, we study the effects of imperfect commitment. As in the standard formulation of Milgrom and Weber [26] and Weber [35], we examine the sequential first-price auction of two units of the same good to a group of potential buyers, each of which has single-unit demand and a private valuation that does not change over time. All buyers other than the winner of the first auction are present in both auctions; and prior to submitting their second-round bids, they are informed of the first-round winning bid.<sup>3</sup> Finally, each unit must be sold in the period it becomes available.<sup>4</sup>

We depart from this setting, first, by allowing the seller to set reserve prices  $r_1$  and  $r_2$

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<sup>1</sup>Typically, this would be a seller with no concerns about reputation.

<sup>2</sup>There are several cases in which the seller tried to break the auction rules in reaction to revealed information. For example, in the sale of the General Motors Building in Manhattan, in 2003, the owners allegedly used their knowledge of the submitted bids so as to bargain with the eventual buyer (who was the last bidder to submit a bid to the auction). For more details, see McAdams and Schwarz [21, 22].

<sup>3</sup>In section 4.2, we will modify our setting by restricting the buyers to observing only whether the first auction has resulted in a sale. However, our results would not change, if the buyers could also observe the winning bid.

<sup>4</sup>Without the constraint of sequential allocation, if the seller could keep the first unit so as to sell it along with the second one in a single auction, the commitment issue would be trivially bypassed.

in each of the two auctions, and second, by restricting the time at which the seller can choose the reserve price  $r_2$ . More specifically, the seller cannot credibly commit at the beginning of the game to any reserve price, or rule for determining the reserve price, for the second auction. Rather, he must choose the price  $r_2$  at the beginning of the second period, after having observed (part of) the bidding behavior in the first auction.<sup>5</sup>

The type of information that the seller receives at the end of the first round is determined exogenously, prior to the beginning of the sale. We examine three policies, observing the first-round winning bid, all submitted bids or whether the first item has been sold. The last policy clearly requires the employment of an auctioneer other than the seller; on the other hand, the policy of observing only the winning bid can be implemented by the seller, without an auctioneer, via a sequence of Dutch auctions.

The above setting describes problems, for example, in sequential procurement. In such problems, a number of contracts for similar projects, for example, for highway paving, is procured sequentially. Capacity constraints force the competing firms to limit the supply of their services.<sup>6</sup> If the buyer's commitment not to alter the rules pertaining the future auctions cannot be fully credible, then the issues involved in the design of the optimal procurement process will be identical to the ones we study.

The impossibility of intertemporal commitment has important consequences for the bidders' behavior in the first round of the auction. Since the seller cannot restrict the manner in which he can use the revealed information to set the reserve price in the second round, the bidders have strong incentives to conceal their valuations. In particular, the non-winning bidders are best-off not submitting any bid.

When the seller observes all bids submitted in the first auction, we find that there is no symmetric equilibrium in monotone bidding strategies. In such a strategy profile, the bidders participating in the first round fully reveal their valuations; so, if they fail to win, they can expect no gain in the second round. Therefore, they have to bid aggressively, so much that one can profit from deviating to non-participation. On the other hand, if no bidder type participates in the first round, then, when the reserve price  $r_1$  is sufficiently low, some bidders can profit from participating with a minimal bid.<sup>7</sup>

A symmetric monotone equilibrium exists only when the non-winning bidders can avoid revealing information; in particular, when the seller observes the first-period winning bid. In this equilibrium, the seller sets a first-period reserve price that allows the sale of the first item, if a bidder with a sufficiently high valuation exists. Subsequently,

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<sup>5</sup>In this setting, therefore, imperfect commitment assumes an extreme form, namely, that of non-commitment. In section 4.3, we show that all intermediate situations, in which the seller is allowed to change the reserve price with some positive probability, are qualitatively identical to non-commitment.

<sup>6</sup>The relevance of such constraints has been documented by Jofre-Bonet and Pesendorfer [15], in their study of highway paving contracts in California.

<sup>7</sup>A similar phenomenon occurs in auctions with resale, in which the bidders have incentives to conceal their private information during the auction, so as to gain more from post-auction trade opportunities. (See Garratt and Tröger [9] and Hafalir and Krishna [11].)

given the outcome of the first auction, he updates his beliefs (in particular, he obtains a sharper upper bound for the remaining bidders' valuations) and, accordingly, he sets the reserve price for the second item. Because of the possibility of a lower future reserve price, several bidder types do not submit any bid in the first auction, even though their valuations exceed the requested price.<sup>8</sup> Consequently, those bidders who participate in the first round shade their bids sharply, knowing that they face limited competition. Both the strategic non-participation and the excessive shading of the submitted bids would be absent, if the seller could commit not to change, in particular, not to lower, the reserve price over time.

Overall, the seller suffers a revenue loss. Although he is able to design the second auction in a better informed manner, and therefore to derive a higher revenue from the sale of the second item than in the case of commitment, he cannot prevent severe losses in the first auction. Thus, the intuition favoring commitment<sup>9</sup> is reaffirmed in the setting of sequential auctions. In particular, the seller would be willing to adopt costly measures to enhance his credibility, for example, he would be willing to pay for the services of a trusted intermediary, like a well-established auction house.

Finally, the seller will suffer a more severe revenue loss, if he attempts to restrict partially the information made available to him,<sup>10</sup> in particular, to observe only whether the first item is sold. As it turns out, this informational restriction would not affect the bidders' non-participation decision. On the other hand, it would deprive the seller of the opportunity to design the second auction in a more informed manner. Therefore, it cannot be profitable. Hence, in the absence of commitment, the information policy affects the seller's revenue, in contrast to the case of commitment analyzed by Weber [35].

The literature on sequential auctions has paid relatively little attention to the possibility of a strategic seller. Some of this literature has tried to explain the declining-price anomaly<sup>11</sup>, a problem in which the seller plays no strategic role. Learning in sequential auctions has been studied by Ortega Reichert [28] and Jeitschko [13]. They concentrate, however, on the manner in which the bidders, rather than the seller, can use the information revealed during a sequential auction. A strategically active seller is present in McAfee and Vincent [24], who study the optimal reserve-price path in a sequence of first- and second- price auctions.<sup>12</sup> In particular, in a setting akin to that of the Coase

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<sup>8</sup>This strategic non-participation decision first appeared in the literature of dynamic bargaining; for example, see Hart and Tirole [12]. For examples of its occurrence in sequential auctions, see McAfee and Vincent [24] and Caillaud and Mezzetti [5]. In our case, we shall remark that it differs from the ratchet effect, despite its resemblance to it. A buyer decides not to participate in the first auction even though the seller cannot use against him, in the future, the information revealed by his bid.

<sup>9</sup>This result was formally established by Stokey [32] and Bulow [3].

<sup>10</sup>The result in favor of commitment suggests that the seller will be best-off setting the optimal commitment reserve price  $r_1$  and not receiving any information about the first-round outcome.

<sup>11</sup>Although intertemporal non-arbitrage requires that the expected prices remain constant over time, in practice it has been observed that earlier sales tend to be concluded at higher prices than later ones. For details, see Ashenfelter [1], Jeitschko [14], MacAfee and Vincent [23] and Milgrom and Weber [26].

<sup>12</sup>Sobel and Takahashi [29] have studied a similar problem within the context of dynamic bargaining. Skreta [30] has generalized with respect to the mechanisms that the seller can choose in every period.

conjecture, the seller puts the same object for sale repeatedly, until it is sold.<sup>13</sup> At each round he chooses a reserve price according to his (increasingly pessimistic) beliefs about the buyers' valuations. Since the game ends as soon as the object is sold, that is, as soon as a bid is placed, the buyers do not face the problem of hiding their valuations. Prior to the end of the game, information can be revealed only in a passive manner, by the buyers' refusal to bid for the object at a given reserve price.

The issue of concealing, during the earlier sales, information from the seller appears in Caillaud and Mezzetti [5]. In a sequence of two auctions in which the buyers have multi-unit demands and persistent valuations, the bidders face a problem similar to that in our setting. However, because of the multi-unit demands, this problem eventually concerns only the bidder with the highest valuation, all other bidders realizing that they cannot win either of the two auctions.<sup>14</sup> Therefore, it is the format of the English auction that is employed in this setting, as it allows the winning bidder not to reveal his valuation. Our work complements that of Caillaud and Mezzetti [5] by examining an environment in which the problem of concealing information is faced by the non-winning bidders.

In both settings, it turns out that relevant information can be revealed only indirectly, through the actions of the bidders that have no further interest in the game. In addition, in both cases, imperfect commitment is costly for the seller, because of the bidders' non-participation strategy. In our setting, however, the seller's incapacity to commit has a direct effect upon the participating bidders' strategies, in contrast to the setting in Caillaud and Mezzetti [5], in which each participating bidder bids up to his valuation.

The issue of commitment in a sequential auction is also studied by Zeithammer [36, 37]. In the presence of production costs, Zeithammer examines whether it can be profitable for the seller to base his future supply decisions on the information revealed during the earlier rounds of the auction. He finds that non-commitment can result in higher revenue than certain simple forms of commitment. Unlike our paper (or Caillaud and Mezzetti [5]), Zeithammer's seller does not face the problem of committing to a future reserve price; he only chooses whether to commit into supplying additional units.

Finally, the problem of deciding how much information to make publicly available in a sequential auction appears in Thomas [33] and Tu [34]. In their settings, however, the seller does not set any reserve prices nor make any decisions during the game. Therefore, the issues involved in their analysis are different than the ones in our problem.

In the next two sections, we present the model describing our problem and we find the optimal reserve prices under commitment. In section 4.1, we analyze the case in which the seller observes only the first-period winning bid. In section 4.2, we show that the seller is worse-off when he observes only whether the first item is sold. In section 4.3, we

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<sup>13</sup>Coase [6] conjectured that a durable-good monopolist who is unable to commit to a price schedule over time can have no market power; for more details, see Fudenberg *et al.* [8] or Gul *et al.* [10].

<sup>14</sup>In fact, on the equilibrium path, if there is a winning bidder in the first auction, then the second auction becomes a bargaining problem between the seller and that bidder, simplifying some of the technical issues encountered in our paper.

show that there is no symmetric monotone equilibrium when all bids are observed; and we extend this negative result to a setting in which the seller can change the second-period reserve price with only a small probability. We conclude in section 5.

## 2 General Model

There is one seller with 2 identical (or equivalent) objects for sale. The seller's valuation for the objects is normalized to zero, so that he can derive no benefit from any object that remains unsold.

The seller faces  $N > 2$  potential buyers, indexed by  $i = 1, \dots, N$ . Each buyer has single-unit demand and private valuation  $v_i \in [\underline{v}, \bar{v}]$ , for  $0 \leq \underline{v} < \bar{v}$ , which remains constant throughout the game. The valuations are independently drawn, according to a common distribution function  $F : [\underline{v}, \bar{v}] \rightarrow [0, 1]$ . We assume that the distribution function  $F$  is differentiable and that its derivative,  $f : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}^+$ , has full support.

We denote by  $v_1^{(M)}$  and  $v_2^{(M)}$  the highest and the second-highest order statistics among  $M \geq 2$  independent draws from the distribution  $F$ . We also denote by  $G : [\underline{v}, \bar{v}] \rightarrow [0, 1]$  and  $g : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}^+$  the distribution and the density of the highest order statistic  $v_1^{(N-1)}$ .

The two objects are allocated to the buyers by means of a sequence of two sealed-bid first-price auctions, conducted in periods (or rounds)  $t = 1, 2$ . Immediately after the end of the first auction, the seller and the buyers receive respectively information  $h_s(b^1)$  and  $h_b(b^1)$  about the bids  $b^1 = (b_1^1, \dots, b_N^1)$  submitted in it; to simplify the notation, we have allowed  $b_i^1$  to take the value of "no-bid" or "abstain". We will examine the following cases:<sup>15</sup>

1. Revealing the winning bid,  $h_s(b^1) = h_b(b^1) = b_1^1$ ;
2. Revealing whether the first item was sold,  $h_s(b^1) = h_b(b^1) \in \{s, ns\}$ .
3. Revealing all bids to the seller and only the winning bid to the bidders, that is,  $h_s(b^1) = b^1$  and  $h_b(b^1) = b_1^1$ .

All these information revelation policies can be implemented through the services of a trusted third party, for example, an auctioneer (who differs from the seller). In addition, a sequential sealed-bid first-price auction in which only the winner's bid is revealed is strategically equivalent to a sequential Dutch auction.<sup>16</sup> Finally, if the seller observes all first-round bids, then there is obviously no need for a third party.

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<sup>15</sup>We will also refer to one theoretical case, that in which no information is revealed to any player.

<sup>16</sup>In this auction, the price clock is set high, above  $\bar{v}$ , at the beginning of each round (so that the second item can be sold at a higher price than the first one).



At the beginning of each auction, the seller sets a reserve price  $r_t \in \mathbb{R}^+$ . In particular, the seller must choose  $r_2$  after the end of the first auction and after having observed the information  $h_s(b^1)$  that is released to him. His strategy, therefore, consists of a reserve price for the first auction and a reserve-price rule for the second auction:

$$\begin{aligned} r_1 &\in \mathbb{R}^+; \\ r_2 &: (h_s(b^1), r_1) \longmapsto r_2 \in \mathbb{R}^+. \end{aligned}$$

We have relaxed the usual assumption of intertemporal commitment by making the seller unable to choose a reserve-price rule  $r_2$  at the beginning of the game, in a manner that would allow the bidders to base their first-period behavior upon it.

In each period, every buyer who has not already won a unit either submits a bid or abstains from the auction (but is still present in it). His first-period bid depends on his valuation and the reserve price  $r_1$ . His second-period bid depends on his valuation, the new reserve price, and the information that has been revealed in the first auction. Therefore, for each information revelation scheme  $h_b(\cdot)$ , the strategy of each bidder  $i$  consists of the bidding rules

$$\begin{aligned} \beta_i^1 &: (v_i | r_1) \longmapsto b_i^1 \in \{a\} \cup [r_1, \infty); \\ \beta_i^2 &: (v_i | b_i^1, h_b(b^1), r_1, r_2) \longmapsto b_i^2 \in \{a\} \cup [r_2, \infty), \end{aligned}$$

where  $a$  denotes the action of abstaining from submitting a bid.

The payoff of buyer  $i$ , in case he wins one unit, equals his valuation  $v_i$  minus the price that he pays for it; otherwise, if he does not win any unit, it is equal to zero.<sup>17</sup> The payoff of the seller equals the total revenue from the two auctions.

The solution concept will be that of perfect Bayesian equilibrium. At each decision node, each player must behave optimally, given the other players' strategies and his beliefs.<sup>18</sup> On the equilibrium path, the players' beliefs are determined by Bayes' rule. Off the equilibrium path, we strengthen the equilibrium concept by restricting the bidders' beliefs to remain the same following a deviation by the seller.<sup>19</sup> Finally, we restrict attention to symmetric<sup>20</sup>, pure-strategy equilibria, in weakly increasing bidding strategies.

<sup>17</sup>For simplicity, we assume no discounting of the second-period payoffs. Our results would not change if a discount factor  $\delta \in (0, 1)$ , possibly reflecting the probability with which the second unit becomes available, were introduced; in this case, the seller's loss from non-commitment would be less pronounced.

<sup>18</sup>In particular, non-commitment implies that the optimality of the reserve-price rule  $r_2$  is determined by the seller's updated beliefs and not by his initial ones (as when he could choose  $r_2$  prior to the beginning of the first auction).

<sup>19</sup>We consider this restriction quite plausible, as the seller has no more information than the bidders when he sets the reserve prices. It aims at eliminating certain rather artificial equilibria, for example, equilibria in which each bidder  $i$  believes that his competitors' valuations are  $v_j = \bar{v}$ , for  $j \neq i$ , and abstains from the auction, following any reserve price  $r_t > \bar{v}$ , for  $t = 1, 2$ .

<sup>20</sup>In fact, wherever it is applicable, we impose a stronger symmetry requirement, one that rules out the possibility of using the first-period bids as a labeling device. For example, we will not allow symmetric strategies that prescribe different second-period bidding behavior to the second and the third highest bidders of the first auction.

### 3 Optimal Sequential Auctions under Commitment

If there is only one unit for sale, then, according to Myerson [27], the seller will choose the optimal reserve price by considering the bidders' virtual valuation function

$$\psi(v) = v - \frac{1 - F(v)}{f(v)}.$$

We make the standard regularity assumption that the function  $\psi(v)$  is increasing. For this assumption to hold, it is sufficient that the hazard rate  $\frac{f(v)}{1-F(v)}$  is increasing.

Given the regularity assumption, the seller can maximize his expected revenue by allocating the object by means of any standard auction with reserve price

$$r_0 = \begin{cases} \psi^{-1}(0), & \text{if } \psi(\underline{v}) < 0; \\ \underline{v}, & \text{otherwise.} \end{cases} \quad (1)$$

That is, the seller must exclude all bidders with valuations below  $\psi^{-1}(0)$ .

If the players receive prior information that their opponents' valuations are bounded above by  $\hat{v}$ , then they update their beliefs so that they consider the bidders' valuations i.i.d., according to the distribution function  $F(\cdot)/F(\hat{v})$  on  $[\underline{v}, \hat{v}]$ . In this case, the bidders' virtual valuations will be given by the function

$$\psi(v|\hat{v}) = v - \frac{F(\hat{v}) - F(v)}{f(v)}.$$

It is easy to check that the function  $\psi(v|\hat{v})$  also satisfies the regularity assumption. Therefore, given the knowledge of the bound  $\hat{v}$ , the seller maximizes his expected revenue by setting a reserve price

$$r_0(\hat{v}) = \begin{cases} \psi(\cdot|\hat{v})^{-1}(0), & \text{if } \psi(\underline{v}|\hat{v}) < 0; \\ \underline{v}, & \text{otherwise.} \end{cases} \quad (2)$$

Notice that the reserve price  $r_0(\hat{v})$  is increasing in  $\hat{v}$ . Finally, for any optimal reserve price  $r_0 > \underline{v}$ , the condition  $\psi(r_0|\hat{v}) = 0$  implies that

$$F(\hat{v}) - F(r_0) = f(r_0) r_0, \quad (3)$$

that is, the optimal reserve price  $r_0(\hat{v})$  equals to the inverse hazard rate at  $r_0(\hat{v})$ .

In particular, the seller can implement the optimal mechanism by means of a first-price auction, as in the following lemma.

**Lemma 1.**

Consider a single-unit, first-price auction with  $M$  bidders, whose valuations are i.i.d. according to a distribution function  $F(\cdot)$  on  $[\underline{v}, \bar{v}]$ . Suppose that the seller and the bidders believe that the unknown valuations are bounded above by the value  $\hat{v} \in [\underline{v}, \bar{v}]$ . Then, given any reserve price  $r$ , the following symmetric strategy profile constitutes the unique equilibrium of the auction:

$$\beta^{2,M}(v_i | \hat{v}, r) = \begin{cases} \mathbb{E}[\max\{v_1^{(M-1)}, r\} | v_1^{(M-1)} < v_i], & \text{if } v_i \geq r, v_i \leq \hat{v}; \\ \mathbb{E}[\max\{v_1^{(M-1)}, r\} | v_1^{(M-1)} < \hat{v}], & \text{if } v_i \geq r, v_i > \hat{v}; \\ a, & \text{if } v_i < r. \end{cases} \quad (4)$$

In this auction, the seller's optimal reserve price is given by equation (2).

If there are two units for sale, then, because of the single-unit demand and the regularity assumption, the solution to the seller's revenue optimization problem is similar to that of the single-unit case. By Maskin and Riley [20], the optimal selling mechanism takes the form of any standard multi-unit auction with reserve price  $r_0 \in [\underline{v}, \bar{v}]$ , defined as in the case of a single-unit auction.

If the two units must be allocated sequentially, as in our setting, then the assumptions of risk neutrality and i.i.d. private valuations imply that all allocation-equivalent equilibria of the sequential and the simultaneous auctions are revenue equivalent. Hence, any sequential auction that allocates the two units to the bidders with the two highest valuations, as long as these valuations exceed the price  $r_0$ , is revenue maximizing.<sup>21</sup>

When the seller can commit to a reserve price  $r_2$  at the beginning of the game, then he can implement the optimal outcome by means of a sequential first-price auction in which the bidders are informed of the first-round winning bid.<sup>22</sup>

**Proposition 2.**

The sequential first-price auction in which the first-round winning bid is revealed and the seller commits to reserve prices  $r_1 = r_2 = r_0$  is revenue maximizing.

Finally, the optimal allocation can be also implemented, in equilibria resulting in the same behavior but different second-round beliefs, by sequential sealed-bid auctions in which the seller reveals only whether the first round resulted in a sale.

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<sup>21</sup>Notice, in particular, that the optimality extends over sequential auctions in which the second-period reserve price is determined endogenously by the first-period bids, according to a reserve price schedule  $r_2 = r_2(b_1^1, \dots, b_N^1, r_1)$ .

<sup>22</sup>The equilibrium strategy profile (and the argument establishing it) is simply a modification of the one described in Krishna [17], Propositions 15.1, so as to account for the reserve price  $r_0$ .

## 4 Sequential Auctions under Non-Commitment

When the seller is unable to commit at the beginning of the game to a reserve price  $r_2$  for the second auction, the equilibrium outcome depends on the information revealed in the first round. We treat each case separately.

### 4.1 Revelation of the Winning Bid

When only the highest bid is revealed at the end of the first auction, for example, when the sale is conducted by means of a Dutch auction, the information inferred from the first-round outcome (assuming that the buyers follow monotone strategies) takes the form of an upper bound for the valuations of the remaining bidders.

We show that the following strategies form an equilibrium:

Given a first-period reserve price  $r_1 \leq \bar{r}_1$ , for a certain threshold  $\bar{r}_1$ , each bidder  $i$  follows a bidding strategy  $\beta^1(\cdot | r_1)$  such that he participates in the auction if and only if his valuation is  $v_i \in [\underline{v}(r_1), \bar{v}]$ , for some value  $\underline{v}(r_1) > r_1$ . In addition, in the region of participation,  $[\underline{v}(r_1), \bar{v}]$ , the strategy  $\beta^1(\cdot | r_1)$  is strictly increasing. Thus, the winning bidder fully reveals his valuation. If  $r_1 > \bar{r}_1$ , then, in equilibrium, no bidder participates in the first auction.

If the first-period object is sold at a price  $\hat{b}^1 = \beta^1(\hat{v} | r_1)$ , corresponding to a winning valuation  $\hat{v} \in [\underline{v}(r_1), \bar{v}]$ , then the seller and the bidders update their beliefs, so that the remaining bidders' valuations  $v_i$  are i.i.d., according to the distribution  $F(\cdot)/F(\hat{v})$ . The seller sets a new reserve price  $r_2$ , according to the updated virtual valuation function; and the remaining bidders bid according to the standard first-price auction strategies. If the first-period object remains unsold at the reserve price  $r_1$ , then the same argument applies, with  $\hat{v} = \underline{v}(r_1)$  being the revealed upper bound for the bidders' valuations.

A bidder with valuation  $\underline{v}(r_1)$  is indifferent between winning the first-period object, at the minimal price  $\beta^1(\underline{v}(r_1) | r_1) = r_1$ , and waiting for the second auction, in which the reserve price will be lower. Finally, those bidders who participate in the first auction shade their bids more than in the second auction.<sup>23</sup>

Notice that any bidder with valuation  $v_i \in (r_1, \underline{v}(r_1))$ , for a given reserve price  $r_1$ , does not participate in the first auction. He prefers to wait for the second auction, even if he can buy the first-period object at price  $r_1$ . This strategic non-participation decision, which also appears in McAfee and Vincent [24] and in Caillaud and Mezzetti [5], is entirely the consequence of the seller's inability to commit not to lower the reserve price. It would not occur, if the seller could commit to a second-period reserve price  $r_2 \geq r_1$ . In particular, it does not occur in the subgame following a first-period reserve price

<sup>23</sup>More precisely, given any reserve price  $r_1 < \bar{r}_1$ , suppose that a bidder  $i$  with valuation  $v_i \in [\underline{v}(r_1), \bar{v}_1]$  loses the first-period auction, after bidding  $b_i^1 = \beta^1(v_i, r_1)$ , to a bidder with valuation  $\hat{v} \in [v_i, \bar{v}]$ , who bids  $\hat{b}^1 = \beta^1(\hat{v}, r_1)$ . In addition, suppose that the second-period reserve price is  $r_2 = r_2(\hat{b}^1, r_1)$ . Then, for all  $\hat{v} \in [v_i, \bar{v}]$ , we have  $\beta^2(v_i | b_i^1, \hat{b}^1, r_1, r_2) > \beta^1(v_i, r_1)$ .

$r_1 = 0$ . Therefore, it does not depend on the bidders' expectation of a smaller number of competing bidders in the future.<sup>24</sup>

We start our formal analysis by investigating the second-period auction. Since we are considering monotone first-period bidding strategies, we can abbreviate the notation for the second-period reserve-price and bidding strategies respectively to  $r_{2,M}(\hat{v})$  and  $\beta^{2,M}(v_i | \hat{v}, r_2)$ , where  $M = N - 1$  or  $N$  is the number of bidders remaining in the sequential auction and  $\hat{v} \in [\underline{v}, \bar{v}]$  is the upper bound for these bidders' valuations, as revealed in the first period.

Both on and off the equilibrium path, the seller's optimal reserve price  $r_{2,M}(\hat{v})$  is given by equation (2); since it does not depend on the number of remaining buyers, we can further abbreviate its notation to  $r_2(\hat{v})$ . The bidders' symmetric equilibrium strategy in the continuation game is given by equation (4) in Lemma 1. In particular, off the equilibrium path, a bidder  $i$  with valuation  $v_i > \hat{v}$  will be best-off behaving as if his valuation is  $\hat{v}$ .<sup>25</sup>

Moving backwards, suppose that the seller has set a first-period reserve price  $r_1$  and consider a bidder with valuation  $v_i \geq \underline{v}(r_1)$ , for some value  $\underline{v}(r_1) \geq r_1$  that will be determined later. Suppose that all other bidders follow the strategies  $(\beta^1, \beta^2)$ , where  $\beta^2$  is as in Lemma 1; and the seller follows the strategy  $r_2$ , again described by Lemma 1. Finally, let

$$\bar{v}(v_i) = \begin{cases} (r_2)^{-1}(v_i), & \text{if } r_2(\bar{v}) < v_i; \\ \bar{v}, & \text{otherwise.} \end{cases} \quad (5)$$

That is, for any first-period winning bid  $\hat{b}_1 \geq \beta^1(\bar{v}(v_i) | r_1)$ , the second-period reserve price will be  $r_2 \geq v_i$ .

Then by mimicking a type  $\tilde{v}_i \in [v_i, \bar{v}(v_i)]$ , bidder  $i$  has an expected payoff

$$\begin{aligned} \Pi[\tilde{v}_i; v_i] &= F(\tilde{v}_i)^{N-1} [v_i - \beta^1(\tilde{v}_i | r_1)] \\ &+ (N-1) [F(\bar{v}(v_i)) - F(\tilde{v}_i)] F(v_i)^{N-2} \left[ v_i - \int_{\tilde{v}_i}^{\bar{v}(v_i)} \frac{\beta^{2,N-1}(v_i | \hat{v}, r_2(\hat{v})) f(\hat{v})}{1 - F(\tilde{v}_i)} d\hat{v} \right]. \end{aligned}$$

Similarly, by mimicking a type  $\tilde{v}_i \in [\underline{v}(r_1), v_i)$ , bidder  $i$  has an expected payoff

<sup>24</sup>This expectation affects only how sharply the bidders shade their bids, not their decision to wait.

<sup>25</sup>Obviously, in the first round, such a bidder either bid  $b_i^1 < \beta^1(\hat{v} | r_1)$  or abstained from the auction.

$$\begin{aligned}
\Pi[\tilde{v}_i; v_i] &= F(\tilde{v}_i)^{N-1} [v_i - \beta^1(\tilde{v}_i | r_1)] \\
&+ (N-1) [1 - F(v_i)] F(v_i)^{N-2} \left[ v_i - \int_{v_i}^{\bar{v}(v_i)} \frac{\beta^{2,N-1}(v_i | \hat{v}, r_2(\hat{v})) f(\hat{v})}{F(\bar{v}(v_i)) - F(v_i)} d\hat{v} \right] \\
&+ [F(v_i)^{N-1} - F(\tilde{v}_i)^{N-1}] \left[ v_i - \int_{\tilde{v}_i}^{v_i} \beta^{2,N-1}(\hat{v} | \hat{v}, r_2(\hat{v})) \frac{(N-1)F(\hat{v})^{N-2} f(\hat{v})}{F(v_i)^{N-1} - F(\tilde{v}_i)^{N-1}} d\hat{v} \right].
\end{aligned}$$

The third term corresponds to the possibility in which the first-period object is sold at a price  $\hat{b}^1 \in (\beta^1(\tilde{v}_i | r_1), \beta^1(v_i | r_1))$ . In this case, the winning bidder reveals the valuation  $\hat{v} = \beta^1(\cdot | r_1)^{-1}(\hat{b}^1) \in (\tilde{v}_i, v_i)$ . Therefore, in the second auction, with reserve price  $r_2(\hat{v})$ , bidder  $i$  bids  $\beta^2[\hat{v} | \hat{v}, r_2(\hat{v})]$ .

In either case, by solving the differential equation that results from the necessary first-order condition at the endpoint  $\tilde{v}_i = v_i$  along with the boundary condition  $\beta^1(\underline{v}(r_1) | r_1) = r_1$ , we get the bidding function

$$\beta^1(v_i | r_1) = \frac{1}{G(v_i)} \left[ \int_{\underline{v}(r_1)}^{v_i} \beta^{2,N-1}(v | v, r_2(v)) g(v) dv + G[\underline{v}(r_1)] r_1 \right], \quad (6)$$

where  $G(v)$  and  $g(v)$  denote respectively the distribution and the density of the highest of the competing  $N-1$  bidders' valuations.

To determine the threshold value  $\underline{v}(r_1)$ , notice that a bidder with valuation  $v_i = \underline{v}(r_1)$  must be indifferent between winning the first-period object, at price  $r_1$ , and strategically waiting for one period, to win the second object. This leads to the following result:

**Lemma 3.**

*In a sequential first-price auction in which the first-round winning bid is revealed, if the strategies  $[(\beta^1, \beta^2), (r_1, r_2)]$  form a symmetric monotone equilibrium, then there is a threshold reserve price*

$$\bar{r}_1 = \beta^{2,N}(\bar{v} | \bar{v}, r_0) \quad (7)$$

*such that for any reserve price  $r_1 \leq \bar{r}_1$ , a bidder participates in the first-period auction if and only if his valuation is  $v_i \geq \underline{v}(r_1)$ , for a value  $\underline{v}(r_1) \in [r_1, \bar{v}]$  defined by*

$$r_1 = \beta^{2,N}[\underline{v}(r_1) | \underline{v}(r_1), r_2(\underline{v}(r_1))]; \quad (8)$$

*whereas for any reserve price  $r_1 > \bar{r}_1$ , no bidder participates in the first-period auction.*

Therefore, no bidder participates in the first-period auction, unless the reserve price  $r_1$  is sufficiently low. In this case, for  $r_1 > \underline{v}$ , there are still some bidder types, those in the

interval  $[r_1, \underline{v}(r_1))$  that prefer to wait for the second period, even though their valuation exceeds the reserve price.

The above arguments lead to the following result, regarding the bidders' behavior in the game following a first-period reserve price  $r_1$ :

**Proposition 4.**

*Consider a sequential first-price auction in which the first-round winning bid is revealed. Suppose that the seller has set a reserve price  $r_1 \in \mathbb{R}^+$ . Then in the unique symmetric monotone equilibrium for the continuation game:*

- *The strategies  $r_2$ ,  $\beta^{2,N}$  and  $\beta^{2,N-1}$  are described by the equations (2) and (4).*
- *If  $r_1 \leq \bar{r}_1$ , a bidder submits a bid in the first auction,*

$$\beta^1(v_i | r_1) = \frac{1}{G(v_i)} \left[ \int_{\underline{v}(r_1)}^{v_i} \beta^{2,N-1}(v | v, r_2(v)) g(v) dv + G[\underline{v}(r_1)] r_1 \right],$$

*if his valuation is  $v_i \in [\underline{v}(r_1), \bar{v}]$ . Otherwise, he abstains from it.*

- *If  $r_1 > \bar{r}_1$ , then all bidders abstain from the first auction.*

We can now consider the seller's problem of determining the optimal first-period reserve price  $r_1^*$ . Since there is a bijective relation between a reserve price  $r_1 \in [\underline{v}, \bar{r}_1]$  and the participation threshold  $v(r_1) \in [\underline{v}, \bar{v}]$ , namely,<sup>26</sup>

$$r_1(v) = \beta^{2,N}(v | v, r_2(v)),$$

we can think of the seller's problem as one of determining the revenue maximizing first-period participation threshold  $v^* = \underline{v}(r_1^*)$ .

**Proposition 5.**

*The optimal first-period participation threshold  $v^*$  in a sequence of two Dutch auctions is the unique solution to the equation*

$$\frac{1 - F(v)}{f(v)} G[r_0(v)] \frac{dr_0}{dv}(v) = \int_{r_0(v)}^v \psi(u) g(u) du. \quad (9)$$

*The seller always induces participation by a positive measure of bidders' types, that is,  $v^* < \bar{v}$ . In addition, for distributions  $F(\cdot)$  such that  $r_0 > \underline{v}$ , we have  $r_1(v^*) < r_0 < v^*$ . Finally, for distributions  $F(\cdot)$  such that  $r_0 = \underline{v}$ , we have  $r_1(v^*) = r_0 = v^* = \underline{v}$ .*

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<sup>26</sup>For a function  $\Phi(x) = F[f_1(x), f_2(x)]$ , we have  $\Phi' = F_1 f_1' + F_2 f_2'$ . Hence, if the derivatives  $F_1$ ,  $f_1'$ ,  $F_2$  and  $f_2'$  are all positive, then the function  $\Phi(x)$  is increasing.

It is interesting to notice that the seller's choice of the second-period reserve price  $r_2 = r_2(\hat{b}^1, r_1)$  is better informed under non-commitment. Therefore, he expects a greater revenue in the second auction than in the case of commitment to a reserve price  $r_1 = r_2 = r_0$ . This gain, however, is dominated by the seller's loss in the first period, so that, overall, it is more profitable for him to commit to  $r_0$ . The two revenues are equal only when  $r_0 = \underline{v}$ , which occurs when  $\psi(\underline{v}) \geq 0$ .

The revenue loss in the first period, for  $r_1 > \underline{v}$ , originates from the non-participation decision of the buyers with valuations in  $[r_1, \underline{v}(r_1))$ . First, because of this decision, it is more likely that the first unit remains unsold. Second, the participating bidders shade their bids more, knowing that they face fewer competing types, therefore, sales are made in lower prices. When all bidders participate, following  $r_1 = \underline{v}$ , the seller suffers no loss.

The following result, which follows directly from the characterization of the optimal selling mechanism under commitment and the failure of the optimal non-commitment mechanism to implement the same allocation, expresses the seller's benefit from committing to a reserve price schedule when this is possible:

**Corollary 6.**

*Suppose that the bidders' lowest virtual valuation is  $\psi(\underline{v}) < 0$ . Then, in a sequential first-price auction in which the first-round winning bid is revealed, the seller's revenue is strictly greater under commitment to reserve prices  $r_1 = r_2 = \psi^{-1}(0)$  than in any reserve-price scheme under non-commitment.*

We demonstrate our results in the case of uniformly distributed valuations.

**Example:**

Suppose that the bidders' valuations are uniformly distributed on  $[0,1]$ . In the second round, if the reserve price is  $r_2$ , the revealed upper bound for the bidders' valuations is  $\hat{v}$  and there are  $M$  competing bidders, each bidder acts according to the bidding strategy described by equation (4):

$$\beta^{2,M}(v_i | \hat{v}, r_2) = \begin{cases} \frac{1}{M v_i^{M-1}} [r_2^M + (M-1)v_i^M], & \text{if } r_2 \leq v_i \leq \hat{v}; \\ \frac{1}{M \hat{v}^{M-1}} [r_2^M + (M-1)\hat{v}^M], & \text{if } v_i > \hat{v}; \\ a, & \text{if } v_i < r_2. \end{cases}$$

In this auction, it is optimal for the seller to set a reserve price

$$r_2(\hat{v}) = \frac{1}{2} \hat{v}.$$



In the first auction, there will be a positive measure of participating bidder types if and only if the reserve price is  $r_1 < \bar{r}_1$ , where  $\bar{r}_1 \in [0, 1]$  is given by the equation (7):

$$\bar{r}_1 = \beta^{2,N}(1 | 1, r_2(1)) \implies \bar{r}_1 = \frac{N - 1 + (1/2)^N}{N}.$$

Otherwise, all bidders will wait for the second period.

When  $r_1 < \bar{r}_1$ , bidders with valuations  $v_i \geq \underline{v}(r_1)$  bid according to the strategy

$$\beta^1(v_i | r_1) = \frac{1}{v_i^{N-1}} \int_{\underline{v}(r_1)}^{v_i} r_2(v)^{N-1} + (N-2)v^{N-1} dv + \frac{\underline{v}(r_1)^{N-1}}{v_i^{N-1}} r_1,$$

given by equation (6), while bidders with valuations  $v_i < \underline{v}(r_1)$  abstain from the auction.

By imposing the indifference condition (8) defining the type  $v_i = \underline{v}(r_1)$ , we get

$$r_1 = \beta^{2,N}[\underline{v}(r_1) | \underline{v}(r_1), r_2(\underline{v}(r_1))] \implies \underline{v}(r_1) = \frac{N}{N - 1 + (1/2)^N} r_1 = \frac{r_1}{\bar{r}_1}.$$

Clearly  $\underline{v}(r_1) > r_1$ , because of the possibility of a lower second-round reserve price. In addition,  $\underline{v}(\bar{r}_1) = 1$ , according to the definitions (8) and (7) of  $\underline{v}(r_1)$  and  $\bar{r}_1$ .

By using the expressions for  $r_2(\bar{v})$  and  $\underline{v}(r_1)$  that we derived above, we can simplify the function describing the first-period strategy of a bidder with valuation  $v_i \geq \underline{v}(r_1)$ :

$$\beta^1(v_i | r_1) = \left( \bar{r}_1 - \frac{1 - 2^{-N}}{N} \right) v_i + \frac{1 - 2^{-N}}{N} \left( \frac{r_1}{\bar{r}_1} \right)^N \frac{1}{v_i^{N-1}}.$$

Finally, suppose that the seller chooses a reserve price  $r_1 \leq \bar{r}_1$  corresponding to a participation threshold  $v = \underline{v}(r_1) = r_1/\bar{r}_1$ . Then his expected payoff will be

$$\begin{aligned} R(v) &= \int_v^1 N \left( \bar{r}_1 - \frac{1 - 2^{-N}}{N} \right) v_1^N + (1 - 2^{-N}) v^N dv_1 \\ &+ \int_{v/2}^v \left( \frac{v}{2} \right)^N + (N-1) v_1^N dv_1 \\ &+ \int_v^1 \int_{v_1/2}^{v_1} N \left[ \left( \frac{v_1}{2} \right)^{N-1} + (N-2) v_2^{N-1} \right] dv_2 dv_1. \end{aligned}$$

The necessary condition (9) from Proposition 5 yields:

$$(1 - v) v^{N-1} = 2^N \int_{v/2}^v (2u - 1) (N - 1) u^{N-2} du.$$

Therefore, the seller maximizes his expected revenue by inducing a first-period participation threshold or, equivalently, by setting a first-period reserve price

$$v^* = \frac{N(2^N - 1)}{2(N-1)(2^N - 1) + N} \iff r_1^* = \frac{N(2^N - 1)}{2(N-1)(2^N - 1) + N} \bar{r}_1.$$

Since  $v^* < 1$ , the seller is always willing to sell the first-period object. In fact, the first-period reserve price will be  $r_1^* < \frac{1}{2}$ , the optimal reserve price under commitment. However, since  $v^* > \frac{1}{2}$ , there are strictly fewer bidder types participating in the first auction than in the optimal sequential auction under commitment. □

Finally, similar arguments can be used to construct symmetric monotone equilibria  $[(\beta^1, \beta^2), (r_1, r_2)]$  in sequences of other standard auctions in which only the first-round winning bid is revealed. It is interesting to notice that all such equilibria result in the same (optimal) seller revenue.

For any reserve price  $r_1$ , assuming that the bidding strategy  $\beta^1(\cdot | r_1)$  is monotone, the first-round winning bid reveals the total number of bidders,  $N$  or  $N - 1$ , participating in the second round and an upper bound  $\hat{v} \in [r_1, \bar{v}]$  for their valuations.

In the second round, given the upper bound  $\hat{v}$ , it is optimal for the seller to set a reserve price  $r_2 = r_0(\hat{v})$ , according to equation (2), independently of the number of remaining bidders and the particular (standard) auction format. Therefore, in the second round while facing  $M - 1$  competitors, bidder  $i$  with valuation  $v_i \in [r_0(\hat{v}), \hat{v}]$  expects to pay, conditional on winning,

$$m_{2w}^M(v_i | \hat{v}) = \mathbb{E}[\max\{v_1^{(M-1)}, r_0(\hat{v})\} | v_1^{(M-1)} \leq v_i].$$

In the first round of the auction, for any reserve price  $r_1 \in [\underline{v}, \bar{v}]$ , the lowest participating type,  $\underline{v}(r_1) \in [r_1, \bar{v}]$ , will be such that

$$r_1 = m_{2w}^N(\underline{v}(r_1) | \underline{v}(r_1)),$$

that is, he will be indifferent between winning the first unit at price  $r_1$  or the second unit at price  $m_{2w}^N(\underline{v}(r_1) | \underline{v}(r_1))$ . In particular, for any reserve price  $r_1 > \bar{r}_1$ , where

$$\bar{r}_1 = m_{2w}^N(\bar{v} | \bar{v}),$$

no bidder type will bid in the first auction.

For any reserve price  $r_1$ , therefore, independently of the particular auction format, the continuation game results in the same expected allocation; and by revenue equivalence, in the same expected seller payoff. Hence, the seller obtains the same revenue in all sequential standard auctions in which only the winning bid is revealed.

## 4.2 Sale / Non-sale

The revenue comparison in Corollary 6 suggests that a seller who cannot commit not to change the reserve price may be better off acquiring less information about the outcome of the first auction.<sup>27</sup> In this manner, by making his adjustment of the second-period reserve price less flexible, the seller may succeed in eliminating some of the bidders' first-period incentives to wait. However, this turns out not to be necessarily true.

When the seller observes only whether the first item is sold, the buyers' incentives to wait for a lower reserve price remain unaffected. Indeed, for any reserve price  $r_1 \in [\underline{v}, \bar{v}]$ , the lowest type participating in the first auction,  $\underline{v}(r_1)$ , must be indifferent between winning the first unit at price  $r_1$  and waiting for the second auction, in which he can win the second unit at price  $\beta^2(\underline{v}(r_1) | a, ns, r_1, r_2(r_1, ns))$ ; that is,

$$r_1 = \beta^2(\underline{v}(r_1) | a, ns, r_1, r_2(r_1, ns)).$$

Since both the seller and the buyers act in the second auction in the manner described by Lemma 1, with beliefs  $\hat{v} = \underline{v}(r_1)$ , the last condition reduces to equation (8). Therefore, independently of whether the seller observes the first-round winning bid or only whether the first unit has been sold, a reserve price  $r_1$  results in the same set of participating bidder types.

On the other hand, by not observing the winning bid when the first unit is sold, the seller fails to acquire useful information. In this case, the new reserve price that he sets,  $r_2(r_1, s) \equiv r_0(\bar{v}) = r_0$ , is less informed than the one he would set,  $r_2(\hat{v})$ , if he could infer the winner's valuation  $\hat{v}$ . Therefore, in the second round, the seller suffers revenue loss.

Finally, the only gains for the seller come from the bidding behavior in the first round. If the reserve price  $r_1$  is such that  $\underline{v}(r_1) \in [r_0, \bar{v}]$ , then, for each valuation  $v_i \geq \underline{v}(r_1)$ ,

$$\beta^1(v_i | r_1) = \frac{1}{G(v_i)} \left[ \int_{\underline{v}(r_1)}^{v_i} \beta^{2, N-1}(v | \bar{v}, r_0) g(v) dv + G[\underline{v}(r_1)] r_1 \right]. \quad (10)$$

In addition, for any reserve price  $r_1$  such that  $\underline{v}(r_1) < r_0$ , any buyer with valuation  $v_i \in [\underline{v}(r_1), r_0]$  knows that if he fails to win the first round, he will face a second-period reserve price above their valuation. Therefore, he shades his bid less:

$$\beta_L^1(v_i | r_1) = \frac{1}{G(v_i)} \left[ \int_{\underline{v}(r_1)}^{v_i} v g(v) dv + G[\underline{v}(r_1)] r_1 \right]. \quad (11)$$

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<sup>27</sup>In particular, if the seller could observe no information about the first round, then the commitment outcome would be replicated.

In this case, bidders with valuations  $v_i \geq r_0$  shade their bids more:

$$\beta_H^1(v_i | r_1) = \frac{1}{G(v_i)} \left[ \int_{r_0}^{v_i} \beta^{2,N-1}(v | \bar{v}, r_0) g(v) dv + G(r_0) \beta_L^1(r_0 | r_1) \right]. \quad (12)$$

For any reserve price  $r_1$ , since  $r_0 \geq r_2(v)$  for all  $v \in [\underline{v}, \bar{v}]$ , the participating bidders bid more aggressively than in the case in which the seller observes the winning bid.

First, we describe formally the bidders' participation in the first auction.

**Lemma 7.**

*In the sequential first-price auction in which the seller and the buyers observe only whether the first item is sold, if the strategies  $[(\beta^1, \beta^2), (r_1, r_2)]$  form a symmetric monotone equilibrium, then there is a threshold  $\bar{r}_1 \in [\underline{v}, \bar{v}]$ , defined by equation (7), such that a bidder  $i$  participates in the first-period auction if and only if  $r_1 \leq \bar{r}_1$  and  $v_i \geq \underline{v}(r_1)$ , where  $\underline{v}(r_1) \in [r_1, \bar{v}]$  is defined by equation (8).*

The buyers' bidding behavior is described in the following result:

**Proposition 8.**

*In the sequential first-price auction in which the seller and the buyers observe only whether the first item is sold, following a reserve price  $r_1 \in \mathbb{R}^+$ , there is a unique symmetric monotone equilibrium for the continuation game, such that*

- *If  $\underline{v}(r_1) \geq r_0$ , in the first auction, the participating bidders bid according to the strategy  $\beta^1$ , defined by equation (10).*
- *If  $\underline{v}(r_1) < r_0$ , in the first auction, the participating bidders bid according to the strategies  $\beta_L^1$  and  $\beta_H^1$ , defined by the equations (11) and (12).*
- *Following a sale, the players' second-period beliefs are  $\hat{v} = \bar{v}$ , for all  $r_1 \in \mathbb{R}^+$ . Following no sale, the players' second-period beliefs are  $\hat{v} = \underline{v}(r_1)$ , for  $r_1 \leq \bar{r}_1$ ; and  $\hat{v} = \bar{v}$ , for  $r_1 > \bar{r}_1$ .*
- *The players' second-period strategies are given by Lemma 1 and the beliefs  $\hat{v}$ .*

Finally, the seller's gains from inducing more aggressive bidding in the first round are dominated by his losses from setting a less informed reserve price in the second round. Thus, by observing only whether the first item is sold, the seller suffers a revenue loss.<sup>28</sup>

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<sup>28</sup>Recall that when the seller can commit to the optimal reserve price, the information revelation policy has no effect upon his expected revenue, a difference from the case of non-commitment.

**Proposition 9.**

In the sequential first-price auction in which the seller and the buyers observe only whether the first item is sold, the participation threshold  $v^* = \underline{v}(r_1^*)$  induced by the seller's optimal first-period reserve price  $r_1^*$  is the unique solution to the equation

$$\frac{1 - F(v)}{f(v)} G[r_0(v)] \frac{dr_0}{dv}(v) = \int_{r_0}^v \psi(u) g(u) du. \quad (13)$$

Unless  $r_0 = \underline{v}$ , we have  $v^* > r_0$ ; and the seller's expected revenue is strictly lower than that in the sequential auction in which the players observe the first-period winning bid.

We demonstrate the above results in the case of uniform valuations, as in section 4.1.

**Example:**

Suppose that the bidders' valuations are uniformly distributed on  $[0,1]$  and that the auctioneer reveals only whether the first unit was sold. In the second round, the bidders' and the seller's strategies depend only on the number  $M \in \{N - 1, N\}$  of the buyers that remain in the auction and the revealed upper bound  $\hat{v} \in \{\bar{v}, \underline{v}(r_1)\}$ , in a manner that is identical to that of the sequential auction in which the winning bid is revealed. Therefore, the seller will set  $r_2 = r_2(\hat{v})$  and each buyer  $i$  will bid  $b_i^2 = \beta^{2,M}(v_i | \hat{v}, r_2)$ .

In the first period, the threshold price  $\bar{r}_1$  determining whether some buyer types may participate in the auction is the same as when the first-round winning bid is revealed. In addition, for any reserve price  $r_1 \leq \bar{r}_1$ , the lowest participating type is also the same; therefore,  $\underline{v}(r_1) = r_1/\bar{r}_1$ .

For any reserve price  $r_1 \leq \bar{r}_1$  such that  $\underline{v}(r_1) \geq r_0$ , buyers with valuations  $v_i \geq \underline{v}(r_1)$  bid according to the strategy

$$\beta^1(v_i | r_1) = \frac{1}{v_i^{N-1}} \int_{\underline{v}(r_1)}^{v_i} r_0^{N-1} + (N-2)v^{N-1} dv + \frac{\underline{v}(r_1)^{N-1}}{v_i^{N-1}} r_1,$$

while buyers with valuations  $v_i < \underline{v}(r_1)$  abstain from the auction.

For a reserve price  $r_1 \leq \bar{r}_1$  corresponding to a participation threshold  $v = \underline{v}(r_1) \geq r_0$ , the seller's expected payoff will be

$$\begin{aligned} R(v) &= \int_v^1 (N-2)(v_1^N - v^N) + N \left(\frac{1}{2}\right)^{N-1} (v_i - v) + N v^N \bar{r}_1 dv_1 \\ &+ \int_{v/2}^v \left(\frac{v}{2}\right)^N + (N-1)v_1^N dv_1 \\ &+ \int_v^1 \int_{1/2}^{v_1} N \left[ \left(\frac{1}{2}\right)^{N-1} + (N-2)v_2^{N-1} \right] dv_2 dv_1. \end{aligned}$$

The necessary condition (13) yields:

$$(1 - v) v^{N-1} = 2^N \int_{1/2}^v (2u - 1) (N - 1) u^{N-2} du.$$

Solving numerically the resulting  $N^{\text{th}}$ -degree polynomial, for  $N = 6$ , yields the optimal participation threshold  $v_s^* = 0.631$ , which corresponds to the reserve price  $r_s^* = 0.528$  and the expected seller payoff  $R_s^* = 1.678$ . If the seller observed the winning bid, the corresponding values would be  $v_1^* = 0.571 < v_s^*$  and  $r_1^* = 0.477 < r_s^*$ , with an expected payoff  $R_1^* = 1.719 > R_s^*$ . Therefore, the seller is better off observing the winning bid.

□

### 4.3 Revelation of all Bids

In the absence of commitment, the seller is able to use in an unrestricted manner any information about the bidders' valuations that the first auction may reveal. For example, if a non-winning bidder is revealed to have a valuation  $v_i \geq v_L$ , then the seller will set a second-period reserve price  $r_2 \geq v_L$ . The bidders, therefore, have a strong incentive to conceal their valuations during the first auction. Non-surprisingly, this incentive leads to strong negative results.

We present these results assuming that the seller observes all bids while the buyers are informed of only the winning bid. If the buyers were also to observe all bids, then, even under commitment, as shown in Cai *et al.* [4], a perfect Bayesian equilibrium in symmetric monotone bidding strategies would fail to exist, because of the bidders' incentive to conceal information from one another. Our assumption allows us to concentrate on the bidders' incentive to conceal information from the seller.

**Proposition 10.**

*In the sequential first-price auction in which the seller observes all first-period bids, there does not exist any symmetric perfect Bayesian equilibrium in weakly increasing first-period bidding strategies.*

This non-existence result<sup>29</sup> originates from the asymmetric effects of the marginal deviations to  $\beta^1(\tilde{v}_i)$ , for  $\tilde{v}_i \gtrsim v_i$  and  $\tilde{v}_i \lesssim v_i$ . In particular, a deviation to mimicking a type  $\tilde{v}_i \lesssim v_i$  has a positive marginal effect upon the bidder's payoff, the second-period gain from misinforming the seller, that is not present when a bidder deviates to mimicking a

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<sup>29</sup>Notice that if we adopted a weaker notion of perfect Bayesian equilibrium, one that would not impose any restriction upon the players beliefs off the equilibrium path, then an equilibrium would exist in a rather pathological manner. In particular, on the equilibrium path, the seller would set  $r_1 = \bar{v}$ , so that effectively only the second-period auction, with  $r_2 = r_0 = \psi^{-1}(0)$ , would take place. Off the equilibrium path, for  $r_1 < \bar{v}$ , we could allow each bidder  $i$  to believe that all other bidders' valuations are  $v_j = \bar{v}$ ; thus, all bidders would abstain from the first auction. Therefore, the seller could not benefit from lowering the reserve price  $r_1$ .

type  $\tilde{v}_i \gtrsim v_i$ . Hence, if the bidders shade their first-period bids as much as it is necessary to avoid deviations to  $\tilde{v}_i \lesssim v_i$ , then the deviation to  $\tilde{v}_i \gtrsim v_i$  becomes strictly profitable.

We now consider a variation of the sequential auction that we just analyzed, one in which the seller commits not to change the second-period reserve price; however, he can do so only in a manner that is not fully credible.

At the beginning of the game, the seller announces a reserve price  $r_1 \in [\underline{v}, \bar{v}]$ . After the end of the first auction, the seller observes the submitted bids and, with some commonly known probability  $\rho > 0$ , which reflects the seller's lack of credibility, he can set a new reserve price  $r_2 = r_2(b^1, r_1)$  for the second auction; otherwise, with probability  $1 - \rho$ , we have  $r_2 = r_1$ . We assume that the seller learns his type, whether he can change the reserve price or not, only after the end of the first auction; therefore, his choice of  $r_1$  does not convey any information to the bidders.

One might hope that for  $\rho \approx 0$  there can exist symmetric monotone equilibrium. As the following result shows, this turns out not to be possible.

**Proposition 11.**

*Under imperfect commitment, in the sequential first-price auction in which the seller observes all first-period bids, there does not exist any symmetric perfect Bayesian equilibrium in weakly increasing first-period bidding strategies.*

On the other hand, when the seller observes only the first-period winning bid or whether the first item is sold, it is easy to show that there exists a sequential equilibrium demonstrating similar characteristics to the ones of the equilibrium described in the case of non-commitment. Therefore, the two cases, of imperfect commitment and of non-commitment, are qualitatively identical.

In conclusion, however small the seller's lack of credibility may be, the buyers will be concerned about the information their first-round bids reveal. Because of this concern a symmetric monotone equilibrium exists only when the seller restricts the amount of information he can observe. In the absence of sufficiently strict legal assurances or other means of establishing credibility<sup>30</sup>, the use of a Dutch auction (so that only the winning bid is revealed), or of a single-round, multi-unit auction, is the only manner in which the seller can induce a positive outcome.

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<sup>30</sup>For example, the seller might try to trade through a well-established auction house. One can assume that reputational concerns may be more important for such an institution than for a single individual.

## 5 Conclusions

In a sequential auction in which the buyers have single-unit demands and the seller is unable to commit perfectly to the auction rules, we have shown that a symmetric monotone equilibrium exists only when the non-winning bidders can securely hide their valuations, for example, in a sequence of two Dutch auctions. Our result complements the analysis of Caillaud and Mezzetti [5], in which the multi-unit demands, along with the persistent valuations, place the problem of concealing one's valuation upon the winning bidder and, therefore, force the use of the English auction. In both settings, information can be revealed only indirectly, through the actions of the bidders that do not have any further interest in the game.

In the absence of commitment, we show that the seller can gain in the second round, by choosing the reserve price in a more informed manner; but he loses more during the first round, since some bidders prefer not to participate, thus triggering a decrease in the intensity of the competition among the participating bidders. Therefore, non-commitment results in lower seller revenue.

The seller's gains and losses in the two auction rounds exhibit characteristics that are similar to those observed by Mezzetti *et al.* [25] in a sequential second-price auction with interdependent valuations and affiliated signals. In particular, in both settings, the seller gains in the second round<sup>31</sup> because of the information revealed earlier. Even though the mechanisms generating these revenue effects are quite different,<sup>32</sup> it would be interesting to combine them, by examining a sequential auction with affiliated private values and no commitment. In such a setting, the information revealed in the first round will be of higher value for the seller, so that, in the end, his second-round gains may dominate his first-round losses, countering the result in this paper.

In addition, we have examined what happens when the seller observes only whether the first auction has resulted in a sale. Contrary to Weber [35], who shows that this restriction in the information revealed is of no consequence when the seller is strategically inactive, we have found that under non-commitment, this policy results in revenue loss.

When the non-winning bidders must reveal information about their valuations, such as when a sealed-bid auction is used, we have shown that there is no symmetric monotone equilibrium, however small the imperfection of the seller's commitment is. In this case, it would be interesting to know the equilibrium bidding behavior. The same problem arises in environments in which all bidders try to hide their valuations, independently of whether they win the first object or not, for example, in a sequential auction with multi-unit demands and non-persistent valuations or decreasing marginal returns. Again, in equilibrium, the bidders need to mix their strategies.

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<sup>31</sup>In this paper, relative to commitment; in Mezzetti *et al.* [25], relative to no information revelation.

<sup>32</sup>The effects studied by Mezzetti *et al.* [25] would be absent in an IPV setting, like the one we study; whereas the effects in this paper depend on the presence of a strategically active seller.



Finally, one could study a general sequential allocation problem in which the seller can select, in each period, any single-object selling mechanism. When a seller has only one unit, which he can offer to the buyers again in period 2 if he does not sell it in period 1, Skreta [31], generalizing McAfee and Vincent [24], shows that the optimal mechanism takes the form of a sequence of standard auctions with reserve prices that decline over time, according to the seller's updated beliefs. Prior to the conclusion of the sale, this mechanism reveals no information other than the buyers' refusal to bid above the reserve price. In our setting, it would be interesting to know whether the seller can profit from using a first-period mechanism that extracts information from the non-winning bidders.

## Appendix: Proof of Results

### Proof of Lemma 3:

In any symmetric monotone equilibrium, the strategies  $\beta^2$  and  $r_2$  must be as described in Lemma 1; and the strategy  $\beta^1(\cdot | r_1)$  must be as in equation (6), derived from the necessary FOC.

First, for any reserve price  $r_1 \leq \bar{r}_1$ , we show that no bidder with valuation  $v_i \geq \underline{v}(r_1)$ , where the threshold value  $\underline{v}(r_1)$  is defined by equation (8), can profit from abstaining from the first auction; and that no bidder with valuation  $v_i < \underline{v}(r_1)$  can profit from participating in the first auction.

Suppose that bidder  $i$  has valuation  $v_i \geq \underline{v}(r_1)$ . Since  $\Pi[v_i, v_i] > \Pi[\underline{v}(r_1), v_i]$ , it suffices to show that this bidder's payoff from abstaining does not exceed his payoff from mimicking the type  $\underline{v}(r_1)$  in the first auction. Clearly, these two payoffs differ only if all other bidders' valuations are below  $\underline{v}(r_1)$ . In this case, in the second period, it is optimal for bidder  $i$  to bid

$$\beta^{2,N}[\underline{v}(r_1) | \underline{v}(r_1), r_2(\underline{v}(r_1))] = r_1 = \beta^1(\underline{v}(r_1) | r_1). \quad (14)$$

Therefore, bidder  $i$  is indifferent between abstaining from the first auction and participating with a bid  $\beta^1(\underline{v}(r_1), r_1)$ ; and he is better-off participating with a bid  $\beta^1(v_i | r_1)$ , as prescribed.

Conversely, suppose that bidder  $i$  has valuation  $v_i < \underline{v}(r_1)$ . Since  $\Pi[\tilde{v}_i, v_i] < \Pi[\underline{v}(r_1), v_i]$ , for all  $\tilde{v}_i > \underline{v}(r_1)$ , it suffices to show that this bidder's payoff from mimicking the type  $\underline{v}(r_1)$  does not exceed that from abstaining. His gain from such a deviation is

$$\begin{aligned} \Pi[v_i, v_i] - \Pi[\underline{v}(r_1), v_i] &= F(v_i)^{N-1} [v_i - \beta^{2,N}[v_i | \underline{v}(r_1), r_2(\underline{v}(r_1))]] \\ &\quad - F(\underline{v}(r_1))^{N-1} [v_i - \beta^1(\underline{v}(r_1) | r_1)], \end{aligned}$$

and because of equation (14),

$$\begin{aligned} \Pi[v_i, v_i] - \Pi[\underline{v}(r_1), v_i] &= F(v_i)^{N-1} [v_i - \beta^{2,N}[v_i | \underline{v}(r_1), r_2(\underline{v}(r_1))]] \\ &\quad - F(\underline{v}(r_1))^{N-1} [v_i - \beta^{2,N}[\underline{v}(r_1) | \underline{v}(r_1), r_2(\underline{v}(r_1))]]. \end{aligned}$$

Therefore, bidder  $i$  gains from mimicking the type  $\underline{v}(r_1)$  in the first round equal those from mimicking that type in the second round, following no sale. Since the strategy  $\beta^{2,N}$  constitutes an equilibrium for the continuation game, this deviation is not profitable.

Finally, for any reserve price  $r_1 > \bar{r}_1$ , no bidder submits a first-period bid. Therefore, any bidder can win the first-period object at price  $r_1$ . Since

$$r_1 > \bar{r}_1 = \beta^{2,N}(\bar{v} | \bar{v}, r_2(\bar{v})),$$

winning the first-period object results in a lower payoff than abstaining from the first auction and mimicking, in the second auction, the type  $\bar{v}$ . Again, since the strategy  $\beta^{2,N}$  constitutes an equilibrium for the second auction, following no sale, this deviation is not profitable. □

**Proof of Proposition 4:**

Suppose that all bidders other than bidder  $i$  follow the strategy  $(\beta^1, \beta^2)$  described by equations (4) and (6); while the seller follows the strategy  $r_2$ , described by equation (2). We will show that bidder  $i$  with valuation  $v_i \in [\underline{v}, \bar{v}]$  is best-off also bidding according to the strategies  $(\beta^1, \beta^2)$ .

During the second round, in all continuation games (in which bidder  $i$  has not won the first unit), by Lemma 1, bidder  $i$  cannot profit from deviating from the strategy  $\beta^2$ . So, it suffices to establish the optimality of  $\beta^1$ .

Suppose that  $r_1 < \bar{r}_1$  and consider a bidder  $i$ , with valuation  $v_i \in [\underline{v}(r_1), \bar{v}]$ .

First, we show that he cannot profit from deviating to bidding  $\beta^1(\tilde{v}_i, r_1)$ , for  $\tilde{v}_i \in (v_i, \bar{v}(v_i))$ , where  $\bar{v}(v_i)$  is defined by equation (5). His gain from such a deviation is

$$\begin{aligned} \Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) &= [F(\tilde{v}_i)^{N-1} - F(v_i)^{N-1}] v_i \\ &- (N-1) [F(\tilde{v}_i) - F(v_i)] F(v_i)^{N-1} v_i \\ &- \int_{v_i}^{\tilde{v}_i} \beta^{2,N-1}(\hat{v} | \hat{v}, r_2(\hat{v})) (N-1) F(\hat{v})^{N-2} f(\hat{v}) d\hat{v} \\ &+ \int_{v_i}^{\tilde{v}_i} \beta^{2,N-1}(v_i | \hat{v}, r_2(\hat{v})) (N-1) F(v_i)^{N-2} f(\hat{v}) d\hat{v} \end{aligned}$$

or, after using equation (4), defining the strategy  $\beta^{2,N-1}$ ,

$$\begin{aligned} \Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) &= [F(\tilde{v}_i)^{N-1} - F(v_i)^{N-1}] v_i \\ &- (N-1) [F(\tilde{v}_i) - F(v_i)] F(v_i)^{N-1} v_i \\ &- \int_{v_i}^{\tilde{v}_i} \int_{v_i}^u u (N-1)(N-2) F(u)^{N-3} f(u) f(\hat{v}) du d\hat{v} \end{aligned}$$

Since the variable  $u$  in the last term is  $u \geq v_i$ , it follows that

$$\Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) \geq 0,$$

as required.

Second, a direct calculation shows that

$$\Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) = 0,$$

for all  $\tilde{v}_i \in (\underline{v}(r_1), v_i)$ , so that deviations to mimicking a weaker type are not profitable.

Finally, when  $(\bar{v}(v_i) < \bar{v})$ , to rule out deviations to bidding  $\beta^1(\tilde{v}_i, r_1)$ , for  $\tilde{v}_i \in (\bar{v}(v_i), \bar{v}]$ , notice that

$$\Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) = [\Pi(\tilde{v}_i; v_i) - \Pi(\bar{v}(v_i); v_i)] + [\Pi(\bar{v}(v_i); v_i) - \Pi(v_i; v_i)].$$

Since the second summand has been shown to be negative, it suffices to show that so is the first summand. Notice that

$$\begin{aligned} \Pi(\tilde{v}_i; v_i) - \Pi(\bar{v}(v_i); v_i) &= [F(\tilde{v}_i)^{N-1} - F(\bar{v}(v_i))^{N-1}] v_i \\ &\quad - \int_{\bar{v}(v_i)}^{\tilde{v}_i} \beta^{2, N-1}(\hat{v} | \hat{v}, r_2(\hat{v})) (N-1) F(\hat{v})^{N-2} f(\hat{v}) d\hat{v} \\ &\leq [F(\tilde{v}_i)^{N-1} - F(\bar{v}(v_i))^{N-1}] [v_i - r_2(\bar{v}(v_i))] \end{aligned}$$

Since for all  $\hat{v} \geq \bar{v}(v_i)$ , we have  $r_2(\hat{v}) \geq v_i$ , we can conclude that

$$\Pi(\tilde{v}_i; v_i) - \Pi(\bar{v}(v_i); v_i) \leq 0$$

as required.

To rule out the two remaining deviations, that of abstaining for types  $v_i \geq \underline{v}(r_1)$  and that of bidding when either  $v_i < \underline{v}(r_1)$  or  $r_1 \geq \bar{r}_1$ , we can show, by arguing as above, that, for all  $r_1 < \bar{r}_1$  and  $v_i \leq \underline{v}(r_1) \leq \tilde{v}_i$ ,

$$\Pi(\underline{v}(r_1); v_i) \geq \Pi(\tilde{v}_i; v_i).$$

The argument follows that in the proof of Lemma 3, using our earlier results regarding non-profitable deviations whenever necessary.

Finally, the uniqueness of the constructed equilibrium follows from Lemmata 1 and 3 and the necessity of the FOC establishing the strategy  $\beta^1$ , that is, of equation (6).

□

**Proof of Proposition 5:**

Notice that the seller's revenue from setting a first-period reserve price  $r_1 > \bar{r}_1$  is equal to that from setting  $r_1 = \bar{r}_1$ . We can therefore reduce attention to reserve prices  $r_1 \in [\underline{v}, \bar{r}_1]$ .

By setting a first-period reserve price  $r_1 \in [\underline{v}, \bar{r}_1]$ , corresponding to a participation threshold  $v = v(r_1) \in [\underline{v}, \bar{v}]$ , the seller expects a revenue

$$\begin{aligned} R(v) &= \int_v^{\bar{v}} \beta^1(v_1 | r_1(v)) N f(v_1) G(v_1) dv_1 \\ &+ \int_{r_2(v)}^v \beta^{2,N}[v_1 | v, r_2(v)] N f(v_1) G(v_1) dv_1 \\ &+ \int_v^{\bar{v}} \int_{r_2(v_1)}^{v_1} \beta^{2,N-1}[v_2 | v_1, r_2(v_1)] N f(v_1) g(v_2) dv_2 dv_1. \end{aligned}$$

The first two integrals correspond to the payment of the buyer with the highest valuation, when he wins either the first or the second of the two auctions, while the third integral corresponds to the payment of the bidder with the second-highest valuation, when he can acquire the second-period object.

By differentiating with respect to the threshold valuation  $v$  and noticing that

$$\beta^{2,N}[v|v, r_2(v)] = \beta^1(v|v) = r_1(v)$$

and that

$$\beta^{2,N}[r_2(v)|v, r_2(v)] = r_2(v),$$

we get

$$\begin{aligned} \frac{dR}{dv}(v) &= \int_v^{\bar{v}} \frac{d\beta^1}{dv}(v_1 | r_1(v)) N G(v_1) f(v_1) dv_1 \\ &+ N G(r_2(v)) f(r_2(v)) r_2(v) \frac{dr_2}{dv}(v) \\ &+ \int_{r_2(v)}^v \frac{d\beta^{2,N}}{dv}[v_1 | v, r_2(v)] N G(v_1) f(v_1) dv_1 \\ &- \int_{r_2(v)}^v \beta^{2,N-1}[v_2 | v, r_2(v)] N g(v_2) f(v) dv_2 \end{aligned}$$

Notice that

$$\beta^1(v_1 | r_1(v)) = \frac{1}{G(v_1)} \left[ \int_v^{v_1} \beta^{2,N-1}[u | u, r_2(u)] g(u) du + \frac{1}{G(v_1)} r_1(v) \right];$$

$$\frac{d\beta^1}{dv}(v_1 | r_1(v)) = \frac{1}{G(v_1)} \left[ -g(v) \beta^{2,N-1}[v | v, r_2(v)] + g(v) r_1(v) + G(v) \frac{dr_1}{dv}(v) \right];$$

$$\frac{d\beta^{2,N}}{dv}[v_1 | v, r_2(v)] = \frac{G(r_2(v))}{G(v_1)} \frac{dr_2}{dv}(v).$$

Therefore, we have

$$\begin{aligned} \frac{dR}{dv}(v) &= N [1 - F(v)] \left[ g(v) r_1(v) + G(v) \frac{dr_1}{dv}(v) \right] \\ &\quad - N [1 - F(v)] g(v) \beta^{2,N-1}[v | v, r_2(v)] \\ &\quad - N G[r_2(v)] f[r_2(v)] r_2(v) \frac{dr_2}{dv}(v) \\ &\quad + N [F(v) - F(r_2(v))] G[r_2(v)] \frac{dr_2}{dv}(v) \\ &\quad - N f(v) \int_{r_2(v)}^v \beta^{2,N-1}[v_2 | v, r_2(v)] g(v_2) dv_2. \end{aligned}$$

Since  $\psi(r_2(v) | v) = 0$  and, therefore,  $F(v) - F[r_2(v)] = f[r_2(v)] r_2(v)$ , as in equation (3), we can simplify this sum by eliminating its third and fourth terms. In addition, by substituting the expressions

$$\begin{aligned} r_1(v) &= \beta^{2,N}[v | v, r_2(v)] \\ &= \frac{1}{G(v)} \left[ \int_{r_2(v)}^{v_1} u g(u) du + G(r_2(v)) r_2(v) \right], \end{aligned}$$

$$\begin{aligned} \frac{dr_1}{dv}(v) &= \frac{1}{G(v)} \left[ -\frac{g(v)}{G(v)} \int_{r_2(v)}^{v_1} u g(u) du + g(v) v \right. \\ &\quad \left. - \frac{g(v)}{G(v)} G(r_2(v)) r_2(v) + G(r_2(v)) \frac{dr_2}{dv}(v) \right], \end{aligned}$$

$$\begin{aligned} g(u) \beta^{2,N-1}[u; v, r_2(v)] &= \int_{r_2(v)}^u w (N-1)(N-2)F(w)^{N-3} f(w) dw \\ &\quad + (N-1)F(r_2(u))^{N-2} r_2(u) f(u), \end{aligned}$$

we get

$$\begin{aligned}
\frac{dR}{dv}(v) &= N[1 - F(v)] \left[ g(v)v + G(r_2(v)) \frac{dr_2}{dv}(v) \right] \\
&- N(N-1)f(v) [1 - F(r_2(v))] F(r_2(v))^{N-2} r_2(v) \\
&- N(N-1)f(v) [1 - F(v)] \int_{r_2(v)}^v w (N-2) F(w)^{N-3} f(w) dw \\
&- N(N-1)f(v) \int_{r_2(v)}^v f(v_2) \int_{r_2(v)}^{v_2} w (N-2) F(w)^{N-3} f(w) dw dv_2.
\end{aligned}$$

By integrating the outer integral in the last term by parts and collecting terms together, we get

$$\begin{aligned}
\frac{dR}{dv}(v) &= N[1 - F(v)] \left[ g(v)v + G(r_2(v)) \frac{dr_2}{dv}(v) \right] \\
&- N(N-1)f(v) [1 - F(r_2(v))] F(r_2(v))^{N-2} r_2(v) \\
&- N(N-1)f(v) \int_{r_2(v)}^v w [1 - F(w)] (N-2) F(w)^{N-3} f(w) dw.
\end{aligned}$$

By integrating again by parts, with terms  $w[1 - F(w)]$  and  $(N-2)F(w)^{N-3}f(w)$ ,

$$\begin{aligned}
\frac{dR}{dv}(v) &= N[1 - F(v)] G(r_2(v)) \frac{dr_2}{dv}(v) \\
&+ N(N-1)f(v) \int_{r_2(v)}^v [1 - F(u) - uf(u)] F(u)^{N-2} du.
\end{aligned}$$

Finally, by introducing the virtual valuation function  $\psi(v) = v - \frac{1-F(v)}{f(v)}$ , the derivative becomes

$$\frac{dR}{dv}(v) = N[1 - F(v)] G(r_2(v)) \frac{dr_2}{dv}(v) - Nf(v) \int_{r_2(v)}^v \psi(u) g(u) du.$$

It remains to show that the FOC,  $\frac{dR}{dv}(v) = 0$ , has a root  $v^*$  corresponding to a maximum.

First, suppose that  $r_0 > \underline{v}$ . If  $v \in [\underline{v}, r_0]$ , then, since the function  $\psi(v)$  is increasing, we have  $\psi(u) < 0$  for all  $u \in [\underline{v}, v]$ . Therefore, the derivative  $\frac{dR}{dv}(v) > 0$ , for all  $v \in [\underline{v}, r_0]$ , implying that  $v^* > r_0$ . By the Intermediate Value Theorem, since  $\frac{dR}{dv}(r_0) > 0$  and  $\frac{dR}{dv}(\bar{v}) < 0$ , we conclude that there exists a value  $v^* \in (r_0, \bar{v})$  for which the seller's revenue function  $R(v)$  attains its maximum.

Now, suppose that  $r_0 = \underline{v}$ . Then, for all  $v \in [\underline{v}, \bar{v}]$ , we have  $r_2(v) = \underline{v}$ , implying that  $\frac{dR}{dv}(v) < 0$ . Hence, in this case,  $v^* = \underline{v}$ , as asserted.

□

**Proof of Lemma 7:**

The argument parallels that in the proof of Lemma 3. Thus, in any symmetric monotone equilibrium, the strategies  $\beta^2$  and  $r_2$  must be as described in Lemma 1; and the bidding strategies  $\beta^1$ ,  $\beta_L^1$  and  $\beta_H^1$  must be as in equations (10)-(12) which are derived from the necessary FOC's.

First, let  $r_1 \leq \bar{r}_1$  be such that  $\underline{v}(r_1) \geq r_0$ . Consider a bidder  $i$  has valuation  $v_i \geq \underline{v}(r_1)$ . As in the proof of Lemma 3, since

$$\beta^{2,N}[\underline{v}(r_1) | \underline{v}(r_1), r_2(\underline{v}(r_1))] = r_1 = \beta^1(\underline{v}(r_1) | r_1).$$

bidder  $i$  is indifferent between abstaining from the first auction and mimicking the type  $\underline{v}(r_1)$ ; and he is better-off bidding  $\beta^1(v_i | r_1)$ , as prescribed.

Now, consider a bidder  $i$  with valuation  $v_i < \underline{v}(r_1)$ . Since  $\Pi[\tilde{v}_i, v_i] < \Pi[\underline{v}(r_1), v_i]$ , for all  $\tilde{v}_i > \underline{v}(r_1)$ , it suffices to show that bidder  $i$  cannot profit from mimicking the type  $\underline{v}(r_1)$ . As in the proof of Lemma 3, conditional on the first unit remaining unsold, bidder  $i$ 's gain from mimicking the type  $\underline{v}(r_1)$  in the first auction is equal to his gain from mimicking the same type in the second auction; which is negative, since the strategy  $\beta^{2,N}$  forms a symmetric equilibrium for the continuation game.

Similar arguments establish the result for  $r_1 \leq \bar{r}_1$  be such that  $\underline{v}(r_1) \leq r_0$ .

Finally, for any reserve price  $r_1 > \bar{r}_1$ , any bidder  $i$  who deviates into submitting a first-period bid  $\tilde{b}_i^1 \geq r_1$  will be worse off than abstaining from the first auction and bidding  $\bar{r}_1 = \beta^{2,N}(\bar{v} | \bar{v}, r_2(\bar{v}))$  in the second auction. Again, since the strategy  $\beta^{2,N}$  constitutes an equilibrium for the second auction, following no sale, this deviation is not profitable.

□



### Proof of Proposition 8

Suppose that all bidders other than bidder  $i$  use the bidding strategies  $\beta^1$ ,  $\beta_L^1$ ,  $\beta_H^1$  and  $\beta^2$ , given by the equations (10)-(12) and (4); while the seller follows the strategy  $r_2$ , described by equation (2). We will show that bidder  $i$  with valuation  $v_i \in [\underline{v}, \bar{v}]$  is best-off also bidding according to the strategies  $\beta^1$ ,  $\beta_L^1$ ,  $\beta_H^1$  and  $\beta^2$ .

The optimality of the strategy  $\beta^2$  follows from Lemma 1. So, we only to establish the optimality of the strategies  $\beta^1$ ,  $\beta_L^1$  and  $\beta_H^1$ .

First, suppose that the first-period reserve price is  $r_1 < \bar{r}_1$  such that  $\underline{v}(r_1) > r_0$ . Suppose bidder  $i$  has valuation  $v_i \in [\underline{v}(r_1), \bar{v}]$ . Then his profit from mimicking a type  $\tilde{v}_i > v_i$  is

$$\begin{aligned} \Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) &= [F(\tilde{v}_i)^{N-1} - F(v_i)^{N-1}] v_i \\ &\quad - (N-1) [F(\tilde{v}_i) - F(v_i)] F(v_i)^{N-2} v_i \\ &\quad - [F(\tilde{v}_i)^{N-1} \beta^1(\tilde{v}_i | r_1) - F(v_i)^{N-1} \beta^1(v_i | r_1)] \\ &\quad + (N-1) [F(\tilde{v}_i) - F(v_i)] F(v_i)^{N-2} \beta^{2,N-1}(v_i | \bar{v}, r_0) \end{aligned}$$

or after using equation (10), defining the bidding strategy  $\beta^1$ ,

$$\begin{aligned} \Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) &= [F(\tilde{v}_i)^{N-1} - F(v_i)^{N-1}] v_i \\ &\quad - (N-1) [F(\tilde{v}_i) - F(v_i)] F(v_i)^{N-2} v_i \\ &\quad - \int_{v_i}^{\tilde{v}_i} \beta^{2,N-1}(\hat{v} | \bar{v}, r_0) (N-1) F(\hat{v})^{N-2} f(\hat{v}) d\hat{v} \\ &\quad + (N-1) [F(\tilde{v}_i) - F(v_i)] F(v_i)^{N-2} \beta^{2,N-1}(v_i | \bar{v}, r_0). \end{aligned}$$

By using equation (4), defining the strategy  $\beta^{2,N-1}$  we get

$$\begin{aligned} \Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) &= [F(\tilde{v}_i)^{N-1} - F(v_i)^{N-1}] v_i \\ &\quad - (N-1) [F(\tilde{v}_i) - F(v_i)] F(v_i)^{N-2} v_i \\ &\quad - \int_{v_i}^{\tilde{v}_i} (N-1) f(\hat{v}) \int_{r_0}^{\hat{v}} u (N-2) F(u)^{N-3} f(u) du d\hat{v} \\ &\quad + (N-1) [F(\tilde{v}_i) - F(v_i)] \int_{r_0}^{v_i} u (N-2) F(u)^{N-3} f(u) du. \end{aligned}$$

or

$$\begin{aligned}
\Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) &= [F(\tilde{v}_i)^{N-1} - F(v_i)^{N-1}] v_i \\
&- (N-1) [F(\tilde{v}_i) - F(v_i)] F(v_i)^{N-1} v_i \\
&- \int_{v_i}^{\tilde{v}_i} \int_{v_i}^{\hat{v}} u (N-1)(N-2) F(u)^{N-3} f(u) f(\hat{v}) du d\hat{v},
\end{aligned}$$

which has been shown to be negative in the proof of Proposition 4.

Similarly, deviations to mimicking a type  $\tilde{v}_i \in (\underline{v}(r_1), v_i)$  lead to profit

$$\Pi(\tilde{v}_i; v_i) - \Pi(v_i; v_i) = - \int_{\tilde{v}_i}^{v_i} \int_{\hat{v}}^{v_i} u (N-1)(N-2) F(u)^{N-3} f(u) f(\hat{v}) du d\hat{v},$$

which is clearly negative.

Second, for reserve prices  $r_1 < \bar{r}_1$  such that  $\underline{v}(r)_1 < r_0$ , we can rule out deviations from bidding  $b_i^1 = \beta_H^1(v_i|r_1)$ , for  $v_i \in [r_0, \bar{v}]$ , to  $\tilde{b}_i^1 = \beta_H^1(\tilde{v}_i|r_1)$ , for  $\tilde{v}_i \in [r_0, \bar{v}]$ ,  $\tilde{v}_i \neq v_i$ , by replicating the above argument.

Similarly, we can rule out deviations from bidding  $b_i^1 = \beta_L^1(v_i|r_1)$ , for  $v_i \in [\underline{r}_1, r_0]$ , to  $\tilde{b}_i^1 = \beta_L^1(\tilde{v}_i|r_1)$ , for  $\tilde{v}_i \in [\underline{r}_1, r_0]$ ,  $\tilde{v}_i \neq v_i$ , by replicating the argument establishing the symmetric equilibrium for the single-unit first-price auction.

Finally, we can rule out deviations involving types  $v_i \in [r_0, \bar{v}]$  and  $v_i \in [\underline{r}_1, r_0]$  or types  $v_i \in [\underline{r}_1, r_0]$  and  $v_i \in [r_0, \bar{v}]$ , by noticing that  $\beta_H^1(r_0|r_1) = \beta_L^1(r_0|r_1)$ .

Third, for all reserve prices  $r_1 < \bar{r}_1$ , the deviations to abstaining, for types  $v_i \geq \underline{v}(r_1)$ , and to bidding, for types  $v_i < \underline{v}(r_1)$ , can be ruled out by replicating the argument in Lemma (7) and utilizing the above results.

Finally, for any reserve price  $r_1 \geq \bar{r}_1$ , the deviation to bidding can be ruled out by noticing, as in the proof of Proposition 4, that  $r_1 \geq \beta^{2,N}(\bar{v}, \bar{v}, r_0)$ , by equation (7).

The uniqueness of the constructed equilibrium follows from Lemmata 1 and 7 and the necessity of the FOC establishing the strategies  $\beta^1$ ,  $\beta_H^1$  and  $\beta_L^1$ .

□

### Proof of Proposition 9

The argument establishing the first-order condition (13) parallels that in the proof of Proposition 5, with the terms  $r_0$  and  $\beta^{2,N-1}(v | \bar{v}, r_0)$  replacing respectively the terms  $r_2(v)$  and  $\beta^{2,N-1}(v | v, r_2(v))$  in the description of the seller's second-period payoffs, following a first period sale.

Since the function  $r_1(v) = \beta^{2,N}(v | v, r_2(v))$  is strictly increasing in  $v \in [\underline{v}, \bar{v}]$ , it follows that there is a bijection between reserve prices  $r_1 \in [\underline{v}, \bar{r}_1]$  and the induced participation thresholds  $v = \underline{v}(r_1) \in [\underline{v}, \bar{v}]$ . Therefore, we can think of the seller's problem as that of choosing the optimal threshold  $v^*$ .<sup>33</sup>

The seller's revenue from a threshold  $v \geq r_0$  is

$$\begin{aligned} R(v) &= \int_v^{\bar{v}} \beta^1(v_1 | r_1(v)) N f(v_1) G(v_1) dv_1 \\ &+ \int_{r_0(v)}^v \beta^{2,N}[v_1 | v, r_0(v)] N f(v_1) G(v_1) dv_1 \\ &+ \int_v^{\bar{v}} \int_{r_0}^{v_1} \beta^{2,N-1}[v_2 | v_1, r_0] N f(v_1) g(v_2) dv_2 dv_1. \end{aligned}$$

By differentiating with respect to the threshold valuation  $v$  and noticing that

$$\beta^{2,N}[v | v, r_0(v)] = \beta^1(v | r_1(v)) = r_1(v)$$

and that

$$\beta^{2,N}[r_0(v) | v, r_0(v)] = r_0(v),$$

we get

$$\begin{aligned} \frac{dR}{dv}(v) &= \int_v^{\bar{v}} \frac{d\beta^1}{dv}(v_1 | r_1(v)) N G(v_1) f(v_1) dv_1 \\ &+ N G(r_0(v)) f(r_0(v)) r_0(v) \frac{dr_0}{dv}(v) \\ &+ \int_{r_0(v)}^v \frac{d\beta^{2,N}}{dv}[v_1 | v, r_0(v)] N G(v_1) f(v_1) dv_1 \\ &- \int_{r_0}^v \beta^{2,N-1}[v_2 | \bar{v}, r_0] N g(v_2) f(v) dv_2 \end{aligned}$$

Since

$$\frac{d\beta^1}{dv}(v_1 | r_1(v)) = \frac{1}{G(v_1)} \left[ -g(v) \beta^{2,N-1}[v | \bar{v}, r_0] + g(v) r_1(v) + G(v) \frac{dr_1}{dv}(v) \right];$$

and

$$\frac{d\beta^{2,N}}{dv}[v_1 | v, r_2(v)] = \frac{G(r_2(v))}{G(v_1)} \frac{dr_2}{dv}(v),$$

---

<sup>33</sup>Indirectly, the threshold  $v^* = \bar{v}$  covers the possibility of the seller deciding not to sell the first unit.

we have

$$\begin{aligned}
\frac{dR}{dv}(v) &= N [1 - F(v)] \left[ g(v) r_1(v) + G(v) \frac{dr_1}{dv}(v) \right] \\
&- N [1 - F(v)] g(v) \beta^{2,N-1}[v | \bar{v}, r_0] \\
&- N G[r_0(v)] f[r_0(v)] r_0(v) \frac{dr_0}{dv}(v) \\
&+ N [F(v) - F(r_0(v))] G[r_0(v)] \frac{dr_0}{dv}(v) \\
&- N f(v) \int_{r_0}^v \beta^{2,N-1}[v_2 | \bar{v}, r_0] g(v_2) dv_2.
\end{aligned}$$

Using equation (3), we can simplify this sum by eliminating its third and fourth terms. In addition, by substituting the expressions

$$\begin{aligned}
r_1(v) &= \beta^{2,N}[v | v, r_0(v)] \\
&= \frac{1}{G(v)} \left[ \int_{r_0(v)}^{v_1} u g(u) du + G(r_0(v)) r_0(v) \right], \\
\frac{dr_1}{dv}(v) &= \frac{1}{G(v)} \left[ -\frac{g(v)}{G(v)} \int_{r_0(v)}^{v_1} u g(u) du + g(v) v \right. \\
&\quad \left. - \frac{g(v)}{G(v)} G(r_0(v)) r_0(v) + G(r_0(v)) \frac{dr_0}{dv}(v) \right], \\
g(u) \beta^{2,N-1}[u; \bar{v}, r_0] &= \int_{r_0}^u w (N-1)(N-2)F(w)^{N-3} f(w) dw \\
&+ (N-1)F(r_0)^{N-2} r_0 f(u),
\end{aligned}$$

we get

$$\begin{aligned}
\frac{dR}{dv}(v) &= N [1 - F(v)] \left[ g(v) v + G(r_0(v)) \frac{dr_0}{dv}(v) \right] \\
&- N (N-1) f(v) [1 - F(r_0)] F(r_0)^{N-2} r_0 \\
&- N (N-1) f(v) [1 - F(v)] \int_{r_0}^v w (N-2) F(w)^{N-3} f(w) dw \\
&- N (N-1) f(v) \int_{r_0}^v f(v_2) \int_{r_0}^{v_2} w (N-2) F(w)^{N-3} f(w) dw dv_2.
\end{aligned}$$

By integrating by parts the outer integral in the last term and collecting terms together, we get

$$\begin{aligned}\frac{dR}{dv}(v) &= N[1 - F(v)] \left[ g(v)v + G(r_0(v)) \frac{dr_0}{dv}(v) \right] \\ &\quad - N(N-1)f(v)[1 - F(r_0)]F(r_0)^{N-2}r_0 \\ &\quad - N(N-1)f(v) \int_{r_0}^v w[1 - F(w)](N-2)F(w)^{N-3}f(w)dw.\end{aligned}$$

By integrating again by parts, with terms  $w[1 - F(w)]$  and  $(N-2)F(w)^{N-3}f(w)$ ,

$$\begin{aligned}\frac{dR}{dv}(v) &= N[1 - F(v)]G(r_0(v))\frac{dr_0}{dv}(v) \\ &\quad + N(N-1)f(v) \int_{r_0}^v [1 - F(u) - uf(u)]F(u)^{N-2}du.\end{aligned}$$

Finally, by introducing the virtual valuation function  $\psi(v) = v - \frac{1-F(v)}{f(v)}$ , the derivative becomes

$$\frac{dR}{dv}(v) = N[1 - F(v)]G(r_0(v))\frac{dr_0}{dv}(v) - Nf(v) \int_{r_0}^v \psi(u)g(u)du.$$

The argument showing that the FOC,  $\frac{dR}{dv}(v) = 0$ , has a root  $v^* > r_0$ , corresponding to a local maximum, is identical to that in the proof of Proposition 5; it is therefore omitted.

To compare the expected seller revenues from observing the first-period winning bid,  $R^1$ , and whether the first item is sold,  $R^s$ , we will prove a stronger result, namely, that  $R^1(v) > R^s(v)$ , for all first-period participation thresholds  $v = \underline{v}(r_1^1) = \underline{v}(r_1^s) \in [r_0, \bar{v}]$ .

From the first-order conditions (9) and (13), it follows that, for all  $v \in [r_0, \bar{v}]$ ,

$$\frac{dR^s}{dv}(v) - \frac{dR^1}{dv}(v) = Nf(v) \int_{r_2(v)}^{r_0} \psi(u)g(u)du > 0.$$

Since  $R^1(\bar{v}) = R^s(\bar{v})$ , we can conclude that  $R^1(v) > R^s(v)$ , for all  $v \in [r_0, \bar{v}]$ .

□

**Proof of Proposition 10:**

Suppose that there exists a perfect Bayesian equilibrium  $[(\beta_i^1, \beta_i^2)_{i=1}^N, (r_1, r_2)]$  such that the first-period bidding strategies are symmetric and increasing in the valuation  $v_i$ . To derive a contradiction, it suffices to consider the restriction of this equilibrium to the continuation game following a first-period reserve price  $r_1 = \underline{v}$ , independently of whether this is on the equilibrium path or not. In this continuation game, all bidder types participate to the first auction, possibly bidding  $\underline{v}$ , and have beliefs identical to the ones they had prior the seller's announcement of  $r_1$ .

First, we rule out the existence of an equilibrium involving first-period strategies  $\beta^1(\cdot | \underline{v})$  that are strictly increasing in  $[\underline{v}, \bar{v}]$ . Since such strategies perfectly reveal the bidders' valuations, the bidders expect to make zero profit in the second auction. Hence, on the strategy-realization path, they treat the problem as that of bidding in a single-period, single-unit auction. According to the symmetric equilibrium of the first-price auction, we must have

$$\beta^1(v_i | \underline{v}) = \mathbb{E}[v_1^{(N-1)} | v_1^{(N-1)} < v_i].$$

By splitting cases, according to the realized type-profile, we claim that each bidder  $i$  has a profitable deviation to  $\tilde{\beta}^1(v_i | \underline{v}) \equiv \underline{v}$ .

Notice that, on the equilibrium path of the continuation game, the second-period reserve price  $r_2$  will be

$$r_2(b_1^1, \dots, b_N^1 | r_1 = \underline{v}) = [\beta^1(\cdot | r_1)]^{-1}(b_{(2,N)}^1),$$

where  $b_{(2,N)}^1$  is the second-highest bid submitted in the first auction.

If bidder  $i$ , with valuation  $v_i$ , turns out to have the highest valuation, then he can still win an object, in the second auction, at an expected price

$$p_2 = \mathbb{E}[v_2^{(N-1)} | v_1^{(N-1)} < v_i] + \varepsilon,$$

for any  $\varepsilon > 0$ . That is, bidder  $i$  can win the second-period object at a price equal to the second-highest valuation among his competitors, plus any positive amount  $\varepsilon$  to avoid the tie. For sufficiently small  $\varepsilon > 0$ , the expected price  $p_2$  is lower than  $\beta^1(v_i | \underline{v})$ , the price bidder  $i$  will pay, if he wins the object in the first auction.

If bidder  $i$  turns out to have the second-highest valuation, then he will again win an object in the second auction, at an expected price

$$p_2 = \mathbb{E}[v_2^{(N-1)} | v_2^{(N-1)} < v_i < v_1^{(N-1)}] + \varepsilon,$$

for any  $\varepsilon > 0$ . Again, for sufficiently small  $\varepsilon > 0$ , the expected price  $p_2$  is strictly smaller than  $v_i$ , the price bidder  $i$  would pay in the second auction if he revealed his valuation.

Finally, if bidder  $i$  turns out to have the third-highest, or lower, valuation, then he will win no object, as he would do after bidding  $\beta^1(v_i | r_1)$ .

We conclude the proof by ruling out the existence of intervals of non-increase in  $[\underline{v}, \bar{v}]$ . Suppose that  $\beta^1(v_i | \underline{v}) = b$ , for all  $v_i \in [v_L, v_H] \subset [\underline{v}, \bar{v}]$ . If  $b > v_L$ , then any bidder  $i$  with type  $v_i \in V_1 = [v_L, \min\{b, v_H\}]$  is better off bidding  $\tilde{b}^1(v_i | \underline{v}) = v_i$ , a bid that avoids winning the object at a price above  $v_i$ . If  $b < v_L$ , then there is an  $\varepsilon > 0$  sufficiently small such that the deviation to the strategy  $\beta^1(v_i | \underline{v}) = b + \varepsilon$ , for all  $v_i \in [v_L, v_H]$  is profitable. Finally, if  $b = v_L$ , then we can simply apply the argument for  $b < v_L$  to the interval  $[\frac{1}{2}(v_L + v_H), v_H]$ . Hence, there cannot exist an interval of non-increase of  $\beta^1(\cdot | \underline{v})$ .

□

### Proof of Proposition 11:

The argument essentially extends the proof of Proposition 10. Again, we rule out the existence of a perfect Bayesian equilibrium in the continuation game following a reserve price  $r_1 = \underline{v}$ , with bidder beliefs identical to the ones prior to the announcement of  $r_1$ .

If the strategy  $\beta^1(\cdot | \underline{v})$  is strictly increasing, then the seller learns all valuations at the end of the first auction. Therefore, the reserve price in the second auction is

$$r_2(b_1^1, \dots, b_N^1 | r_1 = \underline{v}) = \begin{cases} \underline{v}, & \text{with probability } 1 - \rho; \\ [\beta^1(\cdot | \underline{v})]^{-1}(b_{(2,N)}^1), & \text{with probability } \rho, \end{cases}$$

where  $b_{(2,N)}^1$  is the second-highest bid submitted in the first auction.

In the second auction, on the realization path of  $(\beta^1, r_2)$ , each bidder  $i$  will bid, with probability  $1 - \rho$ ,

$$\beta^2(v_i | \hat{v}, \underline{v}) = \mathbb{E}[v_1^{(N-2)} | v_1^{(N-2)} \leq v_i];$$

and with probability  $\rho$ , either  $r_2$ , if  $v_i = r_2$  or  $a$ , if  $v_i < r_2$ . Therefore, in the second auction, with probability  $\rho$ , a bidder expects to derive no gain.

In the first auction, to avoid deviations to mimicking a type  $\tilde{v}_i \neq v_i$  during that round and bidding according to the strategy described above during the next round, it is necessary that

$$\beta^1(v_i | \underline{v}) = \mathbb{E}[(1 - \rho)v_1^{(n-1)} + \rho\beta^2(v_1^{(n-1)} | v_1^{(n-1)}, \underline{v})].$$

It is straightforward to show that if bidder  $i$ , with valuation  $v_i$ , deviates to bidding  $\beta_i^1 = \underline{v}$ , then he can win the second object at the expected price

$$p_2 = \mathbb{E}[v_2^{(N-1)} \mid v_1^{(N-1)} < v_i] + \varepsilon,$$

for any  $\varepsilon > 0$ , if he turns out to have the highest valuation among the  $N$  bidders; or at the expected price

$$p_2 = \mathbb{E}[v_2^{(N-1)} \mid v_2^{(N-1)} < v_i < v_1^{(N-1)}] + \varepsilon,$$

for any  $\varepsilon > 0$ , if he turns out to have the second-highest valuation; and his payoff from the auction will not be affected, in any other case. Therefore, for sufficiently small  $\varepsilon > 0$ , bidder  $i$ 's deviation will be profitable, ruling out the existence of a symmetric PBE in strictly increasing strategies  $\beta^1$ .

The argument extending the result to weakly increasing strategies  $\beta^1$  is identical to that used in the proof of Proposition 10 and, therefore, omitted.

□



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